POSITIVITY OF EXTENSIONS OF VECTOR BUNDLES

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ABSTRACT. In this paper, we study when positivity conditions of vector bundles are preserved by extension. We prove that an extension of a big (resp. pseudo-effective) line bundle by an ample (resp. a nef) vector bundle is big (resp. pseudo-effective). We also show that an extension of an ample line bundle by a big line bundle is not necessarily pseudo-effective. In particular, this implies that an almost nef vector bundle is not necessarily pseudo-effective.

1. Introduction

Several positivity conditions defined for line bundles (e.g. ampleness, nefness, bigness, and pseudo-effectivity), which play a key role in the study of algebraic varieties, are naturally extended to vector bundles (see Definitions 2.1–2.4). The importance of such extensions is represented by the study of projective varieties whose tangent bundle satisfies such a positivity condition. Hartshorne [8] conjectured that the projective spaces are characterized as projective varieties having ample tangent bundle, which was solved affirmatively by Mori [14]. Also, the geometric structure of a projective variety with nef (resp. pseudo-effective) tangent bundle was studied in [2, 4] (resp. [10]).

When we consider a property of vector bundles, it is natural to ask whether or not it is preserved by extension.

Problem 1.1. *Consider an exact sequence of vector bundles:*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

If $\mathcal{E}'$ and $\mathcal{E}''$ satisfy a positivity condition (e.g. ampleness, nefness, bigness, or pseudo-effectivity), then does $\mathcal{E}$ satisfy the same?

This problem is known to hold affirmatively for ampleness and nefness (cf. [13, §6]). Using such a property of nefness, Campana and Peternell [2] completed the classifications of smooth projective surfaces and threefolds with nef tangent bundle. As their study implies, an affirmative answer to Problem 1.1 will be useful to know whether a vector bundle satisfies a positivity condition, so the problem has been expected to be solved for bigness and pseudo-effectivity (cf. [10, Problem 4.3], [7, Question 2.23]).

This paper includes two theorems. One of them gives an affirmative and partial answer to Problem 1.1 for bigness and pseudo-effectivity.

Theorem 1.2. *Let X be a normal projective variety over an algebraically closed field. Let $\mathcal{E}$ and $\mathcal{G}$ be vector bundles on X. Let $\mathcal{L}$ be a line bundle on X. Suppose*

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that there exists the following exact sequence:

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{L} \to 0.$$

(1) *If $\mathcal{G}$ is nef and $\mathcal{L}$ is pseudo-effective, then $\mathcal{E}$ is pseudo-effective.*
(2) *If $\mathcal{G}$ is ample and $\mathcal{L}$ is big, then $\mathcal{E}$ is big.*

The other theorem solves negatively Problem 1.1 for bigness and pseudo-effectivity, and also shows that the assumption that $\mathcal{G}$ is nef (resp. ample) of (1) (resp. (2)) in Theorem 1.2 cannot be weakened to that $\mathcal{G}$ is pseudo-effective (resp. big).

Theorem 1.3. *Let k be an algebraically closed field. Then there exist a smooth projective surface S over k and a vector bundle $\mathcal{V}$ on S with the following properties:*

- *there exists an exact sequence*

$$0 \to \mathcal{L} \to \mathcal{V} \to \mathcal{M} \to 0$$

  such that $\mathcal{L}$ is a big line bundle on S and $\mathcal{M}$ is an ample line bundle on S;
- *$\mathcal{V}$ is not pseudo-effective* (*so not big*).

Since a big line bundle is weakly positive, the above theorem also tells us that weak positivity is not necessarily preserved by extension. Here, weak positivity is a notion introduced by Viehweg [15] (see Definition 2.3), which is a stronger condition than pseudo-effectivity. In [1], these positivities were discussed by using the base loci of vector bundles (see also [7]). Note that a pseudo-effective (resp. big) vector bundle in this paper is said to be *V-psef* (resp. *V-big*) in [7, Definition 2.2].

