DIRECT IMAGES OF PLURICANONICAL DIVISORS

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Abstract. We discuss the local freeness and the semipositivity of direct images of pluricanonical divisors for surjective morphisms between smooth projective varieties with connected fibers. We give a desirable semipositivity theorem under the assumption that the geometric generic fiber has a good minimal model.

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1. Introduction

By Griffiths's theory of variations of Hodge structure (see [Gr]), we have:

Theorem 1.1 (Griffiths)**.** *Let $f : X \to Y$ be a smooth morphism between smooth projective varieties. Then $f_*\omega_{X/Y}$ is a semipositive locally free sheaf.*

Before we go further, let us recall the definition of semipositive locally free sheaves.

Definition 1.2 (Semipositive locally free sheaves)**.** Let $\mathcal{E}$ be a locally free sheaf of finite rank on a complete algebraic variety V. Then $\mathcal{E}$ is called *semipositive* if $\mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$ is nef on $\mathbb{P}_V(\mathcal{E})$. A semipositive locally free sheaf $\mathcal{E}$ is sometimes called a *nef* locally free sheaf in the literature.

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Precisely speaking, Griffiths proved that $f_*\omega_{X/Y}$ is semipositive in the sense of Griffiths and his result is sharper than Theorem 1.1. Moreover, Berndtsson proved that $f_*\omega_{X/Y}$ is semipositive in the sense of Nakano by L^2 method (see [Ber, Theorem 1.2]). Unfortunately, Theorem 1.1 is not so useful for various geometric applications since we need the smoothness of f. In [K1], Kawamata proved Theorem 1.3, which is a natural generalization of Theorem 1.1, by using the theory of variations of Hodge structure (see [K1, Theorem 5]).

Theorem 1.3 (Fujita, Zucker, Kawamata, $\cdots$)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then there exists a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y' with the following property. Let X' be any resolution of the main component of $X \times_Y Y'$. Then $f'_*\omega_{X'/Y'}$ is a semipositive locally free sheaf, where $f' : X' \to X \times_Y Y' \to Y'$.*

For the details of Kawamata's original approach and various generalizations, see [F1, Theorems 3.1, 3.4, and 3.9], [FF, Theorem 1.1 and Theorem 1.3], and [FFS, Corollary 2, Theorem 2, and Theorem 3]. Theorem 1.3 has already played crucial roles in the study of higher-dimensional algebraic varieties. For some geometric applications, we have to treat $f_*\omega^{\otimes m}_{X/Y}$ or $f'_*\omega^{\otimes m}_{X'/Y'}$, where m is a positive integer. It is well known that Viehweg proved that $f_*\omega^{\otimes m}_{X/Y}$ is always weakly positive for every positive integer m in Theorem 1.3 (see [V, Theorem III]). His original proof of the weak positivity of $f_*\omega^{\otimes m}_{X/Y}$ uses his mysterious covering trick and Theorem 1.3 (see [V, §5]).

By the way, Theorem 1.1 can be generalized as follows.

Theorem 1.4. *Let $f : X \to Y$ be a smooth morphism between smooth projective varieties. Then $f_*\omega^{\otimes m}_{X/Y}$ is a semipositive locally free sheaf for every positive integer m.*

We give a proof of Theorem 1.4 based on Siu's invariance of plurigenera (see [S, Corollary 0.2] and [Pă, Theorem 1]) and the effective freeness in [PoS] (see [PoS, Theorem 1.4]). Note that Siu's invariance of plurigenera is not Hodge theoretic. It is a very clever application of the Ohsawa–Takegoshi L^2 extension theorem. We have no Hodge theoretic characterization of $f_*\omega^{\otimes m}_{X/Y}$ in Theorem 1.4 when $m \geq 2$. By Theorem 1.3 and Theorem 1.4, it is natural to consider:

Conjecture 1.5 (Semipositivity of direct images of pluricanonical divisors)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Then there exists a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y' with*

the following property. Let X' be any resolution of the main component of $X \times_Y Y'$ sitting in the following commutative diagram:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\tau]{} & Y. \end{array}$$

Then $f'_*\omega^{\otimes m}_{X'/Y'}$ is a semipositive locally free sheaf for every positive integer m.

Conjecture 1.5 can be seen as a correct formulation of Fujita's very naive conjecture: [Ft, Conjecture Wa_m]. Note that [Ft] contains 17 conjectures and that "Wa" means 13th in [Ft].

