INJECTIVITY THEOREM FOR PSEUDO-EFFECTIVE LINE BUNDLES AND ITS APPLICATIONS

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Dedicated to Professor Ichiro Enoki on the occasion of his retirement

Abstract. We formulate and establish a generalization of Kollár's injectivity theorem for adjoint bundles twisted by suitable multiplier ideal sheaves. As applications, we generalize Kollár's torsion-freeness, Kollár's vanishing theorem, and a generic vanishing theorem for pseudo-effective line bundles. Our approach is not Hodge theoretic but analytic, which enables us to treat singular Hermitian metrics with nonalgebraic singularities. For the proof of the main injectivity theorem, we use L^2-harmonic forms on noncompact Kähler manifolds. For applications, we prove a Bertini-type theorem on the restriction of multiplier ideal sheaves to general members of free linear systems.

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1. Introduction

The Kodaira vanishing theorem [Kod] is one of the most celebrated results in complex geometry, and it has been generalized to several significant results; for example, the Kawamata–Viehweg vanishing theorem, the Nadel vanishing theorem, Kollár's injectivity theorem (see [F9, Chapter 3]). Kodaira's original proof is based on the theory of harmonic (differential) forms, and has currently been developed to two approaches from different perspectives: One is the Hodge theoretic approach, which is algebro-geometric theory based on Hodge structures and spectral sequences. The other is the transcendental approach, which is an analytic theory focusing on harmonic forms and L^2-methods for $\overline{\partial}$-equations. These approaches have been nourishing each other in the last decades.

As is well known, the Kawamata–Viehweg vanishing theorem plays a crucial role in the theory of minimal models for higher-dimensional complex algebraic varieties with only

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mild singularities. Now some generalizations of Kollár's injectivity theorem allow us to extend the framework of the minimal model program to highly singular varieties (see [A1], [A2], [EV], [F1], [F2], [F3], [F6], [F7], [F8], [F9], [F10], [F12], [F13], [F14]). The reader can find various vanishing theorems and their applications in the minimal model program in [F9, Chapters 3 and 6]. Kollár's original injectivity theorem, which is one of the most important generalizations of the Kodaira vanishing theorem, was first established by using the Hodge theory (see [Kol1]). The following theorem, which is a special case of [F9, Theorem 3.16.2], is obtained from the theory of mixed Hodge structures on cohomology with compact support.

Theorem 1.1 (Injectivity theorem for log canonical pairs)**.** *Let D be a simple normal crossing divisor on a smooth projective variety X and F be a semiample line bundle on X. Let s be a nonzero global section of a positive multiple $F^{\otimes m}$ such that the zero locus $s^{-1}(0)$ contains no log canonical centers of the log canonical pair (X, D). Then the map*

$$\times s : H^i(X, K_X \otimes D \otimes F) \to H^i(X, K_X \otimes D \otimes F^{\otimes m+1})$$

induced by $\otimes s$ is injective for every i. Here K_X denotes the canonical bundle of X.

The Hodge theoretic approach for Theorem 1.1 is algebro-geometric. For the proof, we first take a suitable resolution of singularities and then take a cyclic cover. After that, we apply the E_1-degeneration of a Hodge to de Rham type spectral sequence coming from the theory of mixed Hodge structures on cohomology with compact support. In this proof, we do not directly use analytic arguments; on the contrary, we have no analytic proof for Theorem 1.1. This indicates that a precise relation between the Hodge theoretic approach and the transcendental method is not clear yet and is still mysterious. There is room for further research from the analytic viewpoint. In this paper, we pursue the transcendental approach for vanishing theorems instead of the Hodge theoretic approach.

A transcendental approach for Kollár's important work (see [Kol1]) was first given by Enoki, which improves Kollár's original injectivity theorem to semipositive line bundles on compact Kähler manifolds as an easy application of the theory of harmonic forms. After Enoki's work, several authors obtained some generalizations of Kollár's injectivity theorem from the analytic viewpoint, based on the theory of L^2-harmonic forms (see, for example, [En], [Ta], [O3], [F4], [F5], [MaS1], [MaS2], and [MaS4]). Based on the same philosophy, it is natural to expect Theorem 1.1 to hold in the complex analytic setting. However, as we mentioned above, there is no analytic proof for Theorem 1.1. Difficulties lie in that the usual L^2-method does not work for log canonical singularities, and that no transcendental methods are corresponding to the theory of mixed Hodge structures (see [MaS8, No, LRW] for some approaches). The transcendental method often provides some powerful tools not only in complex geometry but also in algebraic geometry. Therefore it is of interest to study various vanishing theorems and related topics by using the transcendental method.

In this paper, by developing the transcendental approach for vanishing theorems, we prove Kollár's injectivity, vanishing, torsion-free theorems, and a generic vanishing theorem for $K_X \otimes F \otimes \mathcal{J}(h)$, where K_X is the canonical bundle of X, F is a pseudo-effective line bundle on X, and $\mathcal{J}(h)$ is the multiplier ideal sheaf associated with a singular Hermitian metric h. More specifically, this paper contains three main contributions: The first contribution is to prove a generalization of Kollár's injectivity theorem for adjoint bundles twisted by suitable multiplier ideal sheaves (Theorem A). The second contribution is to establish a Bertini-type theorem on the restriction of multiplier ideal sheaves (Theorem

1.10). Theorem 1.10 provides a useful tool and enables us to use the inductive argument on dimension. The third contribution is to deduce various results related to vanishing theorems as applications of Theorem 1.10 and Theorem A, (Theorems B, C, D, E, and F). Since we adopt the transcendental method, we can formulate all the results for singular Hermitian metrics and (quasi-)plurisubharmonic functions with *arbitrary* singularities. This is one of the main advantages of our approach in this paper. The Hodge theoretic approach explained before does not work for singular Hermitian metrics with nonalgebraic singularities. Furthermore, we sometimes have to deal with singular Hermitian metrics with nonalgebraic singularities for several important applications in birational geometry even when we consider problems in algebraic geometry (see, for example, [Si], [Pa], [DHP], [GM], and [LP]). Therefore, it is worth formulating and proving various results for singular Hermitian metrics with arbitrary singularities although they are much more complicated than singular Hermitian metrics with only algebraic singularities.

1.1. **Main results.** Here, we explain the main results of this paper (Theorems A, B, C, D, E, F, and Theorem 1.10). Theorem A and Theorem 1.10 play important roles in this paper, and other results follow from Theorem A and Theorem 1.10 (see Proposition 1.9). We first recall the definition of pseudo-effective line bundles on compact complex manifolds.

Definition 1.2 (Pseudo-effective line bundles)**.** Let F be a holomorphic line bundle on a compact complex manifold X. We say that F is pseudo-effective if there exists a singular Hermitian metric h on F with $\sqrt{-1}\Theta_h(F) \geq 0$. When X is projective, it is well known that F is pseudo-effective if and only if F is pseudo-effective in the usual sense, that is, $F^{\otimes m} \otimes H$ is big for any ample line bundle H on X and any positive integer m.

The first result is an Enoki-type injectivity theorem.

Theorem A (Enoki-type injectivity)**.** *Let F be a holomorphic line bundle on a compact Kähler manifold X and let h be a singular Hermitian metric on F. Let M be a holomorphic line bundle on X and let h_M be a smooth Hermitian metric on M. Assume that*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \quad \textit{and} \quad \sqrt{-1}(\Theta_h(F) - t\Theta_{h_M}(M)) \geq 0$$

for some $t > 0$. Let s be a nonzero global section of M. Then the map

$$\times s : H^i(X, K_X \otimes F \otimes \mathcal{J}(h)) \to H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes M)$$

induced by $\otimes s$ is injective for every i, where K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

Remark 1.3. Let L be a semipositive line bundle on X, that is, it admits a smooth Hermitian metric with semipositive curvature. Let $F = L^{\otimes m}$ and $M = L^{\otimes k}$ for positive integers m and k. Then we obtain Enoki's original injectivity theorem (see [En, Theorem 0.2]) from Theorem A.

In the case of $M = F$, Theorem A has been proved in [MaS4] under the assumption $\sup_X |s|_h < \infty$. This assumption is a natural condition to guarantee that the multiplication map $\times s$ is well-defined. However, for our applications in this paper, we need to formulate Theorem A for a different (M, h_M) from (F, h). This formulation, which may look slightly artificial, is quite powerful and can produce applications, but raises a new difficulty in the proof: the set of points $x \in X$ with $\nu(h, x) > 0$ is not necessarily contained in a proper Zariski closed set, although such a situation was excluded in [MaS4] thanks to the assumption $\sup_X |s|_h < \infty$, where $\nu(h,x)$ denotes the Lelong number of the local weight of h at x. Compared to [MaS4], Theorem A is novel in the technique to overcome this difficulty (see Section 5 for the technical details), and further, it will be generalized to certain noncompact manifolds along with other techniques (see [MaS5]). Note that Theorem A can be seen as a generalization not only of Enoki's injectivity theorem but also of the Nadel vanishing theorem. In Section 4, we will explain how to reduce Demailly's original formulation of the Nadel vanishing theorem (see Theorem 1.4 below) to Theorem A for the reader's convenience.

Theorem 1.4 (Nadel vanishing theorem due to Demailly: [D2, Theorem 4.5])**.** *Let V be a smooth projective variety equipped with a Kähler form ω. Let L be a holomorphic line bundle on V and let h_L be a singular Hermitian metric on L such that $\sqrt{-1}\Theta_{h_L}(L) \geq \varepsilon\omega$ for some $\varepsilon > 0$. Then*

$$H^i(V, K_V \otimes L \otimes \mathcal{J}(h_L)) = 0$$

for every $i > 0$, where K_V is the canonical bundle of V and $\mathcal{J}(h_L)$ is the multiplier ideal sheaf of h_L.

A semiample line bundle is always semipositive. Thus, as a direct consequence of Theorem A, we obtain Theorem B, which is a generalization of Kollár's original injectivity theorem (see [Kol1]).

Theorem B (Kollár-type injectivity)**.** *Let F be a holomorphic line bundle on a compact Kähler manifold X and let h be a singular Hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Let N_1 and N_2 be semiample line bundles on X and let s be a nonzero global section of N_2. Assume that $N_1^{\otimes a} \simeq N_2^{\otimes b}$ for some positive integers a and b. Then the map*

$$\times s : H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) \to H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2)$$

induced by $\otimes s$ is injective for every i, where K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

Remark 1.5. (1) Let X be a smooth projective variety and (F,h) be a trivial Hermitian line bundle. Then we obtain Kollár's original injectivity theorem (see [Kol1, Theorem 2.2]) from Theorem B.
(2) For the proof of Theorem B, we may assume that $b = 1$, that is, $N_2 \simeq N_1^{\otimes a}$ by replacing s with s^b. We note that the composition

$$\begin{aligned} H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) &\xrightarrow{\times s} H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2) \\ &\xrightarrow{\times s^{b-1}} H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2^{\otimes b}) \end{aligned}$$

is the map $\times s^b$ induced by $\otimes s^b$.

Theorem C is a generalization of Kollár's torsion-free theorem and Theorem D is a generalization of Kollár's vanishing theorem (see [Kol1, Theorem 2.1]).

Theorem C (Kollár-type torsion-freeness)**.** *Let $f\colon X \to Y$ be a surjective morphism from a compact Kähler manifold X onto a projective variety Y. Let F be a holomorphic line bundle on X and let h be a singular Hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Then*

$$R^i f_*(K_X \otimes F \otimes \mathcal{J}(h))$$

is torsion-free for every i, where K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

Theorem D (Kollár-type vanishing theorem)**.** *Let $f\colon X \to Y$ be a surjective morphism from a compact Kähler manifold X onto a projective variety Y. Let F be a holomorphic line bundle on X and let h be a singular Hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Let N be a holomorphic line bundle on X. We assume that there exist positive integers a and b and an ample line bundle H on Y such that $N^{\otimes a} \simeq f^*H^{\otimes b}$. Then we obtain that*

$$H^i(Y, R^j f_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N)) = 0$$

for every $i > 0$ and j, where K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

Remark 1.6. (1) If X is a smooth projective variety and (F, h) is trivial, then Theorem C is nothing but Kollár's torsion-free theorem. Furthermore, if $N \simeq f^*H$, that is, $a = b = 1$, then Theorem D is the Kollár vanishing theorem. For the details, see [Kol1, Theorem 2.1].
(2) There exists a clever proof of Kollár's torsion-freeness by the theory of variations of Hodge structure (see [Ar]).
(3) In [MaS6], the second author obtained a natural analytic generalization of Kollár's vanishing theorem, which corresponds to the case where h is a smooth Hermitian metric and contains Ohsawa's vanishing theorem (see [O2]) as a special case.
(4) In [F15], the first author proved a vanishing theorem containing both Theorem 1.4 and Theorem D as special cases, which is called the vanishing theorem of Kollár–Nadel type.

By combining Theorem D with the Castelnuovo–Mumford regularity, we can easily obtain Corollary 1.7, which is a complete generalization of [Hö, Lemma 3.35 and Remark 3.36]. The proof of [Hö, Lemma 3.35] depends on a generalization of the Ohsawa–Takegoshi L^2 extension theorem. We note that Höring claims the weak positivity of $f_*(K_{X/Y} \otimes F)$ under some extra assumptions by using [Hö, Lemma 3.35]. For the details, see [Hö, 3.H Multiplier ideals].

Corollary 1.7. *Let $f\colon X \to Y$ be a surjective morphism from a compact Kähler manifold X onto a projective variety Y. Let F be a holomorphic line bundle on X and let h be a singular Hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Let H be an ample and globally generated line bundle on Y. Then*

$$R^i f_*(K_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}$$

is globally generated for every $i \geq 0$ and $m \geq \dim Y + 1$, where K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

As a direct consequence of Theorem D, we obtain Theorem E. See Definition 1.8 for the definition of GV-sheaves in the sense of Pareschi and Popa and see [Sc, Theorem 25.5 and Definition 26.3] for the details of GV-sheaves.

Theorem E (GV-sheaves)**.** *Let $f\colon X \to A$ be a morphism from a compact Kähler manifold X to an Abelian variety A. Let F be a holomorphic line bundle on X and let h be a singular Hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Then*

$$R^i f_*(K_X \otimes F \otimes \mathcal{J}(h))$$

is a GV-sheaf for every i, where K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

Definition 1.8 (GV-sheaves in the sense of Pareschi and Popa: [PP]). Let A be an Abelian variety. A coherent sheaf $\mathcal{F}$ on A is said to be a GV-sheaf if

$$\operatorname{codim}_{\operatorname{Pic}^0(A)}\{L \in \operatorname{Pic}^0(A) \,|\, H^i(A, \mathcal{F} \otimes L) \neq 0\} \geq i$$

for every i.

The final one is a generalization of the generic vanishing theorem (see [GL], [Ha], [PP]). The formulation of Theorem F is closer to [Ha] and [PP] than to the original generic vanishing theorem by Green and Lazarsfeld in [GL].

Theorem F (Generic vanishing theorem). *Let $f\colon X \to A$ be a morphism from a compact Kähler manifold X to an Abelian variety A. Let F be a holomorphic line bundle on X and let h be a singular Hermitian metric on F such that $\sqrt{-1}\Theta_h(F) \geq 0$. Then*

$$\operatorname{codim}_{\operatorname{Pic}^0(A)}\{L \in \operatorname{Pic}^0(A) \,|\, H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes f^*L) \neq 0\} \geq i - (\dim X - \dim f(X))$$

for every $i \geq 0$, where K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

The main results explained above are closely related to each other. The following proposition, which is also one of the main contributions in this paper, shows several relations among them. From Proposition 1.9, we see that it is sufficient to prove Theorem A. The proof of Proposition 1.9 will be given in Section 4.