Theorem 1.3 also solves another problem posed by Demailly, Peternell, and Schneider [5, Problem 6.6]. They introduced the notion of almost nefness for vector bundles (see Definition 2.7) as a generalization of nefness, and proved that a pseudo-effective vector bundle is almost nef ([5, Proposition 6.5]), leaving the converse as a problem ([5, Problem 6.6]). I.e., they asked whether almost nefness implies pseudo-effectivity. This is solved negatively by Theorem 1.3, because the preservation of almost nefness by extensions implies that $\mathcal{V}$ in the theorem is almost nef but not pseudo-effective.

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2. Definitions

In this section, we recall several definitions defined for vector bundles. Let k be an algebraically closed field of arbitrary characteristic. A *variety* is an integral separated scheme of finite type over k.

Definition 2.1. Let $\mathcal{E}$ be a vector bundle on a projective variety X. Let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the projectivization of $\mathcal{E}$. Let $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ be the tautological line bundle. We say that $\mathcal{E}$ is *ample* (resp. *nef*) if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample (resp. nef).

Definition 2.2. Let $\mathcal{G}$ be a coherent sheaf on a variety X. Let U be an open subset of X. We say that $\mathcal{G}$ is *globally generated over* U (resp. *generically globally generated*) if the natural map

$$H^0(X, \mathcal{G}) \otimes_k \mathcal{O}_X \to \mathcal{G}$$

is surjective over U (resp. surjective at the generic point of X).

Definition 2.3 ([15, Definition 1.2])**.** Let $\mathcal{G}$ be a vector bundle on a quasi-projective variety X. Let U be an open subset of X. Let H be an ample Cartier divisor on X. We say that $\mathcal{G}$ is *weakly positive over* U (resp. *pseudo-effective*) if for every $\alpha \in \mathbb{Z}_{>0}$, there exists a $\beta \in \mathbb{Z}_{>0}$ such that $S^{\alpha\beta}(\mathcal{G})(\beta H)$ is globally generated over U (resp. generically globally generated). Here, $S^{\alpha\beta}(\mathcal{G})$ denotes the $\alpha\beta$-th symmetric product of $\mathcal{G}$. We say that $\mathcal{G}$ is *weakly positive* if $\mathcal{G}$ is weakly positive over an open subset of X.

We further assume that X is projective in Definition 2.3. In this case, we can easily check that $\mathcal{G}$ is nef if and only if $\mathcal{G}$ is weakly positive over X.

Definition 2.4 ([11, Notation (vii)])**.** Let $\mathcal{G}$ be a vector bundle on a quasi-projective variety X. Let H be an ample Cartier divisor on X. We say that $\mathcal{G}$ is *big* if there exists an $\alpha \in \mathbb{Z}_{>0}$ such that $S^{\alpha}(\mathcal{G})(-H)$ is pseudo-effective.

By [18, Lemma 2.14], Definitions 2.3 and 2.4 are independent of the choice of ample Cartier divisor H. Note that, for a generically surjective morphism $\mathcal{E} \to \mathcal{F}$ between vector bundles, if $\mathcal{E}$ is pseudo-effective (resp. weakly positive, big), then so is $\mathcal{F}$.

Remark 2.5. The terminology "pseudo-effective" (resp. "big") is often used in a different meaning. For example, in other papers, a vector bundle $\mathcal{E}$ on a projective variety X is said to be pseudo-effective (resp. big) if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is pseudo-effective (resp. big). This is weaker than the pseudo-effectivity (resp. bigness) in this paper.

Remark 2.6. To the best knowledge of the authors, the notion of *weakly positive sheaves* was first introduced by Viehweg in [15] (see [15, Definition 1.2]) and that of *big sheaves* originates from [17] (see [17, Lemma 3.6]). We note that the definition of weak positivity in [17] is different from the one in [15] (see also [16, Definition 1.2]) and coincides with that of *pseudo-effectivity.*

Definition 2.7. Let $\mathcal{E}$ be a vector bundle on a projective variety X. We say that $\mathcal{E}$ is *almost nef* if there exists a countable family A_i of proper subvarieties of X such that $\mathcal{E}|_C$ is nef for all curves $C \not\subset \bigcup_i A_i$.