The main purpose of this paper is to prove:

Theorem 1.6 (Main theorem)**.** *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Assume that the geometric generic fiber $X_{\overline{\eta}}$ of $f : X \to Y$ has a good minimal model. Then there exists a generically finite morphism $\tau : Y' \to Y$ from a smooth projective variety Y' with the following property. Let X' be any resolution of the main component of $X \times_Y Y'$ sitting in the following commutative diagram:*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\tau]{} & Y. \end{array}$$

Then $f'_*\omega^{\otimes m}_{X'/Y'}$ is a semipositive locally free sheaf for every positive integer m.

We note that $X_{\overline{\eta}}$ has a good minimal model if $\dim X_{\overline{\eta}} - \kappa(X_{\overline{\eta}}) \leq 3$ (see [BCHM], [La], and Theorem 3.7). In particular, $X_{\overline{\eta}}$ has a good minimal model if $X_{\overline{\eta}}$ is of general type. Theorem 1.6 reduces Conjecture 1.5 to the good minimal model conjecture for geometric generic fibers. Of course, it is highly desirable to prove Conjecture 1.5 without any extra assumptions. Our proof of Theorem 1.6 is geometric and does not use the theory of variations of Hodge structure. We do not even use L^2 method in the proof of Theorem 1.6. Our proof of Theorem 1.6 in this paper is minimal model theoretic. Anyway, Theorem 1.4 and Theorem 1.6 strongly support Conjecture 1.5.

Remark 1.7. If Y is a curve in Conjecture 1.5, then Kawamata proved that $f_*\omega^{\otimes m}_{X/Y}$ is a semipositive locally free sheaf for every positive integer m (see [K2, Theorem 1]). We also note that Viehweg's weak positivity

of $f_*\omega_{X/Y}^{\otimes m}$ (see [V, Theorem III]) implies that $f_*\omega_{X/Y}^{\otimes m}$ is semipositive when Y is a curve.

We sketch the proof of Theorem 1.6 for the reader's convenience.

1.8 (Outline of the proof of Theorem 1.6)**.** We take a weak semistable reduction $f^\dagger : X^\dagger \to Y'$ in the sense of Abramovich–Karu. Then we take a good minimal model $\widetilde{f} : \widetilde{X} \to Y'$ of $f^\dagger : X^\dagger \to Y'$. Let P be an arbitrary point of Y' and let C be a smooth curve on Y' such that $P \in C$ and that $C = H_1 \cap H_2 \cap \cdots \cap H_{\dim Y'-1}$, where H_i is a general very ample Cartier divisor for every i. Then we can prove that $\widetilde{X}_C = \widetilde{X} \times_{Y'} C$ is a normal variety with only canonical singularities. Therefore, we obtain that $\widetilde{f}$ is flat and $\dim H^0(\widetilde{X}_y, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})|_{\widetilde{X}_y})$ is independent of $y \in Y'$ for every positive integer m. This implies that $f'_*\omega_{X'/Y'}^{\otimes m}$ is locally free for every positive integer m. Once we establish the local freeness of $f'_*\omega_{X'/Y'}^{\otimes m}$, the semipositivity of $f'_*\omega_{X'/Y'}^{\otimes m}$ easily follows from the effective freeness by Popa–Schnell and Viehweg's fiber product trick. As explained above, a key point of the proof of Theorem 1.6 is to construct a good minimal model $\widetilde{f} : \widetilde{X} \to Y'$ which behaves well under the base change by $C \hookrightarrow Y'$. Our proof of Theorem 1.6 is not Hodge theoretic.

After the author circulated a preliminary version of this paper, Mihai Păun and Shigeharu Takayama informed him of their new preprint [PăT], where they prove various semipositivity theorems by L^2 method. Their approach is completely different from ours. For the details, we recommend the reader to see [PăT].

We summarize the contents of this paper. In Section 2, we collect some basic definitions and results for the reader's convenience. In Section 3, we discuss the relationship between relative good minimal models and good minimal models of fibers. In Section 4, we prove the local freeness of direct images of pluricanonical divisors in Theorem 1.6 after taking a weak semistable reduction. In order to prove the local freeness, we take a relative good minimal model of the weak semistable reduction. Therefore, we need the assumption that the geometric generic fiber has a good minimal model. In Section 5, we prove the semipositivity in our main theorem: Theorem 1.6. The proof is an easy application of the effective freeness obtained by Popa–Schnell (see [PoS, Theorem 1.4]) and Viehweg's fiber product trick (see [V, (3.4)]).

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. We will freely use the standard notations and results of the minimal model program as in [KM], [F3] and [F5]. We recommend the reader to see [M, §5] and [F2, Section 5] for the details of Theorem 1.3 and various related topics.

