ON DEFORMATIONS OF TERMINAL AND CANONICAL SINGULARITIES

OSAMU FUJINO

1. Introduction

The following result is more or less well known to the experts (see [N, Chapter VI. §5. Deformation of singularities]). Here we give a geometric proof based on the minimal model program for projective morphisms between complex analytic spaces (see [F]). For a treatment of this kind of problem by using extension theorems, we recommend that the interested reader looks at [K] and [N, Chapter VI] (see also Section 3 below). We note that the survey article [K] is one of the triggers for the subsequent great development of the theory of higher-dimensional complex algebraic varieties.

Theorem 1.1 (Deformations of terminal and canonial singularities)**.** *Let X be a complex analytic space and let Y be a Cartier divisor on X. If Y has only terminal singularities* (*resp. canonical singularities*)*, then X has only terminal singularities* (*resp. canonical singularities*) *in a neighborhood of Y.*

Note that X is not necessarily an algebraic variety in Theorem 1.1. It is only a complex analytic space. In this paper, we will freely use [F]. In Section 2, we will prove Theorem 1.1 by using [F]. In Section 3, we will show that Nakayama's extension theorems, by which he proved Theorem 1.1, easily follow from the minimal model program established in [F].

2. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by using [F].

Proof of Theorem 1.1. The problem is local. Hence we can take an arbitrary point $P \in Y$ and freely shrink X around P suitably throughout this proof. Therefore, we may assume that X is Stein. From now on, we will sometimes shrink X around P without mentioning it explicitly.

Step 1 ($\mathbb{Q}$-Gorensteinness)**.** In this step, we will prove that X is $\mathbb{Q}$-Gorenstein, that is, K_X is $\mathbb{Q}$-Cartier.

By assumption, Y has only canonical singularities. Therefore, Y is a Cohen–Macaulay normal complex variety. Hence, by shrinking X around P suitably, we may assume that X is also a Cohen–Macaulay normal complex variety. Since Y is Gorenstein in codimension two, we can find a closed analytic subset Z of X such that X and Y are Gorenstein outside Z, $\operatorname{codim}_X Z \geq 3$, and $\operatorname{codim}_Y (Z \cap Y) \geq 3$ after shrinking X around P suitably again. Here, we used the fact that canonical surface singularities are Gorenstein. Without loss

Date: 2023/3/17, version 0.06.

2010 *Mathematics Subject Classification.* Primary 14E30; Secondary 32S30, 14B07.

Key words and phrases. deformations of complex analytic singularities, terminal singularities, canonical singularities, minimal model program, extension theorems.

This note will be contained in [F].

of generality, we may further assume that $Y = (s = 0)$ for some holomorphic function s on X. We take a positive integer m such that mK_Y is Cartier. By adjunction,

$$0 \longrightarrow \mathcal{O}_{U_X}(m(K_X+Y)-Y) \xrightarrow{\times s} \mathcal{O}_{U_X}(m(K_X+Y)) \longrightarrow \mathcal{O}_{U_Y}(mK_Y) \longrightarrow 0$$

is exact, where $U_X := X \setminus Z$ and $U_Y := Y \setminus (Z \cap Y)$. Let $\iota\colon U_X \hookrightarrow X$ be the natural open immersion. Then we have a long exact sequence:

$$\begin{aligned} &0 \to \iota_*\mathcal{O}_{U_X}(m(K_X+Y)-Y) \to \iota_*\mathcal{O}_{U_X}(m(K_X+Y)) \to \iota_*\mathcal{O}_{U_Y}(mK_Y) \\ &\to R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)-Y) \to R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)) \to R^1\iota_*\mathcal{O}_{U_Y}(mK_Y) \\ &\to \cdots . \end{aligned} \tag{2.1}$$

