BASE POINT FREE THEOREMS —SATURATION, B-DIVISORS, AND CANONICAL BUNDLE FORMULA—

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Abstract. We reformulate base point free theorems. Our formulation is flexible and has some important applications. One of the main purposes of this paper is to prove a generalization of the base point free theorem in Fukuda's paper: On numerically effective log canonical divisors.

Contents

1. Introduction

In this paper, we reformulate base point free theorems by using Shokurov's ideas: *b-divisors, saturation of linear systems* (cf. [S]). Combining the refined Kawamata–Shokurov base point free theorem (cf. Theorem 2.1) or its generalization (cf. Theorem 6.1) with Ambro's formulation of Kodaira's canonical bundle formula, we obtain various new base point free theorems (cf. Theorems 4.4, 6.2). They are flexible and have some important applications (cf. Theorem 7.11). One of the main purposes of this paper is to prove a generalization of the base

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point free theorem in Fukuda's paper [Fk2]: On numerically effective log canonical divisors. See [Fk2, Proposition 3.3].

Theorem 1.1 (cf. Corollary 6.7)**.** *Let (X, B) be an lc pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(A) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X,*
(B) *$H - (K_X + B)$ is π-nef and π-abundant,*
(C) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S,*
(D) *there is a positive integer c such that cH is Cartier and*
$$\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$$
is π-generated, where $T = \mathrm{Nklt}(X, B)$ is the non-klt locus of (X, B).

Then H is π-semi-ample.

As an application of Theorem 1.1, we prove the following theorem in [FG2].

Theorem 1.2 (cf. [FG2, Theorem 4.12])**.** *Let $\pi : X \to S$ be a projective morphism between projective varieties. Let (X, B) be an lc pair such that $K_X + B$ is nef and log abundant over S. Then $K_X + B$ is f-semi-ample.*

We also need Theorem 1.1 to prove the finite generation of the log canonical ring for log canonical 4-folds in [F7]. See [F7, Remark 3.4]. As we explained in [F4, Remark 3.10.3] and [F9, 5.1], the proof of [K1, Theorem 4.3] contains a gap. There are no rigorous proofs of [K1, Theorem 5.1] by the gap in the proof of [K1, Theorem 4.3], and the proof of [Fk2, Proposition 3.3] depends on [K1, Theorem 5.1]. Therefore, the proof of Corollary 6.7 in this paper is the first rigorous proof of Fukuda's important result (cf. [Fk2, Proposition 3.3]). Another purpose of this paper is to show how to use Shokurov's ideas: *b-divisors*, *saturation of linear systems*, various kinds of *adjunction*, and so on, by reproving some known results by our new formulation. We recommend this paper as Chapter $8\frac{1}{2}$ of the book: Flips for 3-folds and 4-folds. We note that this paper is a complement of the paper [F9]. We do not use the powerful new method developed in [A1], [F5], [F6], [F8], [F10], [F11], and [F12]. For related topics and applications, see [F7], [G, 6. Applications], [Ca], and [FG2].

Remark 1.3. In his new preprint [K5], Professor Yujiro Kawamata claims that he corrects the error in the proof of [K1, Theorem 4.3]. The proof in [K5] seems to depend heavily on arguments in his preprints [K3] and [K4]. If we accept his correction, then Theorem 1.1 holds under the assumptions that (X, Δ) is dlt and that S is a point by [Fk2, Proposition 3.3]. As Kawamata says in the introduction of [K5], our arguments are simpler. We note that our approach is completely different from Kawamata's original one. Anyway, Theorem 1.1 plays crucial roles in our study of the log abundance conjecture for log canonical pairs (see [FG2, Section 4]). Therefore, this paper is indispensable for the minimal model program for log canonical pairs.

Let us explain the motivation for our formulation.

