ON DIOPHANTINE APPROXIMATION (PRIVATE NOTE)

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Abstract. This note is a supplement of [S, §5].

The following theorem is one of the key lemmas in the proof of Shokurov's PL flips [S]. It is [S, 5.13 Corollary] but seems to be slightly better than the original one. In this paper, we will give a detailed proof.

Theorem 1. *Let D be a semi-ample $\mathbb{R}$-divisor on X/S. If D is not a $\mathbb{Q}$-divisor, then there exists an increasing sequence of positive integers $m_1 < m_2 < \cdots$ and $\mathbb{Q}$-divisors $\{Q_m\}_{m=m_1}^{\infty}$ such that*

(1) *$Q_m = D$ in the rational part of D,*
(2) *$Q_m \not\leq D$ for all m,*
(3) *$|mQ_m|$ is base point free $/S$, in particular, mQ_m is integral,*
(4) *$||mD - mQ_m|| < \frac{\epsilon}{m^r}$ for some positive real numbers ϵ, r.*

Remark 2. The norm in Theorem 1 (4) is the maximal norm of the vector space $\sum \mathbb{R}F_i$. See the proof of Theorem 1.

Definition 3. Let $\alpha = (\alpha_1, \cdots, \alpha_l) \in \mathbb{R}^l$ be a vector such that $\alpha_i \notin \mathbb{Q}$ for every i. We define

$$R_m(\alpha) := (m\alpha_1 - n_1, \cdots, m\alpha_l - n_l),$$

where n_i is the integer that makes $|m\alpha_i - n_i|$ minimal for every i. We define

$$\delta_m(\alpha) := \max |m\alpha_i - n_i|,$$

and put

$$SR_m(\alpha) := (\mathrm{sign}(m\alpha_1 - n_1), \cdots, \mathrm{sign}(m\alpha_l - n_l)).$$

Lemma 4. *For any positive real number $q > 1$, there exists $m \in \mathbb{Z}_{>0}$ such that $m \leq q$ and*

$$\delta_m(\alpha) \leq q^{-\frac{1}{l}}.$$

Proof. This follows from [C, Chapter I Theorem VI] easily. It is essentially Minkowski's theorem. We omit the details. □

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Construction 5. *We construct an increasing sequence of positive integers $m_1 < m_2 < \cdots$ as follows: we put $m_1 := 1$. We define*

$$m_{c+1} := \min\{m \in \mathbb{Z} | \delta_m(\alpha) < \delta_{m_c}(\alpha), m > m_c\}$$

inductively. By Lemma 4, we can take $m_1 < m_2 < \cdots$.

Proof. First, take $q > (\frac{1}{\delta_{m_c}(\alpha)})^l$ and apply Lemma 4. We obtain $m \in \mathbb{Z}_{>0}$ such that $\delta_m(\alpha) < \delta_{m_c}(\alpha)$. Next, we check $m > m_c$. We assume $m \leq m_c$ by contradiction. Then there exists $d \in \mathbb{Z}_{>0}$ such that $d+1 \leq c$ and $m_d \leq m \leq m_{d+1}$. Therefore, $\delta_m(\alpha) < \delta_{m_c}(\alpha) \leq \delta_{m_{d+1}}(\alpha) < \delta_{m_d}(\alpha)$. This contradicts the choice of m_{d+1}. So m_{c+1} is well-defined. Therefore, we obtained the required increasing sequence. □

Lemma 6. $\delta_{m_c}(\alpha) < m_c^{-\frac{1}{l}}$.