2. Preliminaries

In this section, we collect some basic notations and results for the reader's convenience. For the details, see [KM], [F3], and [F5].

2.1 (Dualizing sheaves and canonical divisors)**.** Let X be a normal quasi-projective variety. Then we put $\omega_X = \mathcal{H}^{-\dim X}(\omega_X^\bullet)$, where $\omega_X^\bullet$ is the dualizing complex of X, and call ω_X the *dualizing sheaf* of X. We put $\omega_X \simeq \mathcal{O}_X(K_X)$ and call K_X the *canonical divisor* of X. Note that K_X is a well-defined Weil divisor on X up to the linear equivalence. Let $f : X \to Y$ be a morphism between Gorenstein varieties. Then we put $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{\otimes -1}$.

2.2 (Singularities of pairs)**.** Let X be a normal variety and let Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a resolution of singularities. We write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i$$

and $a(E_i, X, \Delta) = a_i$. Note that the *discrepancy* $a(E, X, \Delta) \in \mathbb{Q}$ can be defined for every prime divisor E *over* X. If $a(E, X, \Delta) > -1$ for every exceptional divisor E over X, then (X, Δ) is called a *plt* pair. If $a(E, X, \Delta) > -1$ for every divisor E over X, then (X, Δ) is called a *klt* pair. In this paper, if $\Delta = 0$ and $a(E, X, 0) \geq 0$ for every divisor E over X, then we say that X has only *canonical singularities*.

Remark 2.3. Although $\mathbb{R}$-divisors play crucial roles in the recent developments of the minimal model program, we do not use $\mathbb{R}$-divisors in this paper.

We need the following lemma in the proof of the local freeness in the main theorem: Theorem 1.6.

Lemma 2.4. *Let X be a normal variety with only canonical singularities. Then $\mathcal{O}_X(mK_X)$ is Cohen–Macaulay for every integer m.*

Proof. We note that X has only rational singularities when X is canonical. Let r be the smallest positive integer such that rK_X is Cartier. Since the problem is local, we may assume that $rK_X \sim 0$ by shrinking X. If $r = 1$, then $\mathcal{O}_X(mK_X) \simeq \mathcal{O}_X$ for every integer m. In this case, $\mathcal{O}_X(mK_X)$ is Cohen–Macaulay for every integer m since X has only rational singularities. From now on, we assume that $r \geq 2$. Let $\pi : \widetilde{X} \to X$ be the index one cover. Then we have

$$\pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \simeq \bigoplus_{i=1}^{r} \mathcal{O}_X(iK_X).$$

Since $\widetilde{X}$ has only canonical singularities and $K_{\widetilde{X}}$ is Cartier, $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ is Cohen–Macaulay. Since π is finite, $\mathcal{O}_X(iK_X)$ is Cohen–Macaulay for $1 \leq i \leq r$. By $rK_X \sim 0$, we obtain that $\mathcal{O}_X(mK_X)$ is Cohen–Macaulay for every integer m. □

3. Relative good minimal models

In this section, we discuss the relationship between relative good minimal models and good minimal models of fibers for the reader's convenience. The results in this section are more or less known to the experts.

Let us recall the definition of sufficiently general fibers.

Definition 3.1 (Sufficiently general fibers)**.** Let $f : X \to Y$ be a morphism between algebraic varieties. Then a *sufficiently general fiber* F of $f : X \to Y$ means that $F = f^{-1}(y)$ where y is any point contained in a countable intersection of Zariski dense open subsets of Y.

A sufficiently general fiber is sometimes called a *very general fiber* in the literature.

Definition 3.2 (Good minimal models)**.** Let $f : X \to Y$ be a projective morphism between normal quasi-projective varieties. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. A pair (X', Δ') sitting in a diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Y & \end{array}$$

is called a *minimal model of* (X, Δ) *over* Y if

(i) X' is $\mathbb{Q}$-factorial,
(ii) f' is projective,
(iii) ϕ is birational and ϕ^{-1} has no exceptional divisors,

(iv) $\phi_*\Delta = \Delta'$,
(v) $K_{X'} + \Delta'$ is f'-nef, and
(vi) $a(E, X, \Delta) < a(E, X', \Delta')$ for every ϕ-exceptional divisor $E \subset X$.

Furthermore, if $K_{X'} + \Delta'$ is f'-semi-ample, then (X', Δ') is called a *good minimal model of* (X, Δ) *over* Y. When Y is a point, we usually omit "over Y" in the above definitions. We sometimes simply say that (X', Δ') is a *relative* (*good*) *minimal model* of (X, Δ).