3. Proof of Theorem 1.2

Before starting the proof of Theorem 1.2, we recall the following lemma.

Lemma 3.1 ([18, Lemma 2.15])**.** *Let X be a smooth projective variety. Let $\mathcal{E}$ be a vector bundle on X. Then $\mathcal{E}$ is pseudo-effective if and only if for every finite surjective morphism $\pi : X' \to X$ from a smooth projective variety X' and for every ample divisor H' on X', the vector bundle $\pi^*\mathcal{E}(H')$ is pseudo-effective.*

Proof of Theorem 1.2. First, we prove (1). When $\mathrm{char}(k) = 0$ (resp. $\mathrm{char}(k) > 0$), we take a resolution of singularities (resp. a smooth alteration constructed in [3, 4.1. Theorem]). Then, by [6, Lemma 2.4 (2)], we may assume that X is smooth. We replace X with any finite cover of X. By Lemma 3.1, it is enough to show that $\mathcal{E}(H)$ is pseudo-effective for every ample divisor H on X. Note that $\mathcal{L}(H)$ is big. When $\mathrm{char}(k) = 0$, by taking a resolution of a suitable cyclic cover and using [6, Lemma 2.4 (2)], we may assume that there is an injective morphism $\mathcal{O}_X \hookrightarrow \mathcal{L}(H)$. When $\mathrm{char}(k) > 0$, by using the Frobenius morphism and [6, Lemma 2.4 (2)], we can replace $\mathcal{L}(H)$ by $\mathcal{L}(H)^{p^e}$ and obtain an injective morphism $\mathcal{O}_X \hookrightarrow \mathcal{L}(H)$. Let $\mathcal{F}$ be the inverse image of $\mathcal{O}_X$ by $\mathcal{E}(H) \twoheadrightarrow \mathcal{L}(H)$. Then we have the morphism between exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(H) \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d, hook, "\tau"] & \mathcal{O}_X \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{G}(H) \arrow[r] & \mathcal{E}(H) \arrow[r] & \mathcal{L}(H) \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathcal{G}(H)$ and $\mathcal{O}_X$ are nef, we see that $\mathcal{F}$ is also nef. By the generic surjectivity of τ, we see that $\mathcal{E}(H)$ is pseudo-effective.

Next, we prove (2). Let H be an ample Cartier divisor on X. Take $m \in \mathbb{Z}_{>0}$ such that $\mathrm{char}(k) \nmid m$, $S^m(\mathcal{G})(-H)$ is nef, and $\mathcal{L}^m(-H)$ is pseudo-effective. By [12, Theorem 4.1.10], there are a surjective finite morphism $\pi : X' \to X$ from a normal projective variety X' and an ample Cartier divisor H' on X' such that $mH' \sim \pi^*H$. Consider the exact sequence

$$0 \to \pi^*\mathcal{G}(-H') \to \pi^*\mathcal{E}(-H') \to \pi^*\mathcal{L}(-H') \to 0.$$

Since $S^m(\pi^*\mathcal{G}(-H')) \cong \pi^*\big(S^m(\mathcal{G})(-H)\big)$ is nef, so is $\pi^*\mathcal{G}(-H')$. Also, $\pi^*\mathcal{L}(-H')$ is pseudo-effective, so we see that $\pi^*\mathcal{E}(-H')$ is pseudo-effective by (1). Then

$$\begin{aligned}
S^{(m+1)\beta}(\pi^*\mathcal{E}(-H'))(\beta H') &\cong S^{(m+1)\beta}(\pi^*\mathcal{E})(-m\beta H') \\
&\cong S^{(m+1)\beta}(\pi^*\mathcal{E})(-\beta\pi^*H) \cong \pi^*\left(S^{(m+1)\beta}(\mathcal{E})(-\beta H)\right)
\end{aligned}$$

is generically globally generated for some $\beta \in \mathbb{Z}_{>0}$, so $S^{(m+1)\beta}(\mathcal{E})(-\beta H)$ is pseudo-effective by [6, Lemma 2.4 (2)], which means that $\mathcal{E}$ is big. $\square$

If the following question is answered affirmatively, then we can generalize Theorem 1.2 to the case of higher rank by an argument similar to that of the above proof.