1.4 (Motivation)**.** Let (X, B) be a projective klt pair and let D be a nef Cartier divisor on X such that $D-(K_X+B)$ is nef and big. Then the Kawamata–Shokurov base point free theorem means that $|mD|$ is free for every $m \gg 0$. Let $f : Y \to X$ be a projective birational morphism from a normal projective variety Y such that $K_Y + B_Y = f^*(K_X + B)$. We note that f^*D is a nef Cartier divisor on Y and that $f^*D - (K_Y + B_Y)$ is nef and big. It is obvious that $|mf^*D|$ is free for every $m \gg 0$ because $|mD|$ is free for every $m \gg 0$. In general, we can not directly apply the Kawamata–Shokurov base point free theorem to f^*D and (Y, B_Y). It is because (Y, B_Y) is sub klt but is not always klt. Note that a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on X is nef, big, or semi-ample if and only if so is f^*L. However, the notion of klt is not stable under birational pull-backs. By adding a *saturation condition*, which is trivially satisfied for klt pairs, we can apply the Kawamata–Shokurov base point free theorem for sub klt pairs (see Theorem 2.1). By this new formulation, the base point free theorem becomes more flexible and has some important applications.

1.5 (Background)**.** A key result we need from Ambro's papers is [A2, Theorem 0.2], which is a generalization of [F3, §4. Pull-back of $L^{ss}_{X/Y}$]. It originates from *Kawamata's positivity theorem* in [K2] and Shokurov's idea on *adjunction*. For the details, see [A2, §0. Introduction]. The formulation and calculation we borrow from [A4] and [A5] grew out from Shokurov's *saturation of linear systems* (cf. [S, 4.32 Saturation of linear systems]).

We summarize the contents of this paper. In Section 2, we reformulate the Kawamata–Shokurov base point free theorem for sub klt pairs with a saturation condition. To state our theorem, we use the notion of b-divisors. It is very useful to discuss linear systems with some base conditions. In Section 3, we collect basic properties of b-divisors and prove some elementary properties. In Section 4, we discuss a slight generalization of the main theorem of [K1]. We need this generalization in Section 7. The main ingredient of our proof is Ambro's formulation of Kodaira's canonical bundle formula. By this formula and the refined Kawamata–Shokurov base point free theorem obtained in Section 2, we can quickly prove Kawamata's theorem in [K1] and its generalization without appealing to the notion of generalized normal crossing varieties. In Section 5, we treat the base point free theorem of Reid–Fukuda type. In this case, the saturation condition behaves very well for inductive arguments. It helps us understand the saturation condition of linear systems. In Section 6, we prove some variants of base point free theorems. They are mainly due to Fukuda [Fk2]. We reformulate them by using b-divisors and saturation conditions. Then we use Ambro's canonical bundle formula to reduce them to the easier case instead of proving them directly by the X-method. In Section 7, we generalize the Kawamata–Shokurov base point free theorem and Kawamata's main theorem in [K1] for *pseudo-klt pairs*. Theorem 7.11, which is new, is the main theorem of this section. It will be useful for the study of lc centers (cf. Theorem 7.13).

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Notation. Let $B = \sum b_i B_i$ be a $\mathbb{Q}$-divisor on a normal variety X such that B_i is prime for every i and that $B_i \neq B_j$ for $i \neq j$. We denote by

$$\ulcorner B \urcorner = \sum \ulcorner b_i \urcorner B_i, \quad \llcorner B \lrcorner = \sum \llcorner b_i \lrcorner B_i, \quad \text{and} \quad \{B\} = B - \llcorner B \lrcorner$$

the *round-up*, the *round-down*, and the *fractional part* of B. Note that we do not use $\mathbb{R}$-divisors in this paper. We make one general remark here. Since the freeness (or semi-ampleness) of a Cartier divisor D on a variety X depends only on the linear equivalence class of D, we can freely replace D by a linearly equivalent divisor to prove the freeness (or semi-ampleness) of D.

We will work over an algebraically closed field k of characteristic zero throughout this paper.

2. Kawamata–Shokurov base point free theorem revisited

Kawamata and Shokurov claimed the following theorem for klt pairs, that is, they assumed that B is effective. In this case, the condition (2) is trivially satisfied. We think that our formulation is useful for some applications. If the readers are not familiar with the notion of b-divisors, we recommend them to see Section 3.

Theorem 2.1 (Base point free theorem)**.** *Let (X, B) be a sub klt pair, let $\pi : X \to S$ be a proper surjective morphism onto a variety S and let D be a π-nef Cartier divisor on X. Assume the following conditions*:

(1) *$rD - (K_X + B)$ is nef and big over S for some positive integer r, and*
(2) (*Saturation condition*) *there exists a positive integer j_0 such that $\pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner + j\overline{D}) \subseteq \pi_*\mathcal{O}_X(jD)$ for every integer $j \geq j_0$.*

Then mD is π-generated for every $m \gg 0$, that is, there exists a positive integer m_0 such that for every $m \geq m_0$ the natural homomorphism $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective.