Although Theorem 3.3 holds for klt pairs, we state it for varieties with only canonical singularities for simplicity. Theorem 3.3 is useful and sufficient for our application in this paper.

Theorem 3.3. *Let $f : X \to Y$ be a projective surjective morphism from a normal quasi-projective variety X with only canonical singularities to a normal quasi-projective variety Y with connected fibers. Then the following conditions are equivalent.*

(i) *X has a good minimal model over Y.*
(ii) *$X_{\overline{\eta}}$ has a good minimal model, where $X_{\overline{\eta}}$ is the geometric generic fiber of $f : X \to Y$.*
(iii) *F has a good minimal model, where F is a sufficiently general fiber of $f : X \to Y$.*
(iv) *G has a good minimal model, where G is a general fiber of $f : X \to Y$.*

In order to understand Theorem 3.3, we give some supplementary results.

Theorem 3.4. *Let (X, Δ) be a projective klt pair such that Δ is a $\mathbb{Q}$-divisor. Then (X, Δ) has a good minimal model if and only if $K_X + \Delta$ is pseudo-effective, equivalently, $\kappa_\sigma(X, K_X + \Delta) \geq 0$, and*

$$\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta),$$

where κ_σ denotes Nakayama's numerical Kodaira dimension and κ denotes Iitaka's D-dimension.

Proof. For the proof, see [GL, Theorem 4.3] or [DHP, Remark 2.6]. □

Lemma 3.5. *Let $f : X \to Y$ be a projective surjective morphism between normal varieties with connected fibers and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Let $X_{\overline{\eta}}$ be the geometric generic fiber of $f : X \to Y$. We put $\Delta_{\overline{\eta}} = \Delta|_{X_{\overline{\eta}}}$. Then we have*

$$\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}) = \kappa(F, K_F + \Delta|_F)$$

and

$$\kappa_\sigma(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}) = \kappa_\sigma(F, K_F + \Delta|_F)$$

where F is a sufficiently general fiber of $f : X \to Y$.

Proof. This is obvious by the definitions of Iitaka's D-dimension κ and Nakayama's numerical Kodaira dimension κ_σ. For the details, see [N2] and [Le]. □

By combining Theorem 3.4 with Lemma 3.5, we have:

Corollary 3.6. *Let $f : X \to Y$ be a projective surjective morphism between normal varieties and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Then $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ has a good minimal model if and only if $(F, \Delta|_F)$ has a good minimal model, where F is a sufficiently general fiber of $f : X \to Y$.*

Proof. This statement is obvious by Theorem 3.4 and Lemma 3.5. □

Let us give a proof of Theorem 3.3 for the reader's convenience.

Proof of Theorem 3.3. We divide the proof into several steps.

Step 1 ((ii)$\Longleftrightarrow$(iii))**.** This step is a special case of Corollary 3.6.

Step 2 ((iv)$\Longrightarrow$(iii))**.** This is obvious since a sufficiently general fiber of $f : X \to Y$ is a general fiber of $f : X \to Y$.

Step 3 ((i)$\Longrightarrow$(iv))**.** Let $f' : X' \to Y$ be a good minimal model of X over Y. Let $y \in Y$ be a general point. We consider the fibers $G = f^{-1}(y)$ and $G' = f'^{-1}(y)$. If G' is not $\mathbb{Q}$-factorial, then we take a small projective $\mathbb{Q}$-factorization of G'. Then we see that G' is a good minimal model of G.

Step 4 ((iii)$\Longrightarrow$(i))**.** This is a special case of [HX, Theorem 2.12] (see also the proof of [Bir, Theorem 5.1]).

We complete the proof of Theorem 3.3. □

We close this section with a useful result, which follows from [La, Theorem 4.4] (see also [Bir, Theorem 1.5] and [HX, Theorem 2.12]).

Theorem 3.7. *Let X be a smooth projective variety with non-negative Kodaira dimension. Then X has a good minimal model if and only if the geometric generic fiber of the Iitaka fibration of X has a good minimal model.*

Proof. See [La, Theorem 4.4], [Bir, Theorem 5.1], and [HX, Theorem 2.12]. □

By Theorem 3.7, we know that any smooth projective variety X with $\dim X - \kappa(X) \leq 3$ has a good minimal model.

