CONE THEOREM AND MORI HYPERBOLICITY

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Abstract

We discuss the cone theorem for quasi-log schemes and the Mori hyperbolicity. In particular, we establish that the log canonical divisor of a Mori hyperbolic projective normal pair is nef if it is nef when restricted to the non-lc locus. This answers Svaldi's question completely. We also treat the uniruledness of the degenerate locus of an extremal contraction morphism for quasi-log schemes. Furthermore, we prove that every fiber of a relative quasi-log Fano scheme is rationally chain connected modulo the non-qlc locus.

Contents

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1. Introduction

This paper gives not only new results around the cone theorem and Mori hyperbolicity of quasi-log schemes but also a new framework and some techniques to treat higher-dimensional complex algebraic varieties based on the theory of mixed Hodge structures. It also shows that the theory of quasi-log schemes is very powerful even for the study of log canonical pairs. We note that this paper heavily depends on [**F11**, Chapter 6] and [**F14**].

In his epoch-making paper [**Mo**], Shigefumi Mori established the following cone theorem for smooth projective varieties.

Theorem 1.1 (Cone theorem for smooth projective varieties)**.** *Let X be a smooth projective variety defined over an algebraically closed field.*

(i) *There are countably many (possibly singular) rational curves $C_j \subset X$ such that*
$$0 < -(C_j \cdot K_X) \leq \dim X + 1$$
and
$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum_j \mathbb{R}_{\geq 0}[C_j].$$

(ii) *For any $\varepsilon > 0$ and any ample Cartier divisor H on X,*
$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\varepsilon H) \geq 0} + \sum_{\textit{finite}} \mathbb{R}_{\geq 0}[C_j].$$

In particular, we have:

Theorem 1.2. *Let X be a smooth projective variety defined over an algebraically closed field. Assume that there are no rational curves on X. Then K_X is nef.*

Precisely speaking, Mori proved the existence of rational curves on X under the assumption that K_X is not nef (see Theorem 1.2) by his ingenious method of *bend and break*. Then he obtained the above cone theorem for smooth projective varieties (see Theorem 1.1). For the details, see [**Mo**], [**KM**, Sections 1.1, 1.2, and 1.3], [**D**], [**Ko1**], [**Ma**, Chapter 10], and so on.

From now on, we will work over $\mathbb{C}$, the complex number field. Our arguments in this paper heavily depend on Hironaka's resolution of singularities and its generalizations and several Kodaira type vanishing theorems. Hence they do not work over a field of characteristic $p > 0$. Let us recall the notion of *Mori hyperbolicity* following [**LZ**] and [**S**].

Definition 1.3 (Mori hyperbolicity)**.** Let (X, Δ) be a normal pair such that Δ is effective. This means that X is a normal variety and Δ is an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let W be an lc stratum of (X, Δ). This means that W is an lc center of (X, Δ) or W is X itself. We put

$$U := W \setminus \left\{ (W \cap \mathrm{Nlc}(X, \Delta)) \cup \bigcup_{W'} W' \right\},$$

where W' runs over lc centers of (X, Δ) strictly contained in W and $\mathrm{Nlc}(X, \Delta)$ denotes the non-lc locus of (X, Δ), and call it the *open lc stratum of* (X, Δ) *associated to* W. We say that (X, Δ) is *Mori hyperbolic* if there is no non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow U$$

for any open lc stratum U of (X, Δ).

The following theorem is a generalization of Theorem 1.2 for normal pairs and is an answer to [**S**, Question 6.6].

Theorem 1.4. *Let X be a normal projective variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Assume that (X, Δ) is Mori hyperbolic and that $K_X + \Delta$ is nef when restricted to $\mathrm{Nlc}(X, \Delta)$. Then $K_X + \Delta$ is nef.*

Theorem 1.4 follows from the following cone theorem for normal pairs. We can see it as a generalization of Theorem 1.1 for normal pairs.

Theorem 1.5 (Cone theorem for normal pairs)**.** *Let (X, Δ) be a normal pair such that Δ is effective and let $\pi\colon X \to S$ be a projective morphism between schemes.*

(i) *Then*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{(K_X+\Delta)\geq 0} + \overline{\mathrm{NE}}(X/S)_{-\infty} + \sum_j R_j$$

holds, where R_j's are the $(K_X + \Delta)$-negative extremal rays of $\overline{\mathrm{NE}}(X/S)$ that are rational and relatively ample at infinity. In particular, each R_j is spanned by an integral curve C_j on X such that $\pi(C_j)$ is a point. Note that an extremal ray R of $\overline{\mathrm{NE}}(X/S)$ is rational and relatively ample at infinitely if and only if there exists a π-nef $\mathbb{Q}$-line bundle $\mathcal{L}$ on X such that $R = \overline{\mathrm{NE}}(X/S) \cap \mathcal{L}^{\perp}$ and that $\mathcal{L}|_{\mathrm{Nlc}(X,\Delta)}$ is $\pi|_{\mathrm{Nlc}(X,\Delta)}$-ample.

(ii) *Let H be a π-ample $\mathbb{R}$-divisor on X. Then*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{(K_X+\Delta+H)\geq 0} + \overline{\mathrm{NE}}(X/S)_{-\infty} + \sum_{\mathit{finite}} R_j$$

holds.

(iii) *For each $(K_X+\Delta)$-negative extremal ray R_j of $\overline{\mathrm{NE}}(X/S)$ that is rational and relatively ample at infinity, there are an open lc stratum U_j of (X,Δ) and a non-constant morphism*

$$f_j\colon \mathbb{A}^1 \longrightarrow U_j$$

such that C_j, the closure of $f_j(\mathbb{A}^1)$ in X, spans R_j in $N_1(X/S)$ with

$$0 < -(K_X+\Delta)\cdot C_j \leq 2\dim U_j.$$

More generally, we establish the following cone theorem for quasi-log schemes. We note that Theorem 1.5 is a very special case of Theorem 1.6. Since the precise definition of quasi-log schemes may look technical and artificial, we omit it here. For the details, see Section 4 below. Here, we only explain a typical example of quasi-log schemes. Let (V,Δ) be a log canonical pair which is not kawamata log terminal. Then the non-klt locus $X := \mathrm{Nklt}(V,\Delta)$ of (V,Δ) with $\omega := (K_V+\Delta)|_X$ naturally has a quasi-log scheme structure. In this case, the non-qlc locus $X_{-\infty} = \mathrm{Nqlc}(X,\omega)$ of $[X,\omega]$ is empty and W is a qlc stratum of $[X,\omega]$ if and only if W is an lc center of (V,Δ). We can define open qlc stratum of $[X,\omega]$ similarly to Definition 1.3. In general, X is reducible and is not equidimensional.

Theorem 1.6 (Cone theorem for quasi-log schemes)**.** *Let $[X,\omega]$ be a quasi-log scheme and let $\pi\colon X\to S$ be a projective morphism between schemes.*

(i) *Then*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{\omega\geq 0} + \overline{\mathrm{NE}}(X/S)_{-\infty} + \sum_j R_j$$

holds, where R_j's are the ω-negative extremal rays of $\overline{\mathrm{NE}}(X/S)$ that are rational and relatively ample at infinity. In particular, each R_j is spanned by an integral curve C_j on X such that $\pi(C_j)$ is a point. Note that an extremal ray R of $\overline{\mathrm{NE}}(X/S)$ is rational and relatively ample at infinitely if and only if there exists a π-nef $\mathbb{Q}$-line bundle $\mathcal{L}$ on X such that $R = \overline{\mathrm{NE}}(X/S)\cap \mathcal{L}^{\perp}$ and that $\mathcal{L}|_{\mathrm{Nqlc}(X,\omega)}$ is $\pi|_{\mathrm{Nqlc}(X,\omega)}$-ample.

(ii) *Let H be a π-ample $\mathbb{R}$-line bundle on X. Then*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{(\omega+H)\geq 0} + \overline{\mathrm{NE}}(X/S)_{-\infty} + \sum_{\mathit{finite}} R_j$$

holds.

(iii) *For each ω-negative extremal ray R_j of $\overline{\mathrm{NE}}(X/S)$ that is rational and relatively ample at infinity, there are an open qlc stratum U_j*

of $[X, \omega]$ *and a non-constant morphism*

$$f_j \colon \mathbb{A}^1 \longrightarrow U_j$$

such that C_j*, the closure of* $f_j(\mathbb{A}^1)$ *in* X*, spans* R_j *in* $N_1(X/S)$ *with*

$$0 < -\omega \cdot C_j \leq 2 \dim U_j.$$

We make a remark on U_j in Theorem 1.6.

Remark 1.7. In Theorem 1.6 (iii), let φ_{R_j} be the extremal contraction morphism associated to R_j. Then the proof of Theorem 1.6 shows that U_j is any open qlc stratum of $[X, \omega]$ such that $\varphi_{R_j} \colon \overline{U_j} \to \varphi_{R_j}(\overline{U_j})$ is not finite and that $\varphi_{R_j} \colon W^\dagger \to \varphi_{R_j}(W^\dagger)$ is finite for every qlc center $W^\dagger$ of $[X, \omega]$ with $W^\dagger \subsetneq \overline{U_j}$, where $\overline{U_j}$ is the closure of U_j in X.

The main ingredients of the proof of Theorem 1.6 are the following three results.

Theorem 1.8. *Let* X *be a normal variety and let* Δ *be an effective* $\mathbb{R}$*-divisor on* X *such that* $K_X + \Delta$ *is* $\mathbb{R}$*-Cartier. Let* $\pi \colon X \to S$ *be a projective morphism onto a scheme* S*. Assume that* $(K_X + \Delta)|_{\mathrm{Nklt}(X, \Delta)}$ *is nef over* S*, where* $\mathrm{Nklt}(X, \Delta)$ *denotes the non-klt locus of* (X, Δ)*, and that* $K_X + \Delta$ *is not nef over* S*. Then there exists a non-constant morphism*

$$f \colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

such that $\pi \circ f(\mathbb{A}^1)$ *is a point and that the curve* C*, the closure of* $f(\mathbb{A}^1)$ *in* X*, is a (possibly singular) rational curve with*

$$0 < -(K_X + \Delta) \cdot C \leq 2 \dim X.$$

We prove Theorem 1.8 with the aid of the minimal model theory for higher-dimensional algebraic varieties mainly due to [**BCHM**]. Theorem 1.9 is a slight generalization of [**FLh**, Theorem 1.1], where $[X, \omega]$ is a quasi-log canonical pair. In Theorem 1.9, $[X, \omega]$ is not necessarily quasi-log canonical.

Theorem 1.9. *Let* $[X, \omega]$ *be a quasi-log scheme such that* X *is irreducible. Let* $\nu \colon Z \to X$ *be the normalization. Then there exists a proper surjective morphism* $f' \colon (Y', B_{Y'}) \to Z$ *from a quasi-projective globally embedded simple normal crossing pair* $(Y', B_{Y'})$ *such that every stratum of* Y' *is dominant onto* Z *and that*

$$\bigl(Z, \nu^*\omega, f' \colon (Y', B_{Y'}) \to Z\bigr)$$

naturally becomes a quasi-log scheme with

$$\mathrm{Nqklt}(Z, \nu^*\omega) = \nu^{-1} \mathrm{Nqklt}(X, \omega).$$

More precisely, the following equality

$$\nu_* \mathcal{I}_{\mathrm{Nqklt}(Z, \nu^*\omega)} = \mathcal{I}_{\mathrm{Nqklt}(X, \omega)}$$

holds, where $\mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$ *and* $\mathcal{I}_{\mathrm{Nqklt}(Z,\nu^*\omega)}$ *are the defining ideal sheaves of* $\mathrm{Nqklt}(X,\omega)$ *and* $\mathrm{Nqklt}(Z,\nu^*\omega)$ *respectively.*

Theorem 1.10 is similar to [**F15**, Theorem 1.1]. The proof of Theorem 1.10 needs some deep results on basic slc-trivial fibrations obtained in [**F14**] and [**FFL**]. Therefore, Theorem 1.10 depends on the theory of variations of mixed Hodge structure (see [**FF**] and [**FFS**]).

Theorem 1.10. *Let* $[X,\omega]$ *be a quasi-log scheme such that* X *is a normal quasi-projective variety. Let* H *be an ample* $\mathbb{R}$*-divisor on* X*. Then there exists an effective* $\mathbb{R}$*-divisor* Δ *on* X *such that*

$$K_X+\Delta\sim_{\mathbb{R}}\omega+H$$

and that

$$\mathrm{Nklt}(X,\Delta)=\mathrm{Nqklt}(X,\omega)$$

holds set theoretically, where $\mathrm{Nklt}(X,\Delta)$ *denotes the non-klt locus of* (X,Δ)*. Furthermore, if* $[X,\omega]$ *has a* $\mathbb{Q}$*-structure and* H *is an ample* $\mathbb{Q}$*-divisor on* X*, then we can make* Δ *a* $\mathbb{Q}$*-divisor on* X *such that*

$$K_X+\Delta\sim_{\mathbb{Q}}\omega+H$$

holds.

When X *is a smooth curve, we can take an effective* $\mathbb{R}$*-divisor* Δ *on* X *such that*

$$K_X+\Delta\sim_{\mathbb{R}}\omega$$

and that

$$\mathrm{Nklt}(X,\Delta)=\mathrm{Nqklt}(X,\omega)$$

holds set theoretically. Of course, if we further assume that $[X,\omega]$ *has a* $\mathbb{Q}$*-structure, then we can make* Δ *an effective* $\mathbb{Q}$*-divisor on* X *such that*

$$K_X+\Delta\sim_{\mathbb{Q}}\omega$$

holds.

Let us briefly explain the idea of the proof of Theorem 1.6 (iii), which is one of the main results of this paper. We take an ω-negative extremal ray R_j of $\overline{\mathrm{NE}}(X/S)$ that are rational and relatively ample at infinity. Then, by the contraction theorem, there exists a contraction morphism $\varphi:=\varphi_{R_j}\colon X\to Y$ over S associated to R_j. We take a qlc stratum W of $[X,\omega]$ such that $\varphi\colon W\to\varphi(W)$ is not finite and that $\varphi\colon W^\dagger\to\varphi(W^\dagger)$ is finite for every qlc center $W^\dagger$ with $W^\dagger\subsetneq W$. By adjunction, $W':=W\cup\mathrm{Nqlc}(X,\omega)$ with $\omega|_{W'}$ becomes a quasi-log scheme. Hence we can replace $[X,\omega]$ with $[W',\omega|_{W'}]$. By using Theorem 1.9, we can reduce the problem to the case where X is a normal variety. By Theorem 1.10, we see that it is sufficient to treat normal pairs. For normal pairs, by Theorem 1.8, we can find a non-constant morphism

$$f_j\colon\mathbb{A}^1\longrightarrow X\setminus\mathrm{Nqklt}(X,\omega)$$

with the desired properties.

We also treat an ampleness criterion for Mori hyperbolic normal pairs. It is a generalization of [**S**, Theorem 7.5].

Theorem 1.11 (Ampleness criterion for Mori hyperbolic normal pairs)**.** *Let X be a normal projective variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Assume that (X,Δ) is Mori hyperbolic, $(K_X+\Delta)|_{\mathrm{Nlc}(X,\Delta)}$ is ample, and $K_X+\Delta$ is log big with respect to (X,Δ), that is, $(K_X+\Delta)|_W$ is big for every lc stratum W of (X,Δ). Then $K_X+\Delta$ is ample.*

Theorem 1.11 is a very special case of the ampleness criterion for quasi-log schemes (see Theorem 11.1). We omit the precise statement of Theorem 11.1 here since it looks technical. We note that $K_X+\Delta$ is nef by Theorem 1.4 since (X,Δ) is Mori hyperbolic and $(K_X+\Delta)|_{\mathrm{Nlc}(X,\Delta)}$ is ample. Therefore, $K_X+\Delta$ is nef and log big with respect to (X,Δ) in Theorem 1.11. Hence we can see that $K_X+\Delta$ is semi-ample with the aid of the basepoint-free theorem of Reid–Fukuda type (see [**F10**]). Then we prove that $K_X+\Delta$ is ample.

By using the method established for the proof of Theorem 1.6, we can prove the following theorems. Note that Theorems 1.12, 1.13, and 1.14 are free from the theory of minimal models. Theorem 1.12 is a generalization of Kawamata's famous theorem (see [**Ka**]).

Theorem 1.12. *Let $[X,\omega]$ be a quasi-log scheme and let $\varphi\colon X\to W$ be a projective morphism between schemes such that $-\omega$ is φ-ample. Let P be an arbitrary closed point of W. Let E be any positive-dimensional irreducible component of $\varphi^{-1}(P)$ such that $E\not\subset X_{-\infty}$. Then E is covered by* (*possibly singular*) *rational curves ℓ with*

$$0<-\omega\cdot\ell\leq 2\dim E.$$

In particular, E is uniruled.

For the reader's convenience, let us explain the idea of the proof of Theorem 1.12. We take an effective $\mathbb{R}$-Cartier divisor B on W passing through P such that E is a qlc stratum of $[X,\omega+\varphi^*B]$. Let $\nu\colon\overline{E}\to E$ be the normalization. By adjunction for quasi-log schemes, Theorems 1.9, 1.10, and so on, for any ample $\mathbb{R}$-divisor H on $\overline{E}$, we obtain an effective $\mathbb{R}$-divisor $\Delta_{\overline{E},H}$ on $\overline{E}$ such that

$$\nu^*\omega+H\sim_{\mathbb{R}}K_{\overline{E}}+\Delta_{\overline{E},H}$$

holds. This implies that $C\cdot K_{\overline{E}}<0$ holds for any general curve C on $\overline{E}$. Thus, it is not difficult to see that $\overline{E}$ is covered by rational curves (see [**MM**]). Our approach is different from Kawamata's original one, which uses a relative Kawamata–Viehweg vanishing theorem for projective bimeromorphic morphisms between complex analytic spaces. Kawamata's approach does not work for our setting.

As a direct consequence of Theorem 1.12, we have:

Theorem 1.13 (Lengths of extremal rational curves)**.** *Let $[X, \omega]$ be a quasi-log scheme and let $\pi\colon X \to S$ be a projective morphism between schemes. Let R be an ω-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ that are rational and relatively ample at infinity. Let $\varphi_R\colon X \to W$ be the contraction morphism over S associated to R. We put*

$$d = \min_E \dim E,$$

where E runs over positive-dimensional irreducible components of $\varphi_R^{-1}(P)$ for all $P \in W$. Then R is spanned by a (possibly singular) rational curve ℓ with

$$0 < -\omega \cdot \ell \leq 2d.$$

If (X, Δ) is a log canonical pair, then $[X, K_X + \Delta]$ naturally becomes a quasi-log canonical pair. Hence we can apply Theorems 1.12 and 1.13 to log canonical pairs. Note that Theorems 1.12 and 1.13 are new even for log canonical pairs (see also Corollary 12.3). We can prove the following result on rational chain connectedness for relative quasi-log Fano schemes.

Theorem 1.14 (Rational chain connectedness)**.** *Let $[X, \omega]$ be a quasi-log scheme and let $\pi\colon X \to S$ be a projective morphism between schemes with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$. Assume that $-\omega$ is ample over S. Then $\pi^{-1}(P)$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$ for every closed point $P \in S$. In particular, if further $\pi^{-1}(P) \cap X_{-\infty} = \emptyset$ holds, that is, $[X, \omega]$ is quasi-log canonical in a neighborhood of $\pi^{-1}(P)$, then $\pi^{-1}(P)$ is rationally chain connected.*

Let us see the idea of the proof of Theorem 1.14. We assume that $\pi^{-1}(P) \cap X_{-\infty} \neq \emptyset$ for simplicity. By using the framework of quasi-log schemes, we construct a good finite increasing sequence of closed subschemes

$$Z_{-1} := \mathrm{Nqlc}(X, \omega) \subset Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_k$$

of X such that $\pi^{-1}(P) \subset Z_k$ after shrinking X around $\pi^{-1}(P)$. It is well known that if (V, Δ) is a projective normal pair such that Δ is effective and that $-(K_V + \Delta)$ is ample then V is rationally chain connected modulo $\mathrm{Nklt}(V, \Delta)$ (see [**HM**] and [**BP**]). By this fact, adjunction for quasi-log schemes, Theorems 1.9, 1.10, and so on, we prove that $Z_{i+1} \cap \pi^{-1}(P)$ is rationally chain connected modulo $Z_i \cap \pi^{-1}(P)$ for every $-1 \leq i \leq k-1$. Since $Z_k \cap \pi^{-1}(P) = \pi^{-1}(P)$ and $Z_{-1} \cap \pi^{-1}(P) = \pi^{-1}(P) \cap X_{-\infty}$, we obtain that $\pi^{-1}(P)$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$.

Theorems 1.6, 1.12, and 1.14 are closely related one another. Let us see these theorems for extremal birational contraction morphisms of log canonical pairs. Let (X, Δ) be a projective log canonical pair and let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X)$. Assume that the contraction morphism $\varphi_R\colon X \to W$ associated to R is birational. We take a closed point P of W such that $\dim \varphi_R^{-1}(P) > 0$. Then Theorem 1.14 says that $\varphi_R^{-1}(P)$ is rationally chain connected. However, Theorem 1.14 gives no information on degrees of rational curves on $\varphi_R^{-1}(P)$ with respect to $-(K_X + \Delta)$. On the other hand, Theorem 1.12 shows that every irreducible component of $\varphi_R^{-1}(P)$ is covered by rational curves ℓ with $0 < -(K_X + \Delta)\cdot \ell \leq 2 \dim \varphi_R^{-1}(P)$. In particular, every irreducible component of the exceptional locus of φ_R is uniruled. Note that the rational chain connectedness of $\varphi^{-1}(P)$ does not directly follow from Theorem 1.12. Theorem 1.6 (see also Theorem 1.5) shows that there exist a rational curve C on X and an open lc stratum U of (X, Δ) such that $\varphi_R(C)$ is a point and that the normalization of $C \cap U$ contains $\mathbb{A}^1$.

We pose a conjecture related to [**LZ**, Theorem 3.1].

Conjecture 1.15. *Let $[X, \omega]$ be a quasi-log scheme and let $\pi\colon X \to S$ be a projective morphism between schemes such that $-\omega$ is π-ample and that*

$$\pi\colon \mathrm{Nqklt}(X, \omega) \to \pi(\mathrm{Nqklt}(X, \omega))$$

is finite. Let P be a closed point of S such that there exists a curve $C^\dagger \subset \pi^{-1}(P)$ with $\mathrm{Nqklt}(X, \omega) \cap C^\dagger \neq \emptyset$. Then there exists a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow (X \setminus \mathrm{Nqklt}(X, \omega)) \cap \pi^{-1}(P)$$

such that C, the closure of $f(\mathbb{A}^1)$ in X, satisfies $C \cap \mathrm{Nqklt}(X, \omega) \neq \emptyset$ with

$$0 < -\omega \cdot C \leq 1.$$

In this paper, we solve Conjecture 1.15 under the assumption that any sequence of klt flips terminates.

Theorem 1.16 (see Theorem 14.2)**.** *Assume that any sequence of klt flips terminates after finitely many steps. Then Conjecture 1.15 holds true.*

For the precise statement of Theorem 1.16, see Theorem 14.2. In a joint paper with Kenta Hashizume (see [**FH1**]), we will prove the following theorem, which is a very special case of Conjecture 1.15, by using some deep results in the theory of minimal models for log canonical pairs obtained in [**H2**].

Theorem 1.17 (see [**FH1**, Theorem 1.7])**.** *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X \to S$ be a projective morphism onto a scheme S such that $-(K_X + \Delta)$ is π-ample. We assume that*

$$\pi\colon \mathrm{Nklt}(X, \Delta) \to \pi(\mathrm{Nklt}(X, \Delta))$$

is finite. Let P be a closed point of S such that there exists a curve $C^\dagger \subset \pi^{-1}(P)$ with $\mathrm{Nklt}(X,\Delta) \cap C^\dagger \neq \emptyset$*. Then there exists a non-constant morphism*

$$f\colon \mathbb{A}^1 \longrightarrow (X \setminus \mathrm{Nklt}(X,\Delta)) \cap \pi^{-1}(P)$$

such that the curve C, the closure of $f(\mathbb{A}^1)$ in X, is a (*possibly singular*) *rational curve satisfying $C \cap \mathrm{Nklt}(X,\Delta) \neq \emptyset$ with*

$$0 < -(K_X+\Delta)\cdot C \leq 1.$$

Although Theorem 1.17 looks very similar to Theorem 1.8, the proof of Theorem 1.17 is much harder. By using Theorem 1.17, we will establish:

Theorem 1.18 (see [**FH1**, Theorem 1.8])**.** *Conjecture 1.15 holds true.*

As an application of Theorem 1.18, we will prove the following statement in [**FH1**], which supplements Theorem 1.6 (iii).

Theorem 1.19 (see [**FH1**, Theorem 1.9])**.** *Let $[X,\omega]$ be a quasi-log scheme and let $\pi\colon X \to S$ be a projective morphism between schemes. Let R_j be an ω-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ that are rational and relatively ample at infinity and let φ_{R_j} be the contraction morphism associated to R_j. Let U_j be any open qlc stratum of $[X,\omega]$ such that $\varphi_{R_j}\colon \overline{U_j} \to \varphi_{R_j}(\overline{U_j})$ is not finite and that $\varphi_{R_j}\colon W^\dagger \to \varphi_{R_j}(W^\dagger)$ is finite for every qlc center $W^\dagger$ of $[X,\omega]$ with $W^\dagger \subsetneq \overline{U_j}$, where $\overline{U_j}$ is the closure of U_j in X. Let P be a closed point of $\varphi_{R_j}(U_j)$. If there exists a curve $C^\dagger$ such that $\varphi_{R_j}(C^\dagger)=P$, $C^\dagger \not\subset U_j$, and $C^\dagger \subset \overline{U_j}$, then there exists a non-constant morphism*

$$f_j\colon \mathbb{A}^1 \longrightarrow U_j \cap \varphi_{R_j}^{-1}(P)$$

such that C_j, the closure of $f_j(\mathbb{A}^1)$ in X, spans R_j in $N_1(X/S)$ and satisfies $C_j \not\subset U_j$ with

$$0 < -\omega \cdot C_j \leq 1.$$

We note that Theorem 1.19 is a generalization of [**LZ**, Theorem 3.1]. In this paper, we prove the following simpler statement for dlt pairs for the reader's convenience since Theorems 1.17, 1.18, and 1.19 are difficult. Theorem 1.20 is much weaker than Theorem 1.19. However, it contains a generalization of [**LZ**, Theorem 3.1].

Theorem 1.20. *Let (X,Δ) be a dlt pair and let $\pi\colon X \to S$ be a projective morphism between schemes. Let R_j be a $(K_X+\Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ and let φ_{R_j} be the contraction morphism associated to R_j. Let U_j be any open lc stratum of (X,Δ) such that $\varphi_{R_j}\colon \overline{U_j} \to \varphi_{R_j}(\overline{U_j})$ is not finite and that $\varphi_{R_j}\colon W^\dagger \to \varphi_{R_j}(W^\dagger)$ is finite for every lc center $W^\dagger$ of (X,Δ) with $W^\dagger \subsetneq \overline{U_j}$, where $\overline{U_j}$ is the closure* *of U_j in X. If there exists a curve $C^\dagger$ such that $\varphi_{R_j}(C^\dagger)$ is a point, $C^\dagger \not\subset U_j$, and $C^\dagger \subset \overline{U_j}$, then there exists a non-constant morphism*

$$f_j \colon \mathbb{A}^1 \longrightarrow U_j$$

such that C_j, the closure of $f_j(\mathbb{A}^1)$ in X, spans R_j in $N_1(X/S)$ and satisfies $C_j \not\subset U_j$ with

$$0 < -\omega \cdot C_j \leq 1.$$

Although we need some deep results on the minimal model program for log canonical pairs in [**H1**] in the proof of Theorem 1.20, the proof of Theorem 1.20 is much simpler than that of Theorems 1.17, 1.18 and 1.19 in [**FH1**] and z will help the reader understand [**FH1**].