Proposition 1.9. *We have the following relations among the above theorems.*

(i) *Theorem A implies Theorem B.*
(ii) *Theorem B is equivalent to Theorem C and Theorem D.*
(iii) *Theorem D implies Theorem E.*
(iv) *Theorem C and Theorem E imply Theorem F.*

A key ingredient of Proposition 1.9 is the following theorem, which can be seen as a Bertini-type theorem on the restriction of multiplier ideal sheaves to general members of free linear systems. Theorem 1.10 enables us to use the inductive argument on dimension. We remark that $\mathcal{G}$ in Theorem 1.10 is not always an intersection of countably many Zariski open sets (see Example 3.10). The proof of Theorem 1.10, which is quite technical, will be given in Section 3

Theorem 1.10 (Density of good divisors: Theorem 3.6). *Let X be a compact complex manifold, let Λ be a free linear system on X with $\dim \Lambda \geq 1$, and let φ be a quasi-plurisubharmonic function on X. We put*

$$\mathcal{G} := \{H \in \Lambda \,|\, H \text{ is smooth and } \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}.$$

Then $\mathcal{G}$ is dense in Λ in the classical topology, that is, the Euclidean topology.

Although the above formulation is sufficient for our applications, it is of independent interest to find a more precise formulation. The following problem, posed by Sébastien Boucksom, is reasonable from the viewpoint of Berndtsson's complex Prekopa theorem (see [Be]).

Problem 1.11. *In Theorem 1.10, is the complement $\Lambda \setminus \mathcal{G}$ a pluripolar subset of Λ?*

All the results explained above hold even if we replace K_X with $K_X \otimes E$, where E is any Nakano semipositive vector bundle on X. We will explain Theorem 1.12 in Section 6.

Theorem 1.12 (Twists by Nakano semipositive vector bundles)**.** *Let E be a Nakano semipositive vector bundle on a compact Kähler manifold X. Then Theorems A, B, C, D, E, F, Theorem 1.4, Corollary 1.7, and Proposition 1.9 hold even when K_X is replaced with $K_X \otimes E$.*

In this paper, we assume that all the varieties and manifolds are compact and connected for simplicity. We summarize the contents of this paper. In Section 2, we recall some basic definitions and collect several preliminary lemmas. Section 3 is devoted to the proof of Theorem 1.10. Theorem 1.10 plays a crucial role in the proof of Proposition 1.9. In Section 4, we prove Proposition 1.9 and Corollary 1.7, and explain how to reduce Theorem 1.4 to Theorem A. By these results, we see that all we have to do is to establish Theorem A. In Section 5, we give a detailed proof of Theorem A. In the final section: Section 6, we explain how to modify the arguments used before for the proof of Theorem 1.12.

After the authors put a preprint version of this paper on arXiv, some further generalizations of Theorem A have been studied in [MaS5], [CDM], [ZZ], and a relative version of Theorem 1.10 has been established in [F16], by developing the techniques in this paper. See [Ta], [F5], [MaS5], [CDM], [F16] for some injectivity, torsion-free, and vanishing theorems for noncompact manifolds.

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2. Preliminaries

We briefly review the definition of singular Hermitian metrics, (quasi-)plurisubharmonic functions, and Nadel's multiplier ideal sheaves. See [D3] for the details.

Definition 2.1 (Singular Hermitian metrics and curvatures)**.** Let F be a holomorphic line bundle on a complex manifold X. A *singular Hermitian metric* on F is a metric h which is given in every trivialization $\theta\colon F|_\Omega \simeq \Omega \times \mathbb{C}$ by

$$|\xi|_h = |\theta(\xi)|e^{-\varphi} \text{ on } \Omega,$$

where ξ is a section of F on Ω and $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ is an arbitrary function. Here $L^1_{\mathrm{loc}}(\Omega)$ is the space of locally integrable functions on Ω. We usually call φ the weight function of the metric with respect to the trivialization θ. The curvature of a singular Hermitian metric h is defined by

$$\sqrt{-1}\Theta_h(F) := 2\sqrt{-1}\partial\overline{\partial}\varphi,$$

where φ is a weight function and $\sqrt{-1}\partial\overline{\partial}\varphi$ is taken in the sense of currents. It is easy to see that the right-hand side does not depend on the choice of trivializations.

The notion of multiplier ideal sheaves introduced by Nadel plays an important role in the recent developments of complex geometry and algebraic geometry.

Definition 2.2 ((Quasi-)plurisubharmonic functions and multiplier ideal sheaves)**.** A function $u\colon \Omega \to [-\infty, \infty)$ defined on an open set $\Omega \subset \mathbb{C}^n$ is said to be *plurisubharmonic* if

- u is upper semicontinuous, and
- for every complex line $L \subset \mathbb{C}^n$, the restriction $u|_{\Omega \cap L}$ to L is subharmonic on $\Omega \cap L$, that is, for every $a \in \Omega$ and $\xi \in \mathbb{C}^n$ satisfying $|\xi| < d(a, \Omega^c)$, the function u satisfies the mean inequality

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\xi)\, d\theta.$$

Let X be a complex manifold. A function $\varphi\colon X \to [-\infty, \infty)$ is said to be *plurisubharmonic on* X if there exists an open cover $\{U_i\}_{i \in I}$ of X such that $\varphi|_{U_i}$ is plurisubharmonic on U_i $(\subset \mathbb{C}^n)$ for every i. We can easily see that this definition is independent of the choice of open covers. A *quasi-plurisubharmonic* function is a function φ which is locally equal to the sum of a plurisubharmonic function and of a smooth function. If φ is a quasi-plurisubharmonic function on a complex manifold X, then the multiplier ideal sheaf $\mathcal{J}(\varphi) \subset \mathcal{O}_X$ is defined by

$$\Gamma(U, \mathcal{J}(\varphi)) := \{f \in \mathcal{O}_X(U) \,|\, |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\}$$

for every open set $U \subset X$. Then it is known that $\mathcal{J}(\varphi)$ is a coherent ideal sheaf (see, for example, [D3, (5.7) Lemma]). Let S be a complex submanifold of X. Then the restriction $\mathcal{J}(\varphi)|_S$ of the multiplier ideal sheaf $\mathcal{J}(\varphi)$ to S is defined by the image of $\mathcal{J}(\varphi)$ under the natural surjective morphism $\mathcal{O}_X \to \mathcal{O}_S$, that is,

$$\mathcal{J}(\varphi)|_S = \mathcal{J}(\varphi)/\mathcal{J}(\varphi) \cap \mathcal{I}_S,$$

where $\mathcal{I}_S$ is the defining ideal sheaf of S on X. We note that the restriction $\mathcal{J}(\varphi)|_S$ does not always coincide with $\mathcal{J}(\varphi) \otimes \mathcal{O}_S = \mathcal{J}(\varphi)/\mathcal{J}(\varphi) \cdot \mathcal{I}_S$.

We have already used $\mathcal{J}(h)$ in theorems in Section 1.

Definition 2.3. Let F be a holomorphic line bundle on a complex manifold X and let h be a singular Hermitian metric on F. We assume $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$-form γ on X. We fix a smooth Hermitian metric h_∞ on F. Then we can write $h = h_\infty e^{-2\psi}$ for some $\psi \in L^1_{\mathrm{loc}}(X)$. Then ψ coincides with a quasi-plurisubharmonic function φ on X almost everywhere. We define the multiplier ideal sheaf $\mathcal{J}(h)$ of h by $\mathcal{J}(h) := \mathcal{J}(\varphi)$.

We close this section with the following lemmas, which will be used in the proof of Theorem A in Section 5.

Lemma 2.4 ([O1, Proposition 1.1])**.** *Let ω and $\widetilde{\omega}$ be positive $(1,1)$-forms on an n-dimensional complex manifold with $\widetilde{\omega} \geq \omega$. If u is an (n,q)-form, then $|u|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} \leq |u|^2_{\omega}\, dV_{\omega}$. Furthermore, if u is an $(n,0)$-form, then $|u|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} = |u|^2_{\omega}\, dV_{\omega}$. Here $|u|_{\omega}$ (resp. $|u|_{\widetilde{\omega}}$) is the pointwise norm of u with respect to ω (resp. $\widetilde{\omega}$) and dV_{ω} (resp. $dV_{\widetilde{\omega}}$) is the volume form defined by $dV_{\omega} := \omega^n/n!$ (resp. $dV_{\widetilde{\omega}} := \widetilde{\omega}^n/n!$).*

Proof. This lemma follows from simple computations. Thus, we omit the proof. □

Lemma 2.5. *Let $\varphi\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator (continuous linear map) between Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. If $\{w_k\}_{k=1}^{\infty}$ weakly converges to w in $\mathcal{H}_1$, then $\{\varphi(w_k)\}_{k=1}^{\infty}$ weakly converges to $\varphi(w)$.*

Proof. By taking the adjoint operator φ^*, for every $v \in \mathcal{H}_2$, we have

$$\langle\!\langle \varphi(w_k), v \rangle\!\rangle_{\mathcal{H}_2} = \langle\!\langle w_k, \varphi^*(v) \rangle\!\rangle_{\mathcal{H}_1} \to \langle\!\langle w, \varphi^*(v) \rangle\!\rangle_{\mathcal{H}_1} = \langle\!\langle \varphi(w), v \rangle\!\rangle_{\mathcal{H}_2}.$$

This completes the proof. □

Lemma 2.6. *Let L be a closed subspace in a Hilbert space $\mathcal{H}$. Then L is closed with respect to the weak topology of $\mathcal{H}$, that is, if a sequence $\{w_k\}_{k=1}^{\infty}$ in L weakly converges to w, then the weak limit w belongs to L.*

Proof. By the orthogonal decomposition, there exists a closed subspace M such that $L = M^{\perp}$. Then it follows that $w \in M^{\perp} = L$ since $0 = \langle\!\langle w_k, v \rangle\!\rangle_{\mathcal{H}} \to \langle\!\langle w, v \rangle\!\rangle_{\mathcal{H}}$ for any $v \in M$. □

3. Restriction lemma

This section is devoted to the proof of Theorem 1.10 (see Theorem 3.6), which will play a crucial role in the proof of Proposition 1.9. The following lemma is a direct consequence of the Ohsawa–Takegoshi L^2 extension theorem (see [OT, Theorem]).

Lemma 3.1. *Let X be a complex manifold and let φ be a quasi-plurisubharmonic function on X. We consider a filtration*

$$F_k \subset F_{k-1} \subset \cdots \subset F_1 \subset F_0 := X,$$

where F_i is a smooth hypersurface of F_{i-1} for every i. Then we obtain that

$$\mathcal{J}(\varphi|_{F_k}) \subset \mathcal{J}(\varphi|_{F_{k-1}})|_{F_k} \subset \cdots \subset \mathcal{J}(\varphi|_{F_1})|_{F_k} \subset \mathcal{J}(\varphi)|_{F_k}.$$

Proof. This immediately follows from the Ohsawa–Takegoshi L^2 extension theorem. □

The following lemma is a key ingredient of the proof of Theorem 1.10 (see Theorem 3.6).

Lemma 3.2. *Let X and φ be as in Theorem 1.10. Let H_i be a Cartier divisor on X for $1 \leq i \leq k$. We assume the following condition:*

- ♠ *The divisor $\sum_{i=1}^{k} H_i$ is a simple normal crossing divisor on X. Moreover, for every $1 \leq i_1 < i_2 < \cdots < i_l \leq k$ and any $P \in H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_l}$, the set $\{f_{i_1}, f_{i_2}, \cdots, f_{i_l}\}$ is a regular sequence for $\mathcal{O}_{X,P}/\mathcal{J}(\varphi)_P$, where f_i is a (local) defining equation of H_i for every i.*

Furthermore, we assume that $\mathcal{J}(\varphi|_{F_k}) = \mathcal{J}(\varphi)|_{F_k}$ holds, where $F_i := H_1 \cap H_2 \cap \cdots \cap H_i$ for $1 \leq i \leq k$. Then for every j, the equality $\mathcal{J}(\varphi|_{F_j}) = \mathcal{J}(\varphi)|_{F_j}$ holds on a neighborhood of F_k in F_j.

Before we prove Lemma 3.2, we make some remarks to help the reader understand condition ♠.

Remark 3.3. (1) Let $(A, \mathfrak{m})$ be a local ring and let M be a finitely generated (nonzero) A-module. Let $\{x_1, \ldots, x_r\}$ be a sequence of elements of $\mathfrak{m}$. We put $M_0 = M$ and $M_i = M/x_1M + \cdots + x_iM$. Then $\{x_1, \ldots, x_r\}$ is said to be a *regular sequence* for M if $\times x_{i+1} \colon M_i \to M_i$ is injective for every $0 \leq i \leq r-1$.

(2) Condition ♠ in Lemma 3.2 does not depend on the order of $\{H_1, H_2, \cdots, H_k\}$ (see, for example, [MaH, Theorem 16.3] and [AK, Chapter III, Corollary (3.5)]).

(3) Let $\mathcal{F}$ be a coherent analytic sheaf on a compact complex manifold X. Then there exists a finite family $\{Y_i\}_{i \in I}$ of irreducible analytic subsets of X such that

$$\mathrm{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = \{\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}\},$$

where $\mathfrak{p}_{x,1}, \ldots, \mathfrak{p}_{x,r(x)}$ are prime ideals of $\mathcal{O}_{X,x}$ associated to the irreducible components of the germs $x \in Y_i$ (see, for example [Man, (I.6) Lemma]). Note that Y_i is called an analytic subset associated with $\mathcal{F}$. In this paper, we simply say that Y_i is an *associated prime* of $\mathcal{F}$ if there is no risk of confusion. Then condition ♠ is equivalent to the following condition:

- The divisor $\sum_{i=1}^{k} H_i$ is a simple normal crossing divisor on X. Moreover, for every $1 \le i_1 < i_2 < \cdots < i_{l-1} < i_l \le k$, the divisor H_{i_l} contains no associated primes of $\mathcal{O}_X/\mathcal{J}(\varphi)$ and $\mathcal{O}_{H_{i_1}\cap\cdots\cap H_{i_{l-1}}}/\mathcal{J}(\varphi)|_{H_{i_1}\cap\cdots\cap H_{i_{l-1}}}$.

(4) (3.1) below may be helpful to understand condition ♠. We put $H_{i_1\cdots i_m} := H_{i_1} \cap \cdots \cap H_{i_m}$ for every $1 \le i_1 < \cdots < i_m \le k$. Then we can inductively check that

$$0 \to \mathcal{J}(\varphi)|_{H_{i_1\cdots i_{l-1}}} \otimes \mathcal{O}_{H_{i_1\cdots i_{l-1}}}(-H_{i_l}) \to \mathcal{J}(\varphi)|_{H_{i_1\cdots i_{l-1}}} \to \mathcal{J}(\varphi)|_{H_{i_1\cdots i_l}} \to 0$$

is exact and that

$$\begin{aligned} 0 \to &\left(\mathcal{O}_{H_{i_1\cdots i_{l-1}}}/\mathcal{J}(\varphi)|_{H_{i_1\cdots i_{l-1}}}\right) \otimes \mathcal{O}_{H_{i_1\cdots i_{l-1}}}(-H_{i_l}) \\ &\to \mathcal{O}_{H_{i_1\cdots i_{l-1}}}/\mathcal{J}(\varphi)|_{H_{i_1\cdots i_{l-1}}} \to \mathcal{O}_{H_{i_1\cdots i_l}}/\mathcal{J}(\varphi)|_{H_{i_1\cdots i_l}} \to 0 \end{aligned} \tag{3.1}$$

is also exact (see (3.3) and (3.4) in the proof of Lemma 3.2).