Proof. We put $q := m_{c+1} - \eta$, where η is a sufficiently small positive real number, and apply Lemma 4. Then there exists $m < m_{c+1}$ with $\delta_m(\alpha) \leq (m_{c+1} - \eta)^{-\frac{1}{l}} < m_c^{-\frac{1}{l}}$. On the other hand, $\delta_m(\alpha) \geq \delta_{m_c}(\alpha)$ for $1 \leq m < m_{c+1}$. Thus $\delta_{m_c}(\alpha) < m_c^{-\frac{1}{l}}$. □

Lemma 7. $SR_{m_c}(\alpha) \neq SR_{m_{c+1}}(\alpha)$ *for every c.*

Proof. We assume that $SR_{m_c}(\alpha) = SR_{m_{c+1}}(\alpha)$ by contradiction. We put $m := m_{c+1} - m_c$. Then

$$\begin{aligned} |R_m(\alpha)_i| &\leq |R_{m_{c+1}}(\alpha)_i - R_{m_c}(\alpha)_i| \\ &= ||R_{m_{c+1}}(\alpha)_i| - |R_{m_c}(\alpha)_i|| \\ &\leq \max(|R_{m_{c+1}}(\alpha)_i|, |R_{m_c}(\alpha)_i|) \\ &\leq \delta_{m_c}(\alpha), \end{aligned}$$

where $R_m(\alpha)_i$ means the i-th coordinate of the vector $R_m(\alpha)$. Thus $\delta_m(\alpha) \leq \delta_{m_c}(\alpha)$. Since $m < m_{c+1}$, $\delta_m(\alpha) \geq \delta_{m_c}(\alpha)$. Therefore, $\delta_m(\alpha) = \delta_{m_c}(\alpha)$. By the above inequalities, $\delta_{m_c}(\alpha) = |R_{m_c}(\alpha)_i|$ for some i. This implies $|R_{m_{c+1}}(\alpha)_i| = 0$. This is a contradiction. It is because $\alpha_i \notin \mathbb{Q}$. □

Proof of Theorem 1. Since D is a semi-ample $\mathbb{R}$-divisor, we can write $D = \sum a_i H_i + \sum b_j(f_j)$, where H_i is a free $\mathbb{Z}$-divisor and $a_i > 0$ for every i and f_j is a rational function for every j. We define

$$(\cup_i \mathrm{Supp} H_i) \bigcup (\cup_j \mathrm{Supp}(f_j)) = \sum_{i=1}^{k} F_i,$$

where F_i is an irreducible divisor for every i.

Let $\mathcal{C} := (\sum \mathbb{R}_{\geq 0}H_i + \sum \mathbb{R}(f_j)) \cap \sum \mathbb{Z}F_i$ be a rational polyhedral cone. There exists $I \in \mathbb{Z}_{>0}$ such that ID is free for every $D \in \mathcal{C}$. We fix this I.

We write $D = \sum \alpha_i F_i = \sum_{i=1}^{l} \alpha_i F_i + \sum_{i=l+1}^{k} \alpha_i F_i$, where $\alpha_i \notin \mathbb{Q}$ for $1 \leq i \leq l$ and $\alpha_i \in \mathbb{Q}$ for $l+1 \leq i \leq k$. Applying Construction 5 to $\{\alpha_i\}_{i=1}^{l}$, we obtain a required sequence of positive integers $\{m_j\}_{j=1}^{\infty}$ and $\mathbb{Q}$-divisors.

More precisely, we define $Q_{m'} = \sum_{i=1}^{l}(n_i'/m')F_i + \sum_{i=l+1}^{k} \alpha_i F_i$, where $m' \in \{m_j\}$ and n_i' is as in Definition 3, and throw away m' such that $SR_{m'}(\alpha) = (+, \cdots, +)$. We put $m := IJm'$ and $Q_m = Q_{m'}$, where J is a positive integer such that $J\alpha_i \in \mathbb{Z}$ for $l+1 \leq i \leq k$. Then we can check the properties from (1) to (4) easily by the definition of Q_m and Lemmas 6, 7. We note that it is sufficient to put $r := 1/l$ and $\epsilon := (IJ)^{1+(1/l)}$. □

References

[C] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge University Press, 1957.

[S] V. V. Shokurov, PL flips, preprint (August 3rd, 2000 version).

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