FLIPS VS FLOPS
(PRIVATE NOTE)

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Abstract. In this note, we count the numbers of flipping and flopping contractions in the category of non-singular toric varieties.

Contents

1. Introduction

This note is an answer to my own simple question: *Which is more, flips or flops?* I think that it is the first attempt to *count the number of flips*.

It is well-known that there is no contraction of flipping type on non-singular threefolds by the classification of negative extremal rays (Mori). In dimension four, if $f : X \longrightarrow Y$ is a flipping contraction with X non-singular, then the exceptional locus E is isomorphic to $\mathbb{P}^2$ with $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ (Kawamata). On the other hand, there exists Atiyah's flop on non-singular threefolds. In dimension four, we can make two different flopping contractions on non-singular toric varieties without any difficulties (see [Ma, Example-Claim 14-2-8]). So, I believed that flopping contractions occur much more often than flipping contractions on non-singular n-folds on no evidence.

In this note, I count the numbers of flipping and flopping contractions in the category of non-singular toric varieties (see the table in 2.8, where $a(n)$ (resp. $b(n)$) denotes the number of n-dimensional non-singular toric flipping (resp. flopping) contractions). In dimension ≥ 7,

Date: 2004/4/1, Version 3.3.
1991 *Mathematics Subject Classification.* Primary 14M25; Secondary 14E30.

the number of flipping contractions is larger than that of flopping contractions (see Theorem 2.9). I do not discuss *how to count the numbers of flips and flops* in general settings. I only treat toric varieties. I will also discuss miscellaneous results on non-singular toric varieties.

I will work over an algebraically closed field throughout this paper.

Acknowledgments. I would like to thank Doctor Hiroshi Sato, Professors Shigefumi Mori, and Hiromich Takagi for encouragement. I am grateful to the Institute for Advanced Study for its hospitality. I was partially supported by a grant from the National Science Foundation: DMS-0111298.

2. Flips vs Flops

Let us recall the definitions of flipping and flopping contractions.

Definition 2.1 (Flipping and Flopping Contractions)**.** Let $f : X \longrightarrow Y$ be a small projective toric morphism such that Y is affine. We say that f is a *non-singular toric flipping* (resp. *flopping*) *contraction* if and only if X is non-singular, the relative Picard number $\rho(X/Y) = 1$, and $-K_X$ is f-ample (resp. numerically f-trivial).

We define functions $a(n)$ and $b(n)$.

Definition 2.2 (Numbers of Flipping and Flopping Contractions)**.** Two non-singular toric flipping (resp. flopping) contraction $f : X \longrightarrow Y$ and $f' : X' \longrightarrow Y'$ are isomorphic each other if and only if there exists the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[\nu]{} & Y' \end{array}$$

such that μ and ν are isomorphisms and every morphism is toric.

The function $a(n)$ (resp. $b(n)$) denotes the number of the isomorphism classes of n-dimensional non-singular toric flipping (resp. flopping) contractions for $n \geq 1$.

To express $a(n)$ and $b(n)$ explicitly, we need partition functions. For the details, see http://mathworld.wolfram.com/.

Definition 2.3 (Partition Functions)**.** $P(n, k)$ denotes the number of ways of writing the positive integer n as a sum of exactly $k \geq 0$ terms. It can be computed from the recurrence relation

$$P(n, k) = P(n-1, k-1) + P(n-k, k)$$

with $P(n,k) = 0$ for $k > n$, $P(n,n) = 1$, and $P(n,0) = 0$. The functions $P(n,k)$ can also be given explicitly for the first few values of k in the simple forms

$$P(n,2) = \lfloor \frac{1}{2} n \rfloor,$$

$$P(n,3) = [\frac{1}{12} n^2],$$

where $\lfloor x \rfloor$ is the *round down* of x and $[x]$ is the *nint* function, that is, $[x]$ is the integer closest to x. If x is a half-integer, we assume $[x] \in 2\mathbb{Z}$.

The function $q(n,k)$ denotes the number of partitions of the positive integer n with k or fewer addends. The $q(n,k)$ satisfy the recurrence relation

$$q(n,k) = q(n,k-1) + q(n-k,k),$$

with $q(n,0) = 0$ and $q(1,k) = 1$. It is convenient to put $q(n,k) = 0$ for $n \le 0$. We note that $q(n-k,k) = P(n,k)$.

Let's go to the descriptions of toric flips and flops. It was obtained by Reid.