Proof of Lemma 3.2. By condition ♠, the morphism γ in the following commutative diagram is injective.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \mathcal{J}(\varphi)\otimes\mathcal{O}_X(-H_1) & \xrightarrow{\alpha} & \mathcal{J}(\varphi) & \to & \operatorname{Coker}\alpha & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\beta} & & \\ 0 & \to & \mathcal{O}_X(-H_1) & \to & \mathcal{O}_X & \to & \mathcal{O}_{H_1} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & (\mathcal{O}_X/\mathcal{J}(\varphi))\otimes\mathcal{O}_X(-H_1) & \xrightarrow{\gamma} & \mathcal{O}_X/\mathcal{J}(\varphi) & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.2}$$

Therefore β is also injective. This implies that $\operatorname{Coker}\alpha = \mathcal{J}(\varphi)|_{H_1}$ by definition. Thus, we obtain the following short exact sequence:

$$0 \to \mathcal{J}(\varphi)\otimes\mathcal{O}_X(-H_1) \to \mathcal{J}(\varphi) \to \mathcal{J}(\varphi)|_{H_1} \to 0.$$

We also obtain the following short exact sequence:

$$0 \to (\mathcal{O}_X/\mathcal{J}(\varphi))\otimes\mathcal{O}_X(-H_1) \xrightarrow{\gamma} \mathcal{O}_X/\mathcal{J}(\varphi) \to \mathcal{O}_{H_1}/\mathcal{J}(\varphi)|_{H_1} \to 0$$

by the above big commutative diagram. Similarly, by condition ♠, we can inductively check that

$$0 \to \mathcal{J}(\varphi)|_{F_i} \otimes \mathcal{O}_{F_i}(-H_{i+1}) \to \mathcal{J}(\varphi)|_{F_i} \to \mathcal{J}(\varphi)|_{F_{i+1}} \to 0 \tag{3.3}$$

and

$$0 \to (\mathcal{O}_{F_i}/\mathcal{J}(\varphi)|_{F_i}) \otimes \mathcal{O}_{F_i}(-H_{i+1}) \to \mathcal{O}_{F_i}/\mathcal{J}(\varphi)|_{F_i} \to \mathcal{O}_{F_{i+1}}/\mathcal{J}(\varphi)|_{F_{i+1}} \to 0 \tag{3.4}$$

are exact for every $1 \leq i \leq k-1$. For $0 \leq i \leq k-1$, we consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{J}(\varphi|_{F_i}) \otimes \mathcal{O}_{F_i}(-H_{i+1}) & \longrightarrow & \mathcal{J}(\varphi)|_{F_i} \otimes \mathcal{O}_{F_i}(-H_{i+1}) & \longrightarrow & \operatorname{Coker} b_i \otimes \mathcal{O}_{F_i}(-H_{i+1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_i} & & \downarrow & & \downarrow{\scriptstyle d_i} & & \\
0 & \longrightarrow & \mathcal{J}(\varphi|_{F_i}) & \xrightarrow{b_i} & \mathcal{J}(\varphi)|_{F_i} & \longrightarrow & \operatorname{Coker} b_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Coker} a_i & \xrightarrow{c_i} & \mathcal{J}(\varphi)|_{F_{i+1}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

The assumption $\mathcal{J}(\varphi|_{F_k}) = \mathcal{J}(\varphi)|_{F_k}$ implies that $\mathcal{J}(\varphi|_{F_{k-1}})|_{F_k} = \mathcal{J}(\varphi)|_{F_k}$ holds by $\mathcal{J}(\varphi|_{F_k}) \subset \mathcal{J}(\varphi|_{F_{k-1}})|_{F_k} \subset \cdots \subset \mathcal{J}(\varphi)|_{F_k}$ in Lemma 3.1. If $\mathcal{J}(\varphi|_{F_i})|_{F_{i+1}} = \mathcal{J}(\varphi)|_{F_{i+1}}$ on a neighborhood of F_k in F_{i+1}, then c_i is surjective on a neighborhood of F_k in F_{i+1} by the definition of $\mathcal{J}(\varphi|_{F_i})|_{F_{i+1}}$. Then d_i is also surjective on a neighborhood of F_k in F_i by the above big commutative diagram. By Nakayama's lemma, $\operatorname{Coker} b_i$ is zero on a neighborhood of F_k in F_i. This implies that $\mathcal{J}(\varphi|_{F_i}) = \mathcal{J}(\varphi)|_{F_i}$ on a neighborhood of F_k in F_i. Thus, we obtain that $\mathcal{J}(\varphi|_{F_{i-1}})|_{F_i} = \mathcal{J}(\varphi)|_{F_i}$ on a neighborhood of F_k in F_i since we have $\mathcal{J}(\varphi|_{F_i}) \subset \mathcal{J}(\varphi|_{F_{i-1}})|_{F_i} \subset \mathcal{J}(\varphi)|_{F_i}$ by Lemma 3.1. By repeating this argument, we see that $\mathcal{J}(\varphi|_{F_j}) = \mathcal{J}(\varphi)|_{F_j}$ on a neighborhood of F_k in F_j for every j. This is the desired property. □

Lemma 3.4. *Assume that $\{H_1, \cdots, H_m\}$ satisfies condition ♠ in Lemma 3.2. Let H_{m+1} be a smooth Cartier divisor on X such that $\sum_{i=1}^{m+1} H_i$ is a simple normal crossing divisor on X and that H_{m+1} contains no associated primes of*

$$\mathcal{O}_X/\mathcal{J}(\varphi) \quad \text{and} \quad \mathcal{O}_{H_{i_1}\cap\cdots\cap H_{i_l}}/\mathcal{J}(\varphi)|_{H_{i_1}\cap\cdots\cap H_{i_l}}$$

for every $1 \leq i_1 < \cdots < i_l \leq m$. Then $\{H_1, \cdots, H_m, H_{m+1}\}$ also satisfies condition ♠.

Proof. This is obvious from Remark 3.3 (3). □

Lemma 3.5. *Let Λ_0 be a sublinear system of a free linear system Λ on X with $\dim \Lambda_0 \geq 1$. Assume that $\{H_1, \cdots, H_m\}$ satisfies condition ♠ in Lemma 3.2. We put*

$$\mathcal{F}_0 := \{D \in \Lambda_0 \,|\, \{H_1, \cdots, H_m, D\} \text{ satisfies } ♠\}.$$

Then $\mathcal{F}_0$ is Zariski open in Λ_0. In particular, if $\mathcal{F}_0$ is not empty, then it is a dense Zariski open set of Λ_0.

Moreover, we assume that there exists $D_0 \in \mathcal{F}_0$ such that $\mathcal{J}(\varphi|_V) = \mathcal{J}(\varphi)|_V$, where V is an irreducible component of $H_1 \cap \cdots \cap H_m \cap D_0$. Let D be a member of $\mathcal{F}_0$ such that V is an irreducible component of $H_1 \cap \cdots \cap H_m \cap D$. Then $\mathcal{J}(\varphi|_D) = \mathcal{J}(\varphi)|_D$ holds on a neighborhood of V in D.

Proof. We put

$$\mathcal{F} := \{D \in \Lambda \,|\, \{H_1, \cdots, H_m, D\} \text{ satisfies } ♠\}.$$

Then by Remark 3.3 (3) and Lemma 3.4, it is easy to see that $\mathcal{F}$ is a dense Zariski open set in Λ since Λ is a free linear system on X. Therefore, $\mathcal{F}_0 = \mathcal{F} \cap \Lambda_0$ is Zariski open in Λ_0. By Lemma 3.2, the equality $\mathcal{J}(\varphi|_D) = \mathcal{J}(\varphi)|_D$ holds on a neighborhood of V in D if $D \in \mathcal{F}_0$ and V is an irreducible component of $H_1 \cap \cdots \cap H_m \cap D$. We note that we do not need the compactness of X in the proof of Lemma 3.2. Therefore, we can shrink X and assume that $V = H_1 \cap \cdots \cap H_m \cap D$ in the above argument. □

The following theorem (see Theorem 1.10) is one of the key results of this paper.

Theorem 3.6 (Density of good divisors: Theorem 1.10)**.** *Let X be a compact complex manifold, let Λ be a free linear system on X with $\dim \Lambda \geq 1$, and let φ be a quasi-plurisubharmonic function on X. We put*

$$\mathcal{G} := \{H \in \Lambda \,|\, H \text{ is smooth and } \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}.$$

Then $\mathcal{G}$ is dense in Λ in the classical topology.

Proof. We may assume that $\varphi \not\equiv -\infty$. Throughout this proof, we put $f := \Phi_\Lambda : X \to Y := f(X) \subset \mathbb{P}^N$. Note that $N = \dim \Lambda$. We divide the proof into several steps.

Step 0 (Idea of the proof)**.** In this step, we will explain an idea of the proof.

A general member H of Λ is smooth by Bertini's theorem, and it always satisfies that $\mathcal{J}(\varphi|_H) \subset \mathcal{J}(\varphi)|_H$ by Lemma 3.1. Hence, the problem is to check that the opposite inclusion holds for any member of a dense subset in Λ.

If $\dim \Lambda = 1$, that is, Λ is a pencil, then a member H of Λ is a fiber of the morphism $f = \Phi_\Lambda \colon X \to \mathbb{P}^1$ at a point $P \in \mathbb{P}^1$. By Fubini's theorem, we have $\mathcal{J}(\varphi|_{f^{-1}(P)}) \supset \mathcal{J}(\varphi)|_{f^{-1}(P)}$ for almost all $P \in \mathbb{P}^1$. This is the desired statement when $\dim \Lambda = 1$. In general, we have $H_1 \cap H_2 \neq \emptyset$ for two general members H_1 and H_2 of Λ. For this reason, we choose H_1 and H_2 suitably (see Step 2 and Step 3), take the blow-up $Z \to X$ along $H_1 \cap H_2$, and reduce the problem to the pencil case (see Step 4).

Step 1. In this step, we will prove the theorem when $\dim Y = 1$.

Let $\psi_0, \ldots, \psi_N$ be a basis of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$. We put

$$\mathscr{Y} = \{(y, [a_0 : \cdots : a_N]) \in Y \times \mathbb{P}^N \,|\, a_0\psi_0(y) + \cdots + a_N\psi_N(y) = 0\} \subset Y \times \mathbb{P}^N$$

and consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathscr{X} & \hookrightarrow & X \times \mathbb{P}^N & \longrightarrow & X \\
{\scriptstyle \widetilde{f}}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\
\mathscr{Y} & \hookrightarrow & Y \times \mathbb{P}^N & \longrightarrow & Y \\
& {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle p_2} & & \\
& & \mathbb{P}^N, & &
\end{array}$$

where $\mathscr{X} \hookrightarrow X \times \mathbb{P}^N \to X$ is the base change of $\mathscr{Y} \hookrightarrow Y \times \mathbb{P}^N \to Y$ by $f \colon X \to Y$, p_2 is the second projection, and $\pi = p_2|_{\mathscr{Y}}$. We can easily see that there exists a nonempty Zariski open set U of $\mathbb{P}^N$ such that π and $\widetilde{f}$ are étale and smooth over U, respectively. We note that $\Lambda = f^*|\mathcal{O}_{\mathbb{P}^N}(1)|$ by construction. Let H be a member of Λ corresponding to a point of U. Then H is smooth and $\mathcal{J}(\varphi|_H) \subset \mathcal{J}(\varphi)|_H$ holds by Lemma 3.1. On the other hand, by applying Fubini's theorem to $(\pi \circ \widetilde{f})^{-1}(U) \to U$, the opposite inclusion

$\mathcal{J}(\varphi)|_H \subset \mathcal{J}(\varphi|_H)$ holds for almost all $H \in \Lambda$. This means that $\mathcal{G}$ is dense in Λ in the classical topology.

Step 2. In this step, we will prove the following preparatory lemma.

Lemma 3.7. *Let D_1 and D_2 be two members of Λ such that $\{D_1, D_2\}$ satisfies condition ♠ in Lemma 3.2. Let $\mathcal{P}_0$ be the pencil spanned by D_1 and D_2. Then, for almost all $D \in \mathcal{P}_0$, the member D is smooth, $\{D\}$ satisfies condition ♠, and $\mathcal{J}(\varphi|_D) = \mathcal{J}(\varphi)|_D$ holds outside $D_1 \cap D_2$.*

Proof of Lemma 3.7. Let A_i be a hyperplane in $\mathbb{P}^N$ such that $D_i = f^*A_i$, and $\mathrm{pr}\colon \mathbb{P}^{\mathrm{N}} \dashrightarrow \mathbb{P}^1$ be the linear projection from the subspace $A_1 \cap A_2 \cong \mathbb{P}^{N-2}$. Then the meromorphic map $X \dashrightarrow \mathbb{P}^1$ associated with $\mathcal{P}_0$ is the composition of $f\colon X \to \mathbb{P}^N$ and $\mathrm{pr}\colon \mathbb{P}^{\mathrm{N}} \dashrightarrow \mathbb{P}^1$. Since the blow-up of $\mathbb{P}^N$ along $A_1 \cap A_2$ gives an elimination of the indeterminacy locus of $\mathrm{pr}\colon \mathbb{P}^{\mathrm{N}} \dashrightarrow \mathbb{P}^1$, the blow-up $p\colon Z \to X$ along $D_1 \cap D_2$ satisfies the following commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{\ p\ } & X & \xrightarrow{f=\Phi_\Lambda} & \mathbb{P}^N \\ & {\scriptstyle q}\searrow & \downarrow & \swarrow{\scriptstyle \mathrm{pr}} & \\ & & \mathbb{P}^1. & & \end{array}$$

By applying Fubini's theorem to $q\colon Z \to \mathbb{P}^1$, we obtain that $\mathcal{J}(p^*\varphi|_{q^{-1}(Q)}) = \mathcal{J}(p^*\varphi)|_{q^{-1}(Q)}$ for almost all $Q \in \mathbb{P}^1$. Lemma 3.5 implies that $\{D\}$ satisfies condition ♠ for almost all $D \in \mathcal{P}_0$. The desired properties follow since p is an isomorphism outside $D_1 \cap D_2$. □

Step 3. In this step, we will find a smooth member H of Λ such that $\mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H$ and that $\{H\}$ satisfies condition ♠.

From now on, we assume that $\dim \Lambda \geq 2$ and that the statement of Theorem 3.6 holds for lower dimensional free linear systems. We put $l := \dim Y$. By Step 1, we have a smooth member H of Λ with the desired properties when $l = 1$. Therefore, we may assume that $l \geq 2$. We take two general hyperplanes B_1 and B_2 of $\mathbb{P}^N$. We put $D_1 := f^*B_1$ and $D_2 := f^*B_2$. By Lemma 3.7, we can take a hyperplane A_1 of $\mathbb{P}^N$ such that $X_1 := f^*A_1$ is smooth, $\{X_1\}$ satisfies condition ♠, and $\mathcal{J}(\varphi|_{X_1}) = \mathcal{J}(\varphi)|_{X_1}$ outside $D_1 \cap D_2$. Let $\Lambda|_{X_1}$ be the linear system on X_1 defined by $f_1\colon X_1 = X \cap f^{-1}(A_1) \to Y \cap A_1 \subset A_1 \cong \mathbb{P}^{N-1}$, that is, the set of pull-backs of the hyperplanes in $A_1 \cong \mathbb{P}^{N-1}$ by f_1. By construction, we have $\dim \Lambda|_{X_1} = \dim \Lambda - 1$. Thus, we see that

$$\{H \in \Lambda \mid X_1 \cap H \text{ is smooth and } \mathcal{J}(\varphi|_{X_1 \cap H}) = \mathcal{J}(\varphi|_{X_1})|_{X_1 \cap H}\}$$

is dense in Λ in the classical topology by the induction hypothesis. Then we can take general hyperplanes $A_2, A_3, \cdots, A_l$ of $\mathbb{P}^N$ such that $\dim(A_1 \cap \cdots \cap A_l \cap Y) = 0$ and that $f^{-1}(Q)$ is smooth and

$$\mathcal{J}(\varphi|_{f^{-1}(Q)}) = \mathcal{J}(\varphi|_{X_1})|_{f^{-1}(Q)} \tag{3.5}$$

for every $Q \in A_1 \cap \cdots \cap A_l \cap Y$ by using the induction hypothesis repeatedly. Without loss of generality, we may assume that $f^{-1}(Q) \cap D_1 \cap D_2 = \emptyset$ for every $Q \in A_1 \cap \cdots \cap A_l \cap Y$. Since

$$\mathcal{J}(\varphi|_{X_1}) = \mathcal{J}(\varphi)|_{X_1}$$

holds outside $D_1 \cap D_2$,

$$\mathcal{J}(\varphi|_{X_1})|_{f^{-1}(Q)} = \mathcal{J}(\varphi)|_{f^{-1}(Q)} \tag{3.6}$$

holds for every $Q \in A_1 \cap \cdots \cap A_l \cap Y$. Therefore, we have

$$\mathcal{J}(\varphi|_{f^{-1}(Q)}) = \mathcal{J}(\varphi|_{X_1})|_{f^{-1}(Q)} = \mathcal{J}(\varphi)|_{f^{-1}(Q)}$$

for every $Q \in A_1 \cap \cdots \cap A_l \cap Y$ by (3.5) and (3.6). We may assume that $\{X_1 = f^*A_1, f^*A_2, \cdots, f^*A_l\}$ satisfies condition ♠. We take one point P of $A_1 \cap \cdots \cap A_l \cap Y$ and fix $A_2, \cdots, A_l$. By applying Lemma 3.5 to the linear system

$$\Lambda_0 := \{D \in \Lambda \,|\, f^{-1}(P) \subset D\},$$

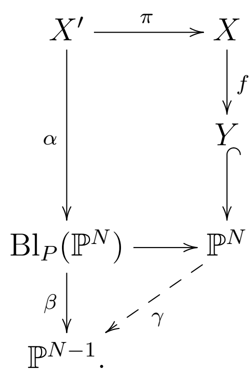
we see that

$$\mathcal{F}_0 := \{D \in \Lambda_0 \,|\, \{D, f^*A_2, \cdots, f^*A_l\} \text{ satisfies } ♠\}$$

is Zariski open in Λ_0. Note that $\mathcal{F}_0$ is nonempty by $X_1 = f^*A_1 \in \mathcal{F}_0$. By the latter conclusion of Lemma 3.5, we have:

Lemma 3.8. *Let A_g be a general hyperplane of $\mathbb{P}^N$ passing through P. We put $X_g := f^*A_g$. Then $\mathcal{J}(\varphi|_{X_g}) = \mathcal{J}(\varphi)|_{X_g}$ holds on a neighborhood of $f^{-1}(P)$ in X_g.*

Let $\pi\colon X' \to X$ be the blow-up along $f^{-1}(P)$ and let $\mathrm{Bl}_P(\mathbb{P}^N) \to \mathbb{P}^N$ be the blow-up of $\mathbb{P}^N$ at P. The induced morphism $\alpha\colon X' \to \mathrm{Bl}_P(\mathbb{P}^N)$ and the linear projection $\gamma\colon \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ from $P \in \mathbb{P}^N$ satisfy the following commutative diagram.