By [BS, Chapter II, Corollary 1.10] and [BS, Chapter II, Theorem 3.6], $R^1\iota_*\mathcal{O}_{U_Y}(mK_Y) = 0$ since $\mathcal{O}_Y(mK_Y)$ is locally free and Y is Cohen–Macaulay. By [BS, Chapter II, Corollary 1.10] and [BS, Chapter II, Corollary 4.4], $R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y))$ is coherent. By taking $\otimes\mathcal{O}_{X,P}/m_P$ in (2.1), we obtain

$$R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)) \otimes \mathcal{O}_{X,P}/m_P = 0,$$

where m_P is the maximal ideal of $\mathcal{O}_{X,P}$. Hence, by Nakayama's lemma,

$$R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)) = 0$$

holds in a neighborhood of P. Therefore,

$$R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)-Y) = 0$$

in a neighborhood of P since

$$R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)-Y) \simeq R^1\iota_*\mathcal{O}_{U_X}(m(K_X+Y)).$$

Thus we obtain the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_X(m(K_X+Y)-Y) \xrightarrow{\times s} \mathcal{O}_X(m(K_X+Y)) \longrightarrow \mathcal{O}_Y(mK_Y) \longrightarrow 0$$

by (2.1) over some open neighborhood of P. Note that $\mathcal{O}_Y(mK_Y)$ is invertible by assumption. This implies that $\mathcal{O}_X(m(K_X+Y))$ is invertible on some open neighborhood of P. This is what we wanted.

Step 2 (Canonical singularities)**.** In this step, we will prove that X has only canonical singularities under the assumption that the singularities of Y are canonical. As mentioned above, we will freely shrink X around P suitably without mentioning it explicitly.

Since X is $\mathbb{Q}$-Gorenstein by Step 1, we can construct a projective bimeromorphic morphism $f\colon X' \to X$ such that X' has only canonical singularities, $K_{X'}$ is f-ample, $K_{X'} = f^*K_X - E$, and E is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X' with $\operatorname{Supp} E = \operatorname{Exc}(f)$, where $\operatorname{Exc}(f)$ is the exceptional locus of f (see [F]). Since Y has only canonical singularities, the pair (X, Y) is purely log terminal in a neighborhood of Y by the well-known inversion of adjunction. Let Y' be the strict transform of Y on X'. Then we have

$$K_{X'} + f^*Y + E = f^*(K_X+Y)$$

with $f^*Y \geq Y'$. We note that $(X', f^*Y + E)$ is purely log terminal. Therefore, Y' is normal. If f is not an isomorphism around Y', then we can easily get a contradiction by $Y' \cap \operatorname{Supp} E \neq \emptyset$. Hence f is an isomorphism in a neighborhood of Y'. This implies that X has only canonical singularities in a neighborhood of Y. This is what we wanted.

Step 3 (Terminal singularities)**.** In this step, we will prove that X has only terminal singularities under the assumption that the singularities of Y are terminal. Note that we will freely shrink X around P.

In Step 2, we have already proved that X has only canonical singularities. Therefore, we can construct a small projective bimeromorphic morphism $f\colon X' \to X$ such that X' is $\mathbb{Q}$-factorial over P (see [F]). Then $K_{X'} = f^*K_X$ and $K_{X'} + Y' = f^*(K_X + Y)$ hold, where Y' is the strict transform of Y on X'. As in Step 2, we see that (X', Y') is purely log terminal with the aid of the inversion of adjunction for (X, Y). In particular, Y' is normal. We note that Y' is Cartier and has only terminal singularities by construction. It is sufficient to prove that X' has only terminal singularities. By [F, Theorem G], we can construct a projective bimeromorphic morphism $f'\colon X'' \to X'$ such that X'' has only terminal singularities, X'' is $\mathbb{Q}$-factorial over P, and $K_{X''} = f'^*K_{X'}$ holds. Let F be the f'-exceptional divisor on X''. Since X' is $\mathbb{Q}$-factorial over P, $\operatorname{Supp} F = \operatorname{Exc}(f')$ holds. By construction,

$$a(F_i, X', 0) = a(F_i, X, 0) = 0$$

holds for every i, where $F = \sum_i F_i$ is the irreducible decomposition of F. Let Y'' be the strict transform of Y on X''. Then