Remark 3.8. In the proof of Theorem 3.7 and Step 4 in the proof of Theorem 3.3, we need the finite generation of canonical rings for (relative) klt pairs, which is established in [BCHM]. We note that the finial step of the proof of the finite generation of canonical rings for klt pairs needs the canonical bundle formula due to Fujino–Mori (see [FM]). We also note that the canonical bundle formula treated in [FM] depends on a special case of Theorem 1.3. Therefore, our proof of Theorem 1.6 in this paper implicitly uses Theorem 1.3.

4. Local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$

In this section, we prove the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$ in Theorem 1.6 by using the minimal model theory and the weak semistable reduction theorem due to Abramovich–Karu (see [AbK]).

Let us start with the proof of the local freeness of $f_*\omega^{\otimes m}_{X/Y}$ in Theorem 1.4. It is a direct consequence of Siu's invariance of plurigenera (see [S, Corollary 0.2] and [Pă, Theorem 1]).

Proof of the local freeness of $f_*\omega^{\otimes m}_{X/Y}$ in Theorem 1.4. By [S, Corollary 0.2], we know that

$$\dim H^0(X_y, \mathcal{O}_{X_y}(mK_{X_y}))$$

is independent of $y \in Y$ for every positive integer m (see also [Pă, Theorem 1]). By the base change theorem (see [H, Chapter III, Corollary 12.9]), this implies that $f_*\omega^{\otimes m}_{X/Y}$ is locally free for every $m \geq 1$. □

Let us recall the following well-known lemma, which is a special case of [N1, Corollary 3].

Lemma 4.1 (cf. [N1, Corollary 3])**.** *Let $g : V \to C$ be a projective surjective morphism from a normal quasi-projective variety V to a smooth quasi-projective curve C. Assume that V has only canonical singularities and that K_V is g-semi-ample. Then $R^ig_*\mathcal{O}_V(mK_V)$ is locally free for every i and every positive integer m.*

Proof. Let $h : V' \to V$ be a resolution of singularities such that $\mathrm{Exc}(h)$ is a simple normal crossing divisor on V'. We write

$$K_{V'} = h^*K_V + E,$$

where E is an effective h-exceptional $\mathbb{Q}$-divisor. Then we have

$$\lceil mh^*K_V + E \rceil - (K_{V'} + \{-(mh^*K_V + E)\}) = (m-1)h^*K_V.$$

We note that the right hand side is semi-ample over C. Therefore,

$$R^i(g\circ h)_*\mathcal{O}_{V'}(\lceil mh^*K_V+E\rceil)$$

is locally free for every i and every positive integer m (see, for example, [F3, Theorem 6.3 (i)]). On the other hand, we have

$$R^ih_*\mathcal{O}_{V'}(\lceil mh^*K_V+E\rceil)=0$$

for every $i>0$ by the relative Kawamata–Viehweg vanishing theorem, and

$$h_*\mathcal{O}_{V'}(\lceil mh^*K_V+E\rceil)\simeq\mathcal{O}_V(mK_V).$$

Therefore, we obtain that

$$R^ig_*\mathcal{O}_V(mK_V)$$

is locally free for every i and every positive integer m. □

Proof of the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$ in Theorem 1.6. Let us divide the proof into several steps.

Step 1 (Weak semistable reduction)**.** By [AbK, Definition 0.1 and Theorem 0.3], there exist a generically finite morphism $\tau:Y'\to Y$ from a smooth projective variety Y' and $f^\dagger:X^\dagger\to Y'$ with the following properties.

- (i) $X^\dagger$ is a normal projective Gorenstein (see [AbK, Lemma 6.1]) variety which is birationally equivalent to $X\times_Y Y'$.
- (ii) $(U_{X^\dagger}\subset X^\dagger)$ and $(U_{Y'}\subset Y')$ are toroidal embeddings without self-intersection, with $U_{X^\dagger}=(f^\dagger)^{-1}(U_{Y'})$.
- (iii) $f^\dagger:(U_{X^\dagger}\subset X^\dagger)\to(U_{Y'}\subset Y')$ is toroidal and equidimensional.
- (iv) all the fibers of the morphism $f^\dagger$ are reduced.