We put $f' := \beta \circ \alpha$ and $Y' := f'(X')$. By applying the induction hypothesis to $f'\colon X' \to Y' \subset \mathbb{P}^{N-1}$, we can take a general hyperplane A of $\mathbb{P}^{N-1}$ such that f'^*A is smooth and that

$$\mathcal{J}(\pi^*\varphi|_{f'^{-1}(A)}) = \mathcal{J}(\pi^*\varphi)|_{f'^{-1}(A)}. \tag{3.7}$$

Let A_0 be the hyperplane of $\mathbb{P}^N$ spanned by P and A. Then we can see that

$$\{f^*A_2, \cdots, f^*A_l, H := f^*A_0\} \tag{3.8}$$

satisfies condition ♠ since A is a general hyperplane of $\mathbb{P}^{N-1}$. We see that $\mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H$ by (3.7) and Lemma 3.8, and that $\{H\}$ satisfies condition ♠ by (3.8). Therefore this H has the desired properties.

Step 4. In this final step, we will prove that $\mathcal{G}$ is dense in Λ in the classical topology.

We will use the induction on $\dim X$. If $\dim X = 1$, then $\dim Y = 1$. Therefore, by Step 1, we see that $\mathcal{G}$ is dense in Λ in the classical topology. Therefore, we assume that $\dim X \geq 2$. If $\dim Y = 1$, then $\mathcal{G}$ is dense by Step 1. Thus, we may assume that $\dim \Lambda \geq \dim Y \geq 2$. By Step 3, we can take a smooth member H_0 of Λ such that $\mathcal{J}(\varphi|_{H_0}) = \mathcal{J}(\varphi)|_{H_0}$ and that $\{H_0\}$ satisfies condition ♠. By applying the induction hypothesis to $\Lambda|_{H_0}$, we see that

$$\mathcal{G}' := \{H' \in \Lambda \,|\, H_0 \cap H' \text{ is smooth and } \mathcal{J}(\varphi|_{H_0 \cap H'}) = \mathcal{J}(\varphi|_{H_0})|_{H_0 \cap H'}\}$$

is dense in Λ in the classical topology. Since Λ is a free linear system, we know that

$$\{H' \in \Lambda \,|\, \{H_0, H'\} \text{ satisfies } \spadesuit\}$$

is a nonempty Zariski open set in Λ. Therefore,

$$\mathcal{G}'' := \{H' \in \mathcal{G}' \,|\, \{H_0, H'\} \text{ satisfies } \spadesuit\}$$

is also dense in Λ in the classical topology. We note that

$$\mathcal{J}(\varphi|_{H_0\cap H'}) = \mathcal{J}(\varphi|_{H_0})|_{H_0\cap H'} = \mathcal{J}(\varphi)|_{H_0\cap H'}$$

for every $H' \in \mathcal{G}'$ since $\mathcal{J}(\varphi|_{H_0}) = \mathcal{J}(\varphi)|_{H_0}$. Therefore, we obtain that

$$\mathcal{J}(\varphi|_{H_0\cap H'}) = \mathcal{J}(\varphi|_{H'})|_{H_0\cap H'} = \mathcal{J}(\varphi)|_{H_0\cap H'} \tag{3.9}$$

for every $H' \in \mathcal{G}''$. By the latter conclusion of Lemma 3.5, (3.9) indicates that $\mathcal{J}(\varphi|_{H'}) = \mathcal{J}(\varphi)|_{H'}$ on a neighborhood of $H_0 \cap H'$ in H' for every $H' \in \mathcal{G}''$. We consider the pencil $\mathcal{P}_{H'}$ spanned by H_0 and $H' \in \mathcal{G}''$, that is, the sublinear system of Λ spanned by H_0 and H'. Let D be a general member of $\mathcal{P}_{H'}$. Then by Lemma 3.5, $\{H_0, D\}$ satisfies $\spadesuit$ and $\mathcal{J}(\varphi|_D) = \mathcal{J}(\varphi)|_D$ holds on a neighborhood of $H_0 \cap H'$ in D. Hence, by Lemma 3.7, we say that almost all members of $\mathcal{P}_{H'}$ are contained in $\mathcal{G}$. By this observation, we obtain that $\mathcal{G}$ is dense in Λ in the classical topology.

Thus, we obtain the desired statement. □

The following examples show that $\mathcal{G}$ in Theorem 1.10 (Theorem 3.6) is not always Zariski open in Λ, or even an intersection of countably many nonempty Zariski open sets of Λ

Example 3.9. We put

$$\psi(z) := \sum_{k=1}^{\infty} 2^{-k} \log \left| z - \frac{1}{k} \right|$$

for $z \in \mathbb{C}$. Then it is easy to see that $\psi(z)$ is smooth for $|z| \geq 2$. By using a suitable partition of unity, we can construct a function $\varphi(z)$ on $\mathbb{P}^1$ such that $\varphi(z) = \psi(z)$ for $|z| \leq 3$ and that $\varphi(z)$ is smooth for $|z| \geq 2$ on $\mathbb{P}^1$. We can see that φ is a quasi-plurisubharmonic function on $\mathbb{P}^1$. Since the Lelong number $\nu(\varphi, 1/n)$ of φ at $1/n$ is 2^{-n} for every positive integer n, we see that $\mathcal{J}(\varphi) = \mathcal{O}_{\mathbb{P}^1}$ by Skoda's theorem (see, for example, [D3, (5.6) Lemma]). Therefore $\mathcal{J}(\varphi)|_P = \mathcal{O}_P$ for every $P \in \mathbb{P}^1$. On the other hand, we have $\varphi(1/n) = -\infty$ for every positive integer n. If $P = 1/n$ for some positive integer n, then $\mathcal{J}(\varphi|_P) = 0$. Thus

$$\mathcal{G} := \{H \in |\mathcal{O}_{\mathbb{P}^1}(1)| \,|\, \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}$$

is not a Zariski open set of $|\mathcal{O}_{\mathbb{P}^1}(1)|$ ($\simeq \mathbb{P}^1$).

Example 3.10. We put $K := \{z \in \mathbb{C} \,|\, |z| \leq 1\}$. Let $\{w_n\}_{n=1}^{\infty}$ be a countable dense subset of K and let $\{a_n\}_{n=1}^{\infty}$ be positive real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. We put

$$\psi(z) := \sum_{n=1}^{\infty} a_n \log |z - w_n|$$

for $z \in \mathbb{C}$. Then we see that

- ψ is subharmonic on $\mathbb{C}$ and $\psi \not\equiv -\infty$,
- $\psi = -\infty$ on an uncountable dense subset of K, and
- ψ is discontinuous almost everywhere on K.

For the details, see [Ra, Theorem 2.5.4]. By using a suitable partition of unity, we can construct a function $\varphi(z)$ on $\mathbb{P}^1$ such that $\varphi(z) = \psi(z)$ for $|z| \leq 3$ and that $\varphi(z)$ is smooth for $|z| \geq 2$ on $\mathbb{P}^1$. Then we can see that φ is a quasi-plurisubharmonic function on $\mathbb{P}^1$. In this case,

$$\mathcal{G} := \{H \in |\mathcal{O}_{\mathbb{P}^1}(1)| \,|\, \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}$$

can not be written as an intersection of countably many nonempty Zariski open sets of $|\mathcal{O}_{\mathbb{P}^1}(1)|$.

As a direct consequence of Theorem 3.6, we have:

Corollary 3.11 (Generic restriction theorem)**.** *Let X be a compact complex manifold and let φ be a quasi-plurisubharmonic function on X. Let Λ be a free linear system on X with* $\dim \Lambda \geq 1$. *We put*

$$\mathcal{H} := \{H \in \mathcal{G} \,|\, H \textit{ contains no associated primes of } \mathcal{O}_X/\mathcal{J}(\varphi)\},$$

where

$$\mathcal{G} := \{H \in \Lambda \,|\, H \textit{ is smooth and } \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}$$

as in Theorem 3.6. Then $\mathcal{H}$ is dense in Λ in the classical topology. Moreover, the following short sequence

$$0 \to \mathcal{J}(\varphi) \otimes \mathcal{O}_X(-H) \to \mathcal{J}(\varphi) \to \mathcal{J}(\varphi|_H) \to 0 \tag{3.10}$$

is exact for any member H of $\mathcal{H}$.

Proof. It is easy to see that

$$\{H \in \Lambda \,|\, H \text{ contains no associated primes of } \mathcal{O}_X/\mathcal{J}(\varphi)\}$$

is a nonempty Zariski open set of Λ since Λ is a free linear system on X. Therefore $\mathcal{H}$ is dense in Λ in the classical topology by Theorem 3.6 (see Theorem 1.10).

Let H be a member of $\mathcal{H}$. Then we obtain the following commutative diagram (see also (3.2)).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}(\varphi) \otimes \mathcal{O}_X(-H) & \xrightarrow{\alpha} & \mathcal{J}(\varphi) & \longrightarrow & \operatorname{Coker} \alpha & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(-H) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_H & \longrightarrow & 0 \end{array}$$

As in the proof of Lemma 3.2, we obtain $\operatorname{Coker} \alpha = \mathcal{J}(\varphi)|_H$. Since $H \in \mathcal{H} \subset \mathcal{G}$, we have $\mathcal{J}(\varphi)|_H = \mathcal{J}(\varphi|_H)$. Therefore, we obtain the desired short exact sequence (3.10). ☐

We will use Corollary 3.11 in Step 3 in the proof of Proposition 1.9 (see Section 4). We close this section with a remark on the multiplier ideal sheaves associated with effective $\mathbb{Q}$-divisors on smooth projective varieties.

Remark 3.12 (Multiplier ideal sheaves for effective $\mathbb{Q}$-divisors)**.** Let X be a smooth projective variety and let D be an effective $\mathbb{Q}$-divisor on X. Let S be a smooth hypersurface in X. We assume that S is not contained in any component of D. Then we obtain the following short exact sequence:

$$0 \to \mathcal{J}(X, D) \otimes \mathcal{O}_X(-S) \to \operatorname{Adj}_S(X, D) \to \mathcal{J}(S, D|_S) \to 0, \tag{3.11}$$

where $\mathcal{J}(X, D)$ (resp. $\mathcal{J}(S, D|_S)$) is the multiplier ideal sheaf associated with D (resp. $D|_S$). Note that $\operatorname{Adj}_S(X, D)$ is the adjoint ideal of D along S (see, for example, [L3, Theorem

3.3]). If S is in general position with respect to D, then we can easily see that $\mathrm{Adj}_S(X, D)$ coincides with $\mathcal{J}(X, D)$. Let H be a general member of a free linear system Λ with $\dim \Lambda \geq 1$. Then we can easily see that

$$\mathcal{J}(H, D|_H) = \mathcal{J}(X, D)|_H \tag{3.12}$$

holds by the definition of the multiplier ideal sheaves for effective $\mathbb{Q}$-divisors (see, for example, [L2, Example 9.5.9]).

By this observation, if X is a smooth projective variety and φ is a quasi-plurisubharmonic function associated with an effective $\mathbb{Q}$-divisor D on X, then $\mathcal{G}$ in Theorem 3.6 (see Theorem 1.10) and $\mathcal{H}$ in Corollary 3.11 are dense Zariski open in Λ by (3.12). Moreover, we can easily check that (3.10) in Corollary 3.11 holds for general members H of Λ by (3.11).

4. Proof of Proposition 1.9

In this section, we prove Proposition 1.9 and explain how to reduce Corollary 1.7 and Theorem 1.4 to Theorem D and Theorem A, respectively.

Proof of Proposition 1.9. Our proof of Proposition 1.9 consists of the following six steps:

Step 1 (Theorem A $\Longrightarrow$ Theorem B)**.** Since N_1 is semiample, we can take a smooth Hermitian metric h_1 on N_1 such that $\sqrt{-1}\Theta_{h_1}(N_1) \geq 0$. We put $h_2 := h_1^{b/a}$. Then

$$\sqrt{-1}(\Theta_{hh_1}(F \otimes N_1) - t\Theta_{h_2}(N_2)) \geq 0$$

for $0 < t \ll 1$. It follows that $\mathcal{J}(hh_1) = \mathcal{J}(h)$ since h_1 is smooth. Therefore, by Theorem A, we obtain the injectivity in Theorem B.

Step 2 (Theorem B $\Longrightarrow$ Theorem C)**.** We assume that $R^i f_*(K_X \otimes F \otimes \mathcal{J}(h))$ has a torsion subsheaf. Then we can find a very ample line bundle H on Y and $0 \neq t \in H^0(Y, H)$ such that

$$\alpha \colon R^i f_*(K_X \otimes F \otimes \mathcal{J}(h)) \to R^i f_*(K_X \otimes F \otimes \mathcal{J}(h)) \otimes H$$

induced by $\otimes t$ is not injective. We take a sufficiently large positive integer m such that $\mathrm{Ker}\, \alpha \otimes H^{\otimes m}$ is generated by global sections. Then we have $H^0(Y, \mathrm{Ker}\, \alpha \otimes H^{\otimes m}) \neq 0$. Without loss of generality, by making m sufficiently large, we may further assume that

$$H^p(Y, R^q f_*(K_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) = 0 \tag{4.1}$$

and

$$H^p(Y, R^q f_*(K_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m+1}) = 0 \tag{4.2}$$

for every $p > 0$ and q by the Serre vanishing theorem. By construction,

$$H^0(Y, R^i f_*(K_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) \to H^0(Y, R^i f_*(K_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m+1}) \tag{4.3}$$

induced by α is not injective. Thus, by (4.1), (4.2), and (4.3), we see that

$$H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes f^*H^{\otimes m}) \to H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes f^*H^{\otimes m+1})$$

induced by $\otimes f^*t$ is not injective. This contradicts Theorem B. Therefore $R^i f_*(K_X \otimes F \otimes \mathcal{J}(h))$ is torsion-free.