Proof. The usual proof of the base point free theorem, that is, the X-method, works without any changes if we note Lemma 3.11. For the details, see, for example, [KMM, §3-1]. See also the remarks in 3.15. ☐

The assumptions in Theorem 2.1 are birational in nature. This point is indispensable in Section 4. We note that we can assume that X is non-singular and Supp$B$ is a simple normal crossing divisor because the conditions (1) and (2) are invariant for birational pull-backs. So, it is easy to see that Theorem 2.1 is equivalent to the following theorem.

Theorem 2.2. *Let X be a non-singular variety and let B be a $\mathbb{Q}$-divisor on X such that $\llcorner B \lrcorner \leq 0$ and* Supp$B$ *is a simple normal crossing divisor. Let $\pi : X \to S$ be a projective morphism onto a variety S and let D be a π-nef Cartier divisor on X. Assume the following conditions*:

(1) *$rD - (K_X + B)$ is nef and big over S for some positive integer r, and*
(2) (*Saturation condition*) *there exists a positive integer j_0 such that $\pi_*\mathcal{O}_X(\ulcorner -B\urcorner + jD) \simeq \pi_*\mathcal{O}_X(jD)$ for every integer $j \geq j_0$.*

Then mD is π-generated for every $m \gg 0$.

The following example says that the original Kawamata–Shokurov base point free theorem does not necessarily hold for *sub* klt pairs.

Example 2.3. Let $X = E$ be an elliptic curve. We take a Cartier divisor H such that $\deg H = 0$ and $lH \not\sim 0$ for every $l \in \mathbb{Z} \setminus \{0\}$. In particular, H is nef. We put $B=-P$, where P is a closed point of X. Then (X,B) is sub klt and $H-(K_X+B)$ is ample. However, H is not semi-ample. In this case, $H^0(X,\mathcal{O}_X(\ulcorner \mathbf{A}(X,B)\urcorner + j\overline{H})) \simeq H^0(X,\mathcal{O}_X(P+jH)) \simeq k$ for every j. However, $H^0(X,\mathcal{O}_X(jH))=0$ for all j. Therefore, the saturation condition in Theorem 2.1 does not hold.

We note that Kollár's effective base point freeness holds under the same assumption as in Theorem 2.1.

Theorem 2.4 (Effective freeness)**.** *We use the same notation and assumption as in* Theorem 2.1. *Then there exists a positive integer l, which depends only on* $\dim X$ *and* $\max\{r,j_0\}$*, such that lD is π-generated, that is, $\pi^*\pi_*\mathcal{O}_X(lD)\to\mathcal{O}_X(lD)$ is surjective.*

Sketch of the proof. We need no new ideas. So, we just explain how to modify the arguments in [Ko1, Section 2]. From now on, we use the notation in [Ko1]. In [Ko1], (X,Δ) is assumed to be klt, that is, (X,Δ) is sub klt and Δ is effective. The effectivity of Δ implies that H' is f-exceptional in [Ko1, (2.1.4.3)]. We need this to prove $H^0(Y,\mathcal{O}_Y(f^*N+H'))=H^0(X,\mathcal{O}_X(N))$ in [Ko1, (2.1.6)]. It is not difficult to see that $0\le H'\le \ulcorner \mathbf{A}(X,\Delta)_Y\urcorner$ in our notation. Therefore, it is sufficient to assume the saturation condition (the assumption (2) in Theorem 2.1) in the proof of Kollár's effective freeness (see Section 2 in [Ko1]). We make one more remark. Applying the argument in the first part of 2.4 in [Ko1] to $\mathcal{O}_X(j\overline{D}+\ulcorner \mathbf{A}(X,B)\urcorner)$ on the generic fiber of $\pi: X\to S$ with the saturation condition (2) in Theorem 2.1, we obtain a positive integer l_0 that depends only on $\dim X$ and $\max\{r,j_0\}$ such that $\pi_*\mathcal{O}_X(l_0D)\neq 0$. As explained above, the arguments in Section 2 in [Ko1] work with only minor modifications in our setting. We leave the details as an exercise for the reader. □

3. B-divisors

3.1 (Singularities of pairs)**.** Let us recall the notion of singularities of pairs. We recommend the readers to see [F4] for more advanced topics on singularities of pairs.