2.4 (Non-singular Toric Flipping and Flopping Contractions)**.** We fix the lattice $N \simeq \mathbb{Z}^n$. Let $e_1, e_2, \cdots, e_n$ form the standard basis of $\mathbb{Z}^n$. By changing the coordinates suitably, we can assume that $X = X(\Delta)$, where

$$\Delta = \{\langle e_1, \cdots, e_n \rangle, \langle e_1, \cdots, e_{n-1}, e_{n+1} \rangle, \text{and their faces}\}$$

such that e_{n+1} is defined by the relation

$$\sum_{i=1}^{n+1} a_i e_i = 0$$

with

$$\begin{cases} a_i < 0 & \text{for} \quad 1 \le i \le \alpha \\ a_i = 0 & \text{for} \quad \alpha + 1 \le i \le \beta \\ a_i > 0 & \text{for} \quad \beta + 1 \le i \le n, \end{cases}$$

and $a_{n+1} = 1$. Since X is non-singular, $a_n = 1$ and $a_i \in \mathbb{Z}$ for every i. In this situation, $Y = X(\langle e_1, \cdots, e_{n+1} \rangle)$. By [R, (2.10) Corollary (i)], the cone $\langle e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_{n+1} \rangle \in \Delta$ for $\beta + 1 \le i \le n+1$. So, non-singularity of X implies $a_i = 1$ for $\beta + 1 \le i \le n+1$. It is easy to check that

$$K_X \cdot V(w) = -\sum_{i=1}^{n+1} a_i = -\sum_{i=1}^{\beta} a_i - (n+1-\beta),$$

where w is the wall $\langle e_1, \cdots, e_{n-1}\rangle$. So, $-K_X \cdot V(w) > 0$ (resp. $= 0$) if and only if $\sum_{i=1}^{\beta} b_i \le n - \beta$ (resp. $\sum_{i=1}^{\beta} b_i = n - \beta + 1$), where $b_i = -a_i \ge 0$. Note that f is small if and only if $\alpha \ge 2$. So, we have $b_1 > 0$ and $b_2 > 0$.

Therefore, we obtain

Theorem 2.5. *We have the following formulas.*

$$\begin{aligned} a(n) &= \sum_{\beta=2}^{n-2} \sum_{k=0}^{n-2-\beta} (q(n-\beta-k, \beta) - 1), \textit{ and} \\ b(n) &= \sum_{\beta=2}^{n-1} (q(n-\beta+1, \beta) - 1). \end{aligned}$$

There is an interesting relation between $a(n)$ and $b(n)$.

Theorem 2.6. *We have the following relation*:

$$a(n+1) - a(n) = b(n)$$

for $n \ge 1$.

Proof. The inequality $\sum_{i=1}^{\beta} b_i \le n-\beta+1$ is equivalent to the condition that $\sum_{i=1}^{\beta} b_i \le n - \beta$ or $\sum_{i=1}^{\beta} b_i = n - \beta + 1$. This implies $a(n+1) = a(n) + b(n)$. □

To compute $a(n)$ and $b(n)$, we introduce a new function $c(n) = b(n+1) - b(n)$. I think that $c(n)$ has no geometric meanings.

Proposition 2.7. *We have*

$$c(n) = \sum_{l=2}^{1+\lfloor \frac{n}{2} \rfloor} P(n+2-l, l).$$

Proof.

$$\begin{aligned} c(n) &= b(n+1)-b(n) \\ &= \sum_{\beta=2}^{n}(q(n+2-\beta,\beta)-1) \\ &\quad -\sum_{\beta=2}^{n-1}(q(n+1-\beta,\beta)-1) \\ &= \sum_{\beta=2}^{n}(q(n+2-\beta,\beta)-1) \\ &\quad -\sum_{l=3}^{n}(q(n+2-l,l-1)-1) \\ &= q(n,2)-1+\sum_{l=3}^{n}P(n+2-l,l) \\ &= \sum_{l=2}^{n}P(n+2-l,l) \\ &= \sum_{l=2}^{1+\lfloor\frac{n}{2}\rfloor}P(n+2-l,l). \end{aligned}$$

□

2.8. The following is a table of the values of $a(n), b(n)$, and $c(n)$ for small n.[1] By Proposition 2.7, we can easily compute $c(n)$. Theorem 2.6 helps us calculate $a(n)$. We note that $a(n)$ (resp. $b(n)$) denotes the number of the isomorphism classes of n-dimensional non-singular toric flipping (resp. flopping) contractions (see Definition 2.2). In dimension 26, the number of non-singular toric flipping contractions ≥ 10000 !