Step 3 (Theorem B $\Longrightarrow$ Theorem D)**.** We use the induction on $\dim Y$. If $\dim Y = 0$, then the statement is obvious. We take a sufficiently large positive integer m and a general divisor $B \in |H^{\otimes m}|$ such that $D := f^{-1}(B)$ is smooth, contains no associated primes of $\mathcal{O}_X/\mathcal{J}(h)$, and satisfies $\mathcal{J}(h|_D) = \mathcal{J}(h)|_D$ by Theorem 3.6 (see Theorem 1.10) and Corollary 3.11. By the Serre vanishing theorem, we may further assume that

$$H^i(Y, R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N) \otimes H^{\otimes m}) = 0 \tag{4.4}$$

for every $i > 0$ and j. By Corollary 3.11 and adjunction, we have the following short exact sequence:

$$\begin{aligned} &0 \to K_X \otimes F \otimes \mathcal{J}(h) \otimes N \to K_X \otimes F \otimes \mathcal{J}(h) \otimes N \otimes f^*H^{\otimes m} \\ &\quad \to K_D \otimes F|_D \otimes \mathcal{J}(h|_D) \otimes N|_D \to 0. \end{aligned} \tag{4.5}$$

Since B is a general member of $|H^{\otimes m}|$, we may assume that B contains no associated primes of $R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N)$ for every j. Hence, by (4.5), we can obtain

$$\begin{aligned} &0 \to R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N) \to R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N) \otimes H^{\otimes m} \\ &\quad \to R^jf_*(K_D \otimes F|_D \otimes \mathcal{J}(h|_D) \otimes N|_D) \to 0 \end{aligned}$$

for every j. By using the long exact sequence and the induction on $\dim Y$, we obtain

$$H^i(Y, R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N)) = H^i(Y, R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N) \otimes H^{\otimes m})$$

for every $i \geq 2$ and j. Thus we have

$$H^i(Y, R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N)) = 0 \tag{4.6}$$

for every $i \geq 2$ and j by (4.4). By Leray's spectral sequence, (4.4), and (4.6), we have the following commutative diagram:

$$\begin{array}{ccc} H^1(Y, \mathcal{S}^j) & \hookrightarrow & H^{j+1}(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ H^1(Y, \mathcal{S}^j \otimes H^{\otimes m}) & \hookrightarrow & H^{j+1}(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N \otimes f^*H^{\otimes m}) \end{array}$$

for every j, where $\mathcal{S}^j$ stands for $R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N)$. Since β is injective by Theorem B, we obtain that α is also injective. By (4.4), we have

$$H^1(Y, R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N) \otimes H^{\otimes m}) = 0$$

for every j. Therefore, we have $H^1(Y, R^jf_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N)) = 0$ for every j. Thus, we obtain the desired vanishing theorem in Theorem D.

Step 4 (Theorems C and D $\Longrightarrow$ Theorem B)**.** By replacing s and N_2 with $s^{\otimes m}$ and $N_2^{\otimes m}$ for some positive integer m (see also Remark 1.5), we may assume that N_2 is globally generated. We consider

$$f := \Phi_{|N_2|} \colon X \to Y.$$

Then $N_2 \simeq f^*H$ for some ample line bundle H on Y and $s = f^*t$ for some $t \in H^0(Y, H)$. We take a smooth Hermitian metric h_1 on N_1 such that $\sqrt{-1}\Theta_{h_1}(N_1) \geq 0$. Then $\sqrt{-1}\Theta_{hh_1}(F \otimes N_1) \geq 0$ and $\mathcal{J}(hh_1) = \mathcal{J}(h)$. By Theorem C, we obtain that

$$R^if_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1)$$

is torsion-free for every i. Therefore, the map

$$R^i f_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) \to R^i f_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) \otimes H$$

induced by $\otimes t$ is injective for every i. By $N_2 \simeq f^*H$, we see that

$$H^0(Y, R^i f_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1)) \to H^0(Y, R^i f_*(K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2)) \tag{4.7}$$

induced by $\otimes t$ is injective for every i. By Theorem D, (4.7) implies that

$$H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1) \to H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes N_1 \otimes N_2)$$

induced by $\otimes s$ is injective for every i.

Step 5 (Theorem D $\Longrightarrow$ Theorem E)**.** The following lemma implies that $R^j f_*(K_X \otimes F \otimes \mathcal{J}(h))$ is a GV-sheaf by [Sc, Theorem 25.5] (see also [Ha] and [PP]). For simplicity, we put $\mathcal{F}^j := R^j f_*(K_X \otimes F \otimes \mathcal{J}(h))$ for every j.

Lemma 4.1. *For every finite étale morphism $p \colon B \to A$ of Abelian varieties and every ample line bundle H on B, we have*

$$H^i(B, H \otimes p^*\mathcal{F}^j) = 0 \tag{4.8}$$

for every $i > 0$ and j.

Proof of Lemma 4.1. We put $Z := B \times_A X$. Then we have the following commutative diagram.

$$\begin{array}{ccc} Z & \xrightarrow{\ q\ } & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\ p\ ]{} & A \end{array} \tag{4.9}$$

By construction, q is also finite and étale. Therefore, we have $q^*K_X = K_Z$ and $q^*\mathcal{J}(h) = \mathcal{J}(q^*h)$. By the flat base change theorem,

$$p^*R^j f_*(K_X \otimes F \otimes \mathcal{J}(h)) \simeq R^j g_*(K_Z \otimes q^*F \otimes \mathcal{J}(q^*h)).$$

By Theorem D, we obtain the desired vanishing (4.8). □

Step 6 (Theorems C and E $\Longrightarrow$ Theorem F)**.** By Theorem C, we have $\mathcal{F}^j := R^j f_*(K_X \otimes F \otimes \mathcal{J}(h)) = 0$ for $j > \dim X - \dim f(X)$. We consider the following spectral sequence:

$$E_2^{pq} = H^p(A, \mathcal{F}^q \otimes L) \Rightarrow H^{p+q}(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes f^*L)$$

for every $L \in \operatorname{Pic}^0(A)$. Note that $\mathcal{F}^j$ is a GV-sheaf for every j and that $\mathcal{F}^j = 0$ for $j > \dim X - \dim f(X)$. Then we obtain

$$\operatorname{codim}_{\operatorname{Pic}^0(A)}\{L \in \operatorname{Pic}^0(A) \,|\, H^i(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes f^*L) \neq 0\} \geq i - (\dim X - \dim f(X))$$

for every $i \geq 0$.

We completed the proof of Proposition 1.9. □

We prove Corollary 1.7 as an application of Theorem D.

Proof of Corollary 1.7 (Theorem D $\Longrightarrow$ Corollary 1.7). By Theorem D, we have

$$H^p(Y, R^i f_*(K_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m-p}) = 0$$

for every $p \geq 1$, $i \geq 0$, and $m \geq \dim Y + 1$. Thus, the Castelnuovo–Mumford regularity (see [L1, Section 1.8]) implies that $R^i f_*(K_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}$ is globally generated for every $i \geq 0$ and $m \geq \dim Y + 1$. □

We close this section with a proof of Theorem 1.4 based on Theorem A for the reader's convenience.

Proof of Theorem 1.4 (Theorem A $\Longrightarrow$ Theorem 1.4). Let A be an ample line bundle on V. Then there exists a sufficiently large positive integer m such that $A^{\otimes m}$ is very ample and that $H^i(V, K_V \otimes L \otimes \mathcal{J}(h_L) \otimes A^{\otimes m}) = 0$ for every $i > 0$ by the Serre vanishing theorem. We can take a smooth Hermitian metric h_A on A such that $\sqrt{-1}\Theta_{h_A}(A)$ is a smooth positive $(1,1)$-form on V. Therefore, we have $\sqrt{-1}\Theta_{h_A^m}(A^{\otimes m}) \geq 0$. By the condition $\sqrt{-1}\Theta_{h_L}(L) \geq \varepsilon\omega$, we see that $\sqrt{-1}(\Theta_{h_L}(L) - t\Theta_{h_A^m}(A^{\otimes m})) \geq 0$ for some $0 < t \ll 1$. We take a nonzero global section s of $A^{\otimes m}$. By Theorem A, we see that

$$\times s \colon H^i(V, K_V \otimes L \otimes \mathcal{J}(h_L)) \to H^i(V, K_V \otimes L \otimes \mathcal{J}(h_L) \otimes A^{\otimes m})$$

is injective for every i. Thus, we obtain that $H^i(V, K_V \otimes L \otimes \mathcal{J}(h_L)) = 0$ for every $i > 0$. □

5. Proof of Theorem A

In this section, we will give the proof of Theorem A.

Theorem 5.1 (Theorem A)**.** *Let F (resp. M) be a line bundle on a compact Kähler manifold X with a singular Hermitian metric h (resp. a smooth Hermitian metric h_M) satisfying*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ and } \sqrt{-1}\Theta_h(F) - b\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ for some } b > 0.$$

Then for a (nonzero) section $s \in H^0(X, M)$, the multiplication map induced by $\otimes s$

$$\times s \colon H^q(X, K_X \otimes F \otimes \mathcal{J}(h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes M)$$

is injective for every q. Here K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

Proof of Theorem 5.1 (Theorem A). The proof can be divided into four steps.

Step 1. Throughout the proof, we fix a Kähler form ω on X. For a given singular Hermitian metric h on F, by applying [DPS, Theorem 2.3] to the weight of h, we obtain a family of singular Hermitian metrics $\{h_\varepsilon\}_{1 \gg \varepsilon > 0}$ on F with the following properties:

(a) h_ε is smooth on $Y_\varepsilon := X \setminus Z_\varepsilon$, where Z_ε is a proper closed analytic subset on X.
(b) $h_{\varepsilon'} \leq h_{\varepsilon''} \leq h$ holds on X when $\varepsilon' > \varepsilon'' > 0$.
(c) $\mathcal{J}(h) = \mathcal{J}(h_\varepsilon)$ on X.
(d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq b\sqrt{-1}\Theta_{h_M}(M) - \varepsilon\omega$ on X.

Here property (d) is obtained from the assumption $\sqrt{-1}\Theta_h(F) \geq b\sqrt{-1}\Theta_{h_M}(M)$.

The main difficulty of the proof is that Z_ε may essentially depend on ε, compared to [MaS4] in which Z_ε is independent of ε. To overcome this difficulty, we consider suitable complete Kähler forms $\{\omega_{\varepsilon,\delta}\}_{\delta > 0}$ on Y_ε such that $\omega_{\varepsilon,\delta}$ converges to ω as $\delta \to 0$. To construct such complete Kähler forms, we first take a complete Kähler form ω_ε on Y_ε with the following properties:

- ω_{ε} is a complete Kähler form on Y_{ε}.
- $\omega_{\varepsilon} \geq \omega$ on Y_{ε}.
- $\omega_{\varepsilon} = \sqrt{-1}\partial\overline{\partial}\Psi_{\varepsilon}$ for some bounded function Ψ_{ε} on a neighborhood of every $p \in X$.

See [F4, Section 3] for the construction of ω_{ε}. For the Kähler form $\omega_{\varepsilon,\delta}$ on Y_{ε} defined to be

$$\omega_{\varepsilon,\delta} := \omega + \delta\omega_{\varepsilon} \text{ for } \varepsilon \text{ and } \delta \text{ with } 0 < \delta \ll \varepsilon,$$

it is easy to see the following properties hold:

(A) $\omega_{\varepsilon,\delta}$ is a complete Kähler form on $Y_{\varepsilon} = X \setminus Z_{\varepsilon}$ for every $\delta > 0$.
(B) $\omega_{\varepsilon,\delta} \geq \omega$ on Y_{ε} for every $\delta > 0$.
(C) $\Psi + \delta\Psi_{\varepsilon}$ is a bounded local potential function of $\omega_{\varepsilon,\delta}$ and converges to Ψ as $\delta \to 0$.

Here Ψ is a local potential function of ω. The first property enables us to consider harmonic forms on the noncompact Y_{ε}, and the third property enables us to construct the de Rham–Weil isomorphism from the $\overline{\partial}$-cohomology on Y_{ε} to the Čech cohomology on X.

Remark 5.2. In the proof of Theorem 5.1, we actually consider only a countable sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ (resp. $\{\delta_{\ell}\}_{\ell=1}^{\infty}$) conversing to zero since we need to apply Cantor's diagonal argument, but we often use the notation ε (resp. δ) for simplicity.

For the proof, it is sufficient to show that an arbitrary cohomology class $\eta \in H^q(X, K_X \otimes F \otimes \mathcal{J}(h))$ satisfying $s\eta = 0 \in H^q(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes M)$ is actually zero. We represent the cohomology class $\eta \in H^q(X, K_X \otimes F \otimes \mathcal{J}(h))$ by a $\overline{\partial}$-closed F-valued (n, q)-form u with $\|u\|_{h,\omega} < \infty$ by using the standard de Rham–Weil isomorphism

$$H^q(X, K_X \otimes F \otimes \mathcal{J}(h)) \cong \frac{\operatorname{Ker}\overline{\partial}\colon L^{n,q}_{(2)}(F)_{h,\omega} \to L^{n,q+1}_{(2)}(F)_{h,\omega}}{\operatorname{Im}\overline{\partial}\colon L^{n,q-1}_{(2)}(F)_{h,\omega} \to L^{n,q}_{(2)}(F)_{h,\omega}}.$$

Here $\overline{\partial}$ is the densely defined closed operator defined by the usual $\overline{\partial}$-operator and $L^{n,q}_{(2)}(F)_{h,\omega}$ is the L^2-space of F-valued (n, q)-forms on X with respect to the L^2-norm $\| \bullet \|_{h,\omega}$ defined by

$$\| \bullet \|^2_{h,\omega} := \int_X | \bullet |^2_{h,\omega} \, dV_{\omega},$$

where $dV_{\omega} := \omega^n/n!$ and $n := \dim X$. Our purpose is to prove that u is $\overline{\partial}$-exact (namely, $u \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(F)_{h,\omega}$) under the assumption that the cohomology class of su is zero in $H^q(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes M)$.

From now on, we mainly consider the L^2-space $L^{n,q}_{(2)}(Y_{\varepsilon}, F)_{h_{\varepsilon},\omega_{\varepsilon,\delta}}$ of F-valued (n, q)-forms on Y_{ε} (not X) with respect to h_{ε} and $\omega_{\varepsilon,\delta}$ (not h and ω). For simplicity we put

$$L^{n,q}_{(2)}(F)_{\varepsilon,\delta} := L^{n,q}_{(2)}(Y_{\varepsilon}, F)_{h_{\varepsilon},\omega_{\varepsilon,\delta}} \quad \text{and} \quad \| \bullet \|_{\varepsilon,\delta} := \| \bullet \|_{h_{\varepsilon},\omega_{\varepsilon,\delta}}.$$

The following inequality plays an important role in the proof.

$$\|u\|_{\varepsilon,\delta} \leq \|u\|_{h,\omega_{\varepsilon,\delta}} \leq \|u\|_{h,\omega} < \infty. \tag{5.1}$$

In particular, the norm $\|u\|_{\varepsilon,\delta}$ is uniformly bounded since the right hand side is independent of ε, δ. The first inequality follows from property (b) of h_{ε}, and the second inequality follows from Lemma 2.4 and property (B) of $\omega_{\varepsilon,\delta}$. Strictly speaking, the left hand side should be $\|u|_{Y_{\varepsilon}}\|_{\varepsilon,\delta}$, but we often omit the symbol of restriction. Now we have the following orthogonal decomposition (for example see [MaS4, Proposition 5.8]).

$$L^{n,q}_{(2)}(F)_{\varepsilon,\delta} = \operatorname{Im}\overline{\partial} \oplus \mathcal{H}^{n,q}_{\varepsilon,\delta}(F) \oplus \operatorname{Im}\overline{\partial}^*_{\varepsilon,\delta}.$$

Here $\overline{\partial}^*_{\varepsilon,\delta}$ is (the maximal extension of) the formal adjoint of the $\overline{\partial}$-operator and $\mathcal{H}^{n,q}_{\varepsilon,\delta}(F)$ is the set of harmonic F-valued (n,q)-forms on Y_ε, namely

$$\mathcal{H}^{n,q}_{\varepsilon,\delta}(F) := \{w \in L^{n,q}_{(2)}(F)_{\varepsilon,\delta} \,|\, \overline{\partial} w = 0 \text{ and } \overline{\partial}^*_{\varepsilon,\delta} w = 0\}.$$

Remark 5.3. The formal adjoint coincides with the Hilbert space adjoint since $\omega_{\varepsilon,\delta}$ is complete for $\delta > 0$ (see, for example, [D4, (3.2) Theorem in Chapter VIII]). The $\overline{\partial}$-operator also depends on h_ε and $\omega_{\varepsilon,\delta}$ in the sense that the domain and range of the closed operator $\overline{\partial}$ depend on them, but we abbreviate $\overline{\partial}_{\varepsilon,\delta}$ to $\overline{\partial}$.