$$K_{X''} + f'^*Y' = f'^*(K_{X'} + Y') \tag{2.2}$$

holds with $f'^*Y' \geq Y''$. Note that Y' has only terminal singularities, $\operatorname{Exc}(f') = F$, and X'' is $\mathbb{Q}$-factorial over P. We also note that if $a(F_i, X', Y') = 0$ holds then we have $f'(F_i) \not\subset Y'$. Hence, if $a(F_i, X', Y') = 0$ with $F_i \cap Y'' \neq \emptyset$, then $F_i \cap Y''$ is a divisor which is exceptional over Y'. Thus, we see that $Y'' \cap F = \emptyset$ by (2.2) since Y' has only terminal singularities. This implies that X' has only terminal singularities in a neighborhood of Y'. Hence X has only terminal singularities in a neighborhood of P. Since P is an arbitrary point of Y, X is terminal in a neighborhood of Y.

We finish the proof of Theorem 1.1. □

3. Extension theorems

In [N, Chapter VI. §5. Deformation of singularities], Nakayama uses the following extension theorem in order to prove Theorem 1.1 although he adopts a different formulation (see [N, Chapter VI. 5.2. Theorem and 5.3. Corollary]).

Theorem 3.1. *Let X be a normal complex variety and let S be a prime divisor on X. Let $\pi\colon Y \to X$ be a projective bimeromorphic morphism from a smooth complex variety Y such that the strict transform T of S on Y is smooth. Then the restriction homomorphism*

$$\pi_*\mathcal{O}_Y(m(K_Y + T)) \to \pi_*\mathcal{O}_T(mK_T) \tag{3.1}$$

is surjective for every positive integer m. Furthermore, if A is a π-ample Cartier divisor on Y, then

$$\pi_*\mathcal{O}_Y(m(K_Y + T) + A) \to \pi_*\mathcal{O}_T(mK_T + A) \tag{3.2}$$

is surjective for every positive integer m.

Theorem 3.1 is a special case of [N, Chapter VI. 3.7. Theorem and 3.9. Theorem]. Here we will show that Theorem 3.1 easily follows from the minimal model program discussed in [F].

Proof of Theorem 3.1. Since the problem is local, we take an arbitrary point $P \in S$ and will prove the restriction maps in (3.1) and (3.2) are surjective over some open neighborhood of P. From now on, we will freely replace X with a relatively compact Stein open neighborhood of P.

Step 1. In this step, we will prove that the restriction map in (3.1) is surjective.

We take a general π-ample Cartier divisor H on Y. Then we can find an effective Cartier divisor B on Y such that $H+B\sim 0$ and that $\operatorname{Supp} B \not\supset T$ since π is bimeromorphic. We consider $(Y, T+\varepsilon H+\varepsilon B)$ for some $0<\varepsilon \ll 1$. Since $0<\varepsilon \ll 1$, $(Y, T+\varepsilon H+\varepsilon B)$ is purely log terminal. By [F], after finitely many flips and divisorial contractions, we get a good log terminal model of $(Y, T+\varepsilon H+\varepsilon B)$ over X. Since $H+B\sim 0$, (Y,T) has a good log terminal model (Y', T') over X, where T' is the pushforward of T on Y'. We note that the pair (Y', T') has only canonical singularities since Y and T are both smooth. In particular, $(K_{Y'}+T')|_{T'}=K_{T'}$ holds by adjunction. By construction, we have the following natural isomorphisms

$$\pi_*\mathcal{O}_Y(m(K_Y+T))\simeq \pi'_*\mathcal{O}_{Y'}(m(K_{Y'}+T'))$$

and

$$\pi_*\mathcal{O}_T(mK_T)\simeq \pi'_*\mathcal{O}_{T'}(mK_{T'})$$

for every positive integer m, where $\pi'\colon Y'\to X$. Since $m(K_{Y'}+T')-T'-K_{Y'}$ is nef and big over X, $R^1\pi'_*\mathcal{O}_{Y'}(m(K_{Y'}+T')-T')=0$ for every positive integer m by the relative Kawamata–Viehweg vanishing theorem. This implies that the restriction map

$$\pi'_*\mathcal{O}_{Y'}(m(K_{Y'}+T'))\to \pi'_*\mathcal{O}_{T'}(mK_{T'})$$

is surjective for every positive integer m. Thus we get the desired surjectivity of the restriction map in (3.1).

