ERRATUM TO “A FOOTNOTE TO A THEOREM OF KAWAMATA”

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Abstract. We give an alternative proof of Theorem A in the paper: M. Mendes Lopes, R. Pardini, S. Tirabassi, A footnote to a theorem of Kawamata. We also explain how to fill a gap in the original proof.

1. Introduction

In this paper, we give an alternative proof of the following theorem, which is the main result of [MPT]:

Theorem 1.1 (see [MPT, Theorem A])**.** *Let X be a smooth variety defined over $\mathbb{C}$ with logarithmic Kodaira dimension $\overline{\kappa}(X) = 0$ and logarithmic irregularity $\overline{q}(X) = \dim X$. Then the quasi-Albanese map $\alpha\colon X \to A$ is birational and there exists a closed subset Z of A with $\mathrm{codim}_A Z \geq 2$ such that $\alpha\colon X \setminus \alpha^{-1}(Z) \to A \setminus Z$ is proper.*

The proof given in [MPT] contains a gap, noticed by the first named author of this paper; in §3 we explain this gap and how to avoid it.

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Conventions: We work over the field $\mathbb{C}$ of complex number and we use freely Iitaka's theory of quasi-Albanese maps and logarithmic Kodaira dimension developed in [I1] and [I2] (see also [F1]).

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. The birationality of $\alpha\colon X \to A$ is a well-known theorem by Kawamata (see [K2]), hence we are going to prove the existence of the desired closed subset Z. The proof given here uses Kawamata's subadditivity formula in [K1]. Before we start the proof of Theorem 1.1, we note the following fact (see also [MPT, Lemma 2.2]):

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Remark 2.1 (Log canonical centers)**.** Let X be a smooth variety and let Δ_X be a simple normal crossings divisor on X, so that (X, Δ_X) is log canonical. Let $\Delta_X = \sum_{i\in I} \Delta_i$ be the irreducible decomposition of Δ_X; then a closed subset W of X is a log canonical center of (X, Δ_X) if and only if W is an irreducible component of $\Delta_{i_1} \cap \ldots \cap \Delta_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$. When $\Delta_{i_1} \cap \ldots \cap \Delta_{i_k}$ is connected for every $\{i_1, \ldots, i_k\} \subset I$, we say that Δ_X is a *strong normal crossings divisor* in the terminology of [MS].

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\alpha\colon X \to A$ be the quasi-Albanese map. By Kawamata's theorem (see [K2, Corollary 29] and [F1, Corollary 10.2]), we see that α is birational.

Step 1. Let

$$0 \longrightarrow \mathbb{G}_m^d \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

be the Chevalley decomposition. Then A is a principal $\mathbb{G}_m^d$-bundle over an abelian variety B in the Zariski topology (see, for example, [BCM, Theorems 4.4.1 and 4.4.2]) and there is a natural completion $\overline{\pi}\colon \overline{A} \to B$ of $\pi\colon A \to B$ where $\overline{A}$ is a $\mathbb{P}^d$-bundle. We set $\Delta_{\overline{A}} := \overline{A} \setminus A$; then $\Delta_{\overline{A}}$ is a simple normal crossings divisor on $\overline{A}$, and $(\overline{A}, \Delta_{\overline{A}})$ is a log canonical pair.

Let $\overline{\alpha}\colon \overline{X} \to \overline{A}$ be a compactification of $\alpha\colon X \to A$, that is, $\overline{X}$ is a smooth complete algebraic variety containing X, $\Delta_{\overline{X}} := \overline{X} \setminus X$ is a simple normal crossing divisor on $\overline{X}$ and $\overline{\alpha}$ is a morphism extending α.

Claim. *Let D be an irreducible component of $\Delta_{\overline{X}}$ such that $\overline{\alpha}(D)$ is a divisor. Then $\overline{\pi}\colon \overline{\alpha}(D) \to B$ is dominant.*

Proof of Claim. We set $D_1 := \overline{\alpha}(D)$. If $\overline{\pi}\colon D_1 \to B$ is not dominant, then we can write $D_1 = \overline{\pi}^* D_2$ for some prime divisor D_2 on B. By Remark 2.1 every log canonical center of $(\overline{A}, \Delta_{\overline{A}})$ dominates B, therefore D_1 does not contain any log canonical center (so in particular it is not a component of $\Delta_{\overline{A}}$). Thus we have

$$\mathrm{mult}_D\left(K_{\overline{X}} + \Delta_{\overline{X}} - \overline{\alpha}^*(K_{\overline{A}} + \Delta_{\overline{A}})\right) = 1.$$