The F-valued (n,q)-form u (representing η) belongs to $L^{n,q}_{(2)}(F)_{\varepsilon,\delta}$ by (5.1), and thus u can be decomposed as follows:

$$u = \overline{\partial} w_{\varepsilon,\delta} + u_{\varepsilon,\delta} \quad \text{for some } w_{\varepsilon,\delta} \in \operatorname{Dom} \overline{\partial} \subset L^{n,q-1}_{(2)}(F)_{\varepsilon,\delta} \text{ and } u_{\varepsilon,\delta} \in \mathcal{H}^{n,q}_{\varepsilon,\delta}(F). \tag{5.2}$$

Note that the orthogonal projection of u to $\operatorname{Im} \overline{\partial}^*_{\varepsilon,\delta}$ must be zero since u is $\overline{\partial}$-closed.

Step 2. The purpose of this step is to prove Proposition 5.7, which reduces the proof to the study of the asymptotic behavior of the norm of $su_{\varepsilon,\delta}$. When we consider a suitable limit of $u_{\varepsilon,\delta}$ in the following proposition, we need to carefully choose the L^2-space since the L^2-space $L^{n,q}_{(2)}(F)_{\varepsilon,\delta}$ depends on ε and δ. We remark that $\{\varepsilon\}_{\varepsilon>0}$ and $\{\delta\}_{\delta>0}$ denote countable sequences converging to zero (see Remark 5.2). Let $\{\delta_0\}_{\delta_0>0}$ denote another countable sequence converging to zero.

Proposition 5.4. *There exist a subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ of $\{\delta\}_{\delta>0}$ and $\alpha_\varepsilon \in L^{n,q}_{(2)}(F)_{h_\varepsilon,\omega}$ with the following properties:*

- *For any $\varepsilon, \delta_0 > 0$, as δ_ν tends to 0,*

  $u_{\varepsilon,\delta_\nu}$ converges to α_ε with respect to the weak L^2-topology in $L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0}$.
- *For any $\varepsilon > 0$,*

$$\|\alpha_\varepsilon\|_{h_\varepsilon,\omega} \leq \varliminf_{\delta_0\to 0} \|\alpha_\varepsilon\|_{\varepsilon,\delta_0} \leq \varliminf_{\delta_\nu\to 0} \|u_{\varepsilon,\delta_\nu}\|_{\varepsilon,\delta_\nu} \leq \|u\|_{h,\omega}.$$

Remark 5.5. The weak limit α_ε does not depend on δ_0, and the subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ does not depend on ε and δ_0.

Proof of Proposition 5.4. For given $\varepsilon, \delta_0 > 0$, by taking a sufficiently small δ with $0 < \delta < \delta_0$, we have

$$\|u_{\varepsilon,\delta}\|_{\varepsilon,\delta_0} \leq \|u_{\varepsilon,\delta}\|_{\varepsilon,\delta} \leq \|u\|_{\varepsilon,\delta} \leq \|u\|_{h,\omega}. \tag{5.3}$$

The first inequality follows from $\omega_{\varepsilon,\delta} \leq \omega_{\varepsilon,\delta_0}$ and Lemma 2.4, the second inequality follows since $u_{\varepsilon,\delta}$ is the orthogonal projection of u with respect to ε, δ, and the last inequality follows from (5.1). Since the right hand side is independent of δ, the family $\{u_{\varepsilon,\delta}\}_{\delta>0}$ is uniformly bounded in $L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0}$. Therefore, there exists a subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ of $\{\delta\}_{\delta>0}$ such that $u_{\varepsilon,\delta_\nu}$ converges to $\alpha_{\varepsilon,\delta_0}$ with respect to the weak L^2-topology in $L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0}$ This subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ may depend on ε, δ_0, but we can choose a subsequence independent of them by applying Cantor's diagonal argument.

Now we show that $\alpha_{\varepsilon,\delta_0}$ does not depend on δ_0. For arbitrary δ_0', δ_0'' with $0 < \delta_0' \leq \delta_0''$, the natural inclusion $L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0'} \to L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0''}$ is a bounded operator (continuous linear map) by $\|\bullet\|_{\varepsilon,\delta_0''} \leq \|\bullet\|_{\varepsilon,\delta_0'}$, and thus $u_{\varepsilon,\delta_\nu}$ weakly converges to $\alpha_{\varepsilon,\delta_0'}$ in not only $L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0'}$

but also $L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0''}$ by Lemma 2.5. Therefore, it follows that $\alpha_{\varepsilon,\delta_0'} = \alpha_{\varepsilon,\delta_0''}$ since the weak limit is unique.

Finally, we consider the norm of α_ε. It is easy to see that

$$\|\alpha_\varepsilon\|_{\varepsilon,\delta_0} \leq \varliminf_{\delta_\nu \to 0} \|u_{\varepsilon,\delta_\nu}\|_{\varepsilon,\delta_0} \leq \varliminf_{\delta_\nu \to 0} \|u_{\varepsilon,\delta_\nu}\|_{\varepsilon,\delta_\nu} \leq \|u\|_{h,\omega}.$$

The first inequality follows since the norm is lower semicontinuous with respect to the weak convergence, the second inequality follows from $\omega_{\varepsilon,\delta_0} \geq \omega_{\varepsilon,\delta_\nu}$, and the last inequality follows from (5.3). Fatou's lemma yields

$$\|\alpha_\varepsilon\|^2_{h_\varepsilon,\omega} = \int_{Y_\varepsilon} |\alpha_\varepsilon|^2_{h_\varepsilon,\omega}\, dV_\omega \leq \varliminf_{\delta_0 \to 0} \int_{Y_\varepsilon} |\alpha_\varepsilon|^2_{h_\varepsilon,\omega_{\varepsilon,\delta_0}}\, dV_{\omega_{\varepsilon,\delta_0}} = \varliminf_{\delta_0 \to 0} \|\alpha_\varepsilon\|^2_{\varepsilon,\delta_0}.$$

These inequalities lead to the desired estimate in the proposition. □

For simplicity, we use the same notation $\{u_{\varepsilon,\delta}\}_{\delta>0}$ for the subsequence $\{u_{\varepsilon,\delta_\nu}\}_{\nu=1}^\infty$ in Proposition 5.4. We fix $\varepsilon_0 > 0$ and consider the weak limit of α_ε in the fixed L^2-space $L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega}$. For a sufficiently small $\varepsilon > 0$, we have

$$\|\alpha_\varepsilon\|_{h_{\varepsilon_0},\omega} \leq \|\alpha_\varepsilon\|_{h_\varepsilon,\omega} \leq \|u\|_{h,\omega}$$

by property (b) and Proposition 5.4. By taking a subsequence of $\{\alpha_\varepsilon\}_{\varepsilon>0}$, we may assume that α_ε weakly converges to some α in $L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega}$.

Proposition 5.6. *If the weak limit α is zero in $L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega}$, then the cohomology class η is zero in $H^q(X, K_X \otimes F \otimes \mathcal{J}(h))$.*

Proof of Proposition 5.6. For every δ with $0 < \delta \leq \delta_0$, we can easily check

$$u - u_{\varepsilon,\delta} \in \operatorname{Im}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta} \subset \operatorname{Im}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0}$$

from the construction of $u_{\varepsilon,\delta}$. As $\delta \to 0$, we obtain

$$u - \alpha_\varepsilon \in \operatorname{Im}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0}$$

by Lemma 2.6 and Proposition 5.4. We remark that $\operatorname{Im}\overline{\partial}$ is a closed subspace (see [MaS4, Proposition 5.8]). On the other hand, we have the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Ker}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0} & \xrightarrow{q_1} & \dfrac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(F)_{\varepsilon,\delta_0} & \xrightarrow[f_1]{\cong} & \check{H}^q(X, K_X \otimes F \otimes \mathcal{J}(h)) \\
\uparrow{\scriptstyle j_1} & & & & \cong \uparrow{\scriptstyle f_2} \\
\operatorname{Ker}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{h_\varepsilon,\omega} & \xrightarrow{j_2} & \operatorname{Ker}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega} & \xrightarrow{q_2} & \dfrac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega}.
\end{array}$$

Here j_1, j_2 are the natural inclusions, q_1, q_2 are the natural quotient maps, and f_1, f_2 are the de Rham–Weil isomorphisms (see [MaS4, Proposition 5.5] for the construction). Strictly speaking, f_1 is an isomorphism to $\check{H}^q(X, K_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$, but which coincides with $\check{H}^q(X, K_X \otimes F \otimes \mathcal{J}(h))$ by property (c). To check that j_2 is well-defined, we have to see that $\overline{\partial} w = 0$ on Y_{ε_0} if $\overline{\partial} w = 0$ on Y_ε. By the L^2-integrability and [D4, (7.3) Lemma, Chapter VIII], the equality $\overline{\partial} w = 0$ can be extended from Y_ε to X (in particular Y_{ε_0}). The key point here is the L^2-integrability with respect to ω (not $\omega_{\varepsilon,\delta}$).

Since $j_2(u-\alpha_\varepsilon)$ weakly converges to $j_2(u-\alpha)$ and the $\overline{\partial}$-cohomology is finite dimensional, we obtain

$$\lim_{\varepsilon\to 0} q_2(u-\alpha_\varepsilon) = q_2(u-\alpha) = q_2(u)$$

by Lemma 2.5 and the assumption $\alpha = 0$. On the other hand, it follows that $q_1(u-\alpha_\varepsilon) = 0$ from the first half argument. Hence, we have $q_2(u) = 0$, that is, $u \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega}$. From $q_2(u) = 0$, we can prove the conclusion, that is, $u \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(F)_{h,\omega}$. Indeed, we can obtain $q_3(u) = 0$ (which leads to the conclusion) by the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Ker}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega} & \xrightarrow{q_2} & \dfrac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega} & \xrightarrow[f_2]{\cong} & \check{H}^q(X, K_X\otimes F\otimes \mathcal{J}(h_{\varepsilon_0})) \\
\uparrow & & & & \Vert \\
\operatorname{Ker}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{h,\omega} & \xrightarrow{q_3} & \dfrac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(F)_{h,\omega} & \xrightarrow[f_3]{\cong} & \check{H}^q(X, K_X\otimes F\otimes \mathcal{J}(h)).
\end{array}$$

□

At the end of this step, we prove Proposition 5.7.

Proposition 5.7. *If we have*

$$\varliminf_{\varepsilon\to 0}\varliminf_{\delta\to 0} \|su_{\varepsilon,\delta}\|_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}} = 0,$$

then the weak limit α is zero. In particular, the cohomology class η is zero by Proposition 5.6.

Proof of Proposition 5.7. In the proof, we compare the norm of $u_{\varepsilon,\delta}$ with the norm of $su_{\varepsilon,\delta}$. For this purpose, we define $Y^k_{\varepsilon_0}$ to be

$$Y^k_{\varepsilon_0} := \{y \in Y_{\varepsilon_0} \mid |s|_{h_M} > 1/k \text{ at } y\}$$

for $k \gg 0$. Note the subset $Y^k_{\varepsilon_0}$ is an open set in Y_{ε_0}. It follows that the restriction $\alpha_\varepsilon|_{Y^k_{\varepsilon_0}}$ also weakly converges to $\alpha|_{Y^k_{\varepsilon_0}}$ in $L^{n,q}_{(2)}(Y^k_{\varepsilon_0}, F)_{h_{\varepsilon_0},\omega}$ since the restriction map $L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega} \to L^{n,q}_{(2)}(Y^k_{\varepsilon_0}, F)_{h_{\varepsilon_0},\omega}$ is a bounded operator and α_ε weakly converges to α in $L^{n,q}_{(2)}(F)_{h_{\varepsilon_0},\omega}$. Since the norm is lower semicontinuous with respect to the weak convergence, we obtain the estimate for the L^2-norm on $Y^k_{\varepsilon_0}$

$$\|\alpha\|_{Y^k_{\varepsilon_0},h_{\varepsilon_0},\omega} \le \varliminf_{\varepsilon\to 0}\|\alpha_\varepsilon\|_{Y^k_{\varepsilon_0},h_{\varepsilon_0},\omega} \le \varliminf_{\varepsilon\to 0}\|\alpha_\varepsilon\|_{Y^k_{\varepsilon_0},h_\varepsilon,\omega}$$

by property (b). By the same argument, the restriction $u_{\varepsilon,\delta}|_{Y^k_{\varepsilon_0}}$ weakly converges to $\alpha_\varepsilon|_{Y^k_{\varepsilon_0}}$ in $L^{n,q}_{(2)}(Y^k_{\varepsilon_0}, F)_{\varepsilon,\delta_0}$, and thus we obtain

$$\|\alpha_\varepsilon\|_{Y^k_{\varepsilon_0},\varepsilon,\delta_0} \le \varliminf_{\delta\to 0}\|u_{\varepsilon,\delta}\|_{Y^k_{\varepsilon_0},\varepsilon,\delta_0} \le \varliminf_{\delta\to 0}\|u_{\varepsilon,\delta}\|_{Y^k_{\varepsilon_0},\varepsilon,\delta}$$

by Lemma 2.4. As $\delta_0 \to 0$ in the above inequality, we have

$$\|\alpha_\varepsilon\|_{Y^k_{\varepsilon_0},h_\varepsilon,\omega} \le \varliminf_{\delta_0\to 0}\|\alpha_\varepsilon\|_{Y^k_{\varepsilon_0},\varepsilon,\delta_0} \le \varliminf_{\delta\to 0}\|u_{\varepsilon,\delta}\|_{Y^k_{\varepsilon_0},\varepsilon,\delta}$$

by Fatou's lemma (see the argument in Proposition 5.4). These inequalities yield

$$\|\alpha\|_{Y^k_{\varepsilon_0},h_{\varepsilon_0},\omega} \le \varliminf_{\varepsilon\to 0}\varliminf_{\delta\to 0}\|u_{\varepsilon,\delta}\|_{Y^k_{\varepsilon_0},\varepsilon,\delta}.$$

On the other hand, it follows that

$$\|u_{\varepsilon,\delta}\|_{Y^k_{\varepsilon_0},\varepsilon,\delta} \le k\|su_{\varepsilon,\delta}\|_{Y^k_{\varepsilon_0},h_\varepsilon h_M,\omega_{\varepsilon,\delta}} \le k\|su_{\varepsilon,\delta}\|_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}}$$

since the inequality $1/k < |s|_{h_M}$ holds on $Y^k_{\varepsilon_0}$. This implies that $\alpha = 0$ on $Y^k_{\varepsilon_0}$ for an arbitrary $k \gg 0$. From $\bigcup_{k\gg 0} Y^k_{\varepsilon_0} = Y_{\varepsilon_0}\setminus\{s=0\}$, we obtain the desired conclusion. □

Step 3. The purpose of this step is to prove the following proposition:

Proposition 5.8.

$$\lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\|\overline{\partial}^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}} = 0.$$

Proof of Proposition 5.8. In the proof, we will often use (5.3). By applying Bochner–Kodaira–Nakano's identity and the density lemma to $u_{\varepsilon,\delta}$ and $su_{\varepsilon,\delta}$ (see [MaS1, Proposition 2.8]), we obtain

$$0 = \langle\!\langle \sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\omega_{\varepsilon,\delta}}u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle\!\rangle_{\varepsilon,\delta} + \|D'^*_{\varepsilon,\delta}u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}, \tag{5.4}$$

$$\|\overline{\partial}^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|^2_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}} = \langle\!\langle \sqrt{-1}\Theta_{h_\varepsilon h_M}(F\otimes M)\Lambda_{\omega_{\varepsilon,\delta}}su_{\varepsilon,\delta}, su_{\varepsilon,\delta}\rangle\!\rangle_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}} + \|D'^*_{\varepsilon,\delta}su_{\varepsilon,\delta}\|^2_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}}, \tag{5.5}$$

where $D'^*_{\varepsilon,\delta}$ is the adjoint operator of the $(1,0)$-part of the Chern connection D_{h_ε}. Here we used the fact that $u_{\varepsilon,\delta}$ is harmonic and $\overline{\partial}(su_{\varepsilon,\delta}) = s\overline{\partial}u_{\varepsilon,\delta} = 0$. Now we have

$$\sqrt{-1}\Theta_{h_\varepsilon}(F) \ge b\sqrt{-1}\Theta_{h_M}(M) - \varepsilon\omega \ge -\varepsilon\omega \ge -\varepsilon\omega_{\varepsilon,\delta}$$

by property (d) and property (B). Hence, the integrand $g_{\varepsilon,\delta}$ of the first term of (5.4) satisfies

$$-\varepsilon q|u_{\varepsilon,\delta}|^2_{\varepsilon,\delta} \le g_{\varepsilon,\delta} := \langle \sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\omega_{\varepsilon,\delta}}u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle_{\varepsilon,\delta}. \tag{5.6}$$

For the precise argument, see [MaS4, Step 2 in the proof of Theorem 3.1]. Then by (5.4), we can easily see

$$\begin{aligned}
\lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\Big(\int_{\{g_{\varepsilon,\delta}\ge 0\}} g_{\varepsilon,\delta}\,dV_{\omega_{\varepsilon,\delta}} + \|D'^*_{\varepsilon,\delta}u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\Big) &= \lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\Big(-\int_{\{g_{\varepsilon,\delta}\le 0\}} g_{\varepsilon,\delta}\,dV_{\omega_{\varepsilon,\delta}}\Big)\\
&\le \lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\Big(\varepsilon q\int_{\{g_{\varepsilon,\delta}\le 0\}} |u_{\varepsilon,\delta}|^2_{\varepsilon,\delta}\,dV_{\omega_{\varepsilon,\delta}}\Big)\\
&\le \lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\Big(\varepsilon q\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\Big) = 0.
\end{aligned}$$

Here we used (5.3) in the last equality.