Finally, we make a conjecture on lengths of extremal rational curves (see [**Ma**, Remark-Question 10-3-6]).

Conjecture 1.21. *If $\varphi_{R_j} \colon U_j \to \varphi_{R_j}(U_j)$ is proper in Theorem 1.6 (iii), where φ_{R_j} is the contraction morphism associated to R_j, then there exists a (possibly singular) rational curve $C_j \subset U_j$ which spans R_j in $N_1(X/S)$ and satisfies*

$$0 < -\omega \cdot C_j \leq d_j + 1$$

with

$$d_j = \min_E \dim E,$$

where E runs over positive-dimensional irreducible components of the fiber $(\varphi_{R_j}|_{U_j})^{-1}(P)$ for all $P \in \varphi_{R_j}(U_j)$.

The following remark on Conjecture 1.21 is obvious.

Remark 1.22. We use the same notation as in Conjecture 1.21. If $\varphi_{R_j} \colon U_j \to \varphi_{R_j}(U_j)$ is proper in Theorem 1.6 (iii), we can make C_j satisfy

$$0 < -\omega \cdot C_j \leq 2d_j$$

by Theorem 1.12.

Of course, we hope that the following sharper estimate

$$0 < -\omega \cdot \ell \leq \dim E + 1$$

should hold true in Theorem 1.12.

In [**FH2**], we will generalize the framework of basic slc-trivial fibrations for $\mathbb{R}$-divisors and establish adjunction and inversion of adjunction for log canonical centers of arbitrary codimension in full generality. We strongly recommend the interested reader to see [**FH1**] and [**FH2**] after reading this paper.

We briefly look at the organization of this paper. In Section 2, we recall some basic definitions and results. Then we treat the notion of uniruledness, rational connectedness, and rational chain connectedness. In Section 3, we treat some basic definitions and results on normal pairs and then discuss dlt blow-ups for quasi-projective normal pairs. In Section 4, we briefly review the theory of quasi-log schemes and prepare some useful and important lemmas. In Section 5, we give a detailed proof of Theorem 1.9. Theorem 1.9 plays a crucial role since a quasi-log scheme is not necessarily normal even when it is a variety. In Section 6, we quickly explain basic slc-trivial fibrations. The results in [**F14**] make the theory of quasi-log schemes very powerful. In Section 7, we prove a very important result on normal quasi-log schemes, which is a slight generalization of [**F14**, Theorem 1.7]. In Section 8, we prove Theorem 1.10 by using the result explained in Section 7. Hence Theorem 1.10 heavily depends on some deep results on the theory of variations of mixed Hodge structure. In Section 9, we prove Theorem 1.8. Note that Theorem 1.8 was essentially obtained in [**LZ**] and [**S**] under some extra assumptions. In Section 10, we prove Theorems 1.4, 1.5, and 1.6. We note that Theorem 1.5 is a special case of Theorem 1.6. In Section 11, we discuss an ampleness criterion for quasi-log schemes. As a very special case, we prove Theorem 1.11. In Section 12, we treat Theorems 1.12 and 1.13. They are generalizations of Kawamata's famous result for quasi-log schemes. In Section 13, we prove Theorem 1.14, which is well known for normal pairs. In Section 14, we discuss several results related to Conjecture 1.15.

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2. Preliminaries

We will work over $\mathbb{C}$, the complex number field, throughout this paper. In this paper, a *scheme* means a separated scheme of finite type over $\mathbb{C}$. A *variety* means an integral scheme, that is, an irreducible and reduced separated scheme of finite type over $\mathbb{C}$. Note that $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of *integers*, *rational numbers*, and *real numbers*, respectively. We also note that $\mathbb{Q}_{>0}$ and $\mathbb{R}_{>0}$ are the set of *positive rational numbers* and *positive real numbers*, respectively.

2.1. Basic definitions. We collect some basic definitions and several useful results. Let us start with the definition of $\mathbb{Q}$-*line bundles* and $\mathbb{R}$-*line bundles.*

Definition 2.1 ($\mathbb{Q}$-line bundles and $\mathbb{R}$-line bundles)**.** Let X be a scheme and let $\mathrm{Pic}(X)$ be the group of line bundles on X, that is, the *Picard group* of X. An element of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) is called an $\mathbb{R}$-*line bundle* (resp. a $\mathbb{Q}$-*line bundle*) on X.

In this paper, we write the group law of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ additively for simplicity of notation. The notion of $\mathbb{R}$-*Cartier divisors* and $\mathbb{Q}$-*Cartier divisors* also plays a crucial role for the study of higher-dimensional algebraic varieties.

Definition 2.2 ($\mathbb{Q}$-Cartier divisors and $\mathbb{R}$-Cartier divisors)**.** Let X be a scheme and let $\mathrm{Div}(X)$ be the group of Cartier divisors on X. An element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$) is called an $\mathbb{R}$-*Cartier divisor* (resp. a $\mathbb{Q}$-*Cartier divisor*) on X. Let Δ_1 and Δ_2 be $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) divisors on X. Then $\Delta_1 \sim_{\mathbb{R}} \Delta_2$ (resp. $\Delta_1 \sim_{\mathbb{Q}} \Delta_2$) means that Δ_1 is $\mathbb{R}$-linearly (resp. $\mathbb{Q}$-linearly) equivalent to Δ_2. Let $f\colon X \to Y$ be a morphism between schemes and let D be an $\mathbb{R}$-Cartier divisor on X. Then $D \sim_{\mathbb{R},f} 0$ means that there exists an $\mathbb{R}$-Cartier divisor G on Y such that $D \sim_{\mathbb{R}} f^*G$.

The following remark is very important.

Remark 2.3 (see [**F11**, Remark 6.2.3])**.** Let X be a scheme. We have the following group homomorphism

$$\mathrm{Div}(X) \to \mathrm{Pic}(X)$$

given by $A \mapsto \mathcal{O}_X(A)$, where A is a Cartier divisor on X. Hence it induces a homomorphism

$$\delta_X\colon \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Note that

$$\mathrm{Div}(X) \to \mathrm{Pic}(X)$$

is not always surjective. We write

$$A + \mathcal{L} \sim_{\mathbb{R}} B + \mathcal{M}$$

for $A, B \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathcal{L}, \mathcal{M} \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. This means that

$$\delta_X(A) + \mathcal{L} = \delta_X(B) + \mathcal{M}$$

holds in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We usually use this type of abuse of notation, that is, the confusion of $\mathbb{R}$-line bundles with $\mathbb{R}$-Cartier divisors. In the theory of minimal models for higher-dimensional algebraic varieties, we sometimes use $\mathbb{R}$-Cartier divisors for ease of notation even when they should be $\mathbb{R}$-line bundles.

On normal varieties or equidimensional reduced schemes, we often treat $\mathbb{R}$-divisors and $\mathbb{Q}$-divisors.

Definition 2.4 (Operations for $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors)**.** Let X be an equidimensional reduced scheme. Note that X is not necessarily regular in codimension one. Let D be an $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-divisor), that is, D is a finite formal sum $\sum_i d_i D_i$, where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is a real number (resp. a rational number) for every i such that $D_i \neq D_j$ for $i \neq j$. We put

$$D^{<c}=\sum_{d_i<c} d_iD_i, \quad D^{\leq c}=\sum_{d_i\leq c} d_iD_i, \quad D^{=1}=\sum_{d_i=1} D_i,$$

and

$$\lceil D\rceil=\sum_i \lceil d_i\rceil D_i,$$

where c is any real number and $\lceil d_i\rceil$ is the integer defined by $d_i \leq \lceil d_i\rceil < d_i+1$. Similarly, we put

$$D^{>c}=\sum_{d_i>c} d_iD_i \quad \text{and} \quad D^{\geq c}=\sum_{d_i\geq c} d_iD_i$$

for any real number c. Moreover, we put $\lfloor D\rfloor = -\lceil -D\rceil$ and $\{D\} = D-\lfloor D\rfloor$.

Let D be an $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-divisor) as above. We call D a *subboundary* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) if $D=D^{\leq 1}$ holds. When D is effective and $D=D^{\leq 1}$ holds, we call D a *boundary* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor).

We further assume that $f\colon X\to Y$ is a surjective morphism onto a variety Y. Then we put

$$D^v=\sum_{f(D_i)\subsetneq Y} d_iD_i \quad \text{and} \quad D^h=D-D^v,$$

and call D^v the *vertical part* and D^h the *horizontal part* of D with respect to $f\colon X\to Y$, respectively.

Since we mainly treat highly singular schemes, we give an important remark.

Remark 2.5. In the theory of minimal models, we are mainly interested in normal quasi-projective varieties. Let X be a normal variety. Then, for $\mathbb{K}=\mathbb{Z},\mathbb{Q}$, and $\mathbb{R}$, the homomorphism

$$\alpha\colon \operatorname{Div}(X)\otimes_{\mathbb{Z}}\mathbb{K}\to \operatorname{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{K}$$

is surjective and the homomorphism

$$\beta\colon \operatorname{Div}(X)\otimes_{\mathbb{Z}}\mathbb{K}\to \operatorname{Weil}(X)\otimes_{\mathbb{Z}}\mathbb{K}$$

is injective, where $\operatorname{Weil}(X)$ is the abelian group generated by Weil divisors on X. We usually use the surjection α and the injection β implicitly. In this paper, however, we frequently treat highly singular schemes X. Hence we have to be careful when we consider $\alpha\colon \operatorname{Div}(X)\otimes_{\mathbb{Z}}\mathbb{K}\to \operatorname{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{K}$ and $\beta\colon \operatorname{Div}(X)\otimes_{\mathbb{Z}}\mathbb{K}\to \operatorname{Weil}(X)\otimes_{\mathbb{Z}}\mathbb{K}$.

Let us recall the following standard notation for the sake of completeness.

Definition 2.6 ($N^1(X/S)$, $N_1(X/S)$, $\rho(X/S)$, and so on)**.** Let $\pi\colon X \to S$ be a proper morphism between schemes. Let $Z_1(X/S)$ be the free abelian group generated by integral complete curves which are mapped to points on S by π. Then we obtain a bilinear form

$$\cdot\colon \operatorname{Pic}(X) \times Z_1(X/S) \to \mathbb{Z},$$

which is induced by the intersection pairing. We have the notion of *numerical equivalence* both in $Z_1(X/S)$ and in $\operatorname{Pic}(X)$, which is denoted by $\equiv$, and we obtain a perfect pairing

$$N^1(X/S) \times N_1(X/S) \to \mathbb{R},$$

where

$$N^1(X/S) = \{\operatorname{Pic}(X)/\equiv\}\otimes_{\mathbb{Z}}\mathbb{R} \quad \text{and} \quad N_1(X/S) = \{Z_1(X/S)/\equiv\}\otimes_{\mathbb{Z}}\mathbb{R}.$$

It is well known that

$$\dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S) < \infty.$$

We write

$$\rho(X/S) = \dim_{\mathbb{R}} N^1(X/S) = \dim_{\mathbb{R}} N_1(X/S)$$

and call it the *relative Picard number* of X over S. When $S = \operatorname{Spec}\mathbb{C}$, we usually drop $/\operatorname{Spec}\mathbb{C}$ from the notation, for example, we simply write $N_1(X)$ instead of $N_1(X/\operatorname{Spec}\mathbb{C})$.

We will freely use the following useful lemma without mentioning it explicitly in the subsequent sections.

Lemma 2.7 (Relative real Nakai–Moishezon ampleness criterion)**.** *Let $\pi\colon X \to S$ be a proper morphism between schemes and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Then $\mathcal{L}$ is π-ample if and only if $\mathcal{L}^{\dim Z} \cdot Z > 0$ for every positive-dimensional closed integral subscheme $Z \subset X$ such that $\pi(Z)$ is a point.*

For the details of Lemma 2.7, see [**FM**]. In the theory of quasi-log schemes, we mainly treat highly singular reducible schemes. Hence Lemma 2.7 is very useful in order to check the ampleness of $\mathbb{R}$-line bundles.

2.2. Uniruledness, rational connectedness, and rational chain connectedness. In this subsection, we quickly recall the notion of uniruledness, rational connectedness, rational chain connectedness, and so on. We need it for Theorems 1.12, 1.13, and 1.14. For the details, see [**Ko1**, Chapter IV.]. We note that a scheme means a separated scheme of finite type over $\mathbb{C}$ in this paper. Let us start with the definition of uniruled varieties.

Definition 2.8 (Uniruledness, see [**Ko1**, Chapter IV. 1.1 Definition])**.** Let X be a variety. We say that X is *uniruled* if there exist a variety Y of dimension $\dim X - 1$ and a dominant rational map

$$\mathbb{P}^1 \times Y \dashrightarrow X.$$

Although the notion of rational connectedness is dispensable for Theorem 1.14, we explain it for the reader's convenience.

Definition 2.9 (Rational connectedness, see [**Ko1**, Chapter IV. 3.6 Proposition])**.** Let X be a projective variety. We say that X is *rationally connected* if for general closed points $x_1, x_2 \in X$ there exists an irreducible rational curve C which contains x_1 and x_2.

The following lemma is almost obvious by definition.

Lemma 2.10. *Let $X \dashrightarrow X'$ be a generically finite dominant rational map between varieties. If X is uniruled, then X' is also uniruled. Furthermore, we assume that $X \dashrightarrow X'$ is a birational map between projective varieties. Then X is rationally connected if and only if X' is rationally connected.*

Let us define rational chain connectedness for projective schemes.

Definition 2.11 (Rational chain connectedness, see [**Ko1**, Chapter IV. 3.5 Corollary and 3.6 Proposition])**.** Let X be a projective scheme. We say that X is *rationally chain connected* if for arbitrary closed points $x_1, x_2 \in X$ there is a connected curve C which contains x_1 and x_2 such that every irreducible component of C is rational.

Note that X may be reducible in Definition 2.11. For projective varieties, we have:

Lemma 2.12. *Let X be a projective variety. If X is rationally connected, then X is rationally chain connected.*

Proof. This follows from [**Ko1**, Chapter IV. 3.6 Proposition]. q.e.d.

We need the following definition for Theorem 1.14.

Definition 2.13 ([**HM**, Definition 1.1])**.** Let X be a projective scheme and let V be any closed subset. We say that X is *rationally chain connected modulo V* if

(1) either $V = \emptyset$ and X is rationally chain connected, or
(2) $V \neq \emptyset$ and, for every $P \in X$, there is a connected pointed curve $0, \infty \in C$ with rational irreducible components and a morphism $h_P \colon C \to X$ such that $h_P(0) = P$ and $h_P(\infty) \in V$.

We close this subsection with a small remark.

Remark 2.14. Let X be a singular normal projective rationally chain connected variety. Then the resolution of X is not always rationally chain connected. Hence the notion of rational chain connectedness is more subtle than that of uniruledness and rational connectedness (see Lemma 2.10).

3. On normal pairs

In this section, we collect some basic definitions and then discuss dlt blow-ups for normal pairs. For the details of normal pairs, see [**BCHM**], [**F6**], and [**F11**]. Let us start with the definition of normal pairs in this paper.

Definition 3.1 (Normal pairs)**.** A *normal pair* (X, Δ) consists of a normal variety X and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Here we do not always assume that Δ is effective.

We note the following definition of *exceptional loci* of birational morphisms between varieties.

Definition 3.2 (Exceptional loci)**.** Let $f\colon X \to Y$ be a birational morphism between varieties. Then the *exceptional locus* $\mathrm{Exc}(f)$ of $f\colon X \to Y$ is the set

$$\{x \in X \mid f \text{ is not biregular at } x\}.$$

3.1. Singularities of pairs. Let us explain singularities of pairs and some related definitions.

Definition 3.3. Let X be a variety and let E be a prime divisor on Y for some birational morphism $f\colon Y \to X$ from a normal variety Y. Then E is called a divisor *over* X.

Definition 3.4 (Singularities of pairs)**.** Let (X, Δ) be a normal pair and let $f\colon Y \to X$ be a projective birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

with

$$f_*\left(\sum_E a(E, X, \Delta)E\right) = -\Delta,$$

where E runs over prime divisors on Y. We call $a(E, X, \Delta)$ the *discrepancy* of E with respect to (X, Δ). Note that we can define the discrepancy $a(E, X, \Delta)$ for any prime divisor E over X by taking a suitable resolution of singularities of X. If $a(E, X, \Delta) \geq -1$ (resp. > -1) for every prime divisor E over X, then (X, Δ) is called *sub log canonical* (resp. *sub kawamata log terminal*). We further assume that Δ is effective. Then (X, Δ) is called *log canonical* and *kawamata log terminal* (*lc* and *klt*, for short) if it is sub log canonical and sub kawamata log terminal, respectively.

Let (X, Δ) be a log canonical pair. If there exists a projective birational morphism $f\colon Y \to X$ from a smooth variety Y such that both $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}\Delta$ are simple normal crossing divisors on

Y and that $a(E, X, \Delta) > -1$ holds for every f-exceptional divisor E on Y, then (X, Δ) is called *divisorial log terminal* (*dlt*, for short).

Let (X, Δ) be a normal pair. If there exist a projective birational morphism $f\colon Y \to X$ from a normal variety Y and a prime divisor E on Y such that (X, Δ) is sub log canonical in a neighborhood of the generic point of $f(E)$ and that $a(E, X, \Delta) = -1$, then $f(E)$ is called a *log canonical center* (an *lc center*, for short) of (X, Δ). A closed subvariety W of X is called a *log canonical stratum* (an *lc stratum*, for short) of (X, Δ) if W is a log canonical center of (X, Δ) or W is X itself.

Although it is well known, we recall the notion of *multiplier ideal sheaves* here for the reader's convenience.

Definition 3.5 (Multiplier ideal sheaves and non-lc ideal sheaves)**.** Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f\colon Y \to X$ be a resolution with

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

such that $\operatorname{Supp} \Delta_Y$ is a simple normal crossing divisor on Y. We put

$$\mathcal{J}(X, \Delta) = f_*\mathcal{O}_Y(-\lfloor \Delta_Y \rfloor).$$

Then $\mathcal{J}(X, \Delta)$ is an ideal sheaf on X and is known as the *multiplier ideal sheaf* associated to the pair (X, Δ). It is independent of the resolution $f\colon Y \to X$. The closed subscheme $\operatorname{Nklt}(X, \Delta)$ defined by $\mathcal{J}(X, \Delta)$ is called the *non-klt locus* of (X, Δ). It is obvious that (X, Δ) is kawamata log terminal if and only if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$. Similarly, we put

$$\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_*\mathcal{O}_X(-\lfloor \Delta_Y \rfloor + \Delta_Y^{=1})$$

and call it the *non-lc ideal sheaf* associated to the pair (X, Δ). We can check that it is independent of the resolution $f\colon Y \to X$. The closed subscheme $\operatorname{Nlc}(X, \Delta)$ defined by $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ is called the *non-lc locus* of (X, Δ). It is obvious that (X, Δ) is log canonical if and only if $\mathcal{J}_{\mathrm{NLC}}(X, \Delta) = \mathcal{O}_X$.

By definition, the natural inclusion

$$\mathcal{J}(X, \Delta) \subset \mathcal{J}_{\mathrm{NLC}}(X, \Delta)$$

always holds. Therefore, we have

$$\operatorname{Nlc}(X, \Delta) \subset \operatorname{Nklt}(X, \Delta).$$

For the details of $\mathcal{J}(X, \Delta)$ and $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$, see [**F4**], [**F6**, Section 7], and [**L**, Chapter 9]. In this paper, we need the notion of *open lc strata*.

Definition 3.6 (Open lc strata)**.** Let (X, Δ) be a normal pair such that Δ is effective. Let W be an lc stratum of (X, Δ). We put

$$U := W \setminus \left\{ (W \cap \operatorname{Nlc}(X, \Delta)) \cup \bigcup_{W'} W' \right\},$$

where W' runs over lc centers of (X, Δ) strictly contained in W, and call it the *open lc stratum of* (X, Δ) *associated to* W.

3.2. Dlt blow-ups revisited. Let us discuss dlt blow-ups. We give a slight generalization of [**F11**, Theorem 4.4.21]. Here we use the theory of minimal models mainly due to [**BCHM**]. Let us start with the definition of movable divisors.

Definition 3.7 (Movable divisors and movable cones, see [**F11**, Definition 2.4.4])**.** Let $f\colon X \to Y$ be a projective morphism from a normal variety X onto a variety Y. A Cartier divisor D on X is called *f-movable* or *movable over* Y if $f_*\mathcal{O}_X(D) \neq 0$ and if the cokernel of the natural homomorphism

$$f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$$

has a support of codimension ≥ 2.

We define $\overline{\mathrm{Mov}}(X/Y)$ as the closure of the convex cone in $N^1(X/Y)$ generated by the numerical equivalence classes of f-movable Cartier divisors. We call $\overline{\mathrm{Mov}}(X/Y)$ the *movable cone* of $f\colon X \to Y$.

We prepare a negativity lemma.

Lemma 3.8 (Negativity lemma)**.** *Let $f\colon X \to Y$ be a projective birational morphism between normal varieties such that X is $\mathbb{Q}$-factorial. Let E be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $-f_*E$ is effective and $E \in \overline{\mathrm{Mov}}(X/Y)$. Then $-E$ is effective.*

Proof. We can write $E = E_+ - E_-$ such that E_+ and E_- are effective $\mathbb{R}$-divisors and have no common irreducible components. We assume that $E_+ \neq 0$. Since $-f_*E$ is effective, E_+ is f-exceptional. Without loss of generality, we may assume that Y is affine by taking an affine open covering of Y. Let A be an ample Cartier divisor on X. Then we can find an irreducible component E_0 of E_+ such that

$$E_0 \cdot (f^*A)^k \cdot H^{n-k-2} \cdot E < 0$$

when $\dim X = n$ and $\mathrm{codim}_Y f(E_+) = k$. This is a contradiction. Note that

$$E_0 \cdot (f^*A)^k \cdot H^{n-k-2} \cdot E \geq 0$$

always hold since $E \in \overline{\mathrm{Mov}}(X/Y)$. Therefore, $-E$ is effective. q.e.d.

By Lemma 3.8, we can prove the existence of dlt blow-ups for quasi-projective normal pairs. We note that Δ is assumed to be a boundary $\mathbb{R}$-divisor in [**F11**, Theorem 4.4.21].

Theorem 3.9 (Dlt blow-ups)**.** *Let X be a normal quasi-projective variety and let $\Delta = \sum_i d_i \Delta_i$ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. In this case, we can construct a projective birational morphism $f\colon Y \to X$ from a normal quasi-projective variety Y with the following properties.*

(i) *Y is $\mathbb{Q}$-factorial.*
(ii) *$a(E,X,\Delta)\le -1$ for every f-exceptional divisor E on Y.*
(iii) *We put*

$$\Delta^{\dagger}=\sum_{0<d_i<1} d_i f_*^{-1}\Delta_i+\sum_{d_i\ge 1} f_*^{-1}\Delta_i+\sum_{E:\ f\text{-exceptional}} E.$$

Then $(Y,\Delta^{\dagger})$ is dlt and the following equality

$$K_Y+\Delta^{\dagger}=f^*(K_X+\Delta)+\sum_{a(E,X,\Delta)<-1}(a(E,X,\Delta)+1)E$$

holds.

We only give a sketch of the proof of Theorem 3.9 since the proof of [**F11**, Theorem 4.4.21] works by Lemma 3.8.

Sketch of Proof of Theorem 3.9. Let $g\colon Z\to X$ be a resolution such that $\mathrm{Exc}(g)\cup\operatorname{Supp} g_*^{-1}\Delta$ is a simple normal crossing divisor on X and g is projective. We write

$$K_Z+\widetilde{\Delta}=g^*(K_X+\Delta)+F,$$

where

$$\widetilde{\Delta}=\sum_{0<d_i<1} d_i g_*^{-1}\Delta_i+\sum_{d_i\ge 1} g_*^{-1}\Delta_i+\sum_{E:\ g\text{-exceptional}} E.$$

We note that $-g_*F$ is effective by construction. Then we apply the same argument as in the proof of [**F11**, Theorem 4.4.21], that is, we run a suitable minimal model program with respect to $(Z,\widetilde{\Delta})$ over X. After finitely many steps, we see that the effective part of F is contracted. Note that all we have to do is to use Lemma 3.8 instead of [**F11**, Lemma 2.4.5]. q.e.d.

When Δ is a boundary $\mathbb{R}$-divisor, Lemma 3.10 is nothing but [**S**, Theorem 3.4].

Lemma 3.10. *Let X be a normal quasi-projective variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Then we can construct a projective birational morphism $g\colon Y\to X$ from a normal $\mathbb{Q}$-factorial variety Y with the following properties.*

(i) *$K_Y+\Delta_Y:=g^*(K_X+\Delta)$,*
(ii) *the pair*

$$\left(Y,\Delta'_Y:=\sum_{d_i<1} d_iD_i+\sum_{d_i\ge 1} D_i\right)$$

is dlt, where $\Delta_Y=\sum_i d_iD_i$ is the irreducible decomposition of Δ_Y,
(iii) *every g-exceptional prime divisor is a component of $(\Delta'_Y)^{=1}$, and*

(iv) $g^{-1}\operatorname{Nklt}(X,\Delta)$ *coincides with* $\operatorname{Nklt}(Y,\Delta_Y)$ *and* $\operatorname{Nklt}(Y,\Delta'_Y)$ *set theoretically.*

By Theorem 3.9, the proof of [**S**, Theorem 3.4] works without any changes even when Δ is not a boundary $\mathbb{R}$-divisor. We give a proof for the sake of completeness.

Proof of Lemma 3.10. There exists a dlt blow-up $\alpha\colon Z \to X$ with $K_Z + \Delta_Z := \alpha^*(K_X+\Delta)$ satisfying (i), (ii), and (iii) by Theorem 3.9. Note that $(Z, \Delta_Z^{<1})$ is a $\mathbb{Q}$-factorial kawamata log terminal pair. We take a minimal model $(Z', \Delta_{Z'}^{<1})$ of $(Z, \Delta_Z^{<1})$ over X by [**BCHM**].

$$\begin{array}{ccc} Z & \overset{\varphi}{\dashrightarrow} & Z' \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & X & \end{array}$$

Then $K_{Z'} + \Delta_{Z'}^{<1} \sim_{\mathbb{R}} -\Delta_{Z'}^{\geq 1} + \alpha'^*(K_X+\Delta)$ is nef over X. Of course, we put $\Delta_{Z'} = \varphi_*\Delta_Z$. We take a dlt blow-up $\beta\colon Y \to Z'$ of $(Z', \Delta_{Z'}^{<1} + \operatorname{Supp}\Delta_{Z'}^{\geq 1})$ again by Theorem 3.9 (or [**F11**, Theorem 4.4.21]) and put $g := \alpha' \circ \beta\colon Y \to X$. It is not difficult to see that this birational morphism $g\colon Y \to X$ with $K_Y + \Delta_Y := g^*(K_X+\Delta)$ satisfies the desired properties. It is obvious that $g^{-1}\operatorname{Nklt}(X,\Delta)$ contains the support of $\beta^*\Delta_{Z'}^{\geq 1}$. Since $-\beta^*\Delta_{Z'}^{\geq 1}$ is nef over X, we see that $\beta^*\Delta_{Z'}^{\geq 1}$ coincides with $g^{-1}\operatorname{Nklt}(X,\Delta)$ set theoretically. q.e.d.

