ON FINITE GENERATION OF ADJOINT RINGS

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Abstract. We prove that adjoint rings are finitely generated in the complex analytic setting.

Contents

1. Finite generation of adjoint rings

The following theorem was first obtained in [DHP], whose argument is a complex analytic generalization of [CL]. When $\pi\colon X\to Y$ is a projective morphism between algebraic varieties, Theorems 1.1 and 1.2 below are well known (see [BCHM, Corollary 1.1.9]). In this paper, we show that they follow easily from [F1]. In [F1, Definition 2.23], we defined *locally finitely generated graded $\mathcal{O}_X$-algebras* on a complex analytic space X. Similarly, we can define *locally finitely generated $\mathbb{N}^k$-graded $\mathcal{O}_X$-algebras*.

Theorem 1.1. *Let X be a smooth complex variety, and let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces. Let $B_1,\cdots,B_k$ be $\mathbb{Q}$-divisors on X with $\lfloor B_i\rfloor=0$ for all i such that the support of $\sum_{i=1}^k B_i$ is a simple normal crossing divisor on X. Let A be a π-nef and π-big $\mathbb{Q}$-divisor on X. Set $D_i=K_X+A+B_i$ for every i. Then the relative adjoint ring*

$$R(X/Y,D_1,\cdots,D_k):=\bigoplus_{(m_1,\ldots,m_k)\in\mathbb{N}^k}\pi_*\mathcal{O}_X\left(\left\lfloor\sum_{i=1}^k m_iD_i\right\rfloor\right)$$

is a locally finitely generated $\mathbb{N}^k$-graded $\mathcal{O}_Y$-algebra.

Although Theorem 1.2 is essentially equivalent to Theorem 1.1, the following formulation may be useful for some applications.

Theorem 1.2. *Let X be a normal complex variety, and let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces. Let $B_1,\ldots,B_k$ be $\mathbb{Q}$-divisors on X such that (X,B_i) is divisorial log terminal for every i. Let A be a π-ample $\mathbb{Q}$-divisor on X. Set $D_i=$*

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$K_X + A + B_i$ for every i. Then the relative adjoint ring

$$R(X/Y, D_1, \cdots, D_k) := \bigoplus_{(m_1,\ldots,m_k)\in\mathbb{N}^k} \pi_*\mathcal{O}_X\left(\left\lfloor \sum_{i=1}^{k} m_i D_i \right\rfloor\right)$$

is a locally finitely generated $\mathbb{N}^k$-graded $\mathcal{O}_Y$-algebra.

We make the following easy remark.

Remark 1.3. If (X, B_i) is kawamata log terminal for every i in Theorem 1.2, then it is sufficient to assume that A is π-nef and π-big. This follows immediately from the proof of Theorem 1.2.

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In this paper, we freely use [F1]. We use $\mathbb{N}$ to denote the set of non-negative integers.

2. Proofs of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. We begin with an easy lemma.

Lemma 2.1. *Let X be a smooth complex variety, and let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be line bundles on X. Set $\mathcal{E} := \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$, and consider the projective bundle $f\colon Z := \mathbb{P}_X(\mathcal{E}) \to X$ associated to $\mathcal{E}$. Note that if $k = 1$ then $Z \simeq X$ and we set $T := 0$. Assume $k \geq 2$. For each i, let T_i be the divisor on Z associated to the quotient*

$$\mathcal{E} \to \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{i-1} \oplus \mathcal{L}_{i+1} \oplus \cdots \oplus \mathcal{L}_k.$$

Then $T := \sum_{i=1}^k T_i$ is a simple normal crossing divisor on Z such that

$$\mathcal{O}_Z\left(K_Z + \sum_{i=1}^{k} T_i\right) \simeq f^*\mathcal{O}_X(K_X).$$

In particular, (Z, T) is a divisorial log terminal pair.