On the other hand, by $\sqrt{-1}\Theta_{h_\varepsilon}(F) \ge b\sqrt{-1}\Theta_{h_M}(M) - \varepsilon\omega_{\varepsilon,\delta}$, we have

$$\begin{aligned}
&\langle\!\langle \sqrt{-1}\Theta_{h_\varepsilon h_M}(F\otimes M)\Lambda_{\omega_{\varepsilon,\delta}}su_{\varepsilon,\delta}, su_{\varepsilon,\delta}\rangle\!\rangle_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}}\\
\le&\Big(1+\frac{1}{b}\Big)\int_{Y_\varepsilon}|s|^2_{h_M}g_{\varepsilon,\delta}\,dV_{\omega_{\varepsilon,\delta}} + \frac{\varepsilon q}{b}\int_{Y_\varepsilon}|s|^2_{h_M}|u_{\varepsilon,\delta}|^2_{\varepsilon,\delta}\,dV_{\omega_{\varepsilon,\delta}}\\
\le&\Big(1+\frac{1}{b}\Big)\sup_X|s|^2_{h_M}\Big\{\int_{\{g_{\varepsilon,\delta}\ge 0\}} g_{\varepsilon,\delta}\,dV_{\omega_{\varepsilon,\delta}} + \frac{\varepsilon q}{b}\sup_X|s|^2_{h_M}\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}\Big\}.
\end{aligned}$$

Furthermore, since $D'^{*}_{\varepsilon,\delta}$ can be expressed as $D'^{*}_{\varepsilon,\delta} = -*\overline{\partial}*$ by the Hodge star operator $*$ with respect to $\omega_{\varepsilon,\delta}$, we have

$$\|D'^{*}_{\varepsilon,\delta} s u_{\varepsilon,\delta}\|^2_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}} = \|s D'^{*}_{\varepsilon,\delta} u_{\varepsilon,\delta}\|^2_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}} \leq \sup_X |s|^2_{h_M} \|D'^{*}_{\varepsilon,\delta} u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}.$$

The right-hand side of (5.5) can be shown to converge to zero by the first half argument and these inequalities. □

Step 4. In this step, we construct solutions $v_{\varepsilon,\delta}$ of the $\overline{\partial}$-equation $\overline{\partial} v_{\varepsilon,\delta} = s u_{\varepsilon,\delta}$ with suitable L^2-norm, and we finish the proof of Theorem 5.1. The proof of the following proposition is a slight variant of that of [MaS4, Theorem 5.9].

Proposition 5.9. *There exist F-valued $(n, q-1)$-forms $w_{\varepsilon,\delta}$ on Y_ε with the following properties:*

- $\overline{\partial} w_{\varepsilon,\delta} = u - u_{\varepsilon,\delta}$.
- $\overline{\lim}_{\delta\to 0} \|w_{\varepsilon,\delta}\|_{\varepsilon,\delta}$ *can be bounded by a constant independent of* ε.

Before we begin to prove Proposition 5.9, we recall the content in [MaS4, Section 5] with our notation. For a finite open cover $\mathcal{U} := \{B_i\}_{i\in I}$ of X by sufficiently small Stein open sets B_i, we can construct

$$f_{\varepsilon,\delta}\colon \operatorname{Ker}\overline{\partial} \text{ in } L^{n,q}_{(2)}(F)_{\varepsilon,\delta} \longrightarrow \operatorname{Ker}\mu \text{ in } C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$$

such that $f_{\varepsilon,\delta}$ induces the de Rham–Weil isomorphism

$$\overline{f_{\varepsilon,\delta}}\colon \frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(F)_{\varepsilon,\delta} \xrightarrow{\ \cong\ } \frac{\operatorname{Ker}\mu}{\operatorname{Im}\mu} \text{ of } C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_\varepsilon)). \tag{5.7}$$

Here $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$ is the space of q-cochains calculated by $\mathcal{U}$ and μ is the coboundary operator. We remark that $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$ is a Fréchet space with respect to the seminorm $p_{K_{i_0\ldots i_q}}(\bullet)$ defined to be

$$p_{K_{i_0\ldots i_q}}(\{\beta_{i_0\ldots i_q}\})^2 := \int_{K_{i_0\ldots i_q}} |\beta_{i_0\ldots i_q}|^2_{h_\varepsilon,\omega}\, dV_\omega$$

for a relatively compact set $K_{i_0\ldots i_q} \Subset B_{i_0\ldots i_q} := B_{i_0} \cap \cdots \cap B_{i_q}$ (see [MaS4, Theorem 5.3]). The construction of $f_{\varepsilon,\delta}$ is essentially the same as in the proof of [MaS4, Proposition 5.5]. The only difference is that we use Lemma 5.12 instead of [MaS4, Lemma 5.4] when we locally solve the $\overline{\partial}$-equation to construct $f_{\varepsilon,\delta}$. Lemma 5.12 will be given at the end of this step. We prove Proposition 5.9 by replacing some constants appearing in the proof of [MaS4, Theorem 5.9] with $C_{\varepsilon,\delta}$ appearing in Lemma 5.12.

Proof of Proposition 5.9. We put $U_{\varepsilon,\delta} := u - u_{\varepsilon,\delta} \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(F)_{\varepsilon,\delta}$. Then there exist the F-valued $(n, q-k-1)$-forms $\beta^{\varepsilon,\delta}_{i_0\ldots i_k}$ on $B_{i_0\ldots i_k} \setminus Z_\varepsilon$ satisfying

$$(*)\begin{cases} \overline{\partial}\beta^{\varepsilon,\delta}_{i_0} & = U_{\varepsilon,\delta}|_{B_{i_0}\setminus Z_\varepsilon}, \\ \overline{\partial}\{\beta^{\varepsilon,\delta}_{i_0 i_1}\} & = \mu\{\beta^{\varepsilon,\delta}_{i_0}\}, \\ \overline{\partial}\{\beta^{\varepsilon,\delta}_{i_0 i_1 i_2}\} & = \mu\{\beta^{\varepsilon,\delta}_{i_0 i_1}\}, \\ & \vdots \\ \overline{\partial}\{\beta^{\varepsilon,\delta}_{i_0\ldots i_{q-1}}\} & = \mu\{\beta^{\varepsilon,\delta}_{i_0\ldots i_{q-2}}\}, \\ f_{\varepsilon,\delta}(U_{\varepsilon,\delta}) & = \mu\{\beta^{\varepsilon,\delta}_{i_0\ldots i_{q-1}}\}. \end{cases}$$

Here $\beta^{\varepsilon,\delta}_{i_0\dots i_k}$ is the solution of the above equation whose norm is minimum among all the solutions (see the construction of $f_{\varepsilon,\delta}$ in [MaS4, Proposition 5.5]). For example, $\beta^{\varepsilon,\delta}_{i_0}$ is the solution of $\overline{\partial}\beta^{\varepsilon,\delta}_{i_0} = U_{\varepsilon,\delta}$ on $B_{i_0} \setminus Z_\varepsilon$ whose norm $\|\beta^{\varepsilon,\delta}_{i_0}\|_{\varepsilon,\delta}$ is minimum among all the solutions. In particular $\|\beta^{\varepsilon,\delta}_{i_0}\|^2_{\varepsilon,\delta} \le C_{\varepsilon,\delta}\|U_{\varepsilon,\delta}\|^2_{B_{i_0},\varepsilon,\delta} \le C_{\varepsilon,\delta}\|U_{\varepsilon,\delta}\|^2_{\varepsilon,\delta}$ holds for some constant $C_{\varepsilon,\delta}$ by Lemma 5.12, where $C_{\varepsilon,\delta}$ is a constant such that $\overline{\lim}_{\delta\to 0} C_{\varepsilon,\delta}$ (is finite and) is independent of ε. Similarly, $\beta^{\varepsilon,\delta}_{i_0 i_1}$ is the solution of $\overline{\partial}\beta^{\varepsilon,\delta}_{i_0 i_1} = (\beta^{\varepsilon,\delta}_{i_1} - \beta^{\varepsilon,\delta}_{i_0})$ on $B_{i_0 i_1} \setminus Z_\varepsilon$ and the norm

$$\|\beta^{\varepsilon,\delta}_{i_0 i_1}\|^2_{\varepsilon,\delta} := \int_{B_{i_0 i_1}\setminus Z_\varepsilon} |\beta^{\varepsilon,\delta}_{i_0 i_1}|^2_{\varepsilon,\delta}\, dV_{\varepsilon,\delta}$$

is minimum among all the solutions. In particular, $\|\beta^{\varepsilon,\delta}_{i_0 i_1}\|^2_{\varepsilon,\delta} \le D_{\varepsilon,\delta}\|(\beta^{\varepsilon,\delta}_{i_1} - \beta^{\varepsilon,\delta}_{i_0})\|^2_{\varepsilon,\delta}$ holds for some constant $D_{\varepsilon,\delta}$ by Lemma 5.12. Of course $D_{\varepsilon,\delta}$ is a constant such that $\overline{\lim}_{\delta\to 0} D_{\varepsilon,\delta}$ (is finite and) is independent of ε. Hence we have

$$\|\beta^{\varepsilon,\delta}_{i_0 i_1}\|_{\varepsilon,\delta} \le D^{1/2}_{\varepsilon,\delta}\|(\beta^{\varepsilon,\delta}_{i_1} - \beta^{\varepsilon,\delta}_{i_0})\|_{\varepsilon,\delta} \le 2C^{1/2}_{\varepsilon,\delta}D^{1/2}_{\varepsilon,\delta}\|U_{\varepsilon,\delta}\|_{\varepsilon,\delta} \le 4C^{1/2}_{\varepsilon,\delta}D^{1/2}_{\varepsilon,\delta}\|u\|_{h,\omega}$$

by (5.3). From now on, the notation $C_{\varepsilon,\delta}$ denotes a (possibly different) constant such that $\overline{\lim}_{\delta\to 0} C_{\varepsilon,\delta}$ can be bounded by a constant independent of ε. By repeating this process, we have

$$\|\beta^{\varepsilon,\delta}_{i_0\dots i_k}\|^2_{\varepsilon,\delta} \le C_{\varepsilon,\delta}\|u\|^2_{h,\omega}.$$

Moreover, by property (c), we have

$$\alpha_{\varepsilon,\delta} := f_{\varepsilon,\delta}(U_{\varepsilon,\delta}) = \mu\{\beta^{\varepsilon,\delta}_{i_0\dots i_{q-1}}\} \in C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_\varepsilon)) = C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h)).$$

Claim. *There exist subsequences $\{\varepsilon_k\}_{k=1}^\infty$ and $\{\delta_\ell\}_{\ell=1}^\infty$ with the following properties:*

- $\alpha_{\varepsilon_k,\delta_\ell} \to \alpha_{\varepsilon_k,0}$ *in $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h))$ as $\delta_\ell \to 0$.*
- $\alpha_{\varepsilon_k,0} \to \alpha_{0,0}$ *in $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h))$ as $\varepsilon_k \to 0$.*

Moreover, the limit $\alpha_{0,0}$ belongs to $B^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h)) := \operatorname{Im}\mu$.

Proof of Claim. By construction, the norm $\|a_{\varepsilon,\delta}\|_{B_{i_0\dots i_q},\varepsilon,\delta}$ of a component $a_{\varepsilon,\delta} := \alpha^{\varepsilon,\delta}_{i_0\dots i_q}$ of $\alpha_{\varepsilon,\delta} = \{\alpha^{\varepsilon,\delta}_{i_0\dots i_q}\}$ can be bounded by a constant $C_{\varepsilon,\delta}$. Note that $a_{\varepsilon,\delta}$ can be regarded as a holomorphic function on $B_{i_0\dots i_q} \setminus Z_\varepsilon$ with bounded L^2-norm since it is a $\overline{\partial}$-closed F-valued $(n,0)$-form such that $\|a_{\varepsilon,\delta}\|_{B_{i_0\dots i_q},\varepsilon,\delta} < \infty$ (see Lemma 2.4). Hence $a_{\varepsilon,\delta}$ can be extended from $B_{i_0\dots i_q} \setminus Z_\varepsilon$ to $B_{i_0\dots i_q}$ by the Riemann extension theorem. The sup-norm $\sup_K |a_{\varepsilon,\delta}|$ is uniformly bounded with respect to δ for every $K \Subset B_{i_0\dots i_q}$ since the local sup-norm of holomorphic functions can be bounded by the L^2-norm. By Montel's theorem, we can take a subsequence $\{\delta_\ell\}_{\ell=1}^\infty$ with the first property. This subsequence may depend on ε, but we can take $\{\delta_\ell\}_{\ell=1}^\infty$ independent of (countably many) ε. Then the norm of the limit $a_{\varepsilon,0}$ is uniformly bounded with respect to ε since $\overline{\lim}_{\delta\to 0} C_{\varepsilon,\delta}$ can be bounded by a constant independent of ε (see Lemma 5.12). Therefore, by applying Montel's theorem again, we can take a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ with the second property. We remark that the convergence with respect to the sup-norm implies the convergence with respect to the local L^2-norm $p_K(\bullet)$ (see [MaS4, Lemma 5.2]).

It is easy to check the latter conclusion. Indeed, it follows that $\alpha_{\varepsilon,\delta} = f_{\varepsilon,\delta}(U_{\varepsilon,\delta}) \in \operatorname{Im}\mu$ since $U_{\varepsilon,\delta} \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(F)_{\varepsilon,\delta}$ and $f_{\varepsilon,\delta}$ induces the de Rham–Weil isomorphism. By [MaS4, Lemma 5.7], the subspace $\operatorname{Im}\mu$ is closed. Therefore, we obtain the latter conclusion. □

Now, we construct solutions $\gamma_{\varepsilon,\delta}$ of the equation $\mu\gamma_{\varepsilon,\delta} = \alpha_{\varepsilon,\delta}$ with suitable L^2-norm. For simplicity, we continue to use the same notation for the subsequences in Claim. By the latter conclusion of the claim, there exists $\gamma \in C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h))$ such that $\mu\gamma = \alpha_{0,0}$. The coboundary operator

$$\mu\colon C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h)) \to B^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h)) = \operatorname{Im}\mu$$

is a surjective bounded operator between Fréchet spaces (see [MaS4, Lemma 5.7]), and thus it is an open map by the open mapping theorem. Therefore $\mu(\Delta_K)$ is an open neighborhood of the limit $\alpha_{0,0}$ in $\operatorname{Im}\mu$, where Δ_K is the open bounded neighborhood of γ in $C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h))$ defined to be

$$\Delta_K := \{\beta \in C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h)) \mid \ p_{K_{i_0\dots i_{q-1}}}(\beta - \gamma) < 1\}$$

for a family $K := \{K_{i_0\dots i_{q-1}}\}$ of relatively compact sets $K_{i_0\dots i_{q-1}} \Subset B_{i_0\dots i_{q-1}}$. We have $\alpha_{\varepsilon,\delta} \in \mu(\Delta_K)$ for sufficiently small $\varepsilon, \delta > 0$ since $\alpha_{\varepsilon,\delta}$ converges to $\alpha_{0,0}$. Since Δ_K is bounded, we can obtain $\gamma_{\varepsilon,\delta} \in C^{q-1}(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h))$ such that

$$\mu\gamma_{\varepsilon,\delta} = \alpha_{\varepsilon,\delta} \quad \text{and} \quad p_{K_{i_0\dots i_{q-1}}}(\gamma_{\varepsilon,\delta})^2 \leq C_K$$

for some positive constant C_K. The above constant C_K depends on the choice of K, γ, but does not depend on ε, δ.

