ON DEFORMATIONS OF TERMINAL AND CANONICAL SINGULARITIES

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Abstract. We study deformations of terminal and canonical singularities as an application of the minimal model program.

Contents

1. Deformations of terminal and canonical singularities

The following result is well known to experts (see [N, Chapter VI. §5. Deformation of singularities]). Here, we give a geometric proof based on the minimal model program for projective morphisms between complex analytic spaces established in [F1].

Theorem 1.1 (Deformations of terminal and canonical singularities)**.** *Let X be a complex analytic space and let Y be a Cartier divisor on X. If Y has only terminal singularities* (*resp. canonical singularities*)*, then X has only terminal singularities* (*resp. canonical singularities*) *in a neighborhood of Y.*

It is also well known that, by Nakayama's extension theorems, one can obtain a slightly stronger result than Theorem 1.1.

Theorem 1.2 (Deformations of terminal and canonical singularities)**.** *Let X be a complex analytic space and let S be a Cartier divisor on X. If S has only canonical singularities, then the pair (X, S) has only canonical singularities in a neighborhood of S. In particular, X has only canonical singularities in a neighborhood of S. If we further assume that S has only terminal singularities, then X has only terminal singularities in a neighborhood of S.*

We briefly comment on Theorem 1.2 in the case where S has only terminal singularities.

Remark 1.3. Assume that S has only terminal singularities in Theorem 1.2. Let $P \in S$ be an arbitrary point. Then we show that, after shrinking X around P, there exists a projective bimeromorphic morphism $\pi\colon Y \to X$ from a smooth complex variety Y such that

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Date: 2026/5/7, version 0.12.

2020 *Mathematics Subject Classification.* Primary 14E30; Secondary 32S30, 14B07.

Key words and phrases. deformations of complex analytic singularities, terminal singularities, canonical singularities, minimal model program, extension theorems.

(1) T is the strict transform of S on Y and is smooth, and
(2) $K_Y+T-\pi^*(K_X+S)$ is effective and its support coincides with $\mathrm{Exc}(\pi)$.

For details, see (3.8) in the proof of Theorem 1.2.

In this paper, we show that Nakayama's extension theorems follow easily from the minimal model program established in [F1]. Therefore, both Theorems 1.1 and 1.2 can be regarded as consequences of the minimal model program for projective morphisms between complex analytic spaces (see [F1]). Note that X is not necessarily an algebraic variety in Theorems 1.1 and 1.2. It is only assumed to be a complex analytic space.

For details on extension theorems, we recommend that the interested reader consult [K] and [N, Chapter VI]. We note that the survey article [K] is one of the triggers for the subsequent great development of the theory of higher-dimensional complex algebraic varieties.

In this paper, we freely use the results obtained in [F1]. In Section 2, we give a geometric proof of Theorem 1.1 using [F1]. In Section 3, we show that Nakayama's extension theorems (see Theorem 3.1), which were used to prove Theorem 1.2 (and hence Theorem 1.1), follow easily from the minimal model program established in [F1]. We also show that Theorem 1.2 follows from Theorem 3.1.

Acknowledgments. The author was partially supported by JSPS KAKENHI Grant Numbers JP19H01787, JP20H00111, JP21H00974, JP21H04994, and JP23K20787.

2. PROOF OF THEOREM 1.1

In this section, we give a geometric proof of Theorem 1.1 using [F1].

Proof of Theorem 1.1. The problem is local. Hence, we may fix an arbitrary point $P\in Y$ and shrink X around P as needed throughout the proof. In particular, we may assume that X is Stein.

Step 1 ($\mathbb{Q}$-Gorenstein property)**.** In this step, we prove that X is $\mathbb{Q}$-Gorenstein, that is, K_X is $\mathbb{Q}$-Cartier.