For the details of the proof of Lemma 3.10, see [**S**, Theorem 3.4]. In [**FH1**], Theorem 3.9 and Lemma 3.10 will be generalized completely by using the minimal model program for log canonical pairs established in [**H2**].

4. On quasi-log schemes

In this section, we explain some basic definitions and results on quasi-log schemes. For the details of the theory of quasi-log schemes, we recommend the reader to see [**F11**, Chapter 6] and [**F17**].

4.1. Definitions and basic properties of quasi-log schemes. The notion of quasi-log schemes was first introduced by Florin Ambro (see [**A**]) in order to establish the cone and contraction theorem for (X,Δ), where X is a normal variety and Δ is an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Here we use the formulation in [**F11**, Chapter 6], which is slightly different from Ambro's original one. We recommend the interested reader to see [**F12**, Appendix A] for the difference between our definition of quasi-log schemes and Ambro's one (see also [**F17**, Section 8]).

In order to define quasi-log schemes, we use the notion of *globally embedded simple normal crossing pairs.*

Definition 4.1 (Globally embedded simple normal crossing pairs, see [**F11**, Definition 6.2.1])**.** Let Y be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(B+Y)$ is a simple normal crossing divisor on M and that B and Y have no common irreducible components. We put $B_Y = B|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y). A *stratum* of (Y, B_Y) is a log canonical center of $(M, Y+B)$ that is contained in Y.

Let us recall the definition of *quasi-log schemes*.

Definition 4.2 (Quasi-log schemes, see [**F11**, Definition 6.2.2])**.** A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a closed subscheme $X_{-\infty} \subsetneq X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f\colon (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism
$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$
where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of reduced and irreducible subschemes $\{C\}$ coincides with the images of the strata of (Y, B_Y) that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data
$$(X, \omega, f\colon (Y, B_Y) \to X)$$
if there is no risk of confusion. Note that a quasi-log scheme $[X, \omega]$ is the union of $\{C\}$ and $X_{-\infty}$. The reduced and irreducible subschemes C are called the *qlc strata* of $[X, \omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$, and $f\colon (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ or
$$\mathrm{Nqlc}(X, \omega, f\colon (Y, B_Y) \to X)$$
to denote $X_{-\infty}$. If a qlc stratum C of $[X, \omega]$ is not an irreducible component of X, then it is called a *qlc center* of $[X, \omega]$.

We say that $(X, \omega, f\colon (Y, B_Y) \to X)$ or $[X, \omega]$ has a $\mathbb{Q}$-*structure* if B_Y is a $\mathbb{Q}$-divisor, ω is a $\mathbb{Q}$-Cartier divisor (or $\mathbb{Q}$-line bundle), and $f^*\omega \sim_{\mathbb{Q}} K_Y + B_Y$ holds in the above definition.

In Definition 4.1, we note that $f\colon Y \to X$ is not necessarily surjective and that Y may be reducible even when X is irreducible. In this paper, the notion of *open qlc strata* is indispensable.

Definition 4.3 (Open qlc strata)**.** Let W be a qlc stratum of a quasi-log scheme $[X, \omega]$. We put

$$U := W \setminus \left\{ (W \cap \mathrm{Nqlc}(X, \omega)) \cup \bigcup_{W'} W' \right\},$$

where W' runs over qlc centers of $[X, \omega]$ strictly contained in W, and call it the *open qlc stratum* of $[X, \omega]$ associated to W.

In Section 11, we need the notion of *log bigness*. For the details of relatively big $\mathbb{R}$-divisors, see [**F11**, Section 2.1].

Definition 4.4 (Log bigness)**.** Let $[X, \omega]$ be a quasi-log scheme and let $\pi\colon X \to S$ be a proper morphism between schemes. Let D be an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) on X. We say that D is *log big over S with respect to* $[X, \omega]$ if $D|_W$ is big over $\pi(W)$ for every qlc stratum W of $[X, \omega]$.

We collect some basic and important properties of quasi-log schemes for the reader's convenience.

Theorem 4.5 ([**F11**, Theorem 6.3.4])**.** *In Definition 4.2, we may assume that the ambient space M of the globally embedded simple normal crossing pair (Y, B_Y) is quasi-projective. In particular, Y is quasi-projective and $f\colon Y \to X$ is projective.*

For the details of Theorem 4.5, see the proof of [**F11**, Theorem 6.3.4]. In the theory of quasi-log schemes, we sometimes need the projectivity of f in order to use the theory of variations of mixed Hodge structure (see [**F14**] and [**FFL**]). Hence Theorem 4.5 plays a crucial role. The most important result in the theory of quasi-log schemes is as follows.

Theorem 4.6 ([**F11**, Theorem 6.3.5])**.** *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty}$ with a* (*possibly empty*) *union of some qlc strata of $[X, \omega]$. Then we have the following properties.*

(i) (Adjunction). *Assume that $X' \neq X_{-\infty}$. Then X' naturally becomes a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.*

(ii) (Vanishing theorem). *Assume that $\pi\colon X \to S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

In this paper, we will repeatedly use adjunction for quasi-log schemes in Theorem 4.6 (i). We strongly recommend the reader to see the proof of [**F11**, Theorem 6.3.5]. Here, we only explain the main idea of the proof of Theorem 4.6 (i) for the reader's convenience.

Idea of Proof of Theorem 4.6 (i). By definition, X' is the union of $X_{-\infty}$ with a union of some qlc strata of $[X, \omega]$ set theoretically. We assume that $X' \neq X_{-\infty}$ holds. By [**F11**, Proposition 6.3.1], we may assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. We note that Y is a simple normal crossing divisor on a smooth variety M (see Definition 4.1). We put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$. We set $f'' = f|_{Y''}$ and $f' = f|_{Y'}$. Then we claim that

$$\bigl(X', \omega', f' \colon (Y', B_{Y'}) \to X'\bigr)$$

becomes a quasi-log scheme satisfying the desired properties. Let us consider the following short exact sequence:

$$\begin{aligned}0 &\to \mathcal{O}_{Y''}(\lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''}) \to \mathcal{O}_Y(\lceil -(B^{<1}_{Y})\rceil - \lfloor B^{>1}_{Y}\rfloor)\\ &\to \mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor) \to 0,\end{aligned}$$

which is induced by

$$0 \to \mathcal{O}_{Y''}(-Y'|_{Y''}) \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

We take the associated long exact sequence:

$$\begin{aligned}0 &\longrightarrow f''_*\mathcal{O}_{Y''}(\lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''})\\ &\longrightarrow f_*\mathcal{O}_Y(\lceil -(B^{<1}_{Y})\rceil - \lfloor B^{>1}_{Y}\rfloor)\\ &\longrightarrow f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor)\\ &\overset{\delta}{\longrightarrow} R^1f''_*\mathcal{O}_{Y''}(\lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''}) \longrightarrow \cdots .\end{aligned}$$

Since

$$\begin{aligned}&\lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B^{=1}_{Y''} - Y'|_{Y''})\\ &= -(K_{Y''} + B_{Y''}) \sim_{\mathbb{R}} -(f'')^*\omega,\end{aligned}$$

no associated prime of $R^1f''_*\mathcal{O}_{Y''}(\lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''})$ is contained in X' by [**F11**, Theorem 5.6.2 (i)], which is a generalization of Kollár's torsion-freeness based on the theory of mixed Hodge structures on cohomology with compact support (see [**F11**, Chapter 5]). Then the connecting homomorphism

$$\delta \colon f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor) \to R^1f''_*\mathcal{O}_{Y''}(\lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''})$$

is zero since $f(Y') \subset X'$. We put

$$\mathcal{I}_{X'} := f''_*\mathcal{O}_{Y''}(\lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''}),$$

which is an ideal sheaf on X since $\mathcal{I}_{X'} \subset \mathcal{I}_{X_{-\infty}}$, and define a scheme structure on X' by $\mathcal{I}_{X'}$. Then we obtain the following big commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & f''_*\mathcal{O}_{Y''}(\lceil -(B_{Y''}^{<1})\rceil - \lfloor B_{Y''}^{>1}\rfloor - Y'|_{Y''}) & \xrightarrow{=} & \mathcal{I}_{X'} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & f_*\mathcal{O}_{Y}(\lceil -(B_{Y}^{<1})\rceil - \lfloor B_{Y}^{>1}\rfloor) = \mathcal{I}_{X_{-\infty}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_{-\infty}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - \lfloor B_{Y'}^{>1}\rfloor) = \mathcal{I}_{X'_{-\infty}} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_{X'_{-\infty}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

by the above arguments. More precisely, by the above big commutative diagram,

$$\mathcal{I}_{X'_{-\infty}} = f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - \lfloor B_{Y'}^{>1}\rfloor)$$

is an ideal sheaf on X' such that $\mathcal{O}_X/\mathcal{I}_{X_{-\infty}} = \mathcal{O}_{X'}/\mathcal{I}_{X'_{-\infty}}$. Thus we obtain that

$$\big(X', \omega', f' \colon (Y', B_{Y'}) \to X'\big)$$

is a quasi-log scheme satisfying the desired properties. q.e.d.

As an obvious corollary, we have:

Corollary 4.7 ([**F11**, Notation 6.3.10])**.** *Let $[X, \omega]$ be a quasi-log scheme. The union of $X_{-\infty}$ with all qlc centers of $[X, \omega]$ is denoted by* $\mathrm{Nqklt}(X, \omega)$*, or, more precisely,*

$$\mathrm{Nqklt}(X, \omega, f \colon (Y, B_Y) \to X).$$

If $\mathrm{Nqklt}(X, \omega) \neq X_{-\infty}$*, then*

$$[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X,\omega)}]$$

naturally becomes a quasi-log scheme by adjunction.

In the framework of quasi-log schemes, $\mathrm{Nqklt}(X, \omega)$ plays an important role by induction on dimension. When $\mathrm{Nqklt}(X, \omega) = \emptyset$, we have the following lemma.

Lemma 4.8 ([**F11**, Lemma 6.3.9])**.** *Let $[X, \omega]$ be a quasi-log scheme with $X_{-\infty} = \emptyset$. Assume that every qlc stratum of $[X, \omega]$ is an irreducible component of X, equivalently,* $\mathrm{Nqklt}(X, \omega) = \emptyset$*. Then X is normal.*

For the proof of Lemma 4.8, see [**F11**, Lemma 6.3.9]. It is convenient to introduce the notion of *quasi-log canonical pairs.*

Definition 4.9 (Quasi-log canonical pairs, see [**F11**, Definition 6.2.9])**.** Let

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

be a quasi-log scheme. If $X_{-\infty} = \emptyset$, then it is called a *quasi-log canonical pair* (*qlc pair*, for short).

By using adjunction, we can prove:

Theorem 4.10 ([**F11**, Theorem 6.3.11 (i)])**.** *Let* $[X, \omega]$ *be a quasi-log canonical pair. Then the intersection of two qlc strata is a union of qlc strata.*

The following example is very important. Example 4.11 shows that we can treat log canonical pairs as quasi-log canonical pairs. In some sense, Ambro introduced the notion of quasi-log schemes in order to treat the following example (see [**A**]).

Example 4.11 ([**F11**, 6.4.1])**.** Let (X, Δ) be a normal pair such that Δ is effective. Let $f\colon Y \to X$ be a resolution of singularities such that

$$K_Y + B_Y = f^*(K_X + \Delta)$$

and that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor on Y. We put $\omega := K_X + \Delta$. Then $K_Y + B_Y \sim_{\mathbb{R}} f^*\omega$ holds. Since Δ is effective, $\lceil -(B_Y^{<1}) \rceil$ is effective and f-exceptional. Therefore, the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$$

is an isomorphism. We put

$$\mathcal{I}_{X_{-\infty}} := \mathcal{J}_{\mathrm{NLC}}(X, \Delta) = f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil - \lfloor B_Y^{>1} \rfloor),$$

where $\mathcal{J}_{\mathrm{NLC}}(X, \Delta)$ is the non-lc ideal sheaf associated to (X, Δ) in Definition 3.5. We put $M = Y \times \mathbb{C}$ and $D = B_Y \times \mathbb{C}$. Then $(Y, B_Y) \simeq (Y \times \{0\}, B_Y \times \{0\})$ is a globally embedded simple normal crossing pair. Thus

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

becomes a quasi-log scheme. By construction, (X, Δ) is log canonical if and only if $[X, \omega]$ is quasi-log canonical. We note that C is a log canonical center of (X, B) if and only if C is a qlc center of $[X, \omega]$. We also note that X itself is a qlc stratum of $[X, \omega]$.

We make a useful remark.

Remark 4.12. Let Y be a smooth variety and let B_Y be an $\mathbb{R}$-divisor on Y such that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor on Y. We put $M' := Y \times \mathbb{P}^1$ and

$$D' := Y \times \{0\} + Y \times \{\infty\} + p^*B_Y,$$

where $p\colon Y \times \mathbb{P}^1 \to Y$ is the first projection. Then $K_{M'} + D' = p^*(K_Y + B_Y)$ holds. We put

$$Z := Y \times \{0\} + p^*B_Y^{=1}$$

and $K_Z+B_Z:=(K_{M'}+D')|_Z$. Then $K_Z+B_Z=g^*(K_Y+B_Y)$ holds, where $g:=p|_Z\colon Z\to Y$. In this case, (Z,B_Z) is a globally embedded simple normal crossing pair. We can check that

$$g_*\mathcal{O}_Z(\lceil -(B_Z^{<1})\rceil - \lfloor B_Z^{>1}\rfloor)\simeq \mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor)$$

holds since $g_*\mathcal{O}_Z\simeq\mathcal{O}_Y$ and

$$B_Z=(D'-Z)|_Z=(Y\times\{\infty\})|_{p^*B_Y^{=1}}+g^*B_Y^{\neq 1},$$

where $B_Y^{\neq 1}:=B_Y-B_Y^{=1}$. Hence, in Example 4.11, $f\circ g\colon (Z,B_Z)\to X$ gives another quasi-log resolution of $[X,\omega]$. Although Z may be reducible, this quasi-log resolution is useful when we use adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]).

Example 4.11 shows that $[X,K_X+\Delta]$ has a natural quasi-log scheme structure. In general, however, $[X,K_X+\Delta]$ has many different quasi-log scheme structures.

Remark 4.13. In Example 4.11, we take an effective $\mathbb{R}$-divisor Δ' on X such that $K_X+\Delta\sim_{\mathbb{R}}K_X+\Delta'$. Let $f'\colon Y'\to X$ be a resolution of singularities such that

$$K_{Y'}+B_{Y'}=(f')^*(K_X+\Delta')$$

and that $\operatorname{Supp}B_{Y'}$ is a simple normal crossing divisor on Y'. Then

$$\big(X,\omega,f'\colon (Y',B_{Y'})\to X\big)$$

is also a quasi-log scheme since $K_{Y'}+B_{Y'}\sim_{\mathbb{R}}(f')^*\omega$. In this case, there is no correspondence between qlc strata of $(X,\omega,f'\colon (Y',B_{Y'})\to X)$ and lc strata of (X,Δ).

By combining Theorem 4.10 with Example 4.11, we have:

Corollary 4.14 ([**F6**, Theorem 9.1 (2)])**.** *Let (X,Δ) be a log canonical pair. Then the intersection of two lc centers is a union of lc centers.*

For the basic properties of quasi-log schemes, see [**F11**, Chapter 6] and [**F17**]. We also recommend the reader to see [**F5**], which is a gentle introduction to the theory of quasi-log schemes. In [**F8**], we establish that every quasi-projective semi-log canonical pair naturally becomes a quasi-log canonical pair. Hence we can use the theory of quasi-log schemes for the study of semi-log canonical pairs. For the details, see [**F8**].

4.2. Kleiman–Mori cones. In this subsection, we discuss basic definitions and results around Kleiman–Mori cones of quasi-log schemes. Let us start with the definition of Kleiman–Mori cones.

Definition 4.15 (Kleiman–Mori cones)**.** Let $\pi\colon X\to S$ be a proper morphism of schemes. Let $NE(X/S)$ be the convex cone in $N_1(X/S)$ generated by effective 1-cycles on X mapped to points by π. Let

$\overline{\mathrm{NE}}(X/S)$ be the closure of $\mathrm{NE}(X/S)$ in $N_1(X/S)$. We call it the *Kleiman–Mori cone* of $\pi\colon X \to S$. As usual, we drop $/\operatorname{Spec}\mathbb{C}$ from the notation when $S = \operatorname{Spec}\mathbb{C}$.

Let us explain some basic definitions.

Definition 4.16 ([**F11**, Definition 6.7.1])**.** Let $[X,\omega]$ be a quasi-log scheme with the non-qlc locus $X_{-\infty}$. Let $\pi\colon X \to S$ be a projective morphism between schemes. We put

$$\overline{\mathrm{NE}}(X/S)_{-\infty} = \operatorname{Im}\left(\overline{\mathrm{NE}}(X_{-\infty}/S) \to \overline{\mathrm{NE}}(X/S)\right).$$

We sometimes use $\overline{\mathrm{NE}}(X/S)_{\mathrm{Nqlc}(X/S)}$ to denote $\overline{\mathrm{NE}}(X/S)_{-\infty}$. For an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) D, we define

$$D_{\geq 0} = \{z \in N_1(X/S) \,|\, D \cdot z \geq 0\}.$$

Similarly, we can define $D_{>0}$, $D_{\leq 0}$, and $D_{<0}$. We also define

$$D^{\perp} = \{z \in N_1(X/S) \,|\, D \cdot z = 0\}.$$

We use the following notation

$$\overline{\mathrm{NE}}(X/S)_{D\geq 0} = \overline{\mathrm{NE}}(X/S) \cap D_{\geq 0},$$

and similarly for > 0, ≤ 0, and < 0.

In order to treat the cone and contraction theorem, we need the following definition.

Definition 4.17 ([**F11**, Definition 6.7.2])**.** An *extremal face* of the Kleiman–Mori cone $\overline{\mathrm{NE}}(X/S)$ is a non-zero subcone $F \subset \overline{\mathrm{NE}}(X/S)$ such that $z, z' \in \overline{\mathrm{NE}}(X/S)$ and $z + z' \in F$ imply that $z, z' \in F$. Equivalently, $F = \overline{\mathrm{NE}}(X/S) \cap H^{\perp}$ for some π-nef $\mathbb{R}$-divisor (or π-nef $\mathbb{R}$-line bundle) H, which is called a *support function* of F. An *extremal ray* is a one-dimensional extremal face.

(1) An extremal face F is called *ω-negative* if $F \cap \overline{\mathrm{NE}}(X/S)_{\omega\geq 0} = \{0\}$.
(2) An extremal face F is called *rational* if we can choose a π-nef $\mathbb{Q}$-divisor (or $\mathbb{Q}$-line bundle) H as a support function of F.
(3) An extremal face F is called *relatively ample at infinity* if $F \cap \overline{\mathrm{NE}}(X/S)_{-\infty} = \{0\}$. Equivalently, $H|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-ample for any supporting function H of F.

The contraction theorem for quasi-log schemes plays an important role in this paper.

Theorem 4.18 (Contraction theorem, see [**F11**, Theorem 6.7.3])**.** *Let $[X,\omega]$ be a quasi-log scheme and let $\pi\colon X \to S$ be a projective morphism between schemes. Let R be an ω-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ that is rational and relatively ample at infinity. Then there exists a projective morphism $\varphi_R\colon X \to Y$ over S with the following properties.*

(i) *Let C be an integral curve on X such that $\pi(C)$ is a point. Then $\varphi_R(C)$ is a point if and only if $[C] \in R$, where $[C]$ denotes the numerical equivalence class of C in $N_1(X/S)$.*
(ii) $\mathcal{O}_Y \simeq (\varphi_R)_*\mathcal{O}_X$.
(iii) *Let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L} \cdot C = 0$ for every curve C with $[C] \in R$. Then there is a line bundle $\mathcal{L}_Y$ on Y such that $\mathcal{L} \simeq \varphi_R^*\mathcal{L}_Y$.*

Proof. Since R is relatively ample at infinity, $\varphi_R \colon X_{-\infty} \to \varphi_R(X_{-\infty})$ is finite. Hence $\mathcal{L}^{\otimes m}|_{X_{-\infty}}$ is $\varphi_R|_{X_{-\infty}}$-generated for every $m \geq 0$. Therefore, this theorem is a special case of [**F11**, Theorem 6.7.3]. q.e.d.

Theorem 4.18 is a generalization of the famous Kawamata–Shokurov basepoint-free theorem.

4.3. Lemmas on quasi-log schemes. In this subsection, we treat useful lemmas on quasi-log schemes. The first two lemmas were already proved in [**F10**]. We will repeatedly use Lemma 4.20 throughout this paper.

Lemma 4.19 ([**F10**, Lemma 3.12])**.** *Let $(X, \omega, f \colon (Y, B_Y) \to X)$ be a quasi-log scheme. Then we can construct a proper morphism*

$$f' \colon (Y', B_{Y'}) \to X$$

from a globally embedded simple normal crossing pair $(Y', B_{Y'})$ such that

(i) *$f' \colon (Y', B_{Y'}) \to X$ gives the same quasi-log scheme structure as one given by $f \colon (Y, B_Y) \to X$, and*
(ii) *every irreducible component of Y' is mapped by f' to $\overline{X \setminus X_{-\infty}}$, the closure of $X \setminus X_{-\infty}$ in X.*

We give the proof for the sake of completeness.

Proof. Let Y'' be the union of all irreducible components of Y that are not mapped to $\overline{X \setminus X_{-\infty}}$. We put $Y' = Y - Y''$ and $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$. Let M be the ambient space of (Y, B_Y). By taking some blow-ups of M, we may assume that the union of all strata of (Y, B_Y) mapped to $\overline{X \setminus X_{-\infty}} \cap X_{-\infty}$ is a union of some irreducible components of Y (see [**F11**, Proposition 6.3.1]). We consider the short exact sequence

$$0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

We put $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$. By applying $\otimes \mathcal{O}_Y(A - N)$, we have

$$0 \to \mathcal{O}_{Y''}(A - N - Y') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{Y'}(A - N) \to 0.$$

By taking $R^i f_*$, we obtain

$$\begin{aligned} 0 &\to f_*\mathcal{O}_{Y''}(A - N - Y') \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{Y'}(A - N) \\ &\to R^1 f_*\mathcal{O}_{Y''}(A - N - Y') \to \cdots . \end{aligned}$$

Note that

$$\begin{aligned}(A-N-Y')|_{Y''}-(K_{Y''}+\{B_{Y''}\}+B^{=1}_{Y''}-Y'|_{Y''})&=-(K_{Y''}+B_{Y''})\\&\sim_{\mathbb{R}}-(f^*\omega)|_{Y''}.\end{aligned}$$

Hence, by [**F11**, Theorem 5.6.2], no associated prime of $R^1f_*\mathcal{O}_{Y''}(A-N-Y')$ is contained in $f(Y')\cap X_{-\infty}$. Therefore, the connecting homomorphism

$$\delta\colon f_*\mathcal{O}_{Y'}(A-N)\to R^1f_*\mathcal{O}_{Y''}(A-N-Y')$$

is zero. This implies that

$$0\to f_*\mathcal{O}_{Y''}(A-N-Y')\to \mathcal{I}_{X_{-\infty}}\to f_*\mathcal{O}_{Y'}(A-N)\to 0$$

is exact. The ideal sheaf $\mathcal{J}=f_*\mathcal{O}_{Y''}(A-N-Y')$ is zero when it is restricted to $X_{-\infty}$ because $\mathcal{J}\subset\mathcal{I}_{X_{-\infty}}$. On the other hand, $\mathcal{J}$ is zero on $X\setminus X_{-\infty}$ because $f(Y'')\subset X_{-\infty}$. Therefore, we obtain $\mathcal{J}=0$. Thus we have $\mathcal{I}_{X_{-\infty}}=f_*\mathcal{O}_{Y'}(A-N)$. So $f'=f|_{Y'}\colon (Y',B_{Y'})\to X$, where $K_{Y'}+B_{Y'}=(K_Y+B_Y)|_{Y'}$, gives the same quasi-log scheme structure as one given by $f\colon (Y,B_Y)\to X$ with the property (ii). q.e.d.

By using Lemma 4.19, we establish the following very useful lemma.

Lemma 4.20 ([**F10**, Lemma 3.14])**.** *Let $[X,\omega]$ be a quasi-log scheme. Let us consider $X^\dagger=\overline{X\setminus X_{-\infty}}$, the closure in X, with the reduced scheme structure. Then $[X^\dagger,\omega^\dagger]$, where $\omega^\dagger=\omega|_{X^\dagger}$, has a natural quasi-log scheme structure induced by $[X,\omega]$. This means that*

(i) *C is a qlc stratum of $[X,\omega]$ if and only if C is a qlc stratum of $[X^\dagger,\omega^\dagger]$, and*
(ii) *$\mathcal{I}_{\mathrm{Nqlc}(X,\omega)}=\mathcal{I}_{\mathrm{Nqlc}(X^\dagger,\omega^\dagger)}$ holds.*

Moreover, we consider a set of some qlc strata $\{C_i\}_{i\in I}$ of $[X,\omega]$. We put

$$\left(X^\dagger\right)'=\mathrm{Nqlc}(X^\dagger,\omega^\dagger)\cup\left(\bigcup_{i\in I}C_i\right)$$

and

$$X'=\mathrm{Nqlc}(X,\omega)\cup\left(\bigcup_{i\in I}C_i\right).$$

Then $\left[(X^\dagger)',\omega^\dagger|_{(X^\dagger)'}\right]$ and $[X',\omega|_{X'}]$ naturally become quasi-log schemes by adjunction and $\mathcal{I}_{(X^\dagger)'}=\mathcal{I}_{X'}$ holds, where $\mathcal{I}_{(X^\dagger)'}$ and $\mathcal{I}_{X'}$ are the defining ideal sheaves of $(X^\dagger)'$ and X' on $X^\dagger$ and X, respectively. In particular, $\mathcal{I}_{\mathrm{Nqklt}(X^\dagger,\omega^\dagger)}=\mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$ holds.

We include the proof for the benefit of the reader.

Proof. In this proof, we use the same notation as in the proof of Lemma 4.19. Let $\mathcal{I}_{X^\dagger}$ be the defining ideal sheaf of $X^\dagger$ on X. Let $f'\colon (Y', B_{Y'}) \to X$ be the quasi-log resolution constructed in the proof of Lemma 4.19. By construction, $f'\colon Y' \to X$ factors through $X^\dagger$. Note that

$$\mathcal{I}_{X_{-\infty}} = f_*\mathcal{O}_Y(A-N) = f'_*\mathcal{O}_{Y'}(A-N) = f'_*\mathcal{O}_{Y'}(-N)$$

and that

$$f'(N) = X_{-\infty} \cap f'(Y') = X_{-\infty} \cap X^\dagger$$

set theoretically, where $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$ (see [**F11**, Remark 6.2.10]). Therefore, we obtain

$$\mathcal{I}_{X^\dagger} \cap \mathcal{I}_{X_{-\infty}} = \mathcal{I}_{X^\dagger} \cap f_*\mathcal{O}_Y(A-N) \subset f_*\mathcal{O}_{Y''}(A-N-Y') = \{0\}.$$

Thus we can construct the following big commutative diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & f'_*\mathcal{O}_{Y'}(A-N) & = & f'_*\mathcal{O}_{Y'}(A-N) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_{X^\dagger} & \to & \mathcal{O}_X & \to & \mathcal{O}_{X^\dagger} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_{X^\dagger} & \to & \mathcal{O}_{X_{-\infty}} & \to & \mathcal{O}_{X^\dagger_{-\infty}} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence $f'\colon (Y', B_{Y'}) \to X^\dagger$ gives the desired quasi-log scheme structure on $[X^\dagger, \omega^\dagger]$.