Question 3.2. Let $\mathcal{E}$ be a *big* vector bundle on a normal projective variety X over an algebraically closed field. Does there exist a surjective finite morphism $\pi : X' \to X$ from a normal projective variety X' such that $\pi^*\mathcal{E}$ is generically globally generated?

For example, one can easily check that the question holds affirmatively if $\mathcal{E}$ is a direct sum of big line bundles.

4. Proof of Theorem 1.3

Set $X := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Let $f : X \to \mathbb{P}^1$ be the projection. Let $C \subset X$ be the section of f corresponding to the quotient $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-2)$. Then

$\mathcal{O}_X(1) \cong \mathcal{O}_X(C)$. We define the divisor H on X as

$$H := C + 3f^*[y],$$

where $y \in \mathbb{P}^1$ is a closed point. Then we see from [9, V, Theorem 2.17] that H is very ample.

Since $f_*\mathcal{O}_X(C) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, we have $H^1(\mathbb{P}^1, f_*\mathcal{O}_X(C)) \cong k$. Thus, from the Leray spectral sequence, we obtain $H^1(X, \mathcal{O}_X(C)) \cong k$. Take $0 \neq \xi \in \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X(C))$. Let

$$(\flat) \qquad 0 \to \mathcal{O}_X(C) \to \mathcal{E} \to \mathcal{O}_X \to 0$$

be the exact sequence corresponding to ξ. Since $H^1(X, \mathcal{O}_X) = 0$, we see that the natural morphism

$$H^1(X, \mathcal{O}_X(C)) \to H^1(C, \mathcal{O}_C(C))$$

is injective, so the exact sequence

$$0 \to \mathcal{O}_C(C) \to \mathcal{E}|_C \to \mathcal{O}_C \to 0$$

does not split. Since $\mathcal{O}_C(C) \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, we see that $\mathcal{E}|_C \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

From now on, we divide the proof into the case of $\mathrm{char}(k) = 0$ and the case of $\mathrm{char}(k) = p > 0$.

<u>Case of $\mathrm{char}(k) = 0$</u>. By [12, Theorem 4.1.10], there is a surjective finite morphism $\pi : X' \to X$ from a smooth projective surface and an ample Cartier divisor H' on X such that $\pi^*H \sim 4H'$. Put $\mathcal{G} := \pi^*\mathcal{E}(H')$. Taking the pullback of $(\flat)$ and the tensor product with $\mathcal{O}_{X'}(H')$, we obtain the exact sequence

$$0 \to \mathcal{O}_{X'}(\pi^*C + H') \to \mathcal{G} \to \mathcal{O}_{X'}(H') \to 0.$$

We prove that $\mathcal{G}$ is not pseudo-effective. From

$$S^4(\mathcal{G}) \cong S^4(\pi^*\mathcal{E})(4H') \cong S^4(\pi^*\mathcal{E})(\pi^*H) \cong \pi^*\left(S^4(\mathcal{E})(H)\right),$$

it is enough to show that $\mathcal{F} := S^4(\mathcal{E})(H)$ is not pseudo-effective by [18, Corollary 2.20] and [6, Lemma 2.4 (2)]. For this purpose, we check that

$$S^{4\beta}(\mathcal{F})(\beta H) \cong S^{4\beta}\left(S^4(\mathcal{E})\right)(5\beta H)$$

is not generically globally generated for each $\beta \in \mathbb{Z}_{>0}$. By $(\flat)$, we have the following surjective morphism

$$\sigma_\beta : S^{4\beta}\left(S^4(\mathcal{E})\right)(5\beta H) \twoheadrightarrow \mathcal{O}_X(5\beta H).$$