By assumption, Y has only canonical singularities. Therefore, Y is a Cohen–Macaulay normal complex variety, since it has only rational singularities. By shrinking X around P if necessary, we may assume that X is also a Cohen–Macaulay normal complex variety. Since Y is Gorenstein in codimension two, we can find a closed analytic subset $Z\subset X$ such that X and Y are Gorenstein outside Z, $\operatorname{codim}_X Z\geq 3$, and $\operatorname{codim}_Y(Z\cap Y)\geq 3$, after possibly shrinking X again. Here we used the fact that canonical surface singularities are Gorenstein. We may assume that $Y=(s=0)$ for some holomorphic function s on X. Take a positive integer m such that mK_Y is Cartier. By adjunction, we have an exact sequence

$$0\longrightarrow \mathcal{O}_{U_X}(m(K_X+Y)-Y)\xrightarrow{\times s}\mathcal{O}_{U_X}(m(K_X+Y))\longrightarrow \mathcal{O}_{U_Y}(mK_Y)\longrightarrow 0,$$

where $U_X:=X\setminus Z$ and $U_Y:=Y\setminus(Z\cap Y)$. Let $\iota\colon U_X\hookrightarrow X$ and $\iota\colon U_Y\hookrightarrow Y$ be the natural open immersions. Then we obtain a long exact sequence

$$\begin{aligned}&0\to \iota_*\mathcal{O}_{U_X}(m(K_X+Y)-Y)\to \iota_*\mathcal{O}_{U_X}(m(K_X+Y))\to \iota_*\mathcal{O}_{U_Y}(mK_Y)\\ &\to R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)-Y)\to R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y))\to R^1\iota_*\mathcal{O}_{U_Y}(mK_Y)\\ &\to\cdots.\end{aligned}\tag{2.1}$$

By [BS, Chapter II, Corollary 1.10] and [BS, Chapter II, Theorem 3.6], we have

$$R^1\iota_*\mathcal{O}_{U_Y}(mK_Y)=0,$$

since $\mathcal{O}_Y(mK_Y)$ is locally free and Y is Cohen–Macaulay. By [BS, Chapter II, Corollary 1.10] and [BS, Chapter II, Corollary 4.4], $R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y))$ is coherent. Taking $\otimes\mathcal{O}_{X,P}/\mathfrak{m}_P$ in (2.1), we obtain

$$R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y))\otimes\mathcal{O}_{X,P}/\mathfrak{m}_P=0,$$

where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{X,P}$. Hence, by Nakayama's lemma,

$$R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y))=0$$

in a neighborhood of P. Therefore,

$$R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)-Y)=0$$

in a neighborhood of P since

$$R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)-Y)\simeq R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)).$$

Thus, by (2.1), we obtain a short exact sequence

$$0\longrightarrow\mathcal{O}_X(m(K_X+Y)-Y)\xrightarrow{\times s}\mathcal{O}_X(m(K_X+Y))\longrightarrow\mathcal{O}_Y(mK_Y)\longrightarrow 0$$

over some open neighborhood of P. Since $\mathcal{O}_Y(mK_Y)$ is invertible, it follows that $\mathcal{O}_X(m(K_X+Y))$ is also invertible near P. Hence K_X is $\mathbb{Q}$-Cartier in a neighborhood of P.

Step 2 (Canonical singularities)**.** In this step, we prove that X has only canonical singularities in a neighborhood of Y under the assumption that Y has only canonical singularities. We may shrink X around P as needed without further mention.

Since X is $\mathbb{Q}$-Gorenstein by Step 1, we can construct a projective bimeromorphic morphism $f\colon X'\to X$ such that X' has only canonical singularities, $K_{X'}$ is f-ample,

$$K_{X'}=f^*K_X-E,$$

and E is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X' with $\operatorname{Supp}E=\operatorname{Exc}(f)$, where $\operatorname{Exc}(f)$ is the exceptional locus of f (see [F1, Theorem 1.16]). Since Y has only canonical singularities, the pair (X,Y) is purely log terminal in a neighborhood of Y by inversion of adjunction. Let Y' be the strict transform of Y on X'. Then we have

$$K_{X'}+f^*Y+E=f^*(K_X+Y), \tag{2.2}$$

with $f^*Y\geq Y'$. In particular, (X',f^*Y+E) is purely log terminal. Therefore, Y' is normal.