We know that $[(X^\dagger)', \omega^\dagger|_{(X^\dagger)'}]$ and $[X', \omega|_{X'}]$ naturally become quasi-log schemes by adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]). Thus it is sufficient to prove the equality $\mathcal{I}_{(X^\dagger)'} = \mathcal{I}_{X'}$. As usual, by [**F11**, Proposition 6.3.1], we may further assume that the union of all strata of (Y, B_Y) that are mapped to X', which is denoted by Z, is a union of some irreducible components of Y. We note that $Z \geq Y''$. We put $Z' = Y - Z$. Then it is obvious that $Z' \leq Y'$ holds. By the proof of adjunction (see the idea of the proof of Theorem 4.6 (i) and the proof of [**F11**, Theorem 6.3.5 (i)]), we see that

$$\mathcal{I}_{(X^\dagger)'} = f'_*\mathcal{O}_{Z'}(A-N-(Z-Y'')|_{Z'}) = f'_*\mathcal{O}_{Z'}(A-N-Z|_{Z'}) = \mathcal{I}_{X'}$$

holds. q.e.d.

By Lemmas 4.19 and 4.20, we can abandon unnecessary components from $f\colon (Y, B_Y) \to X$. The following examples may help the reader understand Lemmas 4.19 and 4.20.

Example 4.21. Let L be a line on $\mathbb{P}^3$. We take general hyperplanes H_i with $L \subset H_i$ for $1 \leq i \leq 4$. Let H_0 be a general hyperplane of $\mathbb{P}^3$. We put $X := \mathbb{P}^3$ and

$$\Delta := H_0 + \frac{1}{2}H_1 + \frac{1}{2}H_2 + \frac{2}{3}H_3 + \frac{2}{3}H_4$$

and consider the normal pair (X, Δ). Then the pair $[X, \omega := K_X + \Delta]$ naturally becomes a quasi-log scheme by Example 4.11. We can easily check that $\mathrm{Nqlc}(X, \omega) = \mathrm{Nlc}(X, \Delta) = L$. Let $p\colon X^\flat \to X$ be the blow-up along L. Then we have

$$K_{X^\flat} + p_*^{-1}\Delta + \frac{4}{3}E = p^*(K_X + \Delta),$$

where E is the p-exceptional divisor on $X^\flat$. By construction, the support of $p_*^{-1}\Delta + E$ is a simple normal crossing divisor on $X^\flat$. We put $X' := H_0 \cup L$ and $\omega' := \omega|_{X'}$. By adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]), $[X', \omega']$ naturally has a quasi-log scheme structure induced by $[X, \omega]$. More precisely, by using $p\colon X^\flat \to X$, we can construct a quasi-log scheme

$$\big(X', \omega', f\colon (Y, B_Y) \to X'\big)$$

such that Y is irreducible (see also Remark 4.12). In this case, $f\colon Y \to X'$ is not surjective. Let $q\colon X^\sharp \to X^\flat$ be the blow-up along $E \cap H_4^\flat$, where $H_4^\flat$ is the strict transform of H_4 on $X^\flat$. Then we have

$$K_{X^\sharp} + (p \circ q)_*^{-1}\Delta + \frac{4}{3}E^\sharp + F = (p \circ q)^*(K_X + \Delta),$$

where $E^\sharp$ is the strict transform of E on $X^\sharp$ and F is the q-exceptional divisor on $X^\sharp$. By construction, the support of $(p \circ q)_*^{-1}\Delta + E^\sharp + F$ is a simple normal crossing divisor on $X^\sharp$. We can easily check that $p \circ q(F) = L$. By using $p \circ q\colon X^\sharp \to X$, we can construct another quasi-log resolution $\tilde{f}\colon (\tilde{Y}, B_{\tilde{Y}}) \to X'$ of $[X', \omega']$ such that $\tilde{f}\colon \tilde{Y} \to X'$ is surjective. In particular, $\tilde{Y}$ is reducible and there exists an irreducible component of $\tilde{Y}$ which is dominant onto L.

Example 4.22. Let M be a smooth variety and let D_1 and D_2 be prime divisors on M such that $D_1 + D_2$ is a simple normal crossing divisor with $D_1 \neq D_2$ and $D_1 \cap D_2 \neq \emptyset$. We consider the normal pair $(M, D_1 + 2D_2)$. Then the pair $[M, K_M + D_1 + 2D_2]$ naturally becomes a quasi-log scheme as explained in Example 4.11. We put $X := D_1 + 2D_2$. Then, by adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]), $[X, \omega]$ is a quasi-log scheme, where $\omega := (K_M + D_1 + 2D_2)|_X$. More precisely, we put $Y := D_1$ and consider $K_Y + B_Y := (K_M + D_1 + 2D_2)|_Y$.

Then $f\colon (Y, B_Y) \to X$ gives a natural quasi-log scheme structure on $[X, \omega]$ by adjunction, where $f\colon Y \to X$ is a natural closed embedding. We note that $f\colon Y \to X$ is not surjective in this case. We put $X^\dagger := D_1$ and consider $\omega^\dagger := (K_M + D_1 + 2D_2)|_{X^\dagger}$. Then $[X^\dagger, \omega^\dagger]$ has a natural quasi-log scheme structure. We can see $f'\colon (Y, B_Y) \to X^\dagger$ as a quasi-log resolution of $[X^\dagger, \omega^\dagger]$, where f' is the identity morphism of $Y = X^\dagger$. We note that $\mathcal{I}_{\mathrm{Nqlc}(X^\dagger,\omega^\dagger)} = \mathcal{I}_{\mathrm{Nqlc}(X,\omega)}$ obviously holds.

Lemma 4.23 is almost obvious by definition.

Lemma 4.23. *Let*

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

be a quasi-log scheme and let B be an effective $\mathbb{R}$-Cartier divisor on X, that is, a finite $\mathbb{R}_{>0}$-linear combination of effective Cartier divisors on X. Let X' be the union of $\mathrm{Nqlc}(X, \omega)$ and all qlc centers of $[X, \omega]$ contained in $\mathrm{Supp}\, B$. Assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. We put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $f'' = f|_{Y''}$. We further assume that

$$\big(Y'', B_{Y''} + (f'')^*B\big)$$

is a globally embedded simple normal crossing pair. Then

$$\big(X, \omega + B, f''\colon (Y'', B_{Y''} + (f'')^*B) \to X\big)$$

is a quasi-log scheme.

Proof. Since $K_Y + B_Y \sim_{\mathbb{R}} f^*\omega$, we have $K_{Y''} + B_{Y''} \sim_{\mathbb{R}} (f'')^*\omega$. Therefore, $K_{Y''} + B_{Y''} + (f'')^*B \sim_{\mathbb{R}} (f'')^*(\omega + B)$ holds true. By the proof of adjunction (see the idea of the proof of Theorem 4.6 (i) and the proof of [**F11**, Theorem 6.3.5 (i)]), we have

$$\mathcal{I}_{X'} = f''_*\mathcal{O}_{X''}(\lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''}),$$

where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X. Note that the following key inequality

$$\lceil -(B_{Y''}+(f'')^*B)^{<1}\rceil - \lfloor (B_{Y''}+(f'')^*B)^{>1}\rfloor \leq \lceil -(B^{<1}_{Y''})\rceil - \lfloor B^{>1}_{Y''}\rfloor - Y'|_{Y''}$$

holds. Therefore, we put

$$\begin{aligned}\mathcal{I}_{\mathrm{Nqlc}(X,\omega+B)} &:= f''_*\mathcal{O}_{Y''}(\lceil -(B_{Y''} + (f'')^*B)^{<1}\rceil - \lfloor (B_{Y''} + (f'')^*B)^{>1}\rfloor) \\ &\subset \mathcal{I}_{X'} \subset \mathcal{O}_X\end{aligned}$$

and define the closed subscheme $\mathrm{Nqlc}(X, \omega + B)$ of X by $\mathcal{I}_{\mathrm{Nqlc}(X,\omega+B)}$. Then

$$\big(X, \omega + B, f''\colon (Y'', B_{Y''} + (f'')^*B) \to X\big)$$

is a quasi-log scheme. Let W be a reduced and irreducible subscheme of X. As usual, we say that W is a qlc stratum of $[X, \omega + B]$ when W is not contained in $\mathrm{Nqlc}(X, \omega + B)$ and is the f''-image of a stratum of

$(Y'', B_{Y''} + (f'')^*B)$. By construction, we have $X' \subset \mathrm{Nqlc}(X, \omega + B)$. We note that $(X, \omega + B, f''\colon (Y'', B_{Y''} + (f'')^*B) \to X)$ coincides with $(X, \omega, f\colon (Y, B_Y) \to X)$ outside $\mathrm{Supp}\, B$. q.e.d.

By using Lemma 4.23, we can prove the following lemma.

Lemma 4.24. *Let $[X, \omega]$ be a quasi-log scheme and let G be an effective $\mathbb{R}$-Cartier divisor on X, that is, a finite $\mathbb{R}_{>0}$-linear combination of effective Cartier divisors on X. Then, for every $0 < \varepsilon \ll 1$, $[X, \omega + \varepsilon G]$ naturally becomes a quasi-log scheme such that $\mathrm{Nqklt}(X, \omega + \varepsilon G) = \mathrm{Nqklt}(X, \omega)$ holds. More precisely, $\mathcal{I}_{\mathrm{Nqklt}(X, \omega+\varepsilon G)} = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}$ holds.*

Note that Lemma 4.24 is almost obvious for normal pairs by the definition of multiplier ideal sheaves.

Proof of Lemma 4.24. Let $f\colon (Y, B_Y) \to X$ be a proper morphism from a globally embedded simple normal crossing pair (Y, B_Y) as in Definition 4.2. Let X' be the union of $\mathrm{Nqlc}(X, \omega)$ and all qlc centers of $[X, \omega]$ contained in $\mathrm{Supp}\, G$. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35], we may assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35] again, we may further assume that the union of all strata of (Y, B_Y) mapped to $\mathrm{Nqklt}(X, \omega)$ by f, which is denoted by Z', is a union of some irreducible components of Y. By construction, $Y' \leq Z'$ obviously holds. As in Lemma 4.23, we put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $f'' = f|_{Y''}$. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35], we further assume that $(Y'', (f'')^*G + \mathrm{Supp}\, B_{Y''})$ is a globally embedded simple normal crossing pair. By Lemma 4.23, we know that

$$\big(X, \omega + \varepsilon G, f''\colon (Y'', B_{Y''} + \varepsilon (f'')^*G) \to X\big)$$

is a quasi-log scheme for every $\varepsilon > 0$. We put $Z'' = Y - Z'$, $K_{Z''} + B_{Z''} = (K_Y + B_Y)|_{Z''}$, and $h = f|_{Z''}$. Thus, by the proof of adjunction (see the idea of the proof of Theorem 4.6 (i) and the proof of [**F11**, Theorem 6.3.5 (i)]), we have

$$\mathcal{I}_{\mathrm{Nqklt}(X,\omega)} = h_*\mathcal{O}_{Z''}(\lceil -(B^{<1}_{Z''})\rceil - \lfloor B^{>1}_{Z''}\rfloor - Z'|_{Z''}).$$

We note that

$$\lceil -(B^{<1}_{Z''})\rceil - \lfloor B^{>1}_{Z''}\rfloor - Z'|_{Z''} = \lfloor B_{Z''}\rfloor$$

holds by definition. On the other hand, by the proof of adjunction again (see the idea of the proof of Theorem 4.6 (i) and the proof of [**F11**, Theorem 6.3.5 (i)]),

$$\begin{aligned} &\mathcal{I}_{\mathrm{Nqklt}(X,\omega+\varepsilon G)} \\ &= h_*\mathcal{O}_{Z''}(\lceil -(B_{Z''} + \varepsilon h^*G)^{<1}\rceil - \lfloor (B_{Z''} + \varepsilon h^*G)^{>1}\rfloor - (Z' - Y')|_{Z''}) \end{aligned}$$

for every $0 < \varepsilon \ll 1$. By direct calculation, for $0 < \varepsilon \ll 1$,

$$\begin{aligned}&\lceil -(B_{Z''} + \varepsilon h^*G)^{<1}\rceil - \lfloor (B_{Z''} + \varepsilon h^*G)^{>1}\rfloor - (Z' - Y')|_{Z''}\\ &= -\lfloor B_{Z''}\rfloor\\ &= \lceil -(B_{Z''}^{<1})\rceil - \lfloor B_{Z''}^{>1}\rfloor - Z'|_{Z''}.\end{aligned}$$

Hence we obtain

$$\mathcal{I}_{\mathrm{Nqklt}(X,\omega+\varepsilon G)} = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)}.$$

This means that

$$\bigl(X, \omega + \varepsilon G, f'' \colon (Y'', B_{Y''} + \varepsilon (f'')^*G) \to X\bigr)$$

is a quasi-log scheme with

$$\mathrm{Nqklt}(X, \omega + \varepsilon G) = \mathrm{Nqklt}(X, \omega)$$

for $0 < \varepsilon \ll 1$. We finish the proof of Lemma 4.24. q.e.d.

We need the following lemma in order to reduce some problems to the case where quasi-log schemes have $\mathbb{Q}$-structures.

Lemma 4.25. *Let $(X, \omega, f \colon (Y, B_Y) \to X)$ be a quasi-log scheme. Then we obtain a $\mathbb{Q}$-divisor D_i on Y, a $\mathbb{Q}$-line bundle ω_i on X, and a positive real number r_i for $1 \le i \le k$ such that*

(i) $\sum_{i=1}^{k} r_i = 1$,
(ii) $\operatorname{Supp} D_i = \operatorname{Supp} B_Y$, $D_i^{=1} = B_Y^{=1}$, $\lceil -(D_i^{<1})\rceil = \lceil -(B_Y^{<1})\rceil$, *and* $\lfloor D_i^{>1}\rfloor = \lfloor B_Y^{>1}\rfloor$ *for every i,*
(iii) $\omega = \sum_{i=1}^{k} r_i \omega_i$ *and* $B_Y = \sum_{i=1}^{k} r_i D_i$, *and*
(iv) $(X, \omega_i, f \colon (Y, D_i) \to X)$ *is a quasi-log scheme with* $K_Y + D_i \sim_{\mathbb{Q}} f^*\omega_i$ *for every i.*

We note that

$$\mathrm{Nqlc}(X, \omega_i) = \mathrm{Nqlc}(X, \omega)$$

holds for every i. We also note that W is a qlc stratum of $[X, \omega]$ if and only if W is a qlc stratum of $[X, \omega_i]$ for every i.

Proof. Without loss of generality, we may assume that ω is an $\mathbb{R}$-line bundle. We put $B_Y = \sum_j b_j B_j$, where B_j is a simple normal crossing divisor on Y for every j, $b_{j_1} \neq b_{j_2}$ for $j_1 \neq j_2$, and $\operatorname{Supp} B_{j_1}$ and $\operatorname{Supp} B_{j_2}$ have no common irreducible components for $j_1 \neq j_2$. We may assume that $b_j \in \mathbb{R} \setminus \mathbb{Q}$ for $1 \le j \le l$ and $b_j \in \mathbb{Q}$ for $j \ge l+1$. We put $\omega = \sum_{p=1}^{m} a_p \omega_p$, where $a_p \in \mathbb{R}$ and ω_p is a line bundle on X for every p. We can write

$$K_Y + B_Y = \sum_{p=1}^{m} a_p f^*\omega_p$$

in $\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$. We consider the following linear map

$$\psi \colon \mathbb{R}^{l+m} \longrightarrow \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$$

defined by

$$\psi(x_1, \ldots, x_{l+m}) = \sum_{\alpha=1}^{m} x_\alpha f^*\omega_\alpha - \sum_{\beta=1}^{l} x_{m+\beta} B_\beta.$$

We note that ψ is defined over $\mathbb{Q}$. By construction,

$$\mathcal{A} := \psi^{-1}\left(K_Y + \sum_{j\geq l+1} b_j B_j\right)$$

is a nonempty affine subspace of $\mathbb{R}^{l+m}$ defined over $\mathbb{Q}$. We put

$$P := (a_1, \ldots, a_m, b_1, \ldots, b_l) \in \mathcal{A}.$$

We can take $P_1, \ldots, P_k \in \mathcal{A} \cap \mathbb{Q}^{l+m}$ and $r_1, \ldots, r_k \in \mathbb{R}_{>0}$ such that $\sum_{i=1}^{k} r_i = 1$ and $\sum_{i=1}^{k} r_i P_i = P$ in $\mathcal{A}$. Note that we can make P_i arbitrary close to P for every i. So we may assume that P_i is sufficiently close to P for every i. For each P_i, we obtain

$$K_Y + D_i \sim_{\mathbb{Q}} f^*\omega_i \tag{4.1}$$

which satisfies (ii) by using ψ. By construction, (i) and (iii) hold. By (4.1) and (ii),

$$(X, \omega_i, f\colon (Y, D_i) \to X)$$

is a quasi-log scheme with the desired properties for every i. Therefore, we get (iv). q.e.d.

5. Proof of Theorem 1.9

In this section, we prove Theorem 1.9. In some sense, Theorem 1.9 is a generalization of [**FLh**, Theorem 1.1].

Proof of Theorem 1.9. Let $f\colon (Y, B_Y) \to X$ be a proper surjective morphism from a quasi-projective globally embedded simple normal crossing pair (Y, B_Y) as in Definition 4.2 (see Theorem 4.5). By [**F11**, Proposition 6.3.1], we may assume that Y is quasi-projective and that the union of all strata of (Y, B_Y) mapped to $\mathrm{Nqklt}(X, \omega)$, which is denoted by Y'', is a union of some irreducible components of Y. We put $Y' = Y - Y''$ and $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} Y' & \overset{\iota}{\hookrightarrow} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V & \underset{p}{\longrightarrow} & X \end{array}$$

where $\iota\colon Y' \to Y$ is a natural closed immersion and

$$Y' \xrightarrow{f'} V \xrightarrow{p} X$$

is the Stein factorization of $f \circ \iota \colon Y' \to X$. By construction, $\iota \colon Y' \to Y$ is an isomorphism over the generic point of X. By construction again, the natural map $\mathcal{O}_V \to f'_*\mathcal{O}_{Y'}$ is an isomorphism and every stratum of Y' is dominant onto V. Therefore, p is birational.

Claim 1. *V is normal.*

Proof of Claim 1. Let $\pi \colon V^n \to V$ be the normalization. Since every stratum of Y' is dominant onto V, there exists a closed subset Σ of Y' such that $\operatorname{codim}_{Y'} \Sigma \geq 2$ and that $\pi^{-1} \circ f' \colon Y' \dashrightarrow V^n$ is a morphism on $Y' \setminus \Sigma$. Let $\widetilde{Y}$ be the graph of $\pi^{-1} \circ f' \colon Y' \dashrightarrow V^n$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{q} & Y' \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f'} \\ V^n & \xrightarrow[\pi]{} & V \end{array}$$

where q and $\widetilde{f}$ are natural projections. Note that $q \colon \widetilde{Y} \to Y'$ is an isomorphism over $Y' \setminus \Sigma$ by construction. Since Y' is a simple normal crossing divisor on a smooth variety, Y' satisfies Serre's S_2 condition. Hence, by $\operatorname{codim}_{Y'} \Sigma \geq 2$, the natural map $\mathcal{O}_{Y'} \to q_*\mathcal{O}_{\widetilde{Y}}$ is an isomorphism. Therefore, the composition

$$\mathcal{O}_V \to \pi_*\mathcal{O}_{V^n} \to \pi_*\widetilde{f}_*\mathcal{O}_{\widetilde{Y}} = f'_*q_*\mathcal{O}_{\widetilde{Y}} \simeq \mathcal{O}_V$$

is an isomorphism. Thus we have $\mathcal{O}_V \simeq \pi_*\mathcal{O}_{V^n}$. This implies that V is normal. q.e.d.

Therefore, $p \colon V \to X$ is nothing but the normalization $\nu \colon Z \to X$. So we have the following commutative diagram.

$$\begin{array}{ccc} Y' & \overset{\iota}{\hookrightarrow} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[\nu]{} & X \end{array}$$

Claim 2. *The natural map*

$$\alpha \colon \mathcal{O}_Z \to f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil)$$

is an isomorphism outside $\nu^{-1}\operatorname{Nqlc}(X, \omega)$.

Proof of Claim 2. Note that $\nu \colon Z \to X$ is an isomorphism over $X \setminus \operatorname{Nqklt}(X, \omega)$ by Lemma 4.8. Moreover, $f' \colon Y' \to Z$ is nothing but $f \colon Y \to X$ over $Z \setminus \nu^{-1}\operatorname{Nqklt}(X, \omega)$ by construction. Therefore, α is an isomorphism outside $\nu^{-1}\operatorname{Nqklt}(X, \omega)$. By replacing X with $X \setminus \operatorname{Nqlc}(X, \omega)$, we may assume that $\operatorname{Nqlc}(X, \omega) = \emptyset$. Hence the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$ is an isomorphism. Therefore, we have $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. Since Z is normal and $f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1}) \rceil)$ is torsion-free, it is sufficient to see that α is an isomorphism in codimension one. Let P be any prime divisor on Z such that $P \subset \nu^{-1}\,\mathrm{Nqklt}(X, \omega)$. We note that every fiber of f is connected by $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. Then, by construction, there exists an irreducible component of $B^{=1}_{Y'}$ which maps onto P. Therefore, the effective divisor $\lceil -(B^{<1}_{Y'})\rceil$ does not contain the whole fiber of f' over the generic point of P. Thus, α is an isomorphism at the generic point of P. This means that α is an isomorphism. q.e.d.

We put $\mathcal{S} := f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor - Y''|_{Y'})$. Then we have:

Claim 3. *$\mathcal{S}$ is an ideal sheaf on Z.*

Proof of Claim 3. By definition, $\mathcal{S}$ is a torsion-free coherent sheaf on Z. By the proof of [**F11**, Theorem 6.3.5 (i)] (see also the idea of the proof of Theorem 4.6 (i)), we have

$$\nu_*\mathcal{S} = f_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor - Y''|_{Y'}) = \mathcal{I}_{\mathrm{Nqklt}(X,\omega)} \subset \mathcal{O}_X.$$

Since ν is finite,

$$\nu^*\nu_*\mathcal{S} \to \mathcal{S}$$

is surjective. This implies that $\mathcal{S}$ is an ideal sheaf on Z. q.e.d.

We put $\mathcal{T} := f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor)$. Then we have:

Claim 4. *$\mathcal{T}$ is an ideal sheaf on Z.*

Proof of Claim 4. Outside $\nu^{-1}\,\mathrm{Nqlc}(X, \omega)$, it is obvious that

$$\mathcal{T} = f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil)$$

holds. Therefore, we obtain $\mathcal{T} = \mathcal{O}_Z$ outside $\nu^{-1}\,\mathrm{Nqlc}(X, \omega)$ by Claim 2. Since $\mathcal{T}$ is torsion-free and Z is normal, it is sufficient to show that $\mathcal{T}$ is an ideal sheaf in codimension one. Let Q be any prime divisor on X such that $Q \subset \mathrm{Nqlc}(X, \omega)$. We take a prime divisor P on Z such that $\nu(P) = Q$.

If $\lceil -(B^{<1}_{Y'})\rceil$ does not contain the whole fiber of f' over the generic point of P, then the natural map

$$\alpha\colon \mathcal{O}_Z \to f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil)$$

is an isomorphism at the generic point of P since the natural map $\mathcal{O}_Z \to f'_*\mathcal{O}_{Y'}$ is an isomorphism by construction. Then $f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor)$ is an ideal sheaf at the generic point of P.

If $\lceil -(B^{<1}_{Y'})\rceil$ contains the whole fiber of f' over the generic point of P, then $\mathcal{S} = \mathcal{T}$ holds over the generic point of P because $\lceil -(B^{<1}_{Y'})\rceil$ and $Y''|_{Y'}$ have no common irreducible components. Therefore, $\mathcal{T}$ is an ideal sheaf at the generic point of P by Claim 3

Hence $\mathcal{T}$ is an ideal sheaf on Z. This is what we wanted. q.e.d.

By construction,

$$K_{Y'}+B_{Y'}\sim_{\mathbb{R}} f'^*\nu^*\omega$$

obviously holds. We can define $\mathrm{Nqlc}(Z,\nu^*\omega)$ by the ideal sheaf

$$f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor)$$

(see Claim 4). Hence

$$\big(Z,\nu^*\omega, f'\colon (Y',B_{Y'})\to Z\big)$$

naturally becomes a quasi-log scheme. By Claim 3 and its proof and [**F11**, Propositions 6.3.1 and 6.3.2],

$$\mathcal{I}_{\mathrm{Nqklt}(Z,\nu^*\omega)}=f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil - \lfloor B^{>1}_{Y'}\rfloor - Y''|_{Y'})$$

satisfies

$$\nu_*\mathcal{I}_{\mathrm{Nqklt}(Z,\nu^*\omega)}=\mathcal{I}_{\mathrm{Nqklt}(X,\omega)}.$$

Hence

$$\big(Z,\nu^*\omega, f'\colon (Y',B_{Y'})\to Z\big)$$

is a quasi-log scheme with the desired properties. q.e.d.

6. On basic slc-trivial fibrations

In this section, we quickly explain *basic slc-trivial fibrations*. For the details, see [**F14**] and [**FFL**]. Let us start with the definition of *potentially nef divisors*.

Definition 6.1 (Potentially nef divisors, see [**F14**, Definition 2.5])**.** Let X be a normal variety and let D be a divisor on X. If there exist a completion $X^\dagger$ of X, that is, $X^\dagger$ is a complete normal variety and contains X as a dense Zariski open set, and a nef divisor $D^\dagger$ on $X^\dagger$ such that $D=D^\dagger|_X$, then D is called a *potentially nef* divisor on X. A finite $\mathbb{Q}_{>0}$-linear (resp. $\mathbb{R}_{>0}$-linear) combination of potentially nef divisors is called a *potentially nef* $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor).

It is convenient to use *b-divisors* to explain several results on basic slc-trivial fibrations. Here we do not repeat the definition of b-divisors. For the details, see [**C**, 2.3.2 b-divisors] and [**F14**, Section 2].

Definition 6.2 (Canonical b-divisors)**.** Let X be a normal variety and let ω be a top rational differential form of X. Then (ω) defines a b-divisor $\mathbf{K}$. We call $\mathbf{K}$ the *canonical b-divisor* of X.

Definition 6.3 ($\mathbb{Q}$-Cartier closures)**.** The $\mathbb{Q}$-*Cartier closure* of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on a normal variety X is the $\mathbb{Q}$-b-divisor $\overline{D}$ with trace

$$\overline{D}_Y=f^*D,$$

where $f\colon Y\to X$ is a proper birational morphism from a normal variety Y.

We use the following definition in order to state the main result of [**F14**].