Definition 3.2 (Singularities of pairs)**.** Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X such that K_X+B is $\mathbb{Q}$-Cartier. Let $f: Y\to X$ be a resolution of singularities such that $\mathrm{Exc}(f)\cup f_*^{-1}B$ has a simple normal crossing support, where $\mathrm{Exc}(f)$ is the exceptional locus of f. We write

$$K_Y = f^*(K_X+B)+\sum a_iA_i.$$

We note that a_i is called the *discrepancy* of A_i. Then the pair (X, B) is *sub klt* (resp. *sub lc*) if $a_i > -1$ (resp. $a_i \geq -1$) for every i. The pair (X, B) is *klt* (resp. *lc*) if (X, B) is sub klt (resp. sub lc) and B is effective. In some literature, sub klt (resp. sub lc) is sometimes called klt (resp. lc). Let (X, B) be an lc pair. If there exists a resolution $f : Y \to X$ such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup f_*^{-1}B$ are simple normal crossing divisors on Y and

$$K_Y = f^*(K_X + B) + \sum a_i A_i$$

with $a_i > -1$ for all f-exceptional A_i's, then (X, B) is called *dlt*.

Remark 3.3. Let (X, B) be a klt (resp. lc) pair and let $f : Y \to X$ be a proper birational morphism of normal varieties. We put $K_Y + B_Y = f^*(K_X + B)$. Then (Y, B_Y) is not necessarily klt (resp. lc) but sub klt (resp. sub lc).

Let us recall the definition of *log canonical centers*.

Definition 3.4 (Log canonical center)**.** Let (X, B) be a sub lc pair. A subvariety $W \subset X$ is called a *log canonical center* or an *lc center* of (X, B) if there is a resolution $f : Y \to X$ such that $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}B$ is a simple normal crossing divisor on Y and a divisor E with discrepancy -1 such that $f(E) = W$. A log canonical center $W \subset X$ of (X, B) is called *exceptional* if there is a unique divisor E_W on Y with discrepancy -1 such that $f(E_W) = W$ and $f(E) \cap W = \emptyset$ for every other divisor $E \neq E_W$ on Y with discrepancy -1 (cf. [Ko2, 8.1 Introduction]).

3.5 (b-divisors)**.** In this paper, we adopt the notion of *b-divisors*, which was introduced by Shokurov. For the details of b-divisors, we recommend the readers to see [A4, 1-B] and [Co, 2.3.2]. The readers can find various examples of b-divisors in [I].

Definition 3.6 (b-divisor)**.** Let X be a normal variety and let $\mathrm{Div}(X)$ be the free abelian group generated by Weil divisors on X. A *b-divisor* on X is an element:

$$\mathbf{D} \in \mathbf{Div}(X) = \mathrm{projlim}_{Y \to X} \mathrm{Div}(Y),$$

where the projective limit is taken over all proper birational morphisms $f : Y \to X$ of normal varieties, under the push forward homomorphism $f_* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$. A $\mathbb{Q}$-*b-divisor* on X is an element of $\mathbf{Div}_{\mathbb{Q}}(X) = \mathbf{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 3.7 (Discrepancy $\mathbb{Q}$-b-divisor)**.** Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then the *discrepancy* $\mathbb{Q}$-*b-divisor* of the pair (X, B) is the $\mathbb{Q}$-b-divisor $\mathbf{A} = \mathbf{A}(X, B)$ with the trace $\mathbf{A}_Y$ defined by the formula:

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties.

Definition 3.8 (Cartier closure)**.** Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal variety X. Then the $\mathbb{Q}$-b-divisor $\overline{D}$ denotes the *Cartier closure* of D, whose trace on Y is $\overline{D}_Y = f^*D$, where $f : Y \to X$ is a proper birational morphism of normal varieties.

Definition 3.9. Let $\mathbf{D}$ be a $\mathbb{Q}$-b-divisor on X. The round up $\ulcorner \mathbf{D} \urcorner \in \mathbf{Div}(X)$ is defined componentwise. The restriction of $\mathbf{D}$ to an open subset $U \subset X$ is a well-defined $\mathbb{Q}$-b-divisor on U, denoted by $\mathbf{D}|_U$. Then $\mathcal{O}_X(\mathbf{D})$ is an $\mathcal{O}_X$-module whose sections on an open subset $U \subset X$ are given by

$$H^0(U, \mathcal{O}_X(\mathbf{D})) = \{a \in k(X)^\times ; (\overline{(a)} + \mathbf{D})|_U \geq 0\} \cup \{0\},$$

where $k(X)$ is the function field of X. Note that $\mathcal{O}_X(\mathbf{D})$ is not necessarily coherent.