Assume that f is not an isomorphism around Y'. Then $Y'\cap\operatorname{Supp}E\neq\emptyset$. By (2.2), we have

$$K_{Y'}+\Delta=f^*K_Y$$

by adjunction, where $\Delta\neq 0$ since E has a nontrivial contribution along Y'. This contradicts the assumption that Y has only canonical singularities. Hence f is an isomorphism in a neighborhood of Y'.

Therefore, X has only canonical singularities in a neighborhood of Y.

Step 3 (Terminal singularities)**.** In this step, we prove that X has only terminal singularities in a neighborhood of Y under the assumption that Y has only terminal singularities. We may shrink X around P as needed without further mention.

By Step 2, X has only canonical singularities. Therefore, we can take a small projective bimeromorphic morphism $f\colon X'\to X$ such that X' is $\mathbb{Q}$-factorial over P (see [F1, Theorem 1.24]). Then $K_{X'}=f^*K_X$ and

$$K_{X'}+Y'=f^*(K_X+Y) \tag{2.3}$$

hold, where Y' is the strict transform of Y on X'. Since $Y' = f^*Y$ and Y is Cartier, Y' is Cartier. We note that X' has only canonical singularities since $K_{X'} = f^*K_X$. We have already proved that (X, Y) is purely log terminal in Step 2. Hence (X', Y') is also purely log terminal by (2.3). In particular, Y' is normal. By adjunction applied to (2.3), we have $K_{Y'} = f^*K_Y$. Therefore, Y' has only terminal singularities. Since $K_{X'} = f^*K_X$, it suffices to prove that X' has only terminal singularities.

By [F1, Theorem G], we can construct a projective bimeromorphic morphism $f' \colon X'' \to X'$ such that X'' has only terminal singularities, X'' is $\mathbb{Q}$-factorial over P, and $K_{X''} = f'^*K_{X'}$. Let F be the f'-exceptional divisor on X''. Since X' is $\mathbb{Q}$-factorial over P, we have $\operatorname{Supp} F = \operatorname{Exc}(f')$. Write $F = \sum_i F_i$ for its irreducible decomposition. By construction,

$$a(F_i, X', 0) = a(F_i, X, 0) = 0$$

for every i. Let Y'' be the strict transform of Y on X''. Then

$$K_{X''} + f'^*Y' = f'^*(K_{X'} + Y') \tag{2.4}$$

with $f'^*Y' \geq Y''$.

Suppose for contradiction that f' is not an isomorphism around Y''. Then $Y'' \cap F = Y'' \cap \operatorname{Exc}(f') \neq \emptyset$. Thus there exists F_{i_0} such that $Y'' \cap F_{i_0} \neq \emptyset$. If $a(F_{i_0}, X', Y') < 0$, then, as in Step 2, (2.4) and adjunction yield

$$K_{Y''} + \Delta = f'^*K_{Y'}$$

with $\Delta \neq 0$, since F_{i_0} has a nontrivial contribution along Y''. This contradicts the fact that Y' has only terminal singularities. If $a(F_{i_0}, X', Y') = 0$, then $f'(F_{i_0}) \not\subset Y'$ since $a(F_{i_0}, X', 0) = 0$. Hence, by adjunction, $F_{i_0} \cap Y''$ defines an exceptional divisor G over Y' with $a(G, Y', 0) = 0$, which contradicts the terminality of Y'. Therefore, $Y'' \cap F = \emptyset$, that is, f' is an isomorphism around Y''. This implies that X' has only terminal singularities in a neighborhood of Y'. Hence X has only terminal singularities in a neighborhood of P. Since P is an arbitrary point of Y, X is terminal in a neighborhood of Y.