Let E be any $\overline{\alpha}$-exceptional divisor on $\overline{X}$ such that $\overline{\alpha}(E)$ is not a log canonical center of $(\overline{A}, \Delta_{\overline{A}})$. Then

$$\mathrm{mult}_E\left(K_{\overline{X}} + \Delta_{\overline{X}} - \overline{\alpha}^*(K_{\overline{A}} + \Delta_{\overline{A}})\right) \geq 1$$

holds since $K_{\overline{A}} + \Delta_{\overline{A}}$ is Cartier and $\mathrm{Supp}\overline{\alpha}^*\Delta_{\overline{A}} \subset \mathrm{Supp}\Delta_{\overline{X}}$. Hence

$$K_{\overline{X}} + \Delta_{\overline{X}} - \overline{\alpha}^*(K_{\overline{A}} + \Delta_{\overline{A}}) \geq \varepsilon\overline{\alpha}^* D_1$$

for some $0 < \varepsilon \ll 1$ since the support of D_1 does not contain any log canonical center of $(\overline{A}, \Delta_{\overline{A}})$. By construction, we have $K_{\overline{A}} + \Delta_{\overline{A}} \sim 0$. Thus we obtain

$$0 = \overline{\kappa}(X) = \kappa(\overline{X}, K_{\overline{X}} + \Delta_{\overline{X}}) \geq \kappa(\overline{X}, \overline{\alpha}^* D_1) = \kappa(\overline{A}, D_1) = \kappa(B, D_2) > 0,$$

where the last inequality follows from the fact that D_2 is a nonzero effective divisor on the abelian variety B. This contradiction proves the claim. □

Step 2. We assume that there exists an irreducible component D of $\Delta_{\overline{X}}$ such that $\overline{\alpha}(D)$ is a divisor with $\overline{\alpha}(D) \not\subset \overline{A} \setminus A$ and we set $D' := \overline{\alpha}(D) \cap A$. By the Claim in Step 1, D' dominates B, therefore we can find a subgroup $\mathbb{G}_m$ of A such that $\varphi|_{D'}\colon D' \to A_1$ is dominant, where

$$0 \longrightarrow \mathbb{G}_m \longrightarrow A \xrightarrow{\varphi} A_1 \longrightarrow 0.$$

Note that A is a principal $\mathbb{G}_m$-bundle over A_1 in the Zariski topology (see, for example, [BCM, Theorems 4.4.1 and 4.4.2]). We take a compactification

$$f^\dagger\colon X^\dagger \xrightarrow{\alpha^\dagger} A^\dagger \xrightarrow{\varphi^\dagger} A_1^\dagger$$

of

$$f\colon X \xrightarrow{\alpha} A \xrightarrow{\varphi} A_1,$$

where $X^\dagger$, $A^\dagger$, and $A_1^\dagger$ are smooth complete algebraic varieties such that $X^\dagger \setminus X$, $A^\dagger \setminus A$, and $A_1^\dagger \setminus A_1$ are simple normal crossing divisors. We note that $\overline{A}$ never coincides with $A^\dagger$ when $A_1 \neq B$. The general fiber of $f^\dagger$ is obviously $\mathbb{P}^1$ by construction. Let F be a general fiber of f. Since $\varphi|_{D'}\colon D' \to A_1$ is dominant, we have $\#(\mathbb{P}^1 \setminus F) \geq 3$. This implies $\overline{\kappa}(F) = 1$. Note that A_1 is a quasi-abelian variety, hence we have $\overline{\kappa}(A_1) = 0$. By Kawamata's theorem (see [K1, Theorem 1] and [F2, Chapter 8]), we obtain

$$0 = \overline{\kappa}(X) \geq \overline{\kappa}(F) + \overline{\kappa}(A_1) = 1.$$

This is a contradiction, showing that every irreducible component of $\Delta_{\overline{X}}$ which is not contracted by $\overline{\alpha}$ is mapped to $\Delta_{\overline{A}}$. Let Δ' be the sum of the irreducible components of $\Delta_{\overline{X}}$ which are not mapped to $\Delta_{\overline{A}}$. It is obvious that $\operatorname{codim}_{\overline{A}}\overline{\alpha}(\Delta') \geq 2$ holds since Δ' is $\overline{\alpha}$-exceptional. We put $Z := \overline{\alpha}(\Delta') \cap A$. Then Z is a closed subset of A with $\operatorname{codim}_A Z \geq 2$. It is easy to see that

$$\alpha\colon X \setminus \alpha^{-1}(Z) \to A \setminus Z$$

is proper.