Proof of Lemma 2.1. We may assume that $k \geq 2$. For each i, set

$$\mathcal{E}_i := \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{i-1} \oplus \mathcal{L}_{i+1} \oplus \cdots \oplus \mathcal{L}_k.$$

Then $T_i = \mathbb{P}_X(\mathcal{E}_i)$ by definition. It is straightforward to see that $\sum_{i=1}^k T_i$ is a simple normal crossing divisor on Z. Set $\mathcal{O}_Z(1) := \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ and $\mathcal{O}_{T_i}(1) := \mathcal{O}_{\mathbb{P}_X(\mathcal{E}_i)}(1)$. Consider the following short exact sequence:

$$0 \to \mathcal{O}_Z(1) \otimes \mathcal{O}_Z(-T_i) \to \mathcal{O}_Z(1) \to \mathcal{O}_{T_i}(1) \to 0.$$

By applying f_*, we obtain an exact sequence

$$0 \to f_*\big(\mathcal{O}_Z(1) \otimes \mathcal{O}_Z(-T_i)\big) \to \mathcal{E} \to \mathcal{E}_i \to 0.$$

This implies that

$$f_*\big(\mathcal{O}_Z(1) \otimes \mathcal{O}_Z(-T_i)\big) \simeq \mathcal{L}_i, \quad \text{and hence} \quad \mathcal{O}_Z(1) \otimes \mathcal{O}_Z(-T_i) \simeq f^*\mathcal{L}_i.$$

We note that

$$\mathcal{O}_Z(K_Z) \simeq f^*\mathcal{O}_X(K_X) \otimes f^*(\det \mathcal{E}) \otimes \mathcal{O}_Z(-k)$$

since $Z = \mathbb{P}_X(\mathcal{E})$. Therefore,

$$\mathcal{O}_Z(K_Z) \simeq f^*\mathcal{O}_X(K_X) \otimes \mathcal{O}_Z\left(-\sum_{i=1}^{k} T_i\right).$$

Since Z is smooth and T is a reduced simple normal crossing divisor on Z, (Z, T) is divisorial log terminal. □

For the proof of Theorem 1.1, we need the following lemma.

Lemma 2.2. *Let X be a normal complex variety and let $D_1, \dots, D_k$ be $\mathbb{Q}$-divisors on X. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let d be any positive integer. Then the relative adjoint ring*

$$\mathcal{A} := \bigoplus_{(m_1,\dots,m_k)\in\mathbb{N}^k} \pi_*\mathcal{O}_X\left(\left\lfloor \sum_{i=1}^{k} m_i D_i \right\rfloor\right)$$

is locally finitely generated $\mathbb{N}^k$-graded $\mathcal{O}_Y$-algebra if and only if so is the truncation

$$\mathcal{A}^{(d)} := \bigoplus_{(m_1,\dots,m_k)\in\mathbb{N}^k} \pi_*\mathcal{O}_X\left(\left\lfloor \sum_{i=1}^{k} m_i d D_i \right\rfloor\right).$$

Proof of Lemma 2.2. [1] Although [F1, Lemma 2.26] only treats $\mathbb{N}$-graded $\mathcal{O}_Y$-algebras, the proof of [F1, Lemma 2.26] works for this lemma. □

Let us prove Theorem 1.1. The proof given below is essentially the same as **Aliter** in the proof of [BCHM, Corollary 1.1.9].

Proof of Theorem 1.1. The problem is local on Y. Hence, we fix an arbitrary point $P \in Y$ and freely replace Y with a sufficiently small open neighborhood of P. Let U be a relatively compact open neighborhood of P in Y.

By [KM, Proposition 2.36 (1)], we take a suitable finite composite of blow-ups $f\colon X' \to X$. Then, over some open neighborhood of $\overline{U}$, we can write

$$K_{X'} + B'_i = f^*(K_X + B_i) + E_i$$

such that B'_i and E_i have no common irreducible components, $f_*B'_i = B_i$, $f_*E_i = 0$, and $\operatorname{Supp} B'_i$ is smooth for every i. We may further assume that the support of $\sum_{i=1}^{k} B'_i$ is a smooth divisor. Then we have

$$R(X/Y, D_1, \cdots, D_k) \simeq R(X'/Y, D'_1, \cdots, D'_k)$$

over some open neighborhood of $\overline{U}$, where we put $D'_i := K_{X'} + f^*A + B'_i$ for every i. Therefore, after shrinking Y suitably and replacing X, B_i, A, and $\pi\colon X \to Y$ with X', B'_i, f^*A, and $\pi \circ f\colon X' \to Y$, respectively, we may assume that the support of $\sum_{i=1}^{k} B_i$ is a smooth divisor.