Step 2. In this step, we will prove the surjectivity of the restriction map in (3.2).

Let H and B be as in Step 1. We can take an effective $\mathbb{Q}$-divisor Δ on Y such that $T+\frac{1}{m}A\sim_{\mathbb{Q}}\Delta$ and that $(Y, \Delta+\varepsilon H+\varepsilon B)$ is kawamata log terminal for $0<\varepsilon\ll 1$. By [F], after finitely many flips and divisorial contractions, we can obtain a good log terminal model (Y', Δ') of (Y,Δ) over X, where Δ' is the pushforward of Δ on Y'. Let E be any exceptional divisor over Y'. Then, by construction, we have

$$a\left(E, Y, T+\frac{1}{m}A\right)\leq a\left(E, Y', T'+\frac{1}{m}A'\right)\leq a(E, Y', T'),$$

where T' (resp. A') is the pushforward of T (resp. A) on Y'. Therefore, we see that

$$a(E, Y, T+C)\leq a(E, Y', T')$$

holds for any effective $\mathbb{Q}$-divisor C on Y with $C\sim_{\mathbb{Q}}\frac{1}{m}A$ by the above argument. Hence we obtain $a(E, Y', T')\geq 0$. This implies that (Y', T') has only canonical singularities. Therefore, $(K_{Y'}+T')|_{T'}=K_{T'}$ holds. As in Step 1, we have the following natural isomorphisms

$$\pi_*\mathcal{O}_Y(m(K_Y+T)+A)\simeq \pi'_*\mathcal{O}_{Y'}(m(K_{Y'}+T')+A')$$

and

$$\pi_*\mathcal{O}_T(mK_T+A)\simeq \pi'_*\mathcal{O}_{T'}(mK_{T'}+A')$$

for every positive integer m, where $\pi'\colon Y'\to X$. We can take an effective $\mathbb{Q}$-divisor C on Y such that $(Y, T+C)$ is purely log terminal with $C\sim_{\mathbb{Q}}\frac{1}{m}A$ because A is π-ample. In this case, $(Y', T'+C')$ is also purely log terminal, where C' is the pushforward of C on Y'. Thus (Y', C') has only kawamata log terminal singularities. Note that

$$m(K_{Y'}+T')+A'-T'-(K_{Y'}+C') \sim_{\mathbb{Q}} (m-1)(K_{Y'}+\Delta')$$

is nef and big over X. Thus we have

$$R^1\pi'_*\mathcal{O}_{Y'}(m(K_{Y'}+T')+A'-T')=0$$

by the relative Kawamata–Viehweg vanishing theorem. This implies the surjectivity of the restriction map

$$\pi'_*\mathcal{O}_{Y'}(m(K_{Y'}+T')+A') \to \pi'_*\mathcal{O}_{T'}(mK_{T'}+A').$$

This is what we wanted.

We finish the proof of Theorem 3.1. □

We recommend that the reader who is interested in the way how to use Theorem 3.1 looks [N, Chapter VI. §5. Deformation of singularities].

Remark 3.2. In the proof of the existence of flips, we used some more sophisticated extension theorems. Hence it does not look a correct way to prove Theorem 3.1 by using the minimal model program established in [F]. However, it seems to be important to point out that Theorem 3.1 easily follows from the minimal model program for projective bimeromorphic morphisms of complex analytic spaces in [F].

Here we prove the following theorem as an application of Theorem 3.1.