Definition 6.4 ([**F14**, Definition 2.12]). Let X be a normal variety. A $\mathbb{Q}$-b-divisor $\mathbf{D}$ of X is *b-potentially nef* (resp. *b-semi-ample*) if there exists a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$, that is, $\mathbf{D}$ is the $\mathbb{Q}$-Cartier closure of $\mathbf{D}_{X'}$, and that $\mathbf{D}_{X'}$ is potentially nef (resp. semi-ample). A $\mathbb{Q}$-b-divisor $\mathbf{D}$ of X is *$\mathbb{Q}$-b-Cartier* if there is a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

Roughly speaking, a basic slc-trivial fibration is a canonical bundle formula for simple normal crossing pairs.

Definition 6.5 (Simple normal crossing pairs). We say that the pair (X, B) is a *simple normal crossing pair* if (X, B) is Zariski locally a globally embedded simple normal crossing pair at any point $x \in X$. Let (X, B) be a simple normal crossing pair and let $\nu\colon X^\nu \to X$ be the normalization. We define Θ by

$$K_{X^\nu} + \Theta = \nu^*(K_X + B),$$

that is, Θ is the sum of the inverse images of B and the singular locus of X. Then a *stratum* of (X, B) is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ).

We note that a globally embedded simple normal crossing pair is obviously a simple normal crossing pair by definition. We also note that the definition of strata of (X, B) in Definition 6.5 coincides with the one in Definition 4.1 when (X, B) is a globally embedded simple normal crossing pair.

Remark 6.6. Let (X, B) be a simple normal crossing pair. A *stratum* of X means a stratum of $(X, 0)$. Let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. Then we can easily check that W is a stratum of X if and only if W is an irreducible component of $X_{i_1} \cap \cdots \cap X_{i_k}$ for some $\{i_1, \ldots, i_k\} \subset I$.

We introduce the notion of basic slc-trivial fibrations.

Definition 6.7 (Basic slc-trivial fibrations, see [**F14**, Definition 4.1]). A *pre-basic slc-trivial fibration* $f\colon (X, B) \to Y$ consists of a projective surjective morphism $f\colon X \to Y$ and a simple normal crossing pair (X, B) satisfying the following properties:

(1) Y is a normal variety,
(2) every stratum of X is dominant onto Y and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$,
(3) B is a $\mathbb{Q}$-divisor such that $B = B^{\leq 1}$ holds over the generic point of Y, and

(4) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that

$$K_X + B \sim_{\mathbb{Q}} f^*D.$$

If a pre-basic slc-trivial fibration $f\colon (X, B) \to Y$ also satisfies

(5) $\operatorname{rank} f_*\mathcal{O}_X(\lceil -(B^{<1})\rceil) = 1$,

then it is called a *basic slc-trivial fibration*.

If X is irreducible and (X, B) is sub kawamata log terminal (resp. sub log canonical) over the generic point of Y in Definition 6.7, then it is a klt-trivial fibration (resp. an lc-trivial fibration). For the details of lc-trivial fibrations, see [**F9**], [**FG2**], and so on.

In order to define discriminant $\mathbb{Q}$-b-divisors and moduli $\mathbb{Q}$-b-divisors for basic slc-trivial fibrations, we need the notion of *induced* (*pre-*)*basic slc-trivial fibrations*.

Definition 6.8 (Induced (pre-)basic slc-trivial fibrations, see [**F14**, 4.3])**.** Let $f\colon (X, B) \to Y$ be a (pre-)basic slc-trivial fibration and let $\sigma\colon Y' \to Y$ be a generically finite surjective morphism from a normal variety Y'. Then we have an *induced* (*pre-*)*basic slc-trivial fibration* $f'\colon (X', B_{X'}) \to Y'$, where $B_{X'}$ is defined by $\mu^*(K_X + B) = K_{X'} + B_{X'}$, with the following commutative diagram:

$$\begin{CD} (X', B_{X'}) @>{\mu}>> (X, B) \\ @V{f'}VV @VV{f}V \\ Y' @>>{\sigma}> Y, \end{CD}$$

where X' coincides with $X \times_Y Y'$ over a nonempty Zariski open set of Y'. More precisely, $(X', B_{X'})$ is a simple normal crossing pair with a morphism $X' \to X \times_Y Y'$ that is an isomorphism over a nonempty Zariski open set of Y' such that X' is projective over Y' and that every stratum of X' is dominant onto Y'.

Now we are ready to define *discriminant* $\mathbb{Q}$*-b-divisors* and *moduli* $\mathbb{Q}$*-b-divisors* for basic slc-trivial fibrations.

Definition 6.9 (Discriminant and moduli $\mathbb{Q}$-b-divisors, see [**F14**, 4.5])**.** Let $f\colon (X, B) \to Y$ be a (pre-)basic slc-trivial fibration as in Definition 6.7. Let P be a prime divisor on Y. By shrinking Y around the generic point of P, we assume that P is Cartier. We set

$$b_P = \max \left\{ t \in \mathbb{Q} \;\middle|\; \begin{array}{l} (X^\nu, \Theta + t\nu^*f^*P) \text{ is sub log canonical} \\ \text{over the generic point of } P \end{array} \right\},$$

where $\nu\colon X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + B)$, that is, Θ is the sum of the inverse images of B and the singular locus of X, and set

$$B_Y=\sum_P(1-b_P)P,$$

where P runs over prime divisors on Y. Then it is easy to see that B_Y is a well-defined $\mathbb{Q}$-divisor on Y and is called the *discriminant* $\mathbb{Q}$*-divisor* of $f\colon (X,B)\to Y$. We set

$$M_Y=D-K_Y-B_Y$$

and call M_Y the *moduli* $\mathbb{Q}$*-divisor* of $f\colon (X,B)\to Y$. By definition, we have

$$K_X+B\sim_{\mathbb{Q}} f^*(K_Y+B_Y+M_Y).$$

Let $\sigma\colon Y'\to Y$ be a proper birational morphism from a normal variety Y' and let $f'\colon (X', B_{X'})\to Y'$ be an induced (pre-)basic slc-trivial fibration by $\sigma\colon Y'\to Y$. We can define $B_{Y'}$, $K_{Y'}$ and $M_{Y'}$ such that $\sigma^*D=K_{Y'}+B_{Y'}+M_{Y'}$, $\sigma_*B_{Y'}=B_Y$, $\sigma_*K_{Y'}=K_Y$ and $\sigma_*M_{Y'}=M_Y$. We note that $B_{Y'}$ is independent of the choice of $(X', B_{X'})$, that is, $B_{Y'}$ is well defined. Hence there exist a unique $\mathbb{Q}$-b-divisor $\mathbf{B}$ such that $\mathbf{B}_{Y'}=B_{Y'}$ for every $\sigma\colon Y'\to Y$ and a unique $\mathbb{Q}$-b-divisor $\mathbf{M}$ such that $\mathbf{M}_{Y'}=M_{Y'}$ for every $\sigma\colon Y'\to Y$. Note that $\mathbf{B}$ is called the *discriminant* $\mathbb{Q}$*-b-divisor* and that $\mathbf{M}$ is called the *moduli* $\mathbb{Q}$*-b-divisor* associated to $f\colon (X,B)\to Y$. We sometimes simply say that $\mathbf{M}$ is the *moduli part* of $f\colon (X,B)\to Y$.

Let us see the main result of [**F14**].

Theorem 6.10 ([**F14**, Theorem 1.2])**.** *Let $f\colon (X,B)\to Y$ be a basic slc-trivial fibration and let $\mathbf{B}$ and $\mathbf{M}$ be the induced discriminant and moduli $\mathbb{Q}$-b-divisors associated to $f\colon (X,B)\to Y$ respectively. Then we have the following properties:*

(i) *$\mathbf{K}+\mathbf{B}$ is $\mathbb{Q}$-b-Cartier, where $\mathbf{K}$ is the canonical b-divisor of Y, and*
(ii) *$\mathbf{M}$ is b-potentially nef, that is, there exists a proper birational morphism $\sigma\colon Y'\to Y$ from a normal variety Y' such that $\mathbf{M}_{Y'}$ is a potentially nef $\mathbb{Q}$-divisor on Y' and that $\mathbf{M}=\overline{\mathbf{M}_{Y'}}$.*

When $\dim Y=1$ in Theorem 6.10, we have:

Theorem 6.11 ([**FFL**, Corollary 1.4])**.** *In Theorem 6.10, we further assume that $\dim Y=1$. Then the moduli $\mathbb{Q}$-divisor M_Y of $f\colon (X,B)\to Y$ is semi-ample.*

The proof of Theorems 6.10 and 6.11 heavily depends on the theory of variations of mixed Hodge structure discussed in [**FF**] (see also [**FFS**]). For some related topics, see [**F2**], [**F9**], [**FG2**], and so on.

In [**FH2**], we will generalize the framework of basic slc-trivial fibrations for $\mathbb{R}$-divisors and establish a generalization of Theorem 6.10 for $\mathbb{R}$-divisors.

7. On normal quasi-log schemes

In this section, we treat the following deep result on the structure of normal quasi-log schemes. It is a generalization of [**F14**, Theorem 1.7]. The proof of Theorem 7.1 uses Theorems 6.10 and 6.11.

Theorem 7.1. *Let $[X, \omega]$ be a quasi-log scheme such that X is a normal variety. Then there exists a projective birational morphism $p\colon X' \to X$ from a smooth quasi-projective variety X' such that*

$$K_{X'} + B_{X'} + M_{X'} = p^*\omega,$$

where $B_{X'}$ is an $\mathbb{R}$-divisor such that $\operatorname{Supp} B_{X'}$ *is a simple normal crossing divisor and that $B_{X'}^{<0}$ is p-exceptional, and $M_{X'}$ is a potentially nef $\mathbb{R}$-divisor on X'. Furthermore, we can make $B_{X'}$ satisfy $p(B_{X'}^{\geq 1}) =$* $\operatorname{Nqklt}(X, \omega)$ *set theoretically. When X is a curve, we can make $M_{X'}$ semi-ample in the above statement.*

We further assume that $[X, \omega]$ has a $\mathbb{Q}$-structure. Then we can make $B_{X'}$ and $M_{X'}$ $\mathbb{Q}$-divisors in the above statement.

Let us prove Theorem 7.1.

Proof of Theorem 7.1. We divide the proof into several steps.

Step 1. Although this step is essentially the same as the proof of Theorem 1.9, we explain it again with some remarks on $\operatorname{Nqlc}(X, \omega)$ for the reader's convenience. Let $f\colon (Y, B_Y) \to X$ be a proper surjective morphism from a quasi-projective globally embedded simple normal crossing pair (Y, B_Y) as in Definition 4.2 (see Theorem 4.5). By [**F11**, Proposition 6.3.1], we may assume that the union of all strata of (Y, B_Y) mapped to $\operatorname{Nqklt}(X, \omega)$, which is denoted by Y'', is a union of some irreducible components of Y. We put $Y' = Y - Y''$ and $K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$. By the proof of Theorem 1.9, we obtain the following commutative diagram:

$$\begin{array}{ccc} Y' & \xhookrightarrow{\iota} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & = & X \end{array}$$

where $\iota\colon Y' \to Y$ is a natural closed immersion such that the natural map $\mathcal{O}_X \to f'_*\mathcal{O}_{Y'}$ is an isomorphism and that every stratum of Y' is dominant onto X. By Theorem 1.9 and its proof,

$$\bigl(X, \omega, f'\colon (Y', B_{Y'}) \to X\bigr)$$

is a quasi-log scheme with

$$\mathcal{I}_{\operatorname{Nqklt}(X, \omega, f'\colon (Y', B_{Y'}) \to X)} = \mathcal{I}_{\operatorname{Nqklt}(X, \omega, f\colon (Y, B_Y) \to X)}.$$

We note that if

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

has a $\mathbb{Q}$-structure then it is obvious that

$$\big(X, \omega, f' \colon (Y', B_{Y'}) \to X\big)$$

also has a $\mathbb{Q}$-structure by construction. Hence, by replacing $f \colon (Y, B_Y) \to X$ with $f' \colon (Y', B_{Y'}) \to X$, we may assume that every stratum of Y is mapped onto X by f. By construction, we can easily see that

$$\mathrm{Nqlc}(X, \omega, f \colon (Y', B_{Y'}) \to X) \subset \mathrm{Nqlc}(X, \omega, f \colon (Y, B_Y) \to X)$$

holds set theoretically. However, the relationship between

$$\mathrm{Nqlc}(X, \omega, f \colon (Y', B_{Y'}) \to X)$$

and

$$\mathrm{Nqlc}(X, \omega, f \colon (Y, B_Y) \to X)$$

is not clear. We note that all we need in this proof is the fact that

$$\mathrm{Nqklt}(X, \omega, f \colon (Y', B_{Y'}) \to X) = \mathrm{Nqklt}(X, \omega, f \colon (Y, B_Y) \to X)$$

holds set theoretically.

Step 2. By Step 1, we may assume that $f \colon (Y, B_Y) \to X$ is a projective surjective morphism from a simple normal crossing pair (Y, B_Y) such that every stratum of Y is dominant onto X. By taking some more blow-ups, we may further assume that $(B_Y^h)^{=1}$ is Cartier and that every stratum of $(Y, (B_Y^h)^{=1})$ is dominant onto X (see, for example, [**BVP**, Theorem 1.4 and Section 8] and [**F13**, Lemma 2.11]).

Step 3. In this step, we treat the case where $[X, \omega]$ has a $\mathbb{Q}$-structure. We note that

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$$

is an isomorphism outside $\mathrm{Nqlc}(X, \omega)$. Hence $\mathrm{rank}\, f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil) = 1$ holds. Therefore, we can check that $f \colon (Y, B_Y) \to X$ is a basic slc-trivial fibration (see Definition 6.7). Let $\mathbf{B}$ be the discriminant $\mathbb{Q}$-b-divisor and let $\mathbf{M}$ be the moduli $\mathbb{Q}$-b-divisor associated to $f \colon (Y, B_Y) \to X$. By [**F14**, Lemma 11.2], we obtain that $\mathbf{B}_X$ is an effective $\mathbb{Q}$-divisor on X. By definition, we have $f((B_Y^v)^{\geq 1}) = \mathrm{Nqklt}(X, \omega)$. We take a projective birational morphism $p \colon X' \to X$ from a smooth quasi-projective variety X'. Let $f' \colon (Y', B_{Y'}) \to X'$ be an induced basic slc-trivial fibration with the following commutative diagram.

$$\begin{array}{ccc} (Y, B_Y) & \xleftarrow{\ q\ } & (Y', B_{Y'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \xleftarrow[\ p\ ]{} & X' \end{array}$$

By Theorem 6.10, we may assume that there exists a simple normal crossing divisor $\Sigma_{X'}$ on X' such that $\mathbf{M} = \overline{\mathbf{M}_{X'}}$, $\mathrm{Supp}\, \mathbf{M}_{X'}$ and $\mathrm{Supp}\, \mathbf{B}_{X'}$ are contained in $\Sigma_{X'}$, and that every stratum of $(Y', \mathrm{Supp}\, B_{Y'}^h)$ is smooth over $X' \setminus \Sigma_{X'}$. Of course, we may assume that $M_{X'} := \mathbf{M}_{X'}$ is potentially nef by Theorem 6.10. When X is a curve, we may further assume that $M_{X'}$ is semi-ample by Theorem 6.11. We may assume that every irreducible component of $q_*^{-1}\left((B_Y^v)^{\geq 1}\right)$ is mapped onto a prime divisor in $\Sigma_{X'}$ with the aid of the flattening theorem (see [**RG**, Théorème (5.2.2)]). We put $B_{X'} := \mathbf{B}_{X'}$. In the above setup, $f'(q_*^{-1}(B_Y^v)^{\geq 1}) \subset B_{X'}^{\geq 1}$ by the definition of $\mathbf{B}$. Thus, we get $\mathrm{Nqklt}(X, \omega) \subset p(B_{X'}^{\geq 1})$. On the other hand, we can easily see that $p(B_{X'}^{\geq 1}) \subset \mathrm{Nqklt}(X, \omega)$ by definition. Therefore, $p(B_{X'}^{\geq 1}) = \mathrm{Nqklt}(X, \omega)$ holds. Since $p_* B_{X'} = \mathbf{B}_X$ and $\mathbf{B}_X$ is effective, $B_{X'}^{<0}$ is p-exceptional. Hence, $B_{X'}$ and $M_{X'}$ satisfy the desired properties. We note that $B_{X'}$ and $M_{X'}$ are obviously $\mathbb{Q}$-divisors by construction.

Step 4. In this step, we treat the general case. We first use Lemma 4.25 and get positive real numbers r_i and $(X, \omega_i, f\colon (Y, D_i) \to X)$ for $1 \leq i \leq k$ with the properties in Lemma 4.25. Then we apply the argument in Step 3 to

$$(X, \omega_i, f\colon (Y, D_i) \to X)$$

for every i. By Theorem 6.10, we can take a projective birational morphism $p\colon X' \to X$ from a smooth quasi-projective variety X' which works for

$$(X, \omega_i, f\colon (Y, D_i) \to X)$$

for every i. By summing them up with weight r_i, we get $\mathbb{R}$-divisors $B_{X'}$ and $M_{X'}$ with the desired properties.

We finish the proof of Theorem 7.1. q.e.d.

8. Proof of Theorem 1.10

In this section, we prove Theorem 1.10 as an application of Theorem 7.1. Then, by using Theorem 1.10, we prove Corollary 8.2 and Lemma 8.3, which will play an important role in Section 9. Let us start with the following elementary lemma for the proof of Theorem 1.10.

Lemma 8.1. *Let X be a quasi-projective variety and let H be an ample $\mathbb{R}$-divisor on X. Let $p\colon X' \to X$ be a projective birational morphism from a smooth quasi-projective variety X' and let F be an effective p-exceptional Cartier divisor on X' such that $-F$ is p-ample. Let M' be a potentially nef $\mathbb{R}$-divisor on X'. Then $p^*H - \varepsilon F + M'$ is ample for any $0 < \varepsilon \ll 1$.*

Proof. We can write $H = \sum_{i=0}^k a_i H_i$ such that a_i is a positive real number and H_i is an ample Cartier divisor on X for every i. If $0 < \varepsilon \ll 1$, then we can take ε_i such that $p^*H_i - \varepsilon_i F$ is ample for every i with $\sum_{i=0}^k a_i \varepsilon_i = \varepsilon$. Since M' is a potentially nef $\mathbb{R}$-divisor on X', we can construct a smooth projective completion $X^\dagger$ of X' and a nef $\mathbb{R}$-divisor $M^\dagger$ on $X^\dagger$ such that $M^\dagger|_{X'} = X'$. By taking a suitable birational modification of $X^\dagger$, we may further assume that there is an ample $\mathbb{R}$-divisor A on $X^\dagger$ such that $A|_{X'} = p^*H_0 - \varepsilon_0 F$ holds. Hence $a_0 p^*H_0 - a_0\varepsilon_0 F + M'$ is ample. Therefore, $p^*H - \varepsilon F + M'$ is ample for any $0 < \varepsilon \ll 1$. This is what we wanted. q.e.d.

Let us start the proof of Theorem 1.10.

Proof of Theorem 1.10. By Theorem 7.1, there is a projective birational morphism $p\colon X' \to X$ from a smooth quasi-projective variety X' such that

$$K_{X'} + B_{X'} + M_{X'} = p^*\omega,$$

where $B_{X'}$ is an $\mathbb{R}$-divisor on X' whose support is a simple normal crossing divisor, $B_{X'}^{<0}$ is p-exceptional, $M_{X'}$ is a potentially nef $\mathbb{R}$-divisor on X', and $p(B_{X'}^{\geq 1}) = \mathrm{Nqklt}(X, \omega)$. By taking some more blow-ups, we may further assume that there is an effective p-exceptional Cartier divisor F on X' such that $-F$ is p-ample and that $\mathrm{Supp}\, F \cup \mathrm{Supp}\, B_{X'}$ is contained in a simple normal crossing divisor on X'. Then $p^*H - \varepsilon F + M_{X'}$ is ample for any $0 < \varepsilon \ll 1$ by Lemma 8.1. We take a general effective $\mathbb{R}$-divisor G on X' such that $G \sim_{\mathbb{R}} p^*H - \varepsilon F + M_{X'}$ with $0 < \varepsilon \ll 1$, $\mathrm{Supp}\, G \cup \mathrm{Supp}\, B_{X'} \cup \mathrm{Supp}\, F$ is contained in a simple normal crossing divisor on X', and $(B_{X'} + \varepsilon F + G)^{\geq 1} = B_{X'}^{\geq 1}$ holds set theoretically. Then we have

$$\begin{aligned} K_{X'} + B_{X'} + M_{X'} + p^*H &= K_{X'} + B_{X'} + \varepsilon F + p^*H - \varepsilon F + M_{X'} \\ &\sim_{\mathbb{R}} K_{X'} + B_{X'} + \varepsilon F + G. \end{aligned}$$

We put $\Delta := p_*(B_{X'} + \varepsilon F + G)$. By construction, $K_X + \Delta \sim_{\mathbb{R}} \omega + H$. By construction again, we have

$$\mathrm{Nklt}(X, \Delta) = p\left((B_{X'} + \varepsilon F + G)^{\geq 1}\right) = p\left(B_{X'}^{\geq 1}\right) = \mathrm{Nqklt}(X, \omega)$$

set theoretically.

When $[X, \omega]$ has a $\mathbb{Q}$-structure, we can make $B_{X'}$ and $M_{X'}$ $\mathbb{Q}$-divisors by Theorem 7.1. Then it is easy to see that we can make Δ a $\mathbb{Q}$-divisor on X such that $K_X + \Delta \sim_{\mathbb{Q}} \omega + H$ when H is an ample $\mathbb{Q}$-divisor and $[X, \omega]$ has a $\mathbb{Q}$-structure by the above construction of Δ.

Finally, if X is a curve in the above argument, then $p\colon X' \to X$ is an isomorphism and $M_{X'}$ is semi-ample (see Theorem 7.1). Hence we can take Δ such that

$$K_X + \Delta \sim_{\mathbb{R}} \omega$$

with the desired properties. q.e.d.

For some related results, see [**FG1**], [**F15**], and so on. By applying Theorem 1.10 to normal pairs, we have the following useful result.

Corollary 8.2. *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let C be a log canonical center of (X, Δ) such that C is a smooth curve. Then*

$$(K_X+\Delta)|_C \sim_{\mathbb{R}} K_C+\Delta_C$$

holds for some effective $\mathbb{R}$-divisor Δ_C such that

$$\operatorname{Supp}\Delta_C^{\geq 1} = C\cap\left(\operatorname{Nlc}(X,\Delta)\cup\bigcup_{C\not\subset W} W\right),$$

where W runs over lc centers of (X, Δ) which do not contain C, holds set theoretically. When $K_X+\Delta$ is $\mathbb{Q}$-Cartier, we can make Δ_C a $\mathbb{Q}$-divisor such that

$$(K_X+\Delta)|_C \sim_{\mathbb{Q}} K_C+\Delta_C$$

in the above statement.

Proof. As we saw in Example 4.11, $[X, K_X+\Delta]$ naturally becomes a quasi-log scheme. By construction, $\operatorname{Nqlc}(X, K_X+\Delta)=\operatorname{Nlc}(X,\Delta)$, W is a qlc center of $[X, K_X+\Delta]$ if and only if W is a log canonical center of (X,Δ). Hence we can see that C is a qlc center of $[X, K_X+\Delta]$. Therefore, by adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]), $[C', (K_X+\Delta)|_{C'}]$ is a quasi-log scheme, where $C'=C\cup\operatorname{Nlc}(X,\Delta)$. By Lemma 4.20, we see that $[C, (K_X+\Delta)|_C]$ is also a quasi-log scheme such that

$$\operatorname{Nqklt}(C, (K_X+\Delta)|_C) = \operatorname{Nqklt}(C', (K_X+\Delta)|_{C'})\cap C$$

holds set theoretically. By construction, we can easily see that

$$\operatorname{Nqklt}(C', (K_X+\Delta)|_{C'})\cap C = C\cap\left(\operatorname{Nlc}(X,\Delta)\cup\bigcup_{C\not\subset W} W\right),$$

where W runs over lc centers of (X,Δ) which do not contain C, holds set theoretically (see Theorem 4.10 and Corollary 4.14). By applying Theorem 1.10 to $[C, (K_X+\Delta)|_C]$, we can find an effective $\mathbb{R}$-divisor Δ_C on C such that

$$(K_X+\Delta)|_C \sim_{\mathbb{R}} K_C+\Delta_C$$

with

$$\operatorname{Supp}\Delta_C^{\geq 1} = \operatorname{Nqklt}(C, (K_X+\Delta)|_C) = C\cap\left(\operatorname{Nlc}(X,\Delta)\cup\bigcup_{C\not\subset W} W\right).$$

Of course, if $K_X+\Delta$ is $\mathbb{Q}$-Cartier, then we can make Δ_C a $\mathbb{Q}$-divisor such that

$$(K_X+\Delta)|_C \sim_{\mathbb{Q}} K_C+\Delta_C$$

in the above statement. q.e.d.

We will use the following lemma in Section 9.

Lemma 8.3. *Let $\varphi\colon X \to Y$ be a proper surjective morphism between normal varieties such that $R^1\varphi_*\mathcal{O}_X = 0$ and that $\dim \varphi^{-1}(y) \leq 1$ holds for every closed point $y \in Y$. Let C be a projective curve on X such that $\varphi(C)$ is a point. Then*

$$C \simeq \mathbb{P}^1.$$

Let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If $C \not\subset \mathrm{Nlc}(X, \Delta)$ and

$$C \cap \left(\mathrm{Nlc}(X, \Delta) \cup \bigcup_{C \not\subset W} W \right) \neq \emptyset,$$

where W runs over lc centers of (X, Δ) which do not contain C, then the following inequality

$$-(K_X + \Delta) \cdot C \leq 1$$

holds.

Proof. In Step 1, we will prove that $C \simeq \mathbb{P}^1$ holds. In Step 2, we will prove that $-(K_X + \Delta) \cdot C \leq 1$ by Corollary 8.2.

Step 1. Although the argument in this step is well known, we will explain it in detail for the reader's convenience. Let us consider the following short exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

where $\mathcal{I}_C$ is the defining ideal sheaf of C on X. Since $\dim \varphi^{-1}(y) \leq 1$ for every $y \in Y$ by assumption, $R^2\varphi_*\mathcal{I}_C = 0$ holds. Therefore, we get the following surjection

$$R^1\varphi_*\mathcal{O}_X \to R^1\varphi_*\mathcal{O}_C \to 0.$$

By assumption, $R^1\varphi_*\mathcal{O}_X = 0$. Hence $R^1\varphi_*\mathcal{O}_C = 0$ holds. Since $\varphi(C)$ is a point by assumption, $H^1(C, \mathcal{O}_C) = 0$ holds. This means that $C \simeq \mathbb{P}^1$.