3.10 (Basic properties)**.** We recall the first basic property of discrepancy $\mathbb{Q}$-b-divisors. We will treat a generalization of Lemma 3.11 for sub lc pairs below.

Lemma 3.11. *Let (X, B) be a sub klt pair and let D be a Cartier divisor on X. Let $f : Y \to X$ be a proper surjective morphism from a non-singular variety Y. We write $K_Y = f^*(K_X + B) + \sum a_i A_i$. We assume that $\sum A_i$ is a simple normal crossing divisor. Then*

$$\mathcal{O}_X(\ulcorner \mathbf{A}(X, B) \urcorner + j\overline{D}) = f_*\mathcal{O}_Y(\sum \ulcorner a_i \urcorner A_i) \otimes \mathcal{O}_X(jD)$$

for every integer j.

Let E be an effective divisor on Y such that $E \leq \sum \ulcorner a_i \urcorner A_i$. Then

$$\pi_* f_* \mathcal{O}_Y(E + f^*jD) \simeq \pi_* \mathcal{O}_X(jD)$$

if

$$\pi_* \mathcal{O}_X(\ulcorner \mathbf{A}(X, B) \urcorner + j\overline{D}) \subseteq \pi_* \mathcal{O}_X(jD),$$

where $\pi : X \to S$ is a proper surjective morphism onto a variety S.

Proof. For the first equality, see [Co, Lemmas 2.3.14 and 2.3.15] or their generalizations: Lemmas 3.22 and 3.23 below. Since E is effective,

$\pi_*\mathcal{O}_X(jD) \subseteq \pi_* f_*\mathcal{O}_Y(E + f^*jD) \simeq \pi_*(f_*\mathcal{O}_Y(E) \otimes \mathcal{O}_X(jD))$. By the assumption and $E \le \sum \ulcorner a_i \urcorner A_i$,

$$\begin{aligned}\pi_*(f_*\mathcal{O}_Y(E) \otimes \mathcal{O}_X(jD)) &\subseteq \pi_*(f_*\mathcal{O}_Y(\sum \ulcorner a_i \urcorner A_i) \otimes \mathcal{O}_X(jD)) \\ &= \pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X,B) \urcorner + j\overline{D}) \\ &\subseteq \pi_*\mathcal{O}_X(jD).\end{aligned}$$

Therefore, we obtain $\pi_* f_*\mathcal{O}_Y(E + f^*jD) \simeq \pi_*\mathcal{O}_X(jD)$. $\square$

We will use Lemma 3.12 in Section 4. The vanishing theorem in Lemma 3.12 is nothing but the Kawamata–Viehweg–Nadel vanishing theorem.

Lemma 3.12. *Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X such that K_X+B is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a proper birational morphism from a normal variety Y. We put $K_Y + B_Y = f^*(K_X + B)$. Then*

$$f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) = \mathcal{O}_X(\ulcorner \mathbf{A}(X,B) \urcorner)$$

and

$$R^i f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) = 0$$

for every $i > 0$.

Proof. Let $g : Z \to Y$ be a resolution such that $\mathrm{Exc}(g) \cup g_*^{-1}B_Y$ has a simple normal crossing support. We put $K_Z + B_Z = g^*(K_Y + B_Y)$. Then $K_Z + B_Z = h^*(K_X + B)$, where $h = f \circ g : Z \to X$. By Lemma 3.11,

$$\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) = g_*\mathcal{O}_Z(\ulcorner -B_Z \urcorner)$$

and

$$\mathcal{O}_X(\ulcorner \mathbf{A}(X, B) \urcorner) = h_*\mathcal{O}_Z(\ulcorner -B_Z \urcorner).$$