Let us consider the following commutative diagram:

$$\begin{tikzcd}
S^{4\beta}\left(S^4(\mathcal{E})\right)(15\beta f^*[y]) \arrow[r, hook, "\tau_\beta"] \arrow[d, "\lambda_\beta"'] & S^{4\beta}\left(S^4(\mathcal{E})\right)(5\beta H) \arrow[d, "\sigma_\beta"] \\
\mathcal{O}_X(15\beta f^*[y]) \arrow[r, hook] & \mathcal{O}_X(5\beta H).
\end{tikzcd}$$

Here, the horizontal arrows are induced from the morphism $\mathcal{O}_X(-5\beta C) \hookrightarrow \mathcal{O}_X$. In order to prove that $S^{4\beta}\left(S^4(\mathcal{E})\right)(5\beta H)$ is not generically globally generated, it is enough to see that $H^0(\sigma_\beta)$ is the zero-map. For this purpose, it is sufficient to prove $H^0(\tau_\beta)$ is bijective and $H^0(\lambda_\beta)$ is the zero-map.

Claim 1. $H^0(\tau_\beta)$ is bijective.

Proof of Claim 1. Taking the tensor product of

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and $S^{4\beta}\left(S^4(\mathcal{E})\right)(lC + 15\beta f^*[y])$ for $l \in \mathbb{Z}_{\geq 0}$, we obtain the following exact sequence:

$$\begin{aligned}&0 \to H^0\left(X, S^{4\beta}\left(S^4(\mathcal{E})\right)((l-1)C + 15\beta f^*[y])\right)\\&\quad \to H^0\left(X, S^{4\beta}\left(S^4(\mathcal{E})\right)(lC + 15\beta f^*[y])\right) \to H^0\left(C, S^{4\beta}\left(S^4(\mathcal{E})\right)(lC + 15\beta f^*[y])|_C\right).\end{aligned}$$

Since $\mathcal{O}_C(C) \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\mathcal{E}|_C \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, we have

$$\begin{aligned}H^0\left(C, S^{4\beta}\left(S^4(\mathcal{E})\right)(lC + 15\beta f^*[y])|_C\right) &\cong \bigoplus H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-16\beta - 2l + 15\beta)\right)\\&= \bigoplus H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-\beta - 2l)\right) = 0.\end{aligned}$$

From $H^0\left(X, S^{4\beta}\left(S^4(\mathcal{E})\right)(5\beta H)\right) = H^0\left(X, S^{4\beta}\left(S^4(\mathcal{E})\right)(5\beta C + 15\beta f^*[y])\right)$, our claim follows. □

Claim 2. $H^0(\lambda_\beta)$ is the zero-map.

Proof of Claim 2. Consider the following commutative diagram:

$$\begin{array}{ccc} H^0\left(X, S^{4\beta}\left(S^4(\mathcal{E})\right)(15\beta f^*[y])\right) & \longrightarrow & H^0\left(C, S^{4\beta}\left(S^4(\mathcal{E})\right)(15\beta f^*[y])|_C\right) \\ {\scriptstyle H^0(\lambda_\beta)}\downarrow & & \downarrow \\ H^0\left(X, \mathcal{O}_X(15\beta f^*[y])\right) & \longrightarrow & H^0\left(C, \mathcal{O}_C(15\beta f^*[y])\right). \end{array}$$

The bottom horizontal arrow is bijective, since C is a section of f. Hence, our claim follows from

$$H^0\left(C, S^{4\beta}(S^4(\mathcal{E}))(15\beta f^*[y])|_C\right) \cong \bigoplus H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-16\beta + 15\beta)\right) = 0.$$

□

We put $S := X'$, $\mathcal{V} := \mathcal{G}$, $\mathcal{L} := \mathcal{O}_{X'}(\pi^*C + H')$, and $\mathcal{M} := \mathcal{O}_{X'}(H')$. Then they satisfy all the desired properties.