Step 2. By shrinking Y around $\varphi(C)$, we may assume that Y is quasi-projective. Let $B_1, \ldots, B_{n+1}$ be general very ample Cartier divisors on Y passing through $\varphi(C)$ with $n = \dim X$. Then it is well known that

$$\left(X, \Delta + \sum_{i=1}^{n+1} \varphi^* B_i \right)$$

is not log canonical at any point of C (see, for example, [**F6**, Lemma 13.2]) such that

$$\mathrm{Nklt}\left(X, \Delta + (1-\varepsilon) \sum_{i=1}^{n+1} \varphi^* B_i \right) = \mathrm{Nklt}(X, \Delta)$$

holds outside $\varphi^{-1}(\varphi(C))$ for every $0 < \varepsilon \leq 1$. Hence we can take $0 \leq c < 1$ such that C is a log canonical center of $(X, \Delta + \varphi^* B)$, where $B = c \sum_{i=1}^{n+1} B_i$. Since B is effective, we see that

$$C \cap \left(\mathrm{Nlc}(X, \Delta + \varphi^* B) \cup \bigcup_{C \not\subset W} W \right) \neq \emptyset,$$

where W runs over lc centers of $(X, \Delta + \varphi^* B)$ which do not contain C. By Corollary 8.2, we can take an effective $\mathbb{R}$-divisor Δ_C on C such that

$$(K_X + \Delta)|_C \sim_{\mathbb{R}} (K_X + \Delta + \varphi^* B)|_C \sim_{\mathbb{R}} K_C + \Delta_C$$

and that

$$\mathrm{Supp}\, \Delta_C^{\geq 1} = C \cap \left(\mathrm{Nlc}(X, \Delta + \varphi^* B) \cup \bigcup_{C \not\subset W} W \right) \neq \emptyset$$

holds. This implies that

$$-(K_X + \Delta) \cdot C = -\deg(K_C + \Delta_C) = 2 - \deg \Delta_C \leq 1.$$

We finish the proof of Lemma 8.3. q.e.d.

9. Proof of Theorem 1.8

In this section, we prove Theorem 1.8. Let us start with the following proposition, which is a consequence of the cone and contraction theorem for normal pairs (see [**F6**, Theorem 1.1]) with the aid of Lemma 8.3. This is essentially due to [**S**, Proposition 5.2].

Proposition 9.1 ([**S**, Proposition 5.2] and [**F15**, Proposition 7.1])**.** *Let $\pi \colon X \to S$ be a projective morphism from a normal $\mathbb{Q}$-factorial variety X onto a scheme S. Let $\Delta = \sum_i d_i \Delta_i$ be an effective $\mathbb{R}$-divisor on X, where the Δ_i's are the distinct prime components of Δ for all i, such that*

$$\left(X, \Delta' := \sum_{d_i < 1} d_i \Delta_i + \sum_{d_i \geq 1} \Delta_i \right)$$

is dlt. Assume that $(K_X + \Delta)|_{\mathrm{Nklt}(X, \Delta)}$ is nef over S. Then $K_X + \Delta$ is nef over S or there exists a non-constant morphism

$$f \colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point. More precisely, the curve C, the closure of $f(\mathbb{A}^1)$ in X, is a (possibly singular) rational curve with

$$0 < -(K_X + \Delta) \cdot C \leq 2 \dim X.$$

Moreover, if $C \cap \mathrm{Nklt}(X, \Delta) \neq \emptyset$, then we can make C satisfy a sharper estimate

$$0 < -(K_X + \Delta) \cdot C \leq 1.$$

Proof. We note that $\mathrm{Nklt}(X, \Delta)$ coincides with $(\Delta')^{=1} = \lfloor \Delta' \rfloor$, $\Delta^{\geq 1}$, and $\lfloor \Delta \rfloor$ set theoretically because (X, Δ') is dlt by assumption. It is sufficient to construct a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point with the desired properties when $K_X + \Delta$ is not nef over S. When (X, Δ) is kawamata log terminal, that is, $\lfloor \Delta \rfloor = 0$, the statement is well known (see, for example, [**F6**, Theorem 1.1], Theorem 1.12, or Corollary 12.3 below). Therefore, we may assume that (X, Δ) is not kawamata log terminal. By shrinking S suitably, we may assume that S and X are both quasi-projective. By the cone and contraction theorem for normal pairs (see [**F6**, Theorem 1.1]), we can take a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/S)$ and the associated extremal contraction morphism $\varphi := \varphi_R\colon X \to Y$ over S since $(K_X + \Delta)|_{\mathrm{Nklt}(X,\Delta)}$ is nef over S. Note that $(K_X + \Delta^{<1}) \cdot R < 0$ and $(K_X + \Delta') \cdot R < 0$ hold because $(K_X + \Delta)|_{\mathrm{Nklt}(X,\Delta)}$ is nef over S. Since $(X, \Delta^{<1})$ is kawamata log terminal and $-(K_X + \Delta^{<1})$ is φ-ample, we get $R^i\varphi_*\mathcal{O}_X = 0$ for every $i > 0$ by the relative Kawamata–Viehweg vanishing theorem (see [**F11**, Corollary 5.7.7]). By construction, $\varphi\colon \mathrm{Nklt}(X, \Delta) \to \varphi(\mathrm{Nklt}(X, \Delta))$ is finite. We have the following short exact sequence

$$0 \to \mathcal{O}_X(-\lfloor \Delta' \rfloor) \to \mathcal{O}_X \to \mathcal{O}_{\lfloor \Delta' \rfloor} \to 0.$$

Since $-\lfloor \Delta' \rfloor - (K_X + \{\Delta'\}) = -(K_X + \Delta')$ is φ-ample and $(X, \{\Delta'\})$ is kawamata log terminal, $R^i\varphi_*\mathcal{O}_X(-\lfloor \Delta' \rfloor) = 0$ holds for every $i > 0$ by the relative Kawamata–Viehweg vanishing theorem again (see [**F11**, Corollary 5.7.7]). Therefore,

$$0 \to \varphi_*\mathcal{O}_X(-\lfloor \Delta' \rfloor) \to \mathcal{O}_Y \to \varphi_*\mathcal{O}_{\lfloor \Delta' \rfloor} \to 0$$

is exact. This implies that $\mathrm{Supp}\lfloor \Delta' \rfloor = \mathrm{Supp}\, \Delta^{\geq 1}$ is connected in a neighborhood of any fiber of φ.

Case 1. Assume that φ is a Fano contraction, that is, $\dim Y < \dim X$. Then we see that $\Delta^{\geq 1}$ is φ-ample and that $\dim Y = \dim X - 1$. Note that $\mathrm{Supp}\, \Delta^{\geq 1}$ is finite over Y since no curves in $\mathrm{Supp}\, \Delta^{\geq 1}$ are contracted by φ.

Assume that there exists a closed subvariety Σ on X with $\dim \Sigma \geq 2$ such that $\varphi(\Sigma)$ is a point. Then

$$\dim \left(\Sigma \cap \mathrm{Supp}\, \Delta^{\geq 1}\right) \geq 1$$

holds since $\Delta^{\geq 1}$ is φ-ample. This is a contradiction because $\mathrm{Supp}\, \Delta^{\geq 1}$ is finite over Y. Hence we obtain that $\dim \varphi^{-1}(y) = 1$ for every closed point $y \in Y$.

Let C be any projective curve on X such that $\varphi(C)$ is a point. Then (X, Δ) is log canonical at the generic point of C, equivalently,

$C \not\subset \mathrm{Nlc}(X, \Delta)$, since $\mathrm{Supp}\,\Delta^{\geq 1}$ is finite over Y. More precisely, since $\mathrm{Supp}\,\Delta^{\geq 1} = \mathrm{Nklt}(X, \Delta)$ is finite over Y, $C \not\subset \mathrm{Nklt}(X, \Delta)$ and

$$\mathrm{Supp}\,\Delta^{\geq 1} = \mathrm{Nklt}(X, \Delta) = \left(\mathrm{Nlc}(X, \Delta) \cup \bigcup_{C \not\subset W} W\right)$$

holds, where W runs over lc centers of (X, Δ) which do not contain C. On the other hand,

$$C \cap \mathrm{Supp}\,\Delta^{\geq 1} \neq \emptyset$$

because $\Delta^{\geq 1}$ is φ-ample. Hence, by Lemma 8.3, we obtain that $C \simeq \mathbb{P}^1$ and that $-(K_X + \Delta) \cdot C \leq 1$.

By the connectedness of $\mathrm{Supp}\,\Delta^{\geq 1}$ discussed above, $C \cap \mathrm{Supp}\,\Delta^{\geq 1}$ is a point. Therefore, we can find a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point and that $0 < -(K_X + \Delta) \cdot C \leq 1$ holds, where C is the closure of $f(\mathbb{A}^1)$ in X.

Case 2. Assume that φ is a birational contraction and that the exceptional locus $\mathrm{Exc}(\varphi)$ of φ is disjoint from $\mathrm{Nklt}(X, \Delta)$. In this situation, we can find a rational curve C in a fiber of φ with $0 < -(K_X + \Delta) \cdot C \leq 2 \dim X$ by the cone theorem for kawamata log terminal pairs (see [**F6**, Theorem 1.1], Theorem 1.12, or Corollary 12.3 below). It is obviously disjoint from $\mathrm{Nklt}(X, \Delta)$. Therefore, we can take a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

such that the closure of $f(\mathbb{A}^1)$ is C.

Case 3. Assume that φ is a birational contraction and that $\mathrm{Exc}(\varphi) \cap \mathrm{Nklt}(X, \Delta) \neq \emptyset$. In this situation, as in Case 1, we see that $\Delta^{\geq 1}$ is φ-ample and that $\dim \varphi^{-1}(y) \leq 1$ for every $y \in Y$. Let C be any projective curve C on X such that $\varphi(C)$ is a point. Then, $C \cap \mathrm{Supp}\,\Delta^{\geq 1} \neq \emptyset$ holds since $\Delta^{\geq 1}$ is φ-ample, and $C \cap \mathrm{Supp}\,\Delta^{\geq 1}$ is a point by the connectedness of $\mathrm{Supp}\,\Delta^{\geq 1}$ discussed above. In particular, we obtain $C \not\subset \mathrm{Nklt}(X, \Delta)$ and

$$C \cap \mathrm{Supp}\,\Delta^{\geq 1} \neq \emptyset,$$

and

$$\mathrm{Supp}\,\Delta^{\geq 1} = \mathrm{Nklt}(X, \Delta) = \left(\mathrm{Nlc}(X, \Delta) \cup \bigcup_{C \not\subset W} W\right),$$

where W runs over lc centers of (X, Δ) which do not contain C. Hence, by Lemma 8.3, $C \simeq \mathbb{P}^1$ with $-(K_X + \Delta) \cdot C \leq 1$. Since $C \cap \mathrm{Supp}\,\Delta^{\geq 1}$ is a point, we get a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

such that $f(\mathbb{A}^1) = C \cap (X \setminus \mathrm{Nklt}(X, \Delta))$.

Therefore, we get the desired statement. q.e.d.

Let us prove Theorem 1.8 as an application of Proposition 9.1.

Proof of Theorem 1.8. By shrinking S suitably, we may assume that X and S are both quasi-projective. By Lemma 3.10, we can construct a projective birational morphism $g\colon Y \to X$ from a normal $\mathbb{Q}$-factorial variety Y satisfying (i), (ii), and (iv) in Lemma 3.10. Let us consider $\pi \circ g\colon Y \to S$. Note that $K_Y + \Delta_Y$ is not nef over S since $K_Y + \Delta_Y = g^*(K_X + \Delta)$ holds. It is obvious that $(K_Y + \Delta_Y)|_{\mathrm{Nklt}(Y,\Delta_Y)}$ is nef over S by (iv) because so is $(K_X + \Delta)|_{\mathrm{Nklt}(X,\Delta)}$. Therefore, by Proposition 9.1, we have a non-constant morphism

$$h\colon \mathbb{A}^1 \longrightarrow Y \setminus \mathrm{Nklt}(Y, \Delta_Y)$$

such that $(\pi \circ g) \circ h(\mathbb{A}^1)$ is a point and that

$$0 < -(K_Y + \Delta_Y) \cdot C_Y \leq 2 \dim Y = 2 \dim X$$

holds, where C_Y is the closure of $h(\mathbb{A}^1)$ in Y. Since $K_Y + \Delta_Y = h^*(K_X + \Delta)$ holds, g does not contract C_Y to a pont. This implies that

$$f := g \circ h\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

is a desired non-constant morphism such that $\pi \circ f(\mathbb{A}^1)$ is a point by (iv). q.e.d.

For the proof of Theorem 1.6, we prepare the following somewhat artificial statement as an application of Theorem 1.8.

Theorem 9.2. *Let $\pi\colon X \to S$ be a proper surjective morphism from a normal quasi-projective variety X onto a scheme S. Let $\mathcal{P}$ be an $\mathbb{R}$-Cartier divisor on X and let H be an ample Cartier divisor on X. Let Σ be a closed subset of X. Assume that π is not finite, $-\mathcal{P}$ is π-ample, and $\pi\colon \Sigma \to \pi(\Sigma)$ is finite. We further assume*

- *$\{\varepsilon_i\}_{i=1}^{\infty}$ is a set of positive real numbers with $\varepsilon_i \searrow 0$ for $i \nearrow \infty$, and*
- *for every i, there exists an effective $\mathbb{R}$-divisor Δ_i on X such that*
$$\mathcal{P} + \varepsilon_i H \sim_{\mathbb{R}} K_X + \Delta_i$$
and that
$$\Sigma = \mathrm{Nklt}(X, \Delta_i)$$
holds set theoretically.

Then there exists a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \Sigma$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point and that the curve C, the closure of $f(\mathbb{A}^1)$ in X, is a rational curve with

$$0 < -\mathcal{P} \cdot C \leq 2 \dim X.$$

Proof. We take an ample $\mathbb{Q}$-divisor A on X such that $-(\mathcal{P}+A)$ is π-ample. Without loss of generality, we may assume that $-(\mathcal{P}+A+\varepsilon_i H)$ is π-ample for every i because $\varepsilon_i \searrow 0$ for $i \nearrow \infty$. By assumption,

$$\mathcal{P}+\varepsilon_i H \sim_{\mathbb{R}} K_X+\Delta_i$$

with

$$\mathrm{Nklt}(X,\Delta_i)=\Sigma$$

for every i. Hence, by Theorem 1.8, there is a non-constant morphism

$$f_i\colon \mathbb{A}^1 \longrightarrow X\setminus\Sigma$$

such that $\pi\circ f_i(\mathbb{A}^1)$ is a point and that

$$0<-(K_X+\Delta_i)\cdot C_i=-(\mathcal{P}+\varepsilon_i H)\cdot C_i\leq 2\dim X,$$

where C_i is the closure of $f_i(\mathbb{A}^1)$ in X. We note that

$$0<A\cdot C_i=((\mathcal{P}+\varepsilon_i H+A)-(\mathcal{P}+\varepsilon_i H))\cdot C_i<2\dim X.$$

It follows that the curves C_i belong to a bounded family. Thus, possibly passing to a subsequence, we may assume that f_i and C_i are constant, that is, there is a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X\setminus\Sigma$$

such that $C_i=C$ for every i, where C is the closure of $f(\mathbb{A}^1)$ in X. Therefore, we get

$$0<-\mathcal{P}\cdot C=\lim_{i\to\infty}-(\mathcal{P}+\varepsilon_i H)\cdot C=\lim_{i\to\infty}-(\mathcal{P}+\varepsilon_i H)\cdot C_i\leq 2\dim X.$$

We finish the proof of Theorem 9.2. q.e.d.

10. Proof of Theorems 1.4, 1.5, and 1.6

In this section, we prove Theorems 1.4, 1.5 and 1.6. Since Theorem 1.4 is an easy consequence of Theorem 1.5 and Theorem 1.5 can be seen as a very special case of Theorem 1.6 by Example 4.11, it is sufficient to prove Theorem 1.6. Let us start with the proof of Theorem 1.6.

Proof of Theorem 1.6. We note that (i) and (ii) were already established in [**F11**, Theorem 6.7.4]. Therefore, it is sufficient to prove (iii). From Step 1 to Step 4, we will reduce the problem to the case where X is a normal variety. Then, in Step 5, we will obtain a desired non-constant morphism from $\mathbb{A}^1$ by Theorem 9.2.

Step 1. Let $\varphi_{R_j}\colon X\to Y$ be the extremal contraction associated to R_j (see Theorem 4.18 and [**F11**, Theorems 6.7.3 and 6.7.4]). We note that

$$\varphi_{R_j}\colon\ \mathrm{Nqlc}(X,\omega)\to\varphi_{R_j}\left(\mathrm{Nqlc}(X,\omega)\right)$$

is finite. By replacing $\pi\colon X\to S$ with $\varphi_{R_j}\colon X\to Y$, we may assume that $-\omega$ is π-ample and that $\overline{\mathrm{NE}}(X/S)_{-\infty}=\{0\}$.

Step 2. We take a qlc stratum W of $[X, \omega]$ such that $\pi\colon W \to \pi(W)$ is not finite and that $\pi\colon W^\dagger \to \pi(W^\dagger)$ is finite for every qlc center $W^\dagger$ of $[X, \omega]$ with $W^\dagger \subsetneq W$. We put $W' = W \cup \mathrm{Nqlc}(X, \omega)$. Then, by adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]), $[W', \omega|_{W'}]$ naturally becomes a quasi-log scheme. By replacing $[X, \omega]$ with $[W', \omega|_{W'}]$, we may further assume that $X \setminus X_{-\infty}$ is irreducible and that

$$\pi\colon \mathrm{Nqklt}(X, \omega) \to \pi\left(\mathrm{Nqklt}(X, \omega)\right)$$

is finite.

Step 3. By Lemma 4.20, we may replace X with $\overline{X \setminus X_{-\infty}}$ and assume that X is a variety. We note that the finiteness of

$$\pi\colon \mathrm{Nqklt}(X, \omega) \to \pi\left(\mathrm{Nqklt}(X, \omega)\right)$$

still holds.

Step 4. Let $\nu\colon Z \to X$ be the normalization. Then $[Z, \nu^*\omega]$ naturally becomes a quasi-log scheme by Theorem 1.9. Since $\mathrm{Nqklt}(Z, \nu^*\omega) = \nu^{-1}\mathrm{Nqklt}(X, \omega)$ by Theorem 1.9, we may assume that X is normal by replacing $[X, \omega]$ with $[Z, \nu^*\omega]$.

Step 5. By shrinking S suitably, we may further assume that X and S are both quasi-projective. Hence we have the following properties:

(a) $\pi\colon X \to S$ is a projective morphism from a normal quasi-projective variety X to a scheme S,
(b) $-\omega$ is π-ample, and
(c) $\pi\colon \Sigma \to \pi(\Sigma)$ is finite, where $\Sigma := \mathrm{Nqklt}(X, \omega)$.

Let H be an ample Cartier divisor on X and let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a set of positive real numbers such that $\varepsilon_i \searrow 0$ for $i \nearrow \infty$. Then, by Theorem 1.10, we have:

(d) there exists an effective $\mathbb{R}$-divisor Δ_i on X such that

$$K_X + \Delta_i \sim_{\mathbb{R}} \omega + \varepsilon_i H$$

with

$$\mathrm{Nklt}(X, \Delta_i) = \Sigma$$

for every i.

Thus, by Theorem 9.2, we have a desired non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nqklt}(X, \omega).$$

We finish the proof of Theorem 1.6. q.e.d.

As we already mentioned above, Theorem 1.5 is a very special case of Theorem 1.6.

Proof of Theorem 1.5. By Example 4.11, $[X, K_X + \Delta]$ naturally becomes a quasi-log scheme. Then, by Theorem 1.6, the desired cone theorem holds for (X, Δ). q.e.d.

Theorem 1.4 easily follows from Theorem 1.5.

Proof of Theorem 1.4. Since (X, Δ) is Mori hyperbolic by assumption, there is no $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X)$ that is rational and relatively ample at infinity (see Theorem 1.5). By assumption, $(K_X + \Delta)|_{\mathrm{Nlc}(X,\Delta)}$ is nef. Hence the subcone $\overline{\mathrm{NE}}(X)_{-\infty}$ is included in $\overline{\mathrm{NE}}(X)_{(K_X+\Delta)\geq 0}$. This implies that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta)\geq 0}$$

holds by Theorem 1.5. Thus $K_X + \Delta$ is nef. q.e.d.

The author thinks that the proof of Theorems 1.4, 1.5 and 1.6 shows that the framework of quasi-log schemes established in [**F11**, Chapter 6] and [**F14**] is very powerful and useful even for the study of normal pairs.

11. Ampleness criterion for quasi-log schemes

The main purpose of this section is to establish the following ampleness criterion for quasi-log schemes. Then we will see that Theorem 1.11 is a very special case of Theorem 11.1.

Theorem 11.1 (Ampleness criterion for quasi-log schemes)**.** *Let $[X, \omega]$ be a quasi-log scheme and let $\pi\colon X \to S$ be a projective morphism between schemes. Assume that $\omega|_{\mathrm{Nqlc}(X,\omega)}$ is ample over S and that ω is log big over S with respect to $[X, \omega]$. We further assume that there is no non-constant morphism*

$$f\colon \mathbb{A}^1 \longrightarrow U$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point, where U is any open qlc stratum of $[X, \omega]$. Then ω is ample over S.

Let us treat a special case of Theorem 11.1.

Theorem 11.2. *Let $[X, \omega]$ be a quasi-log scheme such that X is a normal variety. Let $\pi\colon X \to S$ be a projective morphism onto a scheme S. Assume that $\omega|_{\mathrm{Nqklt}(X,\omega)}$ is ample over S and that there is no non-constant morphism*

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nqklt}(X, \omega)$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point. We further assume that ω is big over S. Then ω is ample over S.

Proof. We divide the proof into several small steps.

Step 1. By Lemma 4.25, we can obtain quasi-log schemes

$$(X, \omega_i, f\colon (Y, D_i) \to X)$$

for $1 \leq i \leq k$ with the following properties:

(a) $[X, \omega_i]$ has a $\mathbb{Q}$-structure for every i,
(b) $\mathrm{Nqlc}(X, \omega_i) = \mathrm{Nqlc}(X, \omega)$ holds for every i,
(c) W is a qlc stratum of $[X, \omega]$ if and only if W is a qlc stratum of $[X, \omega_i]$ for every i, and
(d) there exist positive real numbers r_i for $1 \leq i \leq k$ such that $\omega = \sum_{i=1}^{k} r_i \omega_i$ with $\sum_{i=1}^{k} r_i = 1$.

By construction, we can make ω_i sufficiently close to ω (see the proof of Lemma 4.25). Therefore, we may assume that $\omega_i|_{\mathrm{Nqklt}(X, \omega_i)}$ is ample over S for every i by (b) and (c). Thus $[X, \omega_i]$ satisfies all the assumptions for $[X, \omega]$ in Theorem 11.2. Hence, by replacing $[X, \omega]$ with $[X, \omega_i]$, it is sufficient to prove the ampleness of ω under the extra assumption that $[X, \omega]$ has a $\mathbb{Q}$-structure by (a) and (d).

Step 2. By assumption and Theorem 1.6 (iii), ω is nef over S. Since $\omega|_{\mathrm{Nqklt}(X, \omega)}$ is ample over S by assumption, ω is nef and log big over S with respect to $[X, \omega]$. Therefore, by [**F10**, Theorem 1.1], we obtain that ω is semi-ample over S. Hence $m\omega$ gives a birational contraction morphism $\Phi \colon X \to Y$ between normal varieties over S, where m is a sufficiently large and divisible positive integer.

Step 3. In this step, we will get a contradiction under the assumption that Φ is not an isomorphism.

By shrinking S, we may assume that S, X, and Y are quasi-projective. By construction,

$$\Phi \colon \mathrm{Nqklt}(X, \omega) \to \Phi(\mathrm{Nqklt}(X, \omega))$$

is finite. Since Φ is birational and Y is quasi-projective, we can take an effective Cartier divisor G on X such that $-G$ is Φ-ample. By Lemma 4.24, for $0 < \varepsilon \ll 1$, $[X, \omega + \varepsilon G]$ is a quasi-log scheme such that

$$\mathrm{Nqklt}(X, \omega + \varepsilon G) = \mathrm{Nqklt}(X, \omega)$$

holds. By the cone theorem (see Theorem 1.6 (iii)), we can find a non-constant morphism

$$f \colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nqklt}(X, \omega + \varepsilon G) = X \setminus \mathrm{Nqklt}(X, \omega)$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point and that $0 < -(\omega + \varepsilon G) \cdot C \leq 2 \dim X$ holds, where C is the closure of $f(\mathbb{A}^1)$ in X. This is a contradiction.

Hence Φ is an isomorphism. Therefore, we obtain that ω is ample over S. This is what we wanted. q.e.d.

Once we know Theorem 11.2, it is not difficult to prove Theorem 11.1.

Proof of Theorem 11.1. By Theorem 1.6 (iii), ω is nef and log big over S with respect to $[X, \omega]$. We put

$$[X_0, \omega_0] := [X, \omega]$$

and

$$[X_{i+1}, \omega_{i+1}] := [\mathrm{Nqklt}(X_i, \omega_i), \omega_i|_{\mathrm{Nqklt}(X_i,\omega_i)}]$$

for $i \geq 0$. Then there exists $k \geq 0$ such that

$$\mathrm{Nqklt}(X_k, \omega_k) = \mathrm{Nqlc}(X_k, \omega_k) = \mathrm{Nqlc}(X, \omega).$$

We note that $\mathrm{Nqlc}(X, \omega)$ may be empty. By assumption, $\omega_k|_{\mathrm{Nqklt}(X_k,\omega_k)}$ is ample over S. We want to show by inverse induction on i that ω_i is ample over S. Therefore, it is sufficient to prove the following claim.

Claim. *Let $[X, \omega]$ be a quasi-log scheme and let $\pi\colon X \to S$ be a projective morphism between schemes such that $\omega|_{\mathrm{Nqklt}(X,\omega)}$ is ample over S and that ω is nef and log big over S with respect to $[X, \omega]$. Assume that there is no non-constant morphism*

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nqklt}(X, \omega)$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point. Then ω is ample over S.

Proof of Claim. By adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]), we may assume that $\overline{X \setminus X_{-\infty}}$ is irreducible. By Lemma 4.20, we may further assume that X is irreducible. Then, by Theorem 1.9, we can reduce the problem to the case where X is a normal variety. Hence ω is ample over S by Theorem 11.2. q.e.d.

As we have already mentioned above, by applying Claim inductively, we obtain the desired relative ampleness of $\omega = \omega_0$. q.e.d.

We close this section with the proof of Theorem 1.11.

Proof of Theorem 1.11. By Example 4.11, $[X, K_X + \Delta]$ naturally becomes a quasi-log scheme. We apply Theorem 11.1 to $[X, K_X + \Delta]$. Then we obtain that $K_X + \Delta$ is ample. This is what we wanted. q.e.d.

The author knows no proof of Theorem 1.11 that does not use the framework of quasi-log schemes. Note that a similar result for dlt pairs was already established in [**F7**, Theorem 5.1], whose proof is much easier than that of Theorem 1.11 and depends on the basepoint-free theorem of Reid–Fukuda type for dlt pairs (see [**F1**, Theorem 0.1]). We recommend the interested reader to see [**F7**, Theorem 5.1] and [**F1**, Theorem 0.1].

12. Proof of Theorems 1.12 and 1.13

In this section, we prove Theorems 1.12 and 1.13, and explain an application for normal pairs. For the basic properties of uniruled varieties, see [**Ko1**, Chapter IV. 1]. Let us start with the following lemma, which is a generalization of [**Ka**, Lemma].