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$a(n)$	0	0	0	1	3	8	16	30	51	83	128	193	281	402	563
$b(n)$	0	0	1	2	5	8	14	21	32	45	65	88	121	161	215
$c(n)$	0	1	1	3	3	6	7	11	13	20	23	33	40	54	65

[1] I computed them by hand. Please check this table by yourself. If you compute $c(n)$ by hand, I recommend you to calculate

$$d(n) := c(n+1)-c(n-1) = 1+\sum_{l=2}^{\lfloor\frac{n}{3}\rfloor+1}P(n+3-2l,l).$$

16	17	18	19	20	21	22	23
778	1058	1425	1896	2503	3274	4254	5486
280	367	471	607	771	980	1232	1551
87	104	136	164	209	252	319	382

24	25	26	$\cdots$
7037	8970	11380	$\cdots$
1933	2410	$\cdots$	$\cdots$
477	$\cdots$	$\cdots$	$\cdots$

Theorem 2.9. *We have $a(n) \geq b(n)$ for $n \geq 6$, and $a(n) > b(n)$ for $n \geq 7$.*

Proof. By the above table, we can assume that $n \geq 10$. We note that

$$\begin{aligned} q(n-\beta+1,\beta) &= q(n-\beta,\beta-1)+q(n+1-2\beta,\beta) \\ &\leq q(n+1-\beta,\beta-1)+q(n-1-\beta,\beta) \end{aligned}$$

for $2 \leq \beta \leq n-1$ since $n+1-2\beta \leq n-1-\beta$. So,

$$\begin{aligned} a(n)-b(n) &= \sum_{\beta=2}^{n-2}\sum_{k=0}^{n-2-\beta}(q(n-\beta-k,\beta)-1) \\ &\quad -\sum_{\beta=2}^{n-1}(q(n-\beta+1,\beta)-1) \\ &= \sum_{\beta=2}^{n-1}\sum_{k=0}^{n-1-\beta}(q(n-\beta-k,\beta)-1) \\ &\quad -\sum_{\beta=2}^{n-1}(q(n-\beta+1,\beta)-1) \\ &\geq \sum_{\beta=2}^{n-1}\sum_{k=2}^{n-1-\beta}(q(n-\beta-k,\beta)-1) \\ &\geq \sum_{\beta=2}^{n-4}(q(n-\beta-2,\beta)-1)+\sum_{\beta=2}^{n-5}(q(n-\beta-3,\beta)-1) \\ &\quad -(n-2) \\ &\geq (n-5)+(n-6)-(n-2) \\ &= n-9>0 \end{aligned}$$

since $n \geq 10$. $\square$

Let us introduce the notion of *non-singular toric flips and flops.*

Definition 2.10 (Non-singular Toric Flips and Flops)**.** Let $f : X \longrightarrow Y$ be a non-singular toric flipping (resp. flopping) contraction and

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

the flip (resp. flop) of $f : X \longrightarrow Y$. This means that $f^+ : X^+ \longrightarrow Y$ is a small projective toric morphism such that K_{X^+} is f^+-ample and $\rho(X^+/Y) = 1$. Note that X^+ is uniquely determined by $f : X \longrightarrow Y$ (see [R, §3]). If X^+ is non-singular, then we call $X \dashrightarrow X^+$ the *non-singular toric flip* (resp. *flop*).

The function $e(n)$ (resp. $f(n)$) denotes the number of the isomorphism classes of n-dimensional non-singular toric flip (resp. flop) for $n \geq 1$.

2.11. From now on, we use the same notation as in 2.4. By the construction of $X^+ = X(\Delta^+)$, the cone $\langle e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_{n+1}\rangle \in \Delta^+$ for $1 \leq i \leq \alpha$ (see [R, (3.4) Theorem]). So, non-singularity of X^+ implies $a_i = -1$ for $1 \leq i \leq \alpha$, and

$$-K_X \cdot V(w) = n + 1 - (\alpha + \beta),$$

where w is the wall $\langle e_1, \cdots, e_{n-1}\rangle$. So, $-K_X \cdot V(w) > 0$ (resp. $= 0$) if and only if $\alpha + \beta \leq n$ (resp. $\alpha + \beta = n + 1$). Note that f is small if and only if $\alpha \geq 2$.

So, we obtain the following formulas.

Theorem 2.12. *We have*

$$\begin{aligned} e(n) &= \left(\lfloor \frac{n}{2} \rfloor - 1\right)\left(n - \lfloor \frac{n}{2} \rfloor - 1\right), \\ f(n) &= \lfloor \frac{n+1}{2} \rfloor - 1 \end{aligned}$$

for any $n \geq 1$. Note that we can express

$$\begin{aligned} e(2m-1) &= (m-1)(m-2), \\ e(2m) &= (m-1)^2, \\ f(2m-1) = f(2m) &= m - 1 \end{aligned}$$

for every $m \geq 1$. And we have the following relation

$$e(n+1) = e(n) + f(n).$$

Proof. We have

$$e(n) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} (n - (2k+1)).$$

The other statements are trivial. □

2.13. The following is a table of the values of $e(n), f(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$e(n)$	0	0	0	1	2	4	6	9	12	16	20	25	$\cdots$
$f(n)$	0	0	1	1	2	2	3	3	4	4	5	5	$\cdots$

It is easy to see that $e(n) > f(n)$ for $n \geq 6$, $e(n) = f(n)$ for $n = 4, 5$, and $f(3) > e(3)$. Note that $e(n) \ll a(n)$ and $f(n) \ll b(n)$ for $n \gg 1$.