This finishes the proof of Theorem 1.1. □

3. On the original proof in [MPT]

The final part (§3.2) of the original proof of Theorem 1.1 given in [MPT] contains the unsubstantiated claim that, if $\overline{X}$ contains a divisor contracted to a log canonical center W of $(\overline{A}, \Delta_{\overline{A}})$, then $\overline{\alpha}$ factors through the blow-up of $\overline{A}$ along W. This claim is then used to reduce the proof to the special case treated in §3.1 of [MPT].

We explain here how to reduce the proof to the special case by a different argument.

Let T be a smooth variety and let Δ_T be a simple normal crossing divisor on T. Let $f\colon T' \to T$ be the blow-up of a log canonical center of (T, Δ_T). We set

$$K_{T'} + \Delta_{T'} := f^*(K_T + \Delta_T).$$

Then it is easy to see that T' is smooth and $\Delta_{T'}$ is a simple normal crossing divisor on T'. We call $f\colon (T', \Delta_{T'}) \to (T, \Delta_T)$ a *crepant pull-back* of (T, Δ_T).

Now, with the same notation and assumptions of §2, let D be a component of Δ_X such that $\overline{\alpha}(D)$ is a divisor not contained in $\Delta_{\overline{A}}$. In the terminology of [MS] the divisor $\Delta_{\overline{A}}$ is a strong normal crossings divisor (see Remark 2.1). Then we are precisely in the situation of §5.1 of [MS] and by Corollary 2, ibid., there is a finite sequence of crepant pull-backs

$$(\overline{A}, \Delta_{\overline{A}}) =: (T_0, \Delta_{T_0}) \xleftarrow{f_1} (T_1, \Delta_{T_1}) \xleftarrow{f_2} \cdots \xleftarrow{f_k} (T_k, \Delta_{T_k})$$

such that the strict transform of $\overline{\alpha}(D)$ on T_k does not contain any log canonical center of (T_k, Δ_{T_k}). In addition, the components of $\Delta_{\overline{A}}$ are preserved by the $\mathbb{G}_m^d$-action and therefore it is easy to check that $\mathbb{G}_m^d$ acts also on T_1 and the action preserves the components of Δ_{T_1}.

An inductive argument then shows that $\mathbb{G}_m^d$ acts on T_k and preserves the components of Δ_{T_k}

So, up to replacing $(\overline{A}, \Delta_{\overline{A}})$ by (T_k, Δ_{T_k}) and modifying $\overline{X}$ accordingly, we may assume that $\overline{\alpha}(D)$ does not contain any log canonical center and that the $\mathbb{G}_m^d$-action extends to $\overline{A}$. We conclude by observing that the argument in [MPT, §3.1] (the "special case") works in this situation.

References

[BCM] A. Białynicki-Birula, J. B. Carrell, W. M. McGovern, *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, Encyclopaedia of Mathematical Sciences, **131**. Invariant Theory and Algebraic Transformation Groups, II. Springer-Verlag, Berlin, 2002.

[F1] O. Fujino, On quasi-Albanese maps, preprint (2015) available at https://www.math.kyoto-u.ac.jp/ fujino/papersandpreprints.html.

[F2] O. Fujino, *Iitaka conjecture—an introduction*, SpringerBriefs in Mathematics. Springer, Singapore, 2020.

[I1] S. Iitaka, Logarithmic forms of algebraic varieties, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **23** (1976), no. 3, 525–544.

[I2] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, *Complex analysis and algebraic geometry*, pp. 175–189, Iwanami Shoten Publishers, Tokyo, 1977.

[K1] Y. Kawamata, Addition formula of logarithmic Kodaira dimensions for morphisms of relative dimension one, *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, 207–217, Kinokuniya Book Store, Tokyo, 1978.

[K2] Y. Kawamata, Characterization of abelian varieties, Compositio Math. **43** (1981), no. 2, 253–276.

[MPT] M. Mendes Lopes, R. Pardini, S. Tirabassi, A footnote to a theorem of Kawamata, Math. Nachr. **296** (2023), no. 10, 4739–4744.

[MS] B. Molina-Samper, Combinatorial aspects of classical resolution of singularities, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM **113** (2019), No. 4, 3931–3948 .

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