We complete the proof of Theorem 1.1. □

3. Extension theorems

In [N, Chapter VI. §5. Deformation of singularities], Nakayama uses the following theorem to prove Theorem 1.1, although he adopts a different formulation (see [N, Chapter VI. 5.2. Theorem and 5.3. Corollary]).

Theorem 3.1 (Extension theorems). *Let X be a normal complex variety and let S be a prime divisor on X. Let $\pi \colon Y \to X$ be a projective bimeromorphic morphism from a smooth complex variety Y such that the strict transform T of S on Y is smooth. Then the restriction homomorphism*

$$\pi_*\mathcal{O}_Y\big(m(K_Y + T)\big) \to \pi_*\mathcal{O}_T(mK_T) \tag{3.1}$$

is surjective for every positive integer m. Furthermore, if A is a π-very ample Cartier divisor on Y, then

$$\pi_*\mathcal{O}_Y\big(m(K_Y + T) + A\big) \to \pi_*\mathcal{O}_T(mK_T + A) \tag{3.2}$$

is surjective for every positive integer m.

Theorem 3.1 is a special case of [N, Chapter VI. 3.7. Theorem and 3.9. Theorem]. Here we show that Theorem 3.1 follows easily from the minimal model program established in [F1].

Proof of Theorem 3.1. Since the problem is local, we fix a point $P \in S$ and prove that the restriction maps in (3.1) and (3.2) are surjective over some open neighborhood of P. Hence, throughout the proof, we may freely replace X with a relatively compact Stein open neighborhood of P without further mention.

Step 1. In this step, we prove that the restriction map in (3.1) is surjective.

Choose a general π-ample effective Cartier divisor H on Y. Then we can find an effective Cartier divisor B on Y such that $H + B \sim 0$ and $\operatorname{Supp} B$ does not contain T, since π is bimeromorphic. We consider the pair $(Y, T + \varepsilon H + \varepsilon B)$ for some $0 < \varepsilon \ll 1$. Then $(Y, T + \varepsilon H + \varepsilon B)$ is purely log terminal.

By the minimal model program with scaling (cf. [F1, Theorem 1.7]), after finitely many flips and divisorial contractions, we obtain a good minimal model

$$(Y', T' + \varepsilon H' + \varepsilon B')$$

of $(Y, T + \varepsilon H + \varepsilon B)$ over X, where T', H', and B' are the pushforwards of T, H, and B on Y', respectively. Since $H + B \sim 0$, (Y', T') is a good minimal model of (Y, T) over X. We note that the pair (Y', T') has only canonical singularities since Y and T are smooth. In particular, by adjunction, we have $(K_{Y'} + T')|_{T'} = K_{T'}$.

By construction, we have natural isomorphisms

$$\pi_*\mathcal{O}_Y(m(K_Y + T)) \simeq \pi'_*\mathcal{O}_{Y'}(m(K_{Y'} + T'))$$

and

$$\pi_*\mathcal{O}_T(mK_T) \simeq \pi'_*\mathcal{O}_{T'}(mK_{T'})$$

for every positive integer m, where $\pi' \colon Y' \to X$.

Since $m(K_{Y'} + T') - T' - K_{Y'} = (m-1)(K_{Y'} + T')$ is nef and big over X, the relative Kawamata–Viehweg vanishing theorem implies that

$$R^1\pi'_*\mathcal{O}_{Y'}\big(m(K_{Y'} + T') - T'\big) = 0$$

for every positive integer m. Therefore, the restriction map

$$\pi'_*\mathcal{O}_{Y'}(m(K_{Y'} + T')) \to \pi'_*\mathcal{O}_{T'}(mK_{T'})$$

is surjective for every positive integer m. Thus we obtain the desired surjectivity of the restriction map in (3.1).

Step 2. In this step, we prove the surjectivity of the restriction map in (3.2).