By the same argument as in [MaS4, Claim 5.11 and Claim 5.13], we can obtain F-valued $(n, q-1)$-forms $w_{\varepsilon,\delta}$ with the desired properties. The strategy is as follows: The inverse map $\overline{g_{\varepsilon,\delta}}$ of $\overline{f_{\varepsilon,\delta}}$ is explicitly constructed by using a partition of unity (see the proof of [MaS4, Proposition 5.5] and [MaS4, Remark 5.6]). We can easily see that $g_{\varepsilon,\delta}(\mu\gamma_{\varepsilon,\delta}) = \overline{\partial} v_{\varepsilon,\delta}$ and $g_{\varepsilon,\delta}(\alpha_{\varepsilon,\delta}) = U_{\varepsilon,\delta} + \overline{\partial}\widetilde{v}_{\varepsilon,\delta}$ hold for some $v_{\varepsilon,\delta}$ and $\widetilde{v}_{\varepsilon,\delta}$ by the de Rham–Weil isomorphism. In particular, we have $U_{\varepsilon,\delta} = \overline{\partial}(v_{\varepsilon,\delta} - \widetilde{v}_{\varepsilon,\delta})$ by $\mu\gamma_{\varepsilon,\delta} = \alpha_{\varepsilon,\delta}$. The important point here is that we can explicitly compute $v_{\varepsilon,\delta}$ and $\widetilde{v}_{\varepsilon,\delta}$ by using the partition of unity, $\beta^{\varepsilon,\delta}_{i_0\dots i_k}$, and $\gamma_{\varepsilon,\delta}$. From this explicit expression, we obtain the L^2-estimate for $v_{\varepsilon,\delta}$ and $\widetilde{v}_{\varepsilon,\delta}$. See [MaS4, Claim 5.11 and 5.13] for the precise argument. □

Proposition 5.10. *There exist $F \otimes M$-valued $(n, q-1)$-forms $v_{\varepsilon,\delta}$ on Y_ε with the following properties:*

- $\overline{\partial} v_{\varepsilon,\delta} = s u_{\varepsilon,\delta}$.
- $\overline{\lim}_{\delta\to 0} \|v_{\varepsilon,\delta}\|_{h_\varepsilon h_M, \omega_{\varepsilon,\delta}}$ *can be bounded by a constant independent of ε.*

Proof of Proposition 5.10. Since the cohomology class of su is assumed to be zero in $H^q(X, K_X \otimes F \otimes \mathcal{J}(h) \otimes M)$, there exists an $F \otimes M$-valued $(n, q-1)$-form v such that $\overline{\partial} v = su$ and $\|v\|_{h,\omega} < \infty$. For $w_{\varepsilon,\delta}$ satisfying the properties in Proposition 5.9, by putting $v_{\varepsilon,\delta} := -s w_{\varepsilon,\delta} + v$, we have $\overline{\partial} v_{\varepsilon,\delta} = s u_{\varepsilon,\delta}$. Furthermore, an easy computation yields

$$\|v_{\varepsilon,\delta}\|_{h_\varepsilon h_M, \omega_{\varepsilon,\delta}} \leq \|s w_{\varepsilon,\delta}\|_{h_\varepsilon h_M, \omega_{\varepsilon,\delta}} + \|v\|_{h_\varepsilon h_M, \omega_{\varepsilon,\delta}} \leq \sup_X |s|_{h_M} \|w_{\varepsilon,\delta}\|_{\varepsilon,\delta} + \|v\|_{h_\varepsilon h_M, \omega_{\varepsilon,\delta}}.$$

By Lemma 2.4, property (b), and property (B), we have $\|v\|_{h_\varepsilon h_M, \omega_{\varepsilon,\delta}} \leq \|v\|_{h,\omega} < \infty$. This completes the proof. □

The following proposition completes the proof of Theorem 5.1 (see Proposition 5.7).

Proposition 5.11.

$$\lim_{\varepsilon\to 0} \varlimsup_{\delta\to 0} \|s u_{\varepsilon,\delta}\|_{h_\varepsilon h_M, \omega_{\varepsilon,\delta}} = 0.$$

Proof of Proposition 5.11. For the solution $v_{\varepsilon,\delta}$ satisfying the properties in Proposition 5.10, it is easy to see

$$\begin{aligned}\lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\|su_{\varepsilon,\delta}\|^2_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}} &= \lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\langle\!\langle \overline{\partial}^*_{\varepsilon,\delta} su_{\varepsilon,\delta}, v_{\varepsilon,\delta}\rangle\!\rangle_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}}\\ &\leq \lim_{\varepsilon\to 0}\varlimsup_{\delta\to 0}\|\overline{\partial}^*_{\varepsilon,\delta} su_{\varepsilon,\delta}\|_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}}\|v_{\varepsilon,\delta}\|_{h_\varepsilon h_M,\omega_{\varepsilon,\delta}}.\end{aligned}$$

Proposition 5.8 and Proposition 5.10 assert that the right-hand side is zero. □

We close this step with the following lemma:

Lemma 5.12 (cf. [D1, 4.1 Théorème])**.** *Assume that B is a Stein open set in X such that $\omega_{\varepsilon,\delta} = \sqrt{-1}\partial\overline{\partial}(\Psi+\delta\Psi_\varepsilon)$ on a neighborhood of $\overline{B}$. Then for an arbitrary $\alpha \in$* $\operatorname{Ker}\overline{\partial} \subset L^{n,q}_{(2)}(B\setminus Z_\varepsilon, F)_{\varepsilon,\delta}$*, there exist $\beta \in L^{n,q-1}_{(2)}(B\setminus Z_\varepsilon, F)_{\varepsilon,\delta}$ and a positive constant $C_{\varepsilon,\delta}$ (independent of α) such that*

- $\overline{\partial}\beta = \alpha$ *and* $\|\beta\|^2_{\varepsilon,\delta} \leq C_{\varepsilon,\delta}\|\alpha\|^2_{\varepsilon,\delta}$,
- $\varlimsup_{\delta\to 0} C_{\varepsilon,\delta}$ *(is finite and) is independent of ε.*

Proof of Lemma 5.12. We may assume $\varepsilon < 1/2$ since $0 < \varepsilon \ll 1$. For the singular Hermitian metric $H_{\varepsilon,\delta}$ on F defined by $H_{\varepsilon,\delta} := h_\varepsilon e^{-(\Psi+\delta\Psi_\varepsilon)}$, the curvature satisfies

$$\sqrt{-1}\Theta_{H_{\varepsilon,\delta}}(F) = \sqrt{-1}\Theta_{h_\varepsilon}(F) + \sqrt{-1}\partial\overline{\partial}(\Psi+\delta\Psi_\varepsilon) \geq -\varepsilon\omega + \omega_{\varepsilon,\delta} \geq (1-\varepsilon)\omega_{\varepsilon,\delta} \geq \frac{1}{2}\omega_{\varepsilon,\delta}$$

by property (B) and $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon\omega$. The L^2-norm $\|\alpha\|_{H_{\varepsilon,\delta},\omega_{\varepsilon,\delta}}$ with respect to $H_{\varepsilon,\delta}$ and $\omega_{\varepsilon,\delta}$ is finite since the function $\Psi+\delta\Psi_\varepsilon$ is bounded and $\|\alpha\|_{\varepsilon,\delta}$ is finite. Therefore, from the standard L^2-method for the $\overline{\partial}$-equation (for example see [D1, 4.1 Théorème]), we obtain a solution β of the $\overline{\partial}$-equation $\overline{\partial}\beta = \alpha$ with

$$\|\beta\|^2_{H_{\varepsilon,\delta},\omega_{\varepsilon,\delta}} \leq \frac{2}{q}\|\alpha\|^2_{H_{\varepsilon,\delta},\omega_{\varepsilon,\delta}}.$$

Then we can easily see that

$$\|\beta\|^2_{\varepsilon,\delta} \leq \frac{2}{q}\frac{\sup_B e^{-(\Psi+\delta\Psi_\varepsilon)}}{\inf_B e^{-(\Psi+\delta\Psi_\varepsilon)}}\|\alpha\|^2_{\varepsilon,\delta}.$$

This completes the proof by property (B). □

Remark 5.13. In Lemma 5.12, we take a solution $\beta_0 \in L^{n,q-1}_{(2)}(B\setminus Z_\varepsilon, F)_{\varepsilon,\delta}$ of the equation $\overline{\partial}\beta = \alpha$. Then β_0 is uniquely decomposed as follows:

$$\beta_0 = \beta_1 + \beta_2 \quad \text{for } \beta_1 \in \operatorname{Ker}\overline{\partial} \text{ and } \beta_2 \in (\operatorname{Ker}\overline{\partial})^\perp.$$

We can easily check that β_2 is a unique solution of $\overline{\partial}\beta = \alpha$ whose norm is the minimum among all the solutions.

Thus we finish the proof of Theorem 5.1. □

6. Twists by Nakano semipositive vector bundles

We have already known that some results for K_X can be generalized for $K_X \otimes E$, where E is a Nakano semipositive vector bundle on X (see, for example, [Ta], [Mo], and [Fs]). Let us recall the definition of Nakano semipositive vector bundles.

Definition 6.1 (Nakano semipositive vector bundles)**.** Let E be a holomorphic vector bundle on a complex manifold X. If E admits a smooth Hermitian metric h_E such that the curvature form $\sqrt{-1}\Theta_{h_E}(E)$ defines a positive semi-definite Hermitian form on each fiber of the vector bundle $E \otimes T_X$, where T_X is the holomorphic tangent bundle of X, then E is called a Nakano semipositive vector bundle.

Example 6.2 (Unitary flat vector bundles)**.** Let E be a holomorphic vector bundle on a complex manifold X. If E admits a smooth Hermitian metric h_E such that (E, h_E) is flat, that is, $\sqrt{-1}\Theta_{h_E}(E) = 0$, then E is Nakano semipositive.

For the proof of Theorem 1.12, we need the following lemmas on Nakano semipositive vector bundles. However, these lemmas easily follow from the definition of Nakano semipositive vector bundles, and thus, we omit the proof.

Lemma 6.3. *Let E be a Nakano semipositive vector bundle on a complex manifold X. Let H be a smooth divisor on X. Then $E|_H$ is a Nakano semipositive vector bundle on H.*

Lemma 6.4. *Let $q \colon Z \to X$ be an étale morphism between complex manifolds. Let (E, h_E) be a Nakano semipositive vector bundle on X. Then (q^*E, q^*h_E) is a Nakano semipositive vector bundle on Z.*

Proposition 6.5. *Proposition 1.9 holds even when K_X is replaced with $K_X \otimes E$, where E is a Nakano semipositive vector bundle on X.*

Proof. By Lemma 6.3 and Lemma 6.4, the proof of Proposition 1.9 in Section 4 works for $K_X \otimes E$. □

Therefore, by Proposition 6.5 and the proof of Theorem 1.4 and Corollary 1.7 in Section 4, it is sufficient to prove the following theorem for Theorem 1.12.

Theorem 6.6 (Theorem A twisted by Nakano semipositive vector bundles)**.** *Let E be a Nakano semipositive vector bundle on a compact Kähler manifold X. Let F (resp. M) be a line bundle on a compact Kähler manifold X with a singular Hermitian metric h (resp. a smooth Hermitian metric h_M) satisfying*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ and } \sqrt{-1}\Theta_h(F) - b\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ for some } b > 0.$$

Then for a (nonzero) section $s \in H^0(X, M)$, the multiplication map induced by $\otimes s$

$$\times s \colon H^q(X, K_X \otimes E \otimes F \otimes \mathcal{J}(h)) \xrightarrow{\otimes s} H^q(X, K_X \otimes E \otimes F \otimes \mathcal{J}(h) \otimes M)$$

is injective for every q. Here K_X is the canonical bundle of X and $\mathcal{J}(h)$ is the multiplier ideal sheaf of h.

We will explain how to modify the proof of Theorem 5.1 for Theorem 6.6.

Proof. We replace (F, h_ε) with $(E \otimes F, h_E h_\varepsilon)$ in the proof of Theorem 5.1, where $\{h_\varepsilon\}_{1 \gg \varepsilon > 0}$ is a family of singular Hermitian metrics on F (constructed in Step 1) and h_E is a smooth Hermitian metric on E such that $\sqrt{-1}\Theta_{h_E}(E)$ is Nakano semipositive. Then it is easy to see that essentially the same proof as in Theorem 5.1 works for Theorem 6.6 thanks to the assumption on the curvature of E. For the reader's convenience, we give several remarks on the differences with the proof of Theorem 5.1.

There is no problem when we construct h_ε and $\omega_{\varepsilon,\delta}$. In Step 4 in the proof of Theorem 5.1, we used the de Rham–Weil isomorphism (see (5.7) and [MaS4, Proposition 5.5]), which was constructed by using Lemma 5.12. Since [D1, 4.1 Théorème] (which yields Lemma 5.12) is formulated for holomorphic vector bundles, Lemma 5.12 can be generalized to $(E\otimes F, h_E h_\varepsilon)$. From this generalization, we can construct the de Rham–Weil isomorphism for $E\otimes F$

$$\overline{f_{\varepsilon,\delta}}\colon \frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(E\otimes F)_{h_E h_\varepsilon,\omega_{\varepsilon,\delta}} \xrightarrow{\cong} \frac{\operatorname{Ker}\mu}{\operatorname{Im}\mu} \text{ of } C^q(\mathcal{U}, K_X\otimes E\otimes F\otimes \mathcal{J}(h_\varepsilon)).$$

In Step 1, we used the orthogonal decomposition of $L^{n,q}_{(2)}(F)_{\varepsilon,\delta}$, which was obtained from the fact that $\operatorname{Im}\overline{\partial}\subset L^{n,q}_{(2)}(F)_{\varepsilon,\delta}$ is closed. To obtain the same conclusion for $L^{n,q}_{(2)}(E\otimes F)_{h_E h_\varepsilon,\omega_{\varepsilon,\delta}}$, it is sufficient to show that $C^q(\mathcal{U}, K_X\otimes E\otimes F\otimes\mathcal{J}(h_\varepsilon))$ is a Fréchet space (see [MaS4, Proposition 5.8]). We can easily check it by using the same argument as in [MaS4, Theorem 5.3] for $\mathbb{C}^{\operatorname{rank}E}$-valued holomorphic functions.

The argument of Step 2 works even if we consider $(E\otimes F, h_E h_\varepsilon)$. In Step 3, we need to prove (5.6), but it is easy to see

$$\begin{aligned}-\varepsilon q|u_{\varepsilon,\delta}|^2_{h_E h_\varepsilon,\omega_{\varepsilon,\delta}} &\le \langle\sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\omega_{\varepsilon,\delta}}u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle_{h_E h_\varepsilon,\omega_{\varepsilon,\delta}}\\ &\le \langle\sqrt{-1}\Theta_{h_E h_\varepsilon}(E\otimes F)\Lambda_{\omega_{\varepsilon,\delta}}u_{\varepsilon,\delta}, u_{\varepsilon,\delta}\rangle_{h_E h_\varepsilon,\omega_{\varepsilon,\delta}}\end{aligned}$$

since $\sqrt{-1}\Theta_{h_E}(E)$ is Nakano semipositive. □

When E is Nakano semipositive and is not flat, there seems to be no Hodge theoretic approach to Theorem 6.6 even if h is smooth.

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