In [AbK], $f^\dagger:X^\dagger\to Y'$ is said to be weakly semistable and is called a weak semistable reduction of $f:X\to Y$. For the details of toroidal embeddings and morphisms, see [AbK, Section 1]. We may further assume that $X^\dagger$ is $\mathbb{Q}$-factorial (see [AbK, Remark 4.3]). Note that $X^\dagger$ has only rational singularities since $X^\dagger$ is toroidal. Therefore, $X^\dagger$ has only canonical Gorenstein singularities and is Cohen–Macaulay. Thus, we have

$$f^\dagger_*\mathcal{O}_{X^\dagger}(mK_{X^\dagger/Y'})\simeq f'_*\omega^{\otimes m}_{X'/Y'}$$

for every positive integer m. Therefore, it is sufficient to prove that $f^\dagger_*\mathcal{O}_{X^\dagger}(mK_{X^\dagger/Y'})$ is locally free for every positive integer m. We also note that $f^\dagger$ is flat since Y' is smooth, $X^\dagger$ is Cohen–Macaulay, and $f^\dagger$ is equidimensional (see [H, Chapter III, Exercise 10.9] and [AlK, Chapter V, Proposition (3.5)]).

Remark 4.2. We may assume that $f^\dagger$ is smooth over $U_{Y'}$ although we do not need this property in this paper. For the details, see the construction of weak semistable reductions in [AbK].

Step 2 (Relative good minimal models)**.** By the assumption of Theorem 1.6, the geometric generic fiber of $f^\dagger : X^\dagger \to Y'$ has a good minimal model. Therefore, $f^\dagger : X^\dagger \to Y'$ has a relative good minimal model $\widetilde{f} : \widetilde{X} \to Y'$ by Theorem 3.3. Note that

$$f^\dagger_* \mathcal{O}_{X^\dagger}(mK_{X^\dagger/Y'}) \simeq \widetilde{f}_* \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})$$

for every positive integer m. Therefore, it is sufficient to prove that $\widetilde{f}_* \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})$ is locally free for every positive integer m.

Step 3 (Local freeness via the flat base change theorem)**.** We take an arbitrary point $P \in Y'$. We take general very ample Cartier divisors $H_1, H_2, \cdots, H_{n-1}$, where $n = \dim Y$, such that $C = H_1 \cap H_2 \cap \cdots \cap H_{n-1}$ is a smooth projective curve passing through P. By [AbK, Lemma 6.2], we see that $X^\dagger_C = X^\dagger \times_{Y'} C \to C$ is weakly semistable. In particular, $X^\dagger_C$ has only rational Gorenstein singularities (see [AbK, Lemma 6.1]). By adjunction, we see that $\widetilde{X}_C = \widetilde{X} \times_{Y'} C$ is normal and has only canonical singularities. More precisely, $(f^\dagger)^* H_1 = X^\dagger \times_{Y'} H_1 = X^\dagger_{H_1}$ has only rational Gorenstein singularities since $X^\dagger_{H_1} \to H_1$ is weakly semistable by [AbK, Lemma 6.1 and Lemma 6.2]. In particular, $(f^\dagger)^* H_1$ has only canonical singularities. Therefore, $(X^\dagger, (f^\dagger)^* H_1)$ is plt by the inversion of adjunction (see [KM, Theorem 5.50]). So we have that $(\widetilde{X}, \widetilde{f}^* H_1)$ is plt by the negativity lemma (see, for example, [KM, Proposition 3.51]). Thus, $\widetilde{X}_{H_1} = \widetilde{X} \times_{Y'} H_1 = \widetilde{f}^* H_1$ is normal (see [KM, Proposition 5.51]). By adjunction and the negativity lemma again, we obtain that $\widetilde{X}_{H_1}$ has only canonical singularities. By repeating this process $(n-1)$-times, we obtain that $\widetilde{X}_C$ has only canonical singularities. Note that $\widetilde{X}_C \to C$ is equidimensional. Therefore, we see that $\widetilde{f} : \widetilde{X} \to Y'$ is equidimensional by the choice of C. Since $\widetilde{X}$ is Cohen–Macaulay and Y' is smooth, $\widetilde{f}$ is flat (see [H, Chapter III, Exercise 10.9] and [AlK, Chapter V, Proposition (3.5)]). Moreover, $\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}})$ is flat over Y' for every integer m since $\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}})$ is Cohen–Macaulay by Lemma 2.4 and $\widetilde{f}$ is equidimensional (see [AlK, Chapter V, Proposition (3.5)]). By applying Lemma 4.1 and the base change theorem (see [H, Chapter III, Theorem 12.11]) to $\widetilde{X}_C \to C$, we obtain that

$$\dim H^0(\widetilde{X}_y, \mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})|_{\widetilde{X}_y})$$

is independent of $y \in Y'$ for every positive integer m. By the base change theorem (see [H, Chapter III, Corollary 12.9]), we obtain that $f'_*\omega^{\otimes m}_{X'/Y'} \simeq \widetilde{f}_*\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})$ is locally free for every positive integer m.