Theorem 3.3 (Deformations of terminal and canonical singularities)**.** *Let X be a complex analytic space and let S be a Cartier divisor on X. If S has only canonical singularities, then the pair (X, S) has only canonical singularities in a neighborhood of S. In particular, X has only canonical singularities. If we further assume that S has only terminal singularities, then X has only terminal singularities in a neighborhood of S.*

Proof. By Step 1 in the proof of Theorem 1.1, we may assume that X is a normal $\mathbb{Q}$-Gorenstein complex variety by shrinking X around S suitably. As usual, we will freely shrink X suitably without mentioning it explicitly throughout this proof since the problem is local.

We take a projective bimeromorphic morphism $\pi\colon Y \to X$ from a smooth complex variety Y such that π is an isomorphism over the smooth locus of X and that the exceptional locus $\mathrm{Exc}(\pi)$ of π is a simple normal crossing divisor on Y. Let T be the strict transform of S on Y. We may assume that the union of $\mathrm{Exc}(\pi)$ and T is a simple normal crossing divisor on Y and that there exists an effective π-exceptional divisor E on Y such that $-E$ is π-ample with $\mathrm{Supp}\, E = \mathrm{Exc}(\pi)$.

We take a positive integer m such that $m(K_X+S)$ and mK_S are both Cartier. Since S has only canonical singularities, we have $\pi_*\mathcal{O}_T(mK_T) \simeq \mathcal{O}_S(mK_S)$. By Theorem 3.1, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_*\mathcal{O}_Y(m(K_Y+T)) & \hookrightarrow & \mathcal{O}_X(m(K_X+S)) \\ \downarrow & & \downarrow \\ \pi_*\mathcal{O}_T(mK_T) & \overset{\sim}{\longrightarrow} & \mathcal{O}_S(mK_S), \end{array}$$

where the vertical homomorphisms are surjective. Hence the natural inclusion

$$\pi_*\mathcal{O}_Y(m(K_Y+T)) \subset \mathcal{O}_X(m(K_X+S))$$

is an isomorphism on some open neighborhood of S. This implies that

$$m(K_Y + T) \geq \pi^*(m(K_X + S))$$

holds. Therefore, the pair (X, S) has only canonical singularities.

From now on, we will prove that X has only terminal singularities under the assumption that S is terminal. We note that S is smooth in codimension two. Hence X is smooth in codimension three since S is Cartier. Therefore, by construction, $E_T := E|_T$ is π_T-exceptional, where $\pi_T := \pi|_T \colon T \to S$. Let m be a sufficiently large and divisible positive integer. Then $m(K_X+S)$ and mK_S are Cartier and $\pi_*\mathcal{O}_T(mK_T - E_T) \simeq \mathcal{O}_S(mK_S)$ holds. By Theorem 3.1, the restriction map

$$\pi_*\mathcal{O}_Y(m(K_Y + T) - E) \to \pi_*\mathcal{O}_T(mK_T - E_T)$$

is surjective. This implies that $\pi_*\mathcal{O}_Y(m(K_Y + T) - E) \simeq \mathcal{O}_X(m(K_X + S))$ holds in a neighborhood of S as in the above argument. Hence we can check that $m(K_Y+T) - E \geq \pi^*(m(K_X + S))$ holds. Thus, we obtain

$$K_Y \geq \pi^* K_X + (\pi^* S - T) + \frac{1}{m} E.$$

Therefore, X has only terminal singularities. We finish the proof. □

References

[BS] C. Bănică, O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*, Translated from the Romanian. Editura Academiei, Bucharest; John Wiley & Sons, London–New York–Sydney, 1976.

[F] O. Fujino, Minimal model program for projective morphisms between complex analytic spaces, preprint (2022). arXiv:2201.11315 [math.AG]

[K] Y. Kawamata, Deformations of canonical singularities, J. Amer. Math. Soc. **12** (1999), no. 1, 85–92.

[N] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, **14**. Mathematical Society of Japan, Tokyo, 2004.

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `fujino@math.kyoto-u.ac.jp`