Therefore, $f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) = \mathcal{O}_X(\ulcorner \mathbf{A}(X,B) \urcorner)$. Since, $-B_Z = K_Z - h^*(K_X + B)$, we have

$$\ulcorner -B_Z \urcorner = K_Z + \{B_Z\} - h^*(K_X + B).$$

Therefore, $R^i g_*\mathcal{O}_Z(\ulcorner -B_Z \urcorner) = 0$ and $R^i h_*\mathcal{O}_Z(\ulcorner -B_Z \urcorner) = 0$ for every $i > 0$ by the Kawamata–Viehweg vanishing theorem. Thus,

$$R^i f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) = 0$$

for every $i > 0$ by Leray's spectral sequence. $\square$

Remark 3.13. We use the same notation as in Remark 3.3. Let (X, B) be a klt pair. Let D be a Cartier divisor on X and let $\pi : X \to S$ be a proper morphism onto a variety S. We put $p = \pi \circ f : Y \to S$. Then $p_*\mathcal{O}_Y(jf^*D) \simeq \pi_*\mathcal{O}_X(jD) \simeq p_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner + j\overline{f^*D})$ for every integer j. It is because $f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) = \mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner) \simeq \mathcal{O}_X$ by Lemma 3.12.

We make a brief comment on the *multiplier ideal sheaf*.

Remark 3.14 (Multiplier ideal sheaf)**.** Let D be an effective $\mathbb{Q}$-divisor on a non-singular variety X. Then $\mathcal{O}_X(\ulcorner \mathbf{A}(X, D)\urcorner)$ is nothing but the *multiplier ideal sheaf* $\mathcal{J}(X, D) \subseteq \mathcal{O}_X$ of D on X. See [L, Definition 9.2.1]. More generally, let X be a normal variety and let Δ be a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then $\mathcal{O}_X(\ulcorner \mathbf{A}(X, \Delta + D)\urcorner) = \mathcal{J}((X, \Delta); D)$, where the right hand side is the *multiplier ideal sheaf* defined (but not investigated) in [L, Definition 9.3.56]. In general, $\mathcal{O}_X(\ulcorner \mathbf{A}(X, \Delta + D)\urcorner)$ is a fractional ideal of $k(X)$.

3.15 (Remarks on Theorem 2.1)**.** The following four remarks help us understand Theorem 2.1.

Remark 3.16 (Non-vanishing theorem)**.** By Shokurov's non-vanishing theorem (see [KMM, Theorem 2-1-1]), we have that $\pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner + j\overline{D}) \neq 0$ for every $j \gg 0$. Thus we have $\pi_*\mathcal{O}_X(jD) \neq 0$ for every $j \gg 0$ by the condition (2) in Theorem 2.1.

Remark 3.17. We know that $\ulcorner \mathbf{A}(X, B)\urcorner \geq 0$ since (X, B) is sub klt. Therefore, $\pi_*\mathcal{O}_X(jD) \subseteq \pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner + j\overline{D})$. This implies that $\pi_*\mathcal{O}_X(jD) \simeq \pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner + j\overline{D})$ for $j \geq j_0$ by the condition (2) in Theorem 2.1.

Remark 3.18. If the pair (X, B) is klt, then $\ulcorner \mathbf{A}(X, B)\urcorner$ is effective and exceptional over X. In this case, it is obvious that $\pi_*\mathcal{O}_X(jD) = \pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner + j\overline{D})$.

Remark 3.19. The condition (2) in Theorem 2.1 is a very elementary case of *saturation of linear systems*. See [Co, 2.3.3] and [A4, 1-D].

3.20. We introduce the notion of *non-klt $\mathbb{Q}$-b-divisor*, which is trivial for sub klt pairs. We will use this in Section 5.

Definition 3.21 (Non-klt $\mathbb{Q}$-b-divisor)**.** Let X be a normal variety and let B be a $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then the *non-klt $\mathbb{Q}$-b-divisor* of the pair (X, B) is the $\mathbb{Q}$-b-divisor $\mathbf{N} = \mathbf{N}(X, B)$ with the trace $\mathbf{N}_Y = \sum_{a_i \leq -1} a_i A_i$ for

$$K_Y = f^*(K_X + B) + \sum a_i A_i,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. It is easy to see that $\mathbf{N}(X, B)$ is a well-defined $\mathbb{Q}$-b-divisor. We put

$\mathbf{A}^*(X,B) = \mathbf{A}(X,B) - \mathbf{N}(X,B)$. Of course, $\mathbf{A}^*(X,B)$ is a well-defined $\mathbb{Q}$-b-divisor and $\ulcorner \mathbf{A}^*(X,B) \urcorner \geq 0$. If (X,B) is sub klt, then $\mathbf{N}(X,B) = 0$ and $\mathbf{A}(X,B) = \mathbf{A}^*(X,B)$.