Case of $\mathrm{char}(k) = p > 0$. Set $e := 1$ (resp. $e := 2$) if $p \geq 5$ (resp. $p < 5$). Then $p^e \geq 4$. Put $\mathcal{G} := (F^{e*}\mathcal{E})(H)$, where F is the absolute Frobenius morphism of X. Taking the pullback of ($\flat$) by F^e and the tensor product with $\mathcal{O}_X(H)$, we obtain the exact sequence

$$0 \to \mathcal{O}_X(p^e C + H) \to \mathcal{G} \to \mathcal{O}_X(H) \to 0.$$

We prove that $\mathcal{G}$ is not pseudo-effective. For this purpose, we check that

$$S^{4\beta}(\mathcal{G})(\beta H) \cong S^{4\beta}(F^{e*}\mathcal{E})(5\beta H)$$

is not generically globally generated for each $\beta \in \mathbb{Z}_{>0}$. We have the following surjective morphism

$$s_\beta : S^{4\beta}(F^{e*}\mathcal{E})(5\beta H) \twoheadrightarrow \mathcal{O}_X(5\beta H).$$

Thus, it is enough to check that $H^0(s_\beta)$ is the zero-map. For each $l \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} H^0\left(C, S^{4\beta}(F^{e*}\mathcal{E})(lC+15\beta f^*[y])|_C\right) &\cong \bigoplus H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-4\beta p^e - 2l + 15\beta)\right) \\ &\cong \bigoplus H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left((15-4p^e)\beta - 2l\right)\right) = 0. \end{aligned}$$

Note that $15-4p^e \leq 15-16 = -1$. Hence, we can prove $H^0(s_\beta) = 0$ by an argument similar to that of $H^0(\sigma_\beta) = 0$ as in the $\mathrm{char}(k) = 0$ case.

We put $S := X$, $\mathcal{V} := \mathcal{G}$, $\mathcal{L} := \mathcal{O}_X(p^eC + H)$, and $\mathcal{M} := \mathcal{O}_X(H)$. Then they satisfy all the desired properties. $\square$

Remark 4.1. The vector bundle $\mathcal{E}$ in ($\flat$) is a simple example of an almost nef but not pseudo-effective vector bundle. That $\mathcal{E}$ is not pseudo-effective is proved implicitly in the proof above, but can also be proved directly. By ($\flat$), we get the surjective morphism

$$t_\beta : S^{4\beta}(\mathcal{E})(\beta H) \twoheadrightarrow \mathcal{O}_X(\beta H)$$

for each $\beta \in \mathbb{Z}_{>0}$. We can prove $H^0(t_\beta) = 0$ by using the commutative diagram

$$\begin{tikzcd} S^{4\beta}(\mathcal{E})(3\beta f^*[y]) \arrow[r, hook] \arrow[d, two heads] & S^{4\beta}(\mathcal{E})(\beta H) \arrow[d, two heads, "t_\beta"] \\ \mathcal{O}_X(3\beta f^*[y]) \arrow[r, hook] & \mathcal{O}_X(\beta H), \end{tikzcd}$$

where the horizontal arrows are induced from the morphism $\mathcal{O}_X(-\beta C) \hookrightarrow \mathcal{O}_X$, and a vanishing as in Claim 1, that is,

$$H^0\left(C, S^{4\beta}(\mathcal{E})(lC + 3\beta f^*[y])|_C\right) = \bigoplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-4\beta - 2l + 3\beta)) = 0$$

for each $l \in \mathbb{Z}_{\geq 0}$.

Remark 4.2. In positive characteristic, we do not know whether the pseudo-effectivity of $\mathcal{E}$ implies that of $S^m(\mathcal{E})$, so we choose to separate the proof into the case of $\mathrm{char}(k) = 0$ and the case of $\mathrm{char}(k) > 0$.

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