We complete the proof of the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$. □

Remark 4.3. In general, $\widetilde{X}_y$ may be non-normal. However, we see that the canonical divisor $K_{\widetilde{X}_y}$ is well-defined, $\widetilde{X}_y$ has only semi log canonical singularities, and $\mathcal{O}_{\widetilde{X}}(mK_{\widetilde{X}/Y'})|_{\widetilde{X}_y} \simeq \mathcal{O}_{\widetilde{X}_y}(mK_{\widetilde{X}_y})$ for every positive integer m, by adjunction. For the details of semi log canonical singularities and pairs, see [F4].

In Step 3 in the proof of the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$ in Theorem 1.6, we have proved:

Theorem 4.4. *Let $\pi : V \to W$ be a projective surjective morphism between quasi-projective varieties with connected fibers. Assume the following conditions:*

(i) *W is smooth,*
(ii) *$(U_V \subset V)$ and $(U_W \subset W)$ are toroidal embeddings without self-intersection, with $U_V = \pi^{-1}(U_W)$,*
(iii) *$f : (U_V \subset V) \to (U_W \subset W)$ is toroidal and equidimensional, and*
(iv) *all the fibers of the morphism π are reduced.*

In this case, $\pi : V \to W$ is said to be weakly semistable. We know that V has only rational Gorenstein singularities. Let V' be a minimal model of V over W sitting in the following diagram.

$$\begin{array}{ccc} V & \overset{\phi}{\dashrightarrow} & V' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & W & \end{array}$$

Let $P \in W$ be an arbitrary point and let C be a smooth curve such that $P \in C$ and that $C = H_1 \cap H_2 \cap \cdots \cap H_{\dim W-1}$, where H_i is a general smooth Cartier divisor on W for every i. Then $V_C = V \times_W C \to C$ is weakly semistable and $V'_C = V' \times_W C$ is normal and has only canonical singularities. This implies that $\pi' : V' \to W$ is equidimensional. In particular, π' is flat.

Theorem 4.4 seems to be useful for various geometric applications. So we wrote it separately for the reader's convenience. Note that Theorem 4.4 (see also Step 3 in the proof of the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$ in Theorem 1.6) is a key point of this paper.

5. Semipositivity of $f'_*\omega^{\otimes m}_{X'/Y'}$

In this section, we prove that $f'_*\omega^{\otimes m}_{X'/Y'}$ in Theorem 1.6 is semipositive by using [PoS]. We do not use the theory of variations of Hodge structure. Theorem 5.1, which is a key ingredient of this section, follows from [PoS, Theorem 1.4].

Theorem 5.1. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with connected fibers. Let $\mathcal{L}$ be an ample and globally generated line bundle on Y and let k be a positive integer. Then*

$$f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes l} \simeq f_*\omega^{\otimes k}_{X/Y} \otimes \omega_Y^{\otimes k} \otimes \mathcal{L}^{\otimes l}$$

is generated by global sections for $l \geq k(\dim Y + 1)$.

Proof. See [PoS, Section 2]. □

Remark 5.2. Theorem 5.1 holds under the weaker assumption that X is a normal projective variety with only rational Gorenstein singularities. Note that X has only rational Gorenstein singularities if and only if X has only canonical Gorenstein singularities.

Lemma 5.3. *Let $\mathcal{E}$ be a locally free sheaf of finite rank on a smooth projective variety V. Assume that there exists a line bundle $\mathcal{M}$ such that $\mathcal{E}^{\otimes s} \otimes \mathcal{M}$ is generated by global sections for every positive integer s. Then $\mathcal{E}$ is semipositive.*

Proof. We put $\pi : W = \mathbb{P}_V(\mathcal{E}) \to V$ and $\mathcal{O}_W(1) = \mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$. Since $\mathcal{E}^{\otimes s} \otimes \mathcal{M}$ is generated by global sections, $\mathrm{Sym}^s\mathcal{E} \otimes \mathcal{M}$ is also generated by global sections for every positive integer s. This implies that $\mathcal{O}_W(s) \otimes \pi^*\mathcal{M}$ is generated by global sections for every positive integer s. Thus, we obtain that $\mathcal{O}_W(1)$ is nef, equivalently, $\mathcal{E}$ is semipositive. □

Let us prove the semipositivity of $f_*\omega^{\otimes m}_{X/Y}$ in Theorem 1.4.