We take a positive integer $d \geq 2$ such that dB_i is integral for every i and that dA is also integral. We put

$$\mathcal{E} := \mathcal{O}_X(dB_1) \oplus \cdots \oplus \mathcal{O}_X(dB_k)$$

[1] It may be better to add some explanations

as in Lemma 2.1. We consider the projective bundle $f\colon Z := \mathbb{P}_X(\mathcal{E}) \to X$ associated to $\mathcal{E}$. We take a global section σ_i of $\mathcal{O}_X(dB_i)$ such that $(\sigma_i = 0) = dB_i$ for every i. Then $\sigma := (\sigma_1, \cdots, \sigma_k)$ is a global section of $\mathcal{E}$. By the natural surjection

$$f^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1) =: \mathcal{O}_Z(1),$$

we obtain a global section σ_Z of $\mathcal{O}_Z(1)$, which is the image of $f^*\sigma$. Let S be the divisor on Z defined by σ_Z. Let T_i be the divisor on Z associated to the quotient

$$\mathcal{E} \to \mathcal{O}_X(dB_1) \oplus \cdots \oplus \mathcal{O}_X(dB_{i-1}) \oplus \mathcal{O}_X(dB_{i+1}) \oplus \cdots \oplus \mathcal{O}_X(dB_k)$$

as in Lemma 2.1. It is easy to see that $T := \sum_{i=1}^k T_i$ is a simple normal crossing divisor on Z. We note that if $k = 0$, then $Z \simeq X$ and we put $T := 0$.

Claim. *After shrinking Y around P, the pair $(Z, T + S/d)$ is divisorial log terminal. Moreover, it is kawamata log terminal outside T.*

Proof of Claim. Throughout the proof, we freely shrink Y around P without explicit mention. By the definition of S, we can directly check that the support of S is a simple normal crossing divisor on Z and that all coefficients of S/d are less than one. We note that although the support of $\sum_{i=1}^k B_i$ is smooth, the divisors B_i may share common irreducible components. Hence, $(Z, T + S/d)$ is kawamata log terminal outside T. Let ε be a sufficiently small positive rational number such that

$$\left\lfloor \frac{1+\varepsilon}{d} S \right\rfloor = 0.$$

It is sufficient to prove that

$$\left(Z, T + \frac{1+\varepsilon}{d} S\right)$$

is log canonical, since Z is smooth and T is a reduced simple normal crossing divisor on Z. We prove this by induction on k. The case $k = 1$ is obvious. By adjunction and the induction hypothesis, for every j we have

$$\left(K_Z + T + \frac{1+\varepsilon}{d} S\right)\big|_{T_j} = K_{T_j} + (T - T_j)\big|_{T_j} + \frac{1+\varepsilon}{d} S\big|_{T_j},$$

and the pair

$$\left(T_j, (T - T_j)\big|_{T_j} + \frac{1+\varepsilon}{d} S\big|_{T_j}\right)$$

is log canonical. By inversion of adjunction (see, for example, [F2, Theorem 1.1]), it follows that

$$\left(Z, T + \frac{1+\varepsilon}{d} S\right)$$

is log canonical around T_j for every j. Since the pair is kawamata log terminal outside T by the above argument, it is log canonical there. Therefore,

$$\left(Z, T + \frac{1+\varepsilon}{d} S\right)$$

is log canonical on Z. This completes the proof. □

Let us return to the proof of Theorem 1.1. We put

$$\Gamma := \sum_{i=1}^{k} T_i + f^*A + \frac{1}{d}S.$$

Then, by Lemma 2.1, we have

$$\begin{aligned}\mathcal{O}_Z(md(K_Z+\Gamma)) &\simeq \mathcal{O}_Z\left(mdf^*(K_X+A)+mS\right) \\ &\simeq \mathcal{O}_Z(m)\otimes f^*\mathcal{O}_X(md(K_X+A)).\end{aligned}$$

Therefore,

$$f_*\mathcal{O}_Z(md(K_Z+\Gamma)) \simeq S^m(\mathcal{E})\otimes \mathcal{O}_X(md(K_X+A)),$$

where $S^m(\mathcal{E})$ denotes the m-th symmetric product of $\mathcal{E}$. Hence,

$$\bigoplus_{m\in\mathbb{N}}(\pi\circ f)_*\mathcal{O}_Z(md(K_Z+\Gamma)) \simeq \bigoplus_{(m_1,\cdots,m_k)\in\mathbb{N}^k} \pi_*\mathcal{O}_X\left(\sum_{i=1}^{k} m_i d(K_X+A+B_i)\right).$$

Thus, by Lemma 2.2, it suffices to prove that

$$\bigoplus_{m\in\mathbb{N}}(\pi\circ f)_*\mathcal{O}_Z(md(K_Z+\Gamma))$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra.