Since A is π-very ample, we may assume that A is smooth, does not contain T, and that $\operatorname{Supp}(A + T)$ is a simple normal crossing divisor on Y. In particular, the pair $(Y, T + \frac{1}{m}A)$ is divisorial log terminal.

By the minimal model program with scaling (cf. [F1, Theorem 1.7]), after finitely many flips and divisorial contractions, we obtain a good minimal model $(Y', T' + \frac{1}{m}A')$ of $(Y, T + \frac{1}{m}A)$ over X, where

$$\begin{tikzcd} Y \arrow[rr, dashed, "\phi"] \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

and $T' = \phi_*T$ and $A' = \phi_*A$.

Let D be a general member of $|A|$. Then $(Y', T' + \frac{1}{m}D')$, where $D' = \phi_*D$, is a good minimal model of $(Y, T + \frac{1}{m}D)$. Hence

$$\pi_*\mathcal{O}_Y(m(K_Y + T) + D) \simeq \pi'_*\mathcal{O}_{Y'}(m(K_{Y'} + T') + D') \tag{3.3}$$

holds.

Let E be any exceptional divisor over Y'. By construction and the negativity lemma, we have

$$a\left(E, Y, T+\frac{1}{m}D\right) \leq a\left(E, Y', T'+\frac{1}{m}D'\right) \leq a(E, Y', T').$$

If D is chosen so that it does not contain the center of E, then

$$a\left(E, Y, T+\frac{1}{m}D\right) = a(E, Y, T) \geq 0.$$

Therefore $a(E, Y', T') \geq 0$, and hence (Y', T') has only canonical singularities. In particular, $(K_{Y'}+T')|_{T'} = K_{T'}$ and T' has only canonical singularities.

Applying the negativity lemma and then restricting to T and T', we obtain

$$a\left(F, T, \frac{1}{m}D|_T\right) \leq a\left(F, T', \frac{1}{m}D'|_{T'}\right) \leq a(F, T', 0) \tag{3.4}$$

for any divisor F over T'. If D is general, then $D|_T$ does not contain the center of F. Thus

$$0 \leq a(F, T, 0) = a\left(F, T, \frac{1}{m}D|_T\right) \leq a\left(F, T', \frac{1}{m}D'|_{T'}\right) \leq a(F, T', 0). \tag{3.5}$$

If F is exceptional over T, then $a(F, T, 0) > 0$. This implies $a(F, T', 0) > 0$ by (3.5). Hence F is also exceptional over T', and therefore the induced map $\phi|_T \colon T \dashrightarrow T'$ does not extract any divisors.

We note that if A is general, then

$$A'|_{T'} = \phi_*(A|_T). \tag{3.6}$$

Therefore, by (3.4) and (3.6), we obtain

$$\pi_*\mathcal{O}_T(mK_T + A|_T) \simeq \pi'_*\mathcal{O}_{T'}(mK_{T'} + A'|_{T'}). \tag{3.7}$$

We can take an effective $\mathbb{Q}$-divisor C on Y such that $(Y, T+C)$ is purely log terminal with $C \sim_{\mathbb{Q}} \frac{1}{m}A$, since A is π-ample. Then $(Y', T'+C')$ is also purely log terminal, where C' is the pushforward of C on Y'. Thus (Y', C') has only kawamata log terminal singularities.

Note that

$$m(K_{Y'}+T') + A' - T' - (K_{Y'}+C') \sim_{\mathbb{Q}} (m-1)\left(K_{Y'}+T'+\frac{1}{m}A'\right)$$

is nef and big over X. Therefore,

$$R^1\pi'_*\mathcal{O}_{Y'}(m(K_{Y'}+T') + A' - T') = 0$$

by the relative Kawamata–Viehweg vanishing theorem. This implies the surjectivity of the restriction map

$$\pi'_*\mathcal{O}_{Y'}(m(K_{Y'}+T') + A') \to \pi'_*\mathcal{O}_{T'}(mK_{T'} + A').$$

By combining this surjectivity with (3.3) and (3.7), we obtain the desired surjectivity (3.2).