After all, threefolds seem to be mysterious. The non-singular toric flop in dimension three, which is sometimes called *Atiyah's flop*, is a very special example of the elementary transformations.

2.14 (Miscellaneous Results on Non-singular Toric Varieties)**.** The following result is a supplementary remark on the minimal length of extremal rays of non-singular toric varieties. For the length of extremal rays of singular toric varieties, see [F1, Theorem 0.1] and [F2, Theorem 3.9].

Proposition 2.15 (Minimal Length of Extremal Rays)**.** *Let V be a non-singular projective toric variety and R an extremal ray of $NE(V)$. Let $\varphi_R : V \longrightarrow W$ be the extremal contraction with respect to R. We put $A := \mathrm{Exc}(\varphi_R)$ and $B := \varphi_R(A)$. Then A is irreducible and $\varphi_R|_A : A \longrightarrow B$ is equi-dimensional by* [R, (2.5), (2.6)]*. We know that $\dim A = n - \alpha$, $\dim B = \beta - \alpha$, and a general fiber F of $\varphi_R|_A$ is $\mathbb{P}^{n-\beta}$. The local description of φ_R is the same as* 2.4*. So, we use the notation α, β, and a_is in* 2.4*. Then, we have*

$$\begin{aligned} l(R) := \min_{[C]\in R}(-K_V \cdot C) &= n - \beta + 1 + \sum_{i=1}^{\alpha} a_i \\ &\leq n + 1 - (\alpha + \beta) \\ &= \dim F + 1 - \mathrm{codim} A, \end{aligned}$$

where C is an integral curve. Note that a_i is a negative integer for $1 \leq i \leq \alpha$. Assume that the equality holds in the above inequality. Then W is non-singular if φ_R is not small. When φ_R is small, the elementary transformation V^+ of $\varphi_R : V \longrightarrow W$ is non-singular.

Proof. Almost all the statements are obvious. If the equality holds, then $a_i = -1$ for $1 \le i \le \alpha$. This implies that W is non-singular when φ_R is not small. I recommend you to check the construction of W. The non-singularity of V^+ is obvious by the construction of V^+ (see also 2.11). □

The next claim follows from the local description in 2.4. This recovers [Mu, Section 4] easily (see [F1, Remark 3.3]). See also [R, (2.10) Corollary].

Proposition 2.16. *We use the same notation as in* 2.4. *Let V be a non-singular complete toric variety and D an torus invariant prime divisor on V. Let $C \simeq \mathbb{P}^1$ be a torus invariant integral curve on V. Then the following conditions are equivalent.*

(i) $D \cdot C > 0$,
(ii) $D \cdot C = 1$,
(iii) $C = V(w)$, $D = V(e_i)$ *for some* $\beta + 1 \le i \le n + 1$.

3. Questions

In this section, I do not assume that flipping (resp. flopping) contractions are toric. Here, it is better to work over $\mathbb{C}$. I list my questions about the numbers of flipping and flopping contractions.

Let $a(n)$ (resp. $b(n)$) denote the *number* of the flipping (resp. flopping) contractions $f : X \longrightarrow Y$, where $\dim X = n$ and X has mild singularities. I do not know *how to count the numbers of flips and flops*. So, the first question is

Question 3.1. *How do we define $a(n)$ and $b(n)$?*

Remark 3.2. We consider threefolds with terminal singularities. If we do not fix the indices of singularities, then it is obvious that there are infinitely many flipping contractions even in the toric category (see [Ma, Example-Claim 14-2-5]).

If we can define $a(n)$ and $b(n)$ suitably, then the next question is

Question 3.3. *Are there any relations between $a(n)$ and $b(n)$?*

See Theorem 2.6.

Question 3.4. *Which is larger, $a(n)$ or $b(n)$?*

See Theorem 2.9. The final question is

Question 3.5. *Do the functions*

$$\sum_{n \ge 1} a(n)t^n, \quad \sum_{n \ge 1} b(n)t^n \in \mathbb{Z}[[t]]$$

have good properties?

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