Lemma 12.1. *Let $[X, \omega]$ be a quasi-log scheme and let $\varphi\colon X \to W$ be a projective morphism between schemes. Let P be an arbitrary closed point of W. Let E be a positive-dimensional irreducible component of $\varphi^{-1}(P)$ such that $E \not\subset X_{-\infty}$ and let $\nu\colon \overline{E} \to E$ be the normalization. Then, for every ample $\mathbb{R}$-divisor H on $\overline{E}$, there exists an effective $\mathbb{R}$-divisor $\Delta_{\overline{E},H}$ on $\overline{E}$ such that*

$$\nu^*\omega + H \sim_{\mathbb{R}} K_{\overline{E}} + \Delta_{\overline{E},H}$$

holds. Therefore,

$$\mathcal{A}^{\dim E-1} \cdot \omega \cdot E \geq (\nu^*\mathcal{A})^{\dim E-1} \cdot K_{\overline{E}}$$

holds for every φ-ample line bundle $\mathcal{A}$ on X.

In the above statement, if $[X, \omega]$ has a $\mathbb{Q}$-structure and H is an ample $\mathbb{Q}$-divisor on $\overline{E}$, then we can make $\Delta_{\overline{E},H}$ an effective $\mathbb{Q}$-divisor on $\overline{E}$ with

$$\nu^*\omega + H \sim_{\mathbb{Q}} K_{\overline{E}} + \Delta_{\overline{E},H}.$$

Proof. Our approach is different from Kawamata's in [**Ka**]. A key ingredient of this proof is Theorem 1.10.

Step 1. If E is a qlc stratum of $[X, \omega]$, then we put $B = 0$ and go to Step 3.

Step 2. By Step 1, we may assume that E is not a qlc stratum of $[X, \omega]$. Without loss of generality, we may assume that W is quasi-projective by shrinking W around P. Let $B_1, \dots, B_{n+1}$ be general very ample Cartier divisors on W passing through P with $n = \dim X$. Let $f\colon (Y, B_Y) \to X$ be a proper morphism from a globally embedded simple normal crossing pair (Y, B_Y) as in Definition 4.2. Let X' be the union of $X_{-\infty} = \mathrm{Nqlc}(X, \omega)$ and all qlc strata of $[X, \omega]$ mapped to P by φ. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35], we may assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. As usual, we put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $f'' = f|_{Y''}$. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35] again, we may further assume that

$$\left(Y'', (f'')^*\varphi^* \sum_{i=1}^{n+1} B_i + \mathrm{Supp}\, B_{Y''}\right)$$

is a globally embedded simple normal crossing pair. By [**F11**, Lemma 6.3.13], we can take $0 < c < 1$ such that

$$f''\left(\left(B_{Y''} + c(f'')^*\varphi^* \sum_{i=1}^{n+1} B_i\right)^{>1}\right) \not\supset E$$

and that there exists an irreducible component G of

$$\left(B_{Y''}+c(f'')^*\varphi^*\sum_{i=1}^{n+1}B_i\right)^{=1}$$

with $f''(G)=E$. By Lemma 4.23, we obtain that

$$\left(X,\omega+B,f''\colon (Y'',B_{Y''}+(f'')^*B)\to X\right),$$

where $B=\varphi^*\left(c\sum_{i=1}^{n+1}B\right)$, is a quasi-log scheme such that E is a qlc stratum of this quasi-log scheme.

Step 3. We put $E'=E\cup \mathrm{Nqlc}(X,\omega+B)$. Then, by adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]), $[E',(\omega+B)|_{E'}]$ is a quasi-log scheme. By Lemma 4.20, $[E,(\omega+B)|_E]$ is also a quasi-log scheme. We note that $(\omega+B)|_E\sim_{\mathbb R}\omega|_E$ since $\varphi(E)=P$. Hence $[E,\omega|_E]$ is a quasi-log scheme. By Theorem 1.9, $[\overline{E},\nu^*\omega]$ naturally becomes a quasi-log scheme. By Theorem 1.10, there exists an effective $\mathbb R$-divisor $\Delta_{\overline{E},H}$ on $\overline{E}$ such that

$$\nu^*\omega+H\sim_{\mathbb R}K_{\overline{E}}+\Delta_{\overline{E},H}.$$

This implies that

$$\begin{aligned}(\nu^*\mathcal A)^{\dim E-1}\cdot(\nu^*\omega+H)\cdot\overline{E}&=(\nu^*\mathcal A)^{\dim E-1}(K_{\overline{E}}+\Delta_{\overline{E},H})\\&\geq(\nu^*\mathcal A)^{\dim E-1}\cdot K_{\overline{E}}.\end{aligned}$$

Since the above inequality holds for every ample $\mathbb R$-divisor H on $\overline{E}$, we obtain

$$\mathcal A^{\dim E-1}\cdot\omega\cdot E=(\nu^*\mathcal A)^{\dim E-1}\cdot\nu^*\omega\cdot\overline{E}\geq(\nu^*\mathcal A)^{\dim E-1}\cdot K_{\overline{E}}.$$

This is what we wanted. By the above proof, it is easy to see that we can make $\Delta_{\overline{E},H}$ an effective $\mathbb Q$-divisor on $\overline{E}$ if $[X,\omega]$ has a $\mathbb Q$-structure and H is an ample $\mathbb Q$-divisor on $\overline{E}$.

We finish the proof of Lemma 12.1. q.e.d.

Remark 12.2. In the proof of [**Ka**, Lemma], Kawamata uses a relative Kawamata–Viehweg vanishing theorem for projective bimeromorphic morphisms between complex analytic spaces. His argument does not work for quasi-log schemes.

Let us prove Theorem 1.12.

Proof of Theorem 1.12. In this proof, we will freely use the notation of Lemma 12.1.

Case 1. We will treat the case where $\dim E=1$.

We take an ample $\mathbb{Q}$-divisor H on $\overline{E}$ such that $-(\nu^*\omega + H)$ is still ample. Then, by Lemma 12.1, $-K_{\overline{E}}$ is ample since $\Delta_{\overline{E},H}$ is effective. This means that $\overline{E} \simeq \mathbb{P}^1$. By Lemma 12.1 again, we have

$$0 < -\omega \cdot E \leq -\deg K_{\overline{E}} = 2.$$

Case 2. We will treat the case where $\dim E \geq 2$.

We take a φ-ample line bundle $\mathcal{A}$ such that $\nu^*\mathcal{A}$ is very ample. We put $C = D_1 \cap \cdots \cap D_{\dim E - 1}$, where D_i is a general member of $|\nu^*\mathcal{A}|$ for every i. Then C is a smooth irreducible curve on $\overline{E}$ such that C lies in the smooth locus of $\overline{E}$. By Lemma 12.1, we obtain

$$C \cdot K_{\overline{E}} \leq \mathcal{A}^{\dim E - 1} \cdot \omega \cdot E < 0$$

because $-\omega$ is φ-ample. We note that

$$\begin{aligned} 0 > \nu^*\omega \cdot C &= \nu^*\omega \cdot (\nu^*\mathcal{A})^{\dim E - 1} \cdot \overline{E} \\ &= \omega \cdot \mathcal{A}^{\dim E - 1} \cdot E \\ &\geq (\nu^*\mathcal{A})^{\dim E - 1} \cdot K_{\overline{E}} \\ &= C \cdot K_{\overline{E}}. \end{aligned}$$

Therefore, for any given point $x \in C$, there exists a rational curve Γ on $\overline{E}$ passing through x with

$$\begin{aligned} 0 < -\nu^*\omega \cdot \Gamma &\leq 2 \dim \overline{E} \cdot \frac{-\nu^*\omega \cdot C}{-K_{\overline{E}} \cdot C} \\ &\leq 2 \dim \overline{E}. \end{aligned}$$

This is essentially due to Miyaoka–Mori (see [**MM**]). We note that $\overline{E}$ is not always smooth but it is smooth in a neighborhood of C. Hence we can use the argument of [**MM**]. For the details, see [**Ko1**, Chapter II. 5.8 Theorem]. Thus, E is covered by rational curves $\ell := \nu_*\Gamma$ with

$$0 < -\omega \cdot \ell \leq 2 \dim E.$$

Hence, by [**Ko1**, Chapter IV. 1.4 Proposition–Definition], E is uniruled. We finish the proof of Theorem 1.12. q.e.d.

We prove Theorem 1.13.

Proof of Theorem 1.13. Since φ_R is the contraction morphism associated to R,

$$\varphi_R \colon \operatorname{Nqlc}(X, \omega) \to \varphi_R(\operatorname{Nqlc}(X, \omega))$$

is finite. We apply Theorem 1.12 to $\varphi_R \colon X \to W$, we can take a rational curve ℓ on X such that $\varphi_R(\ell)$ is a point with

$$0 < -\omega \cdot \ell \leq 2d.$$

We finish the proof of Theorem 1.13. q.e.d.

We explain an application of Theorems 1.12 and 1.13 for normal pairs, which is a generalization of [**Ka**, Theorem 1].

Corollary 12.3. *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X \to S$ be a projective morphism between schemes. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S)$ that are rational and relatively ample at infinity. Let $\varphi_R\colon X \to W$ be the contraction morphism over S associated to R. We put*

$$d = \min_E \dim E,$$

where E runs over positive-dimensional irreducible components of $\varphi_R^{-1}(P)$ for all $P \in W$. Then R is spanned by a (possibly singular) rational curve ℓ with

$$0 < -(K_X + \Delta) \cdot \ell \leq 2d.$$

Furthermore, if φ_R is birational and (X, Δ) is kawamata log terminal, then R is spanned by a (possibly singular) rational curve ℓ with

$$0 < -(K_X + \Delta) \cdot \ell < 2d.$$

Let V be an irreducible component of the degenerate locus

$$\{x \in X \mid \varphi_R \text{ is not an isomorphism at } x\}$$

of φ_R. Then V is uniruled.

Proof. We divide the proof into three small steps.

Step 1. By Example 4.11, $[X, K_X + \Delta]$ naturally becomes a quasi-log scheme. By applying Theorem 1.13 to $[X, K_X + \Delta]$, we see that R is spanned by a rational curve ℓ with

$$0 < -(K_X + \Delta) \cdot \ell \leq 2d.$$

Step 2. When (X, Δ) is kawamata log terminal and φ_R is a birational contraction, we take a d-dimensional irreducible component E of $\varphi_R^{-1}(P)$ for some $P \in W$. By shrinking W around P, we may assume that W is affine. Since φ_R is birational, there exists an effective $\mathbb{Q}$-divisor G on X such that $(X, \Delta + G)$ is kawamata log terminal and that $-G$ is φ_R-ample. By applying Theorem 1.12 to $[X, K_X + \Delta + G]$, we see that E is covered by rational curves ℓ with

$$0 < -(K_X + \Delta + G) \cdot \ell \leq 2d.$$

Since $-G$ is φ_R-ample, we have

$$0 < -(K_X + \Delta) \cdot \ell < 2d.$$

This implies that R is spanned by a rational curve ℓ with

$$0 < -(K_X + \Delta) \cdot \ell < 2d$$

when (X, Δ) is kawamata log terminal and φ_R is birational.

Step 3. From now on, we will check that V is uniruled. We shrink W around the generic point of $\varphi_R(V)$ and assume that W is quasi-projective. Then we take a sufficiently ample Cartier divisor H on W such that $-(K_X+\Delta)+\varphi_R^*H$ is ample. By Theorem 1.12, $V\cap\varphi_R^{-1}(P)$ is covered by rational curves ℓ of $-((K_X+\Delta)+\varphi_R^*H)$-degree at most $2\dim V$ for every $P\in\varphi_R(V)\subset W$. We take a suitable projective completion $\overline{X}$ of X and apply [**Ko1**, Chapter IV. 1.4 Proposition–Definition]. Then we obtain that V is uniruled.

We finish the proof of Corollary 12.3. q.e.d.

13. Proof of Theorem 1.14

In this section, we prove Theorem 1.14. Let us start with the following definition.

Definition 13.1 ([**F11**, Definition 6.8.1])**.** Let $[X,\omega]$ be a quasi-log scheme and let $\pi\colon X\to S$ be a projective morphism between schemes. If $-\omega$ is ample over S, then $[X,\omega]$ is called a *relative quasi-log Fano scheme over* S. When S is a point, we simply say that $[X,\omega]$ is a *quasi-log Fano scheme.*

We recall an easy consequence of the vanishing theorem (see Theorem 4.6 (ii)), which is missing in [**F11**, Section 6.8].

Lemma 13.2. *Let $[X,\omega]$ be a quasi-log scheme and let $\pi\colon X\to S$ be a proper morphism between schemes with $\pi_*\mathcal{O}_X\simeq\mathcal{O}_S$. Assume that $-\omega$ is nef and log big over S with respect to $[X,\omega]$. Then $X_{-\infty}\cap\pi^{-1}(P)$ is connected for every closed point $P\in S$.*

Proof. By Theorem 4.6 (ii), $R^1\pi_*\mathcal{I}_{X_{-\infty}}=0$. Therefore, the restriction map

$$\mathcal{O}_S\simeq\pi_*\mathcal{O}_X\to\pi_*\mathcal{O}_{X_{-\infty}}$$

is surjective. This implies that $X_{-\infty}\cap\pi^{-1}(P)$ is connected for every closed point $P\in S$. q.e.d.

Lemma 13.2 should have been stated in [**F11**, Lemma 6.8.3], which plays an important role in this section. The main ingredient of the proof of Theorem 1.14 is the following theorem.

Theorem 13.3 ([**Z**, Theorem 1], [**HM**], and [**BP**, Corollary 1.4])**.** *Let X be a normal projective variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Assume that $-(K_X+\Delta)$ is ample. Then X is rationally chain connected modulo* $\mathrm{Nklt}(X,\Delta)$.

Proof. We take an effective $\mathbb{Q}$-divisor Δ' on X such that $K_X+\Delta'$ is $\mathbb{Q}$-Cartier, $-(K_X+\Delta')$ is ample, and $\mathrm{Nklt}(X,\Delta')=\mathrm{Nklt}(X,\Delta)$ holds. If $\mathrm{Nklt}(X,\Delta')=\emptyset$, that is, (X,Δ') is kawamata log terminal, then X is rationally connected by [**Z**, Theorem 1]. In particular, X is rationally chain connected by Lemma 2.12. When $\mathrm{Nklt}(X, \Delta') \neq \emptyset$, by applying [**BP**, Corollary 1.4] to (X, Δ'), we obtain that for any general point x of X there exists a rational curve R_x passing through x and intersecting $\mathrm{Nklt}(X, \Delta')$. By [**Ko1**, Chapter II. 2.4 Corollary], for every $x \in X$, we can find a chain of rational curves R_x such that $x \in R_x$ and $R_x \cap \mathrm{Nklt}(X, \Delta') \neq \emptyset$. Hence X is rationally chain connected modulo $\mathrm{Nklt}(X, \Delta)$. We note that if $-(K_X + \Delta)$ is an ample $\mathbb{Q}$-divisor then the proof of [**BP**, Theorem 1.2 and Corollary 1.4] becomes much simpler than the general case. Hence we obtain that X is rationally chain connected modulo $\mathrm{Nklt}(X, \Delta)$. q.e.d.

We prepare one useful lemma.

Lemma 13.4. *Let $[X, \omega]$ be a projective quasi-log canonical pair such that $\mathrm{Nqklt}(X, \omega) = \emptyset$, $-\omega$ is ample, and X is connected. Then X is rationally connected. Hence X is rationally chain connected.*

Proof. By Lemma 4.8 and [**F11**, Theorem 6.3.11 (i)], X is a normal variety. By Theorem 1.10, we can find an effective $\mathbb{R}$-divisor Δ on X such that $-(K_X + \Delta)$ is ample with $\mathrm{Nklt}(X, \Delta) = \emptyset$. Hence X is rationally connected by [**Z**, Theorem 1] (see the proof of Theorem 13.3). q.e.d.

By the framework of quasi-log schemes, we can prove the following lemma as a generalization of Theorem 13.3 without any difficulties. We note that if $\mathrm{Nqlc}(X, \omega) = \emptyset$ in Lemma 13.5 then it is nothing but [**F15**, Theorem 1.7]. For semi-log canonical Fano pairs, we recommend the reader to see [**FLw**].

Lemma 13.5. *Let $[X, \omega]$ be a projective quasi-log scheme such that X is connected. Assume that $-\omega$ is ample. Then X is rationally chain connected modulo $X_{-\infty}$.*

Proof. As in the proof of Theorem 11.1, we put

$$[X_0, \omega_0] := [X, \omega]$$

and

$$[X_{i+1}, \omega_{i+1}] := [\mathrm{Nqklt}(X_i, \omega_i), \omega_i|_{\mathrm{Nqklt}(X_i, \omega_i)}]$$

for $i \geq 0$. Then there exists $k \geq 0$ such that

$$\mathrm{Nqklt}(X_k, \omega_k) = \mathrm{Nqlc}(X_k, \omega_k) = \mathrm{Nqlc}(X, \omega) = X_{-\infty}.$$

We note that if $X_{-\infty} = \emptyset$, that is, $[X, \omega]$ is quasi-log canonical, then X_k is the unique minimal qlc stratum of $[X, \omega]$ by [**F11**, Theorem 6.8.3 (ii)]. By applying Lemma 13.4 to $[X_k, \omega_k]$, we obtain that X_k is rationally connected when $X_{-\infty} = \emptyset$. We want to show by inverse induction on i that X_{i+1} is rationally chain connected modulo $X_{-\infty} = \mathrm{Nqlc}(X, \omega)$. Note that we want to show that X is rationally chain connected modulo X_k when $X_{-\infty} = \emptyset$. We also note that X_i is connected by Lemma 13.2

and [**F11**, Theorem 6.8.3]. Hence it is sufficient to prove the following claim.

Claim. *Let $[X, \omega]$ be a quasi-log scheme such that X is connected,* $\mathrm{Nqklt}(X, \omega) \neq \emptyset$*, and $-\omega$ is ample. Then X is rationally chain connected modulo* $\mathrm{Nqklt}(X, \omega)$.

Proof of Claim. By adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]) and [**F11**, Theorem 6.8.3], we may assume that $\overline{X \setminus X_{-\infty}}$ is irreducible. We note that every qlc stratum of $[X, \omega]$ intersects with $\mathrm{Nqklt}(X, \omega)$ (see [**F11**, Theorem 6.8.3]). By Lemma 4.20, we may further assume that X itself is irreducible. Then, by Theorem 1.9, we can further reduce the problem to the case where X is a normal variety. Then, by Theorem 1.10, we can take an ample $\mathbb{R}$-divisor H on X such that $-(\omega + H)$ is still ample and that

$$K_X + \Delta \sim_{\mathbb{R}} \omega + H$$

holds for some effective $\mathbb{R}$-divisor Δ on X with

$$\mathrm{Nklt}(X, \Delta) = \mathrm{Nqklt}(X, \omega).$$

By applying Theorem 13.3 to (X, Δ), we obtain that X is rationally chain connected modulo $\mathrm{Nqklt}(X, \omega)$. We finish the proof of Claim. q.e.d.

By using Claim inductively, we can check that X is rationally chain connected modulo $X_{-\infty} = \mathrm{Nqlc}(X, \omega)$. q.e.d.

Let us prove Theorem 1.14.

Proof of Theorem 1.14. When $\pi^{-1}(P) \cap X_{-\infty} = \emptyset$, we may assume that $X_{-\infty} = \emptyset$ by shrinking X around $\pi^{-1}(P)$. We divide the proof into several steps.

Step 1. Let X_0 be the union of $X_{-\infty}$ and all qlc strata of $[X, \omega]$ contained in $\pi^{-1}(P)$. By Lemma 13.2 and [**F11**, Theorem 6.8.3], $X_0 \cap \pi^{-1}(P)$ is connected.

Case 1. If $X_0 \neq X_{-\infty}$, then $[X_0, \omega_0]$, where $\omega_0 = \omega|_{X_0}$, is a quasi-log scheme by adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]). Let us consider $X_0^\dagger = \overline{X_0 \setminus \mathrm{Nqlc}(X_0, \omega_0)}$. Then $[X_0^\dagger, \omega_0^\dagger]$, where $\omega_0^\dagger = \omega|_{X_0^\dagger}$, is a quasi-log scheme by Lemma 4.20. By construction, $-\omega_0^\dagger$ is ample since $\pi(X_0^\dagger) = P$. Therefore, by Lemma 13.5, $X_0^\dagger$ is rationally chain connected modulo $\mathrm{Nqlc}(X_0^\dagger, \omega_0^\dagger)$. This means that $X_0 \cap \pi^{-1}(P)$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$.

Case 2. If $X_0 = X_{-\infty}$, that is, there is no qlc stratum of $[X, \omega]$ contained in $\pi^{-1}(P)$, then $X_0 \cap \pi^{-1}(P)$ is obviously rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$ because $X_0 \cap \pi^{-1}(P) = \pi^{-1}(P) \cap X_{-\infty}$. Note that $X_0 = X_{-\infty} = \emptyset$ may happen.

Hence $\pi^{-1}(P)$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$ when $\pi^{-1}(P) \subset X_0$. Thus, from now on, we may assume that $\pi^{-1}(P) \not\subset X_0$.

Step 2. Without loss of generality, we may assume that S is quasi-projective by shrinking S around P. We take general very ample Cartier divisors $B_1, \ldots, B_{n+1}$ passing through P with $n = \dim X$. Let

$$f\colon (Y, B_Y) \to X$$

be a proper morphism from a globally embedded simple normal crossing pair (Y, B_Y) as in Definition 4.2. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35], we may assume that the union of all strata of (Y, B_Y) mapped to X_0 by f, which is denoted by Y', is a union of some irreducible components of Y. As usual, we put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $f'' = f|_{Y''}$. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35] again, we may further assume that

$$\left(Y'', (f'')^*\pi^*\sum_{i=1}^{n+1} B_i + \operatorname{Supp} B_{Y''}\right)$$

is a globally embedded simple normal crossing pair. By [**F11**, Lemma 6.3.13], we can take $0 < c_1 < 1$ such that

$$f''\left(\left(B_{Y''} + c_1 (f'')^*\pi^*\sum_{i=1}^{n+1} B_i\right)^{>1}\right) = X_0$$

holds set theoretically and that there exists an irreducible component G of

$$\left(B_{Y''} + c_1 (f'')^*\pi^*\sum_{i=1}^{n+1} B_i\right)^{=1}$$

with $f''(G) \not\subset X_0$. By Lemma 4.23,

$$\left(X, \omega + c_1 B, f''\colon (Y'', B_{Y''} + c_1 (f'')^* B) \to X\right),$$

where $B = \pi^*\left(\sum_{i=1}^{n+1} B\right)$, is a quasi-log scheme.

Let X_1 be the union of $\operatorname{Nqlc}(X, \omega + c_1 B)$ and all qlc strata of $[X, \omega + c_1 B]$ contained in $\pi^{-1}(P)$. By construction, $\operatorname{Nqlc}(X, \omega + c_1 B) = X_0$ holds set theoretically. Therefore, by Case 1 in Step 1, $X_1 \cap \pi^{-1}(P)$ is rationally chain connected modulo $X_0 \cap \pi^{-1}(P)$. We note that by Step 1 $X_0 \cap \pi^{-1}(P)$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$. This means that $X_1 \cap \pi^{-1}(P)$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$.

Step 3. By repeating the argument in Step 2, we can construct a finite increasing sequence of positive real numbers

$$0 < c_1 < \cdots < c_k < 1$$

and closed subschemes

$$X_1 \subsetneq \cdots \subsetneq X_k$$

of X with the following properties:

(a) $[X_i, \omega_i]$ is a quasi-log scheme, where $\omega_i = (\omega + c_i B)|_{X_i}$, for every i,
(b) $\mathrm{Nqlc}(X_{i+1}, \omega_{i+1}) = X_i$ holds set theoretically for every i,
(c) $\pi^{-1}(P) \subset X_k$ holds, and
(d) $X_{i+1} \cap \pi^{-1}(P)$ is rationally chain connected modulo $X_i \cap \pi^{-1}(P)$ for every i.

Hence we obtain that $\pi^{-1}(P) = \pi^{-1}(P) \cap X_k$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$.

We finish the proof of Theorem 1.14. q.e.d.

14. Towards Conjecture 1.15

In this final section, we treat several results related to Conjecture 1.15. This section needs some deep results on the theory of minimal models for higher-dimensional algebraic varieties. Let us start with the following special case of the flip conjecture II.

Conjecture 14.1 (Termination of flips). *Let (X, Δ) be a $\mathbb{Q}$-factorial klt pair and let $\pi\colon X \to S$ be a projective surjective morphism between normal quasi-projective varieties such that $K_X + \Delta$ is not pseudo-effective over S. Let*

$$(X, \Delta) =: (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i) \dashrightarrow \cdots$$

be a sequence of flips over S starting from (X, Δ). Then it terminates after finitely many steps.

In this section, we establish the following theorem, which is a precise version of Theorem 1.16.

Theorem 14.2 (see Theorem 1.16)**.** *Assume that Conjecture 14.1 holds true in dimension at most* $\dim \pi^{-1}(P)$*. Then Conjecture 1.15 holds true.*

For the proof of Theorem 14.2, we prepare a variant of Theorem 1.8. We need the termination of flips in this theorem.

Theorem 14.3. *Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Assume that Conjecture 14.1 holds true in* $\dim X$*. Let $\pi\colon X \to S$ be a projective morphism onto a scheme S such that $-(K_X + \Delta)$ is π-ample with* $\dim S < \dim X$*. We assume that* $\mathrm{Nklt}(X, \Delta)$ *is not empty such that*

$$\pi\colon \mathrm{Nklt}(X, \Delta) \to \pi(\mathrm{Nklt}(X, \Delta))$$

is finite. Then there exists a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \operatorname{Nklt}(X, \Delta)$$

such that $\pi \circ f(\mathbb{A}^1)$ *is a point and that the curve* C*, the closure of* $f(\mathbb{A}^1)$ *in* X*, is a* (*possibly singular*) *rational curve satisfying* $C \cap \operatorname{Nklt}(X, \Delta) \neq \emptyset$ *with*

$$0 < -(K_X + \Delta) \cdot C \leq 1.$$

Proof. By shrinking S suitably, we may assume that X and S are both quasi-projective. By Lemma 3.10, we can construct a projective birational morphism $g\colon Y \to X$ from a normal $\mathbb{Q}$-factorial variety satisfying (i), (ii), and (iv) in Lemma 3.10. Since $K_Y + \Delta_Y = g^*(K_X + \Delta)$, $(K_Y + \Delta_Y)|_{\operatorname{Nklt}(Y, \Delta_Y)}$ is nef over S by Lemma 3.10 (iv). Let us consider $\pi \circ g\colon Y \to S$. By construction, $(Y, \Delta_Y^{<1})$ is a $\mathbb{Q}$-factorial klt pair. Since $-(K_X + \Delta)$ is π-ample by assumption, $K_Y + \Delta_Y$ is not pseudo-effective over S. Hence $K_Y + \Delta_Y^{<1}$ is not pseudo-effective over S. Since $(K_Y + \Delta_Y)|_{\operatorname{Nklt}(Y, \Delta_Y)}$ is nef over S, the cone theorem

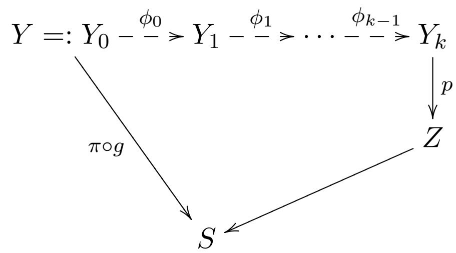
$$\overline{\mathrm{NE}}(Y/S) = \overline{\mathrm{NE}}(Y/S)_{(K_Y + \Delta_Y) \geq 0} + \sum_j R_j$$

holds by [**F6**, Theorem 1.1] (see also Theorem 1.5 (i)). Since $K_Y + \Delta_Y$ is not pseudo-effective over S, $K_Y + \Delta_Y$ is not nef over S. Hence we have a $(K_Y + \Delta_Y)$-negative extremal ray R of $\overline{\mathrm{NE}}(Y/S)$. Then we consider the contraction morphism $\varphi_R\colon Y \to W$ over S associated to R (see [**F6**, Theorem 1.1] and Theorem 4.18). We note that $-(K_Y + \Delta_Y^{<1}) \cdot R > 0$ since $(K_Y + \Delta_Y)|_{\operatorname{Nklt}(Y, \Delta_Y)}$ is nef over S. If φ_R is an isomorphism in a neighborhood of $\operatorname{Nklt}(Y, \Delta_Y)$, then we can run a minimal model program with respect to $K_Y + \Delta_Y$ over S by [**BCHM**]. Thus we run a minimal model program with respect to $K_Y + \Delta_Y$ over S. Then we have a sequence of flips and divisorial contractions

$$Y =: Y_0 \overset{\phi_0}{\dashrightarrow} Y_1 \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} Y_i \overset{\phi_i}{\dashrightarrow} \cdots$$

over S. As usual, we put $(Y_0, \Delta_{Y_0}) := (Y, \Delta_Y)$ and $\Delta_{Y_{i+1}} = \phi_{i*}\Delta_{Y_i}$ for every i.