The next lemma is a generalization of Lemma 3.11.

Lemma 3.22. *Let (X,B) be a sub lc pair and let $f : Y \to X$ be a resolution such that* $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}B$ *is a simple normal crossing divisor on Y. We write $K_Y = f^*(K_X + B) + \sum a_i A_i$. Then*

$$\mathcal{O}_X(\ulcorner \mathbf{A}^*(X,B) \urcorner) = f_*\mathcal{O}_Y(\sum_{a_i \neq -1} \ulcorner a_i \urcorner A_i).$$

In particular, $\mathcal{O}_X(\ulcorner \mathbf{A}^*(X,B) \urcorner)$ is a coherent $\mathcal{O}_X$-module. If (X,B) is lc, then $\mathcal{O}_X(\ulcorner \mathbf{A}^*(X,B) \urcorner) \simeq \mathcal{O}_X$.

Let D be a Cartier divisor on X and let E be an effective divisor on Y such that $E \leq \sum_{a_i \neq -1} \ulcorner a_i \urcorner A_i$. Then

$$\pi_* f_* \mathcal{O}_Y(E + f^* jD) \simeq \pi_* \mathcal{O}_X(jD)$$

if

$$\pi_* \mathcal{O}_X(\ulcorner \mathbf{A}^*(X,B) \urcorner + j\overline{D}) \subseteq \pi_* \mathcal{O}_X(jD),$$

where $\pi : X \to S$ is a proper morphism onto a variety S.

Proof. By definition, $\mathbf{A}^*(X,B)_Y = \sum_{a_i \neq -1} a_i A_i$. If $g : Y' \to Y$ is a proper birational morphism from a normal variety Y', then

$$\ulcorner \mathbf{A}^*(X,B)_{Y'} \urcorner = g^* \ulcorner \mathbf{A}^*(X,B)_Y \urcorner + F,$$

where F is a g-exceptional effective divisor, by Lemma 3.23 below. This implies $f_*\mathcal{O}_Y(\ulcorner \mathbf{A}^*(X,B)_Y \urcorner) = f'_*\mathcal{O}_{Y'}(\ulcorner \mathbf{A}^*(X,B)_{Y'} \urcorner)$, where $f' = f \circ g$, from which it follows that $\mathcal{O}_X(\ulcorner \mathbf{A}^*(X,B) \urcorner) = f_*\mathcal{O}_Y(\sum_{a_i \neq -1} \ulcorner a_i \urcorner A_i)$ is a coherent $\mathcal{O}_X$-module. The latter statement is easy to check. □

Lemma 3.23. *Let (X,B) be a sub lc pair and let $f : Y \to X$ be a resolution as in* Lemma 3.22. *We consider the $\mathbb{Q}$-b-divisor $\mathbf{A}^* = \mathbf{A}^*(X,B) = \mathbf{A}(X,B) - \mathbf{N}(X,B)$. If Y' is a normal variety and $g : Y' \to Y$ is a proper birational morphism, then*

$$\ulcorner \mathbf{A}^*_{Y'} \urcorner = g^* \ulcorner \mathbf{A}^*_Y \urcorner + F,$$

where F is a g-exceptional effective divisor.

Proof. By definition, we have $K_Y = f^*(K_X + B) + \mathbf{A}_Y$. Therefore, we may write,

$$\begin{aligned} K_{Y'} &= g^*f^*(K_X + B) + \mathbf{A}_{Y'} \\ &= g^*(K_Y - \mathbf{A}_Y) + \mathbf{A}_{Y'} \\ &= g^*(K_Y + \{-\mathbf{A}^*_Y\} - \mathbf{N}_Y + \llcorner -\mathbf{A}^*_Y \lrcorner) + \mathbf{A}_{Y'} \\ &= g^*(K_Y + \{-\mathbf{A}^*_Y\} - \mathbf{N}_Y) + \mathbf{A}_{Y'} - g^* \ulcorner \mathbf{A}^*_Y \urcorner. \end{aligned}$$