Claim. *We can find an effective $\mathbb{Q}$-divisor Δ on Z such that $K_Z+\Gamma\sim_{\mathbb{Q}} K_Z+\Delta$ and (Z,Δ) is kawamata log terminal after shrinking Y around P.*

Proof of Claim. Throughout the proof, we freely shrink Y around P without explicit mention. We first note that the non-klt locus of $(Z, T+S/d)$ is T, and every log canonical center of $(Z, T+S/d)$ dominates X by construction. Since A is π-nef and π-big, we can take a sufficiently small effective $\mathbb{Q}$-divisor E on X such that $A-E$ is π-ample, $(Z, T+S/d+f^*E)$ is divisorial log terminal, and the non-klt locus of $(Z, T+S/d+f^*E)$ is still T. Since T is f-ample, for any sufficiently small positive rational number $\varepsilon>0$, the divisor

$$\varepsilon T + f^*(A-E)$$

is $(\pi\circ f)$-ample. We now write

$$\Gamma = T + f^*A + \frac{1}{d}S = (1-\varepsilon)T + \frac{1}{d}S + f^*E + \left(\varepsilon T + f^*(A-E)\right).$$

Since $(Z, T+S/d+f^*E)$ is divisorial log terminal and its non-klt locus is T, it follows that $(Z,(1-\varepsilon)T+S/d+f^*E)$ is kawamata log terminal for sufficiently small $\varepsilon>0$. Hence, by using the $\pi\circ f$-ampleness of $\varepsilon T+f^*(A-E)$, we obtain an effective $\mathbb{Q}$-divisor Δ such that

$$K_Z+\Gamma\sim_{\mathbb{Q}} K_Z+\Delta$$

and (Z,Δ) is kawamata log terminal after possibly shrinking Y around P. This completes the proof of Claim. □

Therefore,

$$\bigoplus_{m\in\mathbb{N}}(\pi\circ f)_*\mathcal{O}_Z(md(K_Z+\Gamma))$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra by [F1, Theorem 1.18] and Lemma 2.2 (see also [F1, Lemma 2.26]). Hence we obtain the desired finite generation. □

We see that Theorem 1.2 follows easily from Theorem 1.1.

Proof of Theorem 1.2. Let $P \in Y$ be an arbitrary point. Throughout the proof, we freely shrink Y around P without explicit mention.

By [KM, Proposition 2.43], we can take an effective $\mathbb{Q}$-divisor $\tilde{B}_i$ such that $(X, \tilde{B}_i)$ is kawamata log terminal and

$$B_i + A \sim_{\mathbb{Q}} \tilde{B}_i + (1-\varepsilon)A$$

for every i, where ε is a sufficiently small positive rational number. Set

$$\tilde{D}_i := K_X + (1-\varepsilon)A + \tilde{B}_i,$$

and consider the relative adjoint ring $R(X/Y, \tilde{D}_1, \dots, \tilde{D}_k)$. Then $R(X/Y, D_1, \dots, D_k)$ and $R(X/Y, \tilde{D}_1, \dots, \tilde{D}_k)$ have isomorphic truncations. Hence, by Lemma 2.2, it suffices to prove the finite generation of $R(X/Y, \tilde{D}_1, \dots, \tilde{D}_k)$. Therefore, replacing B_i and A with $\tilde{B}_i$ and $(1-\varepsilon)A$, respectively, we may assume that (X, B_i) is kawamata log terminal for every i. Take a resolution $f\colon X' \to X$ such that

$$K_{X'} + B'_i = f^*(K_X + B_i) + E_i,$$

where B'_i and E_i have no common irreducible components, $f_*B'_i = B_i$, and $f_*E_i = 0$. We may further assume that the support of $\sum_{i=1}^k B'_i$ is a simple normal crossing divisor on X'. Since (X, B_i) is kawamata log terminal, we have $\lfloor B'_i \rfloor = 0$. As in the proof of Theorem 1.1, by replacing (X, B_i), A, and $\pi\colon X \to Y$ with (X', B'_i), f^*A, and $\pi \circ f\colon X' \to Y$, respectively, we may assume that $\lfloor B_i \rfloor = 0$, the support of $\sum_{i=1}^k B_i$ is a simple normal crossing divisor, and A is π-nef and π-big. Thus, by Theorem 1.1, we obtain the desired finite generation of $R(X/Y, D_1, \dots, D_k)$. □

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