Proof of the semipositivity of $f_*\omega^{\otimes m}_{X/Y}$ in Theorem 1.4. We take the s-fold fiber product

$$f^s : X^s = X \times_Y X \times_Y \cdots \times_Y X \to Y.$$

Since f is smooth, X^s is a smooth projective variety and f^s is smooth. We will check

$$f^s_*\omega^{\otimes m}_{X^s/Y} \simeq \bigotimes^s f_*\omega^{\otimes m}_{X/Y}$$

for every positive integer m by induction on s. We consider the following commutative diagram:

$$\begin{array}{ccc} X^s & \xrightarrow{p} & X^{s-1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f^{s-1}} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

By the base change, we have $\omega_{X^s/X} \simeq p^*\omega_{X^{s-1}/Y}$. Thus we have

$$\begin{aligned} \omega_{X^s/Y} &\simeq \omega_{X^s/X} \otimes q^*\omega_{X/Y} \\ &\simeq p^*\omega_{X^{s-1}/Y} \otimes q^*\omega_{X/Y}. \end{aligned}$$

Therefore, by the flat base change theorem (see [H, Chapter III, Proposition 9.3]) and the projection formula, we obtain

$$\begin{aligned} f^s_*\omega^{\otimes m}_{X^s/Y} &\simeq f^{s-1}_*p_*(p^*\omega^{\otimes m}_{X^{s-1}/Y} \otimes q^*\omega^{\otimes m}_{X/Y}) \\ &\simeq f^{s-1}_*(\omega^{\otimes m}_{X^{s-1}/Y} \otimes p_*q^*\omega^{\otimes m}_{X/Y}) \\ &\simeq f^{s-1}_*(\omega^{\otimes m}_{X^{s-1}/Y} \otimes (f^{s-1})^*f_*\omega^{\otimes m}_{X/Y}) \\ &\simeq f_*\omega^{\otimes m}_{X/Y} \otimes f^{s-1}_*\omega^{\otimes m}_{X^{s-1}/Y} \\ &\simeq \bigotimes^s f_*\omega^{\otimes m}_{X/Y} \end{aligned}$$

for every positive integer m and every positive integer s by induction on s. Note that $f_*\omega^{\otimes m}_{X/Y}$ is locally free for every positive integer m (see Section 4). We put $\mathcal{M} = \omega^{\otimes m}_Y \otimes \mathcal{L}^{\otimes m(\dim Y+1)}$, where $\mathcal{L}$ is an ample and globally generated line bundle on Y. By Theorem 5.1, we obtain that

$$f^s_*\omega^{\otimes m}_{X^s/Y} \otimes \mathcal{M}$$

is generated by global sections for every positive integer s. This means that

$$\bigotimes^s f_*\omega^{\otimes m}_{X/Y} \otimes \mathcal{M}$$

is generated by global sections for every positive integer s. By Lemma 5.3, we obtain that $f_*\omega^{\otimes m}_{X/Y}$ is semipositive. □

In Section 4, we have already proved that $f'_*\omega^{\otimes m}_{X'/Y'}$ is locally free in Theorem 1.6. From now on, we prove the semipositivity of $f'_*\omega^{\otimes m}_{X'/Y'}$ in Theorem 1.6.

Proof of the semipositivity of $f'_*\omega^{\otimes m}_{X'/Y'}$ in Theorem 1.6. By the proof of the local freeness of $f'_*\omega^{\otimes m}_{X'/Y'}$ in Section 4, we may assume that f' :

$X' \to Y'$ is weakly semistable. For simplicity, we denote $f' : X' \to Y'$ by $f : X \to Y$ in this proof. We take the s-fold fiber product

$$f^s : X^s = X \times_Y X \times_Y \cdots \times_Y X \to Y.$$

Then we see that X^s is normal and Gorenstein (cf. [V, Lemma 3.5]). Moreover, X^s has only rational singularities because X^s is local analytically isomorphic to a toric variety. Therefore, X^s has only canonical singularities. By the same argument as in the proof of the semipositivity of $f_*\omega_{X/Y}^{\otimes m}$ in Theorem 1.4, we obtain

$$f^s_*\omega^{\otimes m}_{X^s/Y} \simeq \bigotimes^s f_*\omega^{\otimes m}_{X/Y}$$

for every positive integer s and every positive integer m. By Theorem 5.1 (see also Remark 5.2) and Lemma 5.3, we obtain that the locally free sheaf $f_*\omega_{X/Y}^{\otimes m}$ is semipositive for every positive integer m. This is the same as the proof of the semipositivity of $f_*\omega_{X/Y}^{\otimes m}$ in Theorem 1.4. Anyway, we obtain the desired semipositivity. □

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