We finish the proof of Theorem 3.1. □

We recommend that the reader interested in how to use Theorem 3.1 consults [N, Chapter VI. §5. Deformation of singularities].

Remark 3.2. In the proof of the existence of flips (see, for example, [HM]), more sophisticated extension theorems are used. Therefore, it may seem circular to prove Theorem 3.1 by using the minimal model program established in [F1]. However, it is worth pointing out that Theorem 3.1 follows easily from the minimal model program for projective bimeromorphic morphisms of complex analytic spaces in [F1] (see also [F2]).

We now prove Theorem 1.2 as an application of Theorem 3.1.

Proof of Theorem 1.2. By Step 1 in the proof of Theorem 1.1, we may assume, after shrinking X around S, that X is a normal $\mathbb{Q}$-Gorenstein complex variety. Since the problem is local, we may freely shrink X without further mention throughout the proof.

Choose a projective bimeromorphic morphism $\pi\colon Y \to X$ from a smooth complex variety Y such that π is an isomorphism over the smooth locus of X, and the exceptional locus $\mathrm{Exc}(\pi)$ is a simple normal crossing divisor on Y. Let T be the strict transform of S on Y. We may assume that the union of $\mathrm{Exc}(\pi)$ and T is a simple normal crossing divisor, and that there exists an effective π-exceptional divisor E on Y such that $-E$ is π-very ample and $\operatorname{Supp} E = \mathrm{Exc}(\pi)$.

Fix a positive integer m such that $m(K_X+S)$ and mK_S are Cartier. Since S has only canonical singularities, we have

$$\pi_*\mathcal{O}_T(mK_T) \simeq \mathcal{O}_S(mK_S).$$

By Theorem 3.1, we obtain the following commutative diagram:

$$\begin{tikzcd}
\pi_*\mathcal{O}_Y(m(K_Y+T)) \arrow[r, hook] \arrow[d] & \mathcal{O}_X(m(K_X+S)) \arrow[d] \\
\pi_*\mathcal{O}_T(mK_T) \arrow[r, "\sim"] & \mathcal{O}_S(mK_S),
\end{tikzcd}$$

where the vertical homomorphisms are surjective. Hence the natural inclusion

$$\pi_*\mathcal{O}_Y(m(K_Y+T)) \subset \mathcal{O}_X(m(K_X+S))$$

is an isomorphism on an open neighborhood of S. This implies that

$$m(K_Y+T) \geq \pi^*(m(K_X+S))$$

holds. Therefore, the pair (X, S) has only canonical singularities.

Next, we prove that X has only terminal singularities under the additional assumption that S is terminal. We note that S is smooth in codimension two. Since S is Cartier, it follows that X is smooth in codimension three. Hence, by construction, $E_T := E|_T$ is π_T-exceptional, where $\pi_T := \pi|_T\colon T \to S$.

Let m be a sufficiently large and divisible positive integer. Then $m(K_X+S)$ and mK_S are Cartier, and

$$\pi_*\mathcal{O}_T(mK_T - E_T) \simeq \mathcal{O}_S(mK_S)$$

holds. By Theorem 3.1, the restriction map

$$\pi_*\mathcal{O}_Y(m(K_Y+T)-E) \to \pi_*\mathcal{O}_T(mK_T - E_T)$$

is surjective. Arguing as above, this implies that

$$\pi_*\mathcal{O}_Y(m(K_Y+T)-E) \simeq \mathcal{O}_X(m(K_X+S))$$

holds on a neighborhood of S. Hence we obtain

$$m(K_Y+T) - E \geq \pi^*(m(K_X+S)). \tag{3.8}$$

Therefore,

$$K_Y \geq \pi^*K_X + (\pi^*S - T) + \frac{1}{m}E.$$

This shows that X has only terminal singularities. $\square$

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