Case 1. We assume that ϕ_i is an isomorphism in a neighborhood of $\operatorname{Nklt}(Y_i, \Delta_{Y_i})$ for every i. Then this minimal model program is a minimal model program with respect to $K_Y + \Delta_Y^{<1}$. Hence, by Conjecture 14.1, we finally get the following diagram



where ϕ_i is a flip or a divisorial contraction for every i and $p\colon Y_k \to Z$ is a Fano contraction over S. We note that $(K_{Y_k}+\Delta_{Y_k})|_{\mathrm{Nklt}(Y_k,\Delta_{Y_k})}$ is nef over S. By Case 1 in the proof of Proposition 9.1, we can find a curve $C_{Y_k}\simeq \mathbb{P}^1$ on Y_k such that $p(C_{Y_k})$ is a point, $C_{Y_k}\cap \mathrm{Nklt}(Y_k,\Delta_{Y_k})$ is a point, and $0<-(K_{Y_k}+\Delta_{Y_k})\cdot C_{Y_k}\leq 1$ holds. By using the negativity lemma, we can check that

$$-(K_Y+\Delta_Y)\cdot C_Y \leq -(K_{Y_k}+\Delta_{Y_k})\cdot C_{Y_k}\leq 1$$

holds, where C_Y is the strict transform of C_{Y_k} on Y. Note that $C_Y\cap \mathrm{Nklt}(Y,\Delta_Y)$ is a point since ϕ_i is an isomorphism in a neighborhood of $\mathrm{Nklt}(Y_i,\Delta_{Y_i})$ for every i. Therefore, $C=g(C_Y)$ is a curve on X such that $C\cap \mathrm{Nklt}(X,\Delta)$ is a point by Lemma 3.10 (iv) with $0< -(K_X+\Delta)\cdot C\leq 1$. Hence we can construct a morphism

$$f\colon \mathbb{A}^1 \longrightarrow X\setminus \mathrm{Nklt}(X,\Delta)$$

such that $f(\mathbb{A}^1)=C\cap (X\setminus \mathrm{Nklt}(X,\Delta))$. This is a desired morphism.

Case 2. We assume that there exists i_0 such that ϕ_i is an isomorphism in a neighborhood of $\mathrm{Nklt}(Y_i,\Delta_{Y_i})$ for $0\leq i< i_0$ and ϕ_{i_0} is not an isomorphism in a neighborhood of $\mathrm{Nklt}(Y_{i_0},\Delta_{Y_{i_0}})$. Then, by using Case 3 in the proof of Proposition 9.1, we can find a curve $C_{Y_{i_0}}\simeq\mathbb{P}^1$ on Y_{i_0} such that $C_{Y_{i_0}}\cap\mathrm{Nklt}(Y_{i_0},\Delta_{Y_{i_0}})$ is a point, $C_{Y_{i_0}}$ is mapped to a point on S, and $0<-(K_{Y_{i_0}}+\Delta_{Y_{i_0}})\cdot C_{Y_{i_0}}\leq 1$ holds. By the same argument as in Case 1 above, we get a desired morphism

$$f\colon \mathbb{A}^1 \longrightarrow X\setminus \mathrm{Nklt}(X,\Delta).$$

We finish the proof of Theorem 14.3. q.e.d.

By Theorem 14.3, we have:

Theorem 14.4. *In Theorem 9.2, we further assume that* $\dim S<\dim X$ *and that* $\Sigma\neq\emptyset$. *If Conjecture 14.1 holds true in* $\dim X$, *then there exists a non-constant morphism*

$$f\colon \mathbb{A}^1 \longrightarrow X\setminus \Sigma$$

such that $\pi\circ f(\mathbb{A}^1)$ *is a point and that the curve* C, *the closure of* $f(\mathbb{A}^1)$ *in* X, *is a rational curve satisfying* $C\cap\Sigma\neq\emptyset$ *with*

$$0<-\mathcal{P}\cdot C\leq 1.$$

Proof. We use Theorem 14.3 instead of Theorem 1.8. Then the proof of Theorem 9.2 implies the existence of

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \Sigma$$

with the desired properties. q.e.d.

For the proof of Theorem 14.2, we establish the following somewhat technical lemma.

Lemma 14.5. *Let $\pi\colon X \to S$ be a projective surjective morphism between normal quasi-projective varieties with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$ and $\dim S > 0$ and let $[X, \omega]$ be a quasi-log scheme such that*

$$\pi\colon \operatorname{Nqklt}(X, \omega) \to \pi(\operatorname{Nqklt}(X, \omega))$$

is finite. Let P be a closed point of S such that $\dim \pi^{-1}(P) > 0$. Then there exists an effective $\mathbb{R}$-Cartier divisor B on S such that $[X, \omega+\pi^*B]$ is a quasi-log scheme with the following properties:

(i) $\operatorname{Nqklt}(X, \omega) \subset \operatorname{Nqklt}(X, \omega + \pi^*B)$,
(ii) $\operatorname{Nqklt}(X, \omega) = \operatorname{Nqklt}(X, \omega + \pi^*B)$ *holds outside* $\pi^{-1}(P)$,
(iii) $\pi\colon \operatorname{Nqlc}(X, \omega + \pi^*B) \to \pi(\operatorname{Nqlc}(X, \omega + \pi^*B))$ *is finite, and*
(iv) *there exists a positive-dimensional qlc center of $[X, \omega + \pi^*B]$ in $\pi^{-1}(P)$.*

We further assume that $-\omega$ is π-ample. Let W be a positive-dimensional qlc center of $[X, \omega+\pi^*B]$ with $\pi(W) = P$. Let $\nu\colon W^\nu \to W$ be the normalization. Then $[W^\nu, \nu^*\omega]$ naturally becomes a quasi-log Fano scheme such that

$$\nu^{-1}\left(\operatorname{Nqklt}(X, \omega) \cap \pi^{-1}(P)\right) \subset \operatorname{Nqklt}(W^\nu, \nu^*\omega)$$

holds set theoretically.

Proof. Let $B_1, \dots, B_{n+1}$ be general very ample Cartier divisors on S passing through P with $n = \dim X$. Let $f\colon (Y, B_Y) \to X$ be a proper morphism from a globally embedded simple normal crossing pair (Y, B_Y) as in Definition 4.2. Let X' be the union of $\operatorname{Nqlc}(X, \omega)$ and all qlc centers of $[X, \omega]$ mapped to P by π. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35], we may assume that the union of all strata of (Y, B_Y) mapped to X' by f, which is denoted by Y', is a union of some irreducible components of Y. As usual, we put $Y'' = Y - Y'$, $K_{Y''} + B_{Y''} = (K_Y + B_Y)|_{Y''}$, and $f'' = f|_{Y''}$. By [**F11**, Proposition 6.3.1] and [**Ko2**, Theorem 3.35] again, we may further assume that

$$\left(Y'', (f'')^*\pi^* \sum_{i=1}^{n+1} B_i + \operatorname{Supp} B_{Y''}\right)$$

is a globally embedded simple normal crossing pair. By [**F11**, Lemma 6.3.13], we can take $0 < c < 1$ such that

(a) we have

$$f''\left(\left(B_{Y''}+c(f'')^*\pi^*\sum_{i=1}^{n+1}B_i\right)^{>1}\right)\cap\pi^{-1}(P)=\emptyset$$

or

$$\dim\left(f''\left(\left(B_{Y''}+c(f'')^*\pi^*\sum_{i=1}^{n+1}B_i\right)^{>1}\right)\cap\pi^{-1}(P)\right)=0,$$

and

(b) the following inequality

$$\dim\left(f''\left(\left(B_{Y''}+c(f'')^*\pi^*\sum_{i=1}^{n+1}B_i\right)^{=1}\right)\cap\pi^{-1}(P)\right)\geq 1$$

holds.

By Lemma 4.23, we obtain that

$$\left(X,\omega+\pi^*B, f''\colon (Y'', B_{Y''}+(f'')^*\pi^*B)\to X\right),$$

where $B=c\sum_{i=1}^{n+1}B_i$, is a quasi-log scheme. By construction, we see that (i) holds true and $\mathrm{Nqklt}(X,\omega+\pi^*B)$ coincides with $\mathrm{Nqklt}(X,\omega)$ outside $\pi^{-1}(P)$ since $B_1,\dots,B_{n+1}$ are general very ample Cartier divisors on S. Hence we have (ii). Therefore, (iii) and (iv) follow from (a) and (b), respectively.

From now on, we further assume that $-\omega$ is π-ample. As usual, we put

$$W'=W\cup\mathrm{Nqlc}(X,\omega+\pi^*B).$$

By adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]), $[W',(\omega+\pi^*B)|_{W'}]$ is a quasi-log scheme. By Lemma 4.20, $[W,(\omega+\pi^*B)|_W]$ naturally becomes a quasi-log scheme. Note that $(\pi^*B)|_W\sim_{\mathbb{R}}0$ since $\pi(W)=P$. Therefore, by replacing $(\omega+\pi^*B)|_W$ with $\omega|_W$, we see that $[W,\omega|_W]$ is a quasi-log scheme. By Theorem 1.9, $[W^\nu,\nu^*\omega]$ becomes a quasi-log scheme. We note that $-\nu^*\omega$ is ample since $\pi\circ\nu(W^\nu)=P$.

Claim. *We have*

$$\begin{aligned}&\mathrm{Nqklt}(X,\omega)\cap\pi^{-1}(P)\\&\subset\mathrm{Nqklt}(W',(\omega+\pi^*B)|_{W'})\cap\pi^{-1}(P)\\&=\mathrm{Nqklt}(W,(\omega+\pi^*B)|_W)\\&=\mathrm{Nqklt}(W,\omega|_W)\end{aligned}$$

set theoretically.

Proof of Claim. We divide the proof into several steps.

Step 1. By (iii) and Lemma 13.2, $\mathrm{Nqlc}(X,\omega+\pi^*B)\cap\pi^{-1}(P)$ is empty or a point. By [**F11**, Theorem 6.8.3 (i)], every qlc center of $[X,\omega+\pi^*B]$ in $\pi^{-1}(P)$ contains $\mathrm{Nqlc}(X,\omega+\pi^*B)\cap\pi^{-1}(P)$ when $\mathrm{Nqlc}(X,\omega+\pi^*B)\cap\pi^{-1}(P)\neq\emptyset$. When $\mathrm{Nqlc}(X,\omega+\pi^*B)\cap\pi^{-1}(P)=\emptyset$, the set of all qlc centers intersecting $\pi^{-1}(P)$ has a unique minimal element with respect to the inclusion by [**F11**, Theorem 6.8.3 (ii)].

Step 2. In this step, we will check that

$$\mathrm{Nqklt}(X,\omega)\cap\pi^{-1}(P)\subset\mathrm{Nqklt}(W',(\omega+\pi^*B)|_{W'})\cap\pi^{-1}(P)$$

holds set theoretically.

If $\mathrm{Nqklt}(X,\omega)\cap\pi^{-1}(P)=\emptyset$, then it is obvious. Hence we may assume that $\mathrm{Nqklt}(X,\omega)\cap\pi^{-1}(P)\neq\emptyset$. By assumption, $\mathrm{Nqklt}(X,\omega)\cap\pi^{-1}(P)$ is zero-dimensional. We take $Q\in\mathrm{Nqklt}(X,\omega)\cap\pi^{-1}(P)$. If Q is a qlc center of $[X,\omega]$ or $Q\in\mathrm{Nqlc}(X,\omega)$, then $Q\in\mathrm{Nqlc}(X,\omega+\pi^*B)$ by the construction of the quasi-log scheme structure of $[X,\omega+\pi^*B]$. Then we have

$$Q\in\mathrm{Nqlc}(W',(\omega+\pi^*B)|_{W'})\subset\mathrm{Nqklt}(W',(\omega+\pi^*B)|_{W'}).$$

Therefore, we have

$$Q\in\mathrm{Nqklt}(W',(\omega+\pi^*B)|_{W'})\cap\pi^{-1}(P).$$

From now on, we assume that Q is not a qlc center of $[X,\omega]$ and that $Q\notin\mathrm{Nqlc}(X,\omega)$. Then, there exists a positive-dimensional qlc center V of $[X,\omega]$ such that $Q\in V\cap\pi^{-1}(P)$. Since $\mathrm{Nqklt}(X,\omega)=\mathrm{Nqklt}(X,\omega+\pi^*B)$ holds outside $\pi^{-1}(P)$ (see (ii)), V is also a qlc center of $[X,\omega+\pi^*B]$. If $\mathrm{Nqlc}(X,\omega+\pi^*B)\cap\pi^{-1}(P)\neq\emptyset$, then $\mathrm{Nqlc}(X,\omega+\pi^*B)\cap\pi^{-1}(P)$ is a point by (iii) and Lemma 13.2. In this case, by [**F11**, Theorem 6.8.3 (i)], we have $Q\in V\cap\pi^{-1}(P)\cap\mathrm{Nqlc}(X,\omega+\pi^*B)$. Hence $Q\in\mathrm{Nqlc}(W',(\omega+\pi^*B)|_{W'})\cap\pi^{-1}(P)$. This implies that $Q\in\mathrm{Nqklt}(W',(\omega+\pi^*B)|_{W'})\cap\pi^{-1}(P)$. Thus we further assume that $\mathrm{Nqlc}(X,\omega+\pi^*B)\cap\pi^{-1}(P)=\emptyset$. By shrinking X around $\pi^{-1}(P)$, we may assume that $[X,\omega+\pi^*B]$ is quasi-log canonical. Then $Q\in V\cap W\cap\pi^{-1}(P)$ by Step 1 (see also [**F11**, Theorem 6.8.3 (ii)]). Hence $Q\in\mathrm{Nqklt}(W',(\omega+\pi^*B)|_{W'})\cap\pi^{-1}(P)$ by [**F11**, Theorem 6.3.11 (i)]. More precisely, Q is a qlc center of $[W',(\omega+\pi^*B)|_{W'}]$.

In any case, we obtain

$$\mathrm{Nqklt}(X,\omega)\cap\pi^{-1}(P)\subset\mathrm{Nqklt}(W',(\omega+\pi^*B)|_{W'})\cap\pi^{-1}(P)$$

set theoretically.

Step 3. By Step 1 and Lemma 4.20,

$$\mathrm{Nqklt}(W',(\omega+\pi^*B)|_{W'})\cap\pi^{-1}(P)=\mathrm{Nqklt}(W,(\omega+\pi^*B)|_{W})$$

holds set theoretically. By the definition of the quasi-log scheme structure of $[W,\omega|_W]$,

$$\mathrm{Nqklt}(W,(\omega+\pi^*B)|_W)=\mathrm{Nqklt}(W,\omega|_W)$$

obviously holds.

We finish the proof of Lemma 14.5. q.e.d.

Hence by Claim

$$\nu^{-1}\left(\mathrm{Nqklt}(X,\omega)\cap\pi^{-1}(P)\right)\subset \mathrm{Nqklt}(W^\nu,\nu^*\omega)$$

holds set theoretically since $\nu^{-1}(\mathrm{Nqklt}(W,\omega|_W))=\mathrm{Nqklt}(W^\nu,\nu^*\omega)$ by Theorem 1.9. q.e.d.

Let us prove Theorem 14.2, which is stronger than Theorem 1.16.

Proof of Theorem 14.2. We first use the reduction as in Steps 2, 3, and 4 in the proof of Theorem 1.6. Let us explain it more precisely for the reader's convenience.

Step 1. We take an irreducible component W of X such that $C^\dagger\subset W$. We put $X'=W\cup\mathrm{Nqlc}(X,\omega)$. Then, by adjunction (see Theorem 4.6 (i) and [**F11**, Theorem 6.3.5 (i)]), $[X',\omega'=\omega|_{X'}]$ is a quasi-log scheme. By replacing $[X,\omega]$ with $[X',\omega']$, we may assume that $\overline{X\setminus X_{-\infty}}$ is irreducible. By Lemma 4.20, we may replace X with $\overline{X\setminus X_{-\infty}}$ and assume that X is a variety. Then, by taking the normalization, we may further assume that X is a normal variety (see Theorem 1.9).

Step 2. By taking the Stein factorization, we may further assume that $\pi_*\mathcal{O}_X\simeq\mathcal{O}_S$. We put $\Sigma=\mathrm{Nqklt}(X,\omega)$. It is sufficient to find a non-constant morphism

$$f\colon \mathbb{A}^1\longrightarrow (X\setminus\Sigma)\cap\pi^{-1}(P)$$

such that the curve C, the closure of $f(\mathbb{A}^1)$ in X, is a (possibly singular) rational curve satisfying $C\cap\Sigma\neq\emptyset$ with

$$0<-\omega\cdot C\leq 1.$$

Without loss of generality, we may assume that X and S are quasi-projective by shrinking S suitably.

Step 3. By assumption, $\dim\pi^{-1}(P)>0$ and $\pi^{-1}(P)\cap\Sigma\neq\emptyset$. When $\dim S>0$, by Lemma 14.5, we take an effective $\mathbb{R}$-Cartier divisor B on S such that $[X,\omega+\pi^*B]$ is a quasi-log scheme satisfying the properties (i), (ii), (iii), and (iv) in Lemma 14.5. Then we take a positive-dimensional qlc center W of $[X,\omega+\pi^*B]$ in $\pi^{-1}(P)$ such that there is no positive-dimensional qlc center $W^\dagger\subsetneq W$ of $[X,\omega+\pi^*B]$. By Lemma 14.5, $[W^\nu,\nu^*\omega]$ naturally becomes a quasi-log Fano scheme, where $\nu\colon W^\nu\to W$ is the normalization. When $\dim S=0$, it is sufficient to put $B=0$ and $W=X$. By construction, $\mathrm{Nqklt}(W^\nu,\nu^*\omega)$ is finite. On the other hand, $\mathrm{Nqklt}(W^\nu,\nu^*\omega)$ is connected (see Lemma 13.2 and [**F11**, Theorem 6.8.3]). By Lemma 14.5, we obtain

$$\emptyset\neq\nu^{-1}\left(\Sigma\cap\pi^{-1}(P)\right)\subset\mathrm{Nqklt}(W^\nu,\nu^*\omega).$$

Hence $\mathrm{Nqklt}(W^\nu, \nu^*\omega)$ is a point such that

$$\nu^{-1}\left(\Sigma \cap \pi^{-1}(P)\right) = \mathrm{Nqklt}(W^\nu, \nu^*\omega)$$

holds set theoretically. By applying Theorems 1.10 and 14.4 to $[W^\nu, \nu^*\omega]$ as in Step 5 in the proof of Theorem 1.6, we obtain a non-constant morphism

$$h\colon \mathbb{A}^1 \longrightarrow W^\nu \setminus \mathrm{Nqklt}(W^\nu, \nu^*\omega)$$

such that C', the closure of $h(\mathbb{A}^1)$ in W^ν, is a (possibly singular) rational curve satisfying $C' \cap \mathrm{Nqklt}(W^\nu, \nu^*\omega) \neq \emptyset$ with $0 < -\nu^*\omega \cdot C' \leq 1$. Then

$$f := \iota \circ \nu \circ h\colon \mathbb{A}^1 \longrightarrow (X \setminus \Sigma) \cap \pi^{-1}(P),$$

where $\iota\colon W \hookrightarrow X$ is a natural inclusion, is a desired morphism.

We finish the proof of Theorem 14.2. q.e.d.

For the proof of Theorem 1.20, we prepare the following theorem. The proof of Theorem 14.6 uses a deep result on the existence problem of minimal models in [**H1**].

Theorem 14.6. *Let (X, Δ) be a dlt pair and let $\pi\colon X \to S$ be a projective morphism between normal varieties such that $-(K_X + \Delta)$ is π-ample. We assume that $\mathrm{Nklt}(X, \Delta)$ is not empty such that*

$$\pi\colon \mathrm{Nklt}(X, \Delta) \to \pi(\mathrm{Nklt}(X, \Delta))$$

is finite and that there exists a curve $C^\dagger$ on X such that $\pi(C^\dagger)$ is a point with $C^\dagger \cap \mathrm{Nklt}(X, \Delta) \neq \emptyset$. Then there exists a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

such that $\pi \circ f(\mathbb{A}^1)$ is a point and that the curve C, the closure of $f(\mathbb{A}^1)$ in X, is a (possibly singular) rational curve satisfying $C \cap \mathrm{Nklt}(X, \Delta) \neq \emptyset$ with

$$0 < -(K_X + \Delta) \cdot C \leq 1.$$

Proof. By shrinking S suitably, we may assume that X and S are both quasi-projective. By Lemma 3.10, we can construct a projective birational morphism $g\colon Y \to X$ from a normal $\mathbb{Q}$-factorial variety satisfying (i), (ii), and (iv) in Lemma 3.10. Since $K_Y + \Delta_Y = g^*(K_X + \Delta)$, $(K_Y + \Delta_Y)|_{\mathrm{Nklt}(Y, \Delta_Y)}$ is nef over S by Lemma 3.10 (iv). Let us consider $\pi \circ g\colon Y \to S$. By construction, (Y, Δ_Y) is a $\mathbb{Q}$-factorial dlt pair.

If $\dim S = \dim X$, then $K_Y + \Delta_Y$ is pseudo-effective over S. In this case, we can take an effective $\mathbb{R}$-divisor A on Y such that $K_Y + \Delta_Y + A \sim_{\mathbb{R}, \pi\circ g} 0$ and that $(Y, \Delta_Y + A)$ is dlt since $-(K_Y + \Delta_Y) = -g^*(K_X + \Delta)$ is $(\pi \circ g)$-semi-ample. Hence (Y, Δ_Y) has a good minimal model over S by [**H1**, Theorem 1.1] and any $(K_Y + \Delta_Y)$-minimal model program over S with scaling of an ample divisor terminates (see [**H1**, Theorem 2.11]).

If $\dim S < \dim X$, then $K_Y + \Delta_Y$ is not pseudo-effective over S since $-(K_X+\Delta)$ is ample over S. In this case, we have a $(K_Y+\Delta_Y)$-minimal model program which terminates at a Mori fiber space by [**BCHM**].

Therefore, we have a finite sequence of flips and divisorial contractions

$$Y =: Y_0 \overset{\phi_0}{\dashrightarrow} Y_1 \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} Y_i \overset{\phi_i}{\dashrightarrow} \cdots \overset{\phi_{k-1}}{\dashrightarrow} Y_k$$

starting from $(Y_0, \Delta_{Y_0}) := (Y, \Delta_Y)$ over S such that (Y_k, Δ_{Y_k}) is a good minimal model of (Y, Δ_Y) over S or $p\colon Y_k \to Z$ is a Mori fiber space with respect to $K_{Y_k}+\Delta_{Y_k}$ over S, where $\Delta_{Y_{i+1}} = \phi_{i*}\Delta_{Y_i}$ for every i. By assumption, we can take a curve C' on Y such that $-(K_Y+\Delta_Y)\cdot C' > 0$ with $C' \cap \mathrm{Nklt}(Y, \Delta_Y) \neq \emptyset$. If ϕ_i is an isomorphism in a neighborhood of $\mathrm{Nklt}(Y_i, \Delta_{Y_i})$ for $0 \leq i < l$, then

$$0 < -(K_Y + \Delta_Y)\cdot C' \leq -(K_{Y_l} + \Delta_{Y_l})\cdot C'_{Y_l} \tag{14.1}$$

holds by the negativity lemma, where C'_{Y_l} is the strict transform of C' on Y_l.

Case 1. We assume that ϕ_i is an isomorphism in a neighborhood of $\mathrm{Nklt}(Y_i, \Delta_{Y_i})$ for every i. Then, by (14.1), the final model Y_k has a Mori fiber space structure $p\colon Y_k \to Z$ over S. We note that $(K_{Y_k} + \Delta_{Y_k})|_{\mathrm{Nklt}(Y_k, \Delta_{Y_k})}$ is nef over S. Hence the argument in Case 1 in the proof of Theorem 14.3 works without any changes. Then we get a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

with the desired properties.

Case 2. By Case 1, we may assume that there exists i_0 such that ϕ_i is an isomorphism in a neighborhood of $\mathrm{Nklt}(Y_i, \Delta_{Y_i})$ for $0 \leq i < i_0$ and ϕ_{i_0} is not an isomorphism in a neighborhood of $\mathrm{Nklt}(Y_{i_0}, \Delta_{Y_{i_0}})$. The argument in Case 2 in the proof of Theorem 14.3 works without any changes. Then we get a non-constant morphism

$$f\colon \mathbb{A}^1 \longrightarrow X \setminus \mathrm{Nklt}(X, \Delta)$$

with the desired properties.

We finish the proof of Theorem 14.6. q.e.d.

We close this section with the proof of Theorem 1.20. Since adjunction works well for dlt pairs, Theorem 1.20 directly follows from Theorem 14.6.

Proof of Theorem 1.20. We put $W = \overline{U_j}$. Then W is an lc stratum of (X, Δ). By adjunction, it is well known that we have

$$(K_X + \Delta)|_W = K_W + \Delta_W$$

such that (W, Δ_W) is dlt and that the lc centers of (W, Δ_W) are exactly the lc centers of (X, Δ) that are strictly included in W (see, for example, [**F3**, Proposition 3.9.2]). By replacing $\pi\colon X \to S$ with the Stein factorization of $\varphi_{R_j}\colon W \to \varphi_{R_j}(W)$, we may assume that $\pi\colon \mathrm{Nklt}(X,\Delta) \to \pi(\mathrm{Nklt}(X,\Delta))$ is finite and that there exists a curve $C^\dagger$ on X such that $\pi(C^\dagger)$ is a point with $C^\dagger \cap \mathrm{Nklt}(X,\Delta) \neq \emptyset$. By Theorem 14.6, we obtain a desired non-constant morphism

$$f\colon \mathbb{A}^1 \to X \setminus \mathrm{Nklt}(X,\Delta)$$

with the desired properties. q.e.d.

As we have already mentioned, we will completely prove Conjecture 1.15 in a joint paper with Kenta Hashizume (see [**FH1**]), where we use some deep results on the minimal model program for log canonical pairs. We strongly recommend the interested reader to see [**FH1**].

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