We note that $(Y, \{-\mathbf{A}^*_Y\} - \mathbf{N}_Y)$ is lc and that the set of lc centers of $(Y, \{-\mathbf{A}^*_Y\} - \mathbf{N}_Y)$ coincides with that of $(Y, -\mathbf{A}^*_Y - \mathbf{N}_Y) = (Y, -\mathbf{A}_Y)$. Therefore, the round-up of $\mathbf{A}_{Y'} - g^*\ulcorner \mathbf{A}^*_Y \urcorner - \mathbf{N}_{Y'}$ is effective and g-exceptional. Thus, we can write $\ulcorner \mathbf{A}^*_{Y'} \urcorner = g^* \ulcorner \mathbf{A}^*_Y \urcorner + F$, where F is a g-exceptional effective divisor. □

The next lemma is obvious by Lemma 3.22.

Lemma 3.24. *Let (X, B) be a sub lc pair and let $f : Y \to X$ be a proper birational morphism from a normal variety Y. We put $K_Y + B_Y = f^*(K_X + B)$. Then $f_*\mathcal{O}_Y(\ulcorner \mathbf{A}^*(Y, B_Y) \urcorner) = \mathcal{O}_X(\ulcorner \mathbf{A}^*(X, B) \urcorner)$.*

4. Base point free theorem; nef and abundant case

We recall the definition of *abundant* divisors, which are called *good* divisors in [K1]. See [KMM, §6-1].

Definition 4.1 (Abundant divisor)**.** Let X be a complete normal variety and let D be a $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisor on X. We define the *numerical Iitaka dimension* to be

$$\nu(X, D) = \max\{e; D^e \not\equiv 0\}.$$

This means that $D^{e'} \cdot S = 0$ for any e'-dimensional subvarieties S of X with $e' > e$ and there exists an e-dimensional subvariety T of X such that $D^e \cdot T > 0$. Then it is easy to see that $\kappa(X, D) \leq \nu(X, D)$, where $\kappa(X, D)$ denotes *Iitaka's D-dimension.* A nef $\mathbb{Q}$-divisor D is said to be *abundant* if the equality $\kappa(X, D) = \nu(X, D)$ holds. Let $\pi : X \to S$ be a proper surjective morphism of normal varieties and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then D is said to be π-abundant if $D|_{X_\eta}$ is abundant, where X_η is the generic fiber of π.

The next theorem is the main theorem of [K1]. For the relative statement, see [N, Theorem 5]. We reduced Theorem 4.2 to Theorem 2.1 by using Ambro's results in [A2] and [A5], which is the main theme of [F9]. For the details, see [F9, Section 2].

Theorem 4.2 (cf. [KMM, Theorem 6-1-11])**.** *Let (X, B) be a klt pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(a) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X,*
(b) *$H - (K_X + B)$ is π-nef and π-abundant, and*
(c) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.*

Then H is π-semi-ample.

We recall the definition of the Iitaka fibrations in this paper before we state the main theorem of this section.

Definition 4.3 (Iitaka fibration)**.** Let $\pi : X \to S$ be a proper surjective morphism of normal varieties. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on X such that $\kappa(X_\eta, D_\eta) \geq 0$, where η is the generic point of S. Let $X \dashrightarrow W$ be the rational map over S induced by $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ for a sufficiently large and divisible integer m. We consider a projective surjective morphism $f : Y \to Z$ of non-singular varieties that is birational to $X \dashrightarrow W$. We call $f : Y \to Z$ the *Iitaka fibration* with respect to D over S.

Theorem 4.4 is a slight generalization of Theorem 4.2. It will be used in the proof of Theorem 7.11.

Theorem 4.4. *Let (X, B) be a sub klt pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(a) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X,*
(b) *$H - (K_X + B)$ is π-nef and π-abundant,*
(c) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S,*
(d) *let $f : Y \to Z$ be the Iitaka fibration with respect to $H - (K_X + B)$ over S. We assume that there exists a proper birational morphism $\mu : Y \to X$ and put $K_Y + B_Y = \mu^*(K_X + B)$. In this setting, we assume $\mathrm{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y) \urcorner) = 1$, and*
(e) *(Saturation condition) there exist positive integers b and j_0 such that bH is Cartier and $\pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B) \urcorner + jb\overline{H}) \subseteq \pi_*\mathcal{O}_X(jbH)$ for every positive integer $j \geq j_0$.*

Then H is π-semi-ample.

Proof. The proof of Theorem 4.2 which is given in [F9, Section 2] works without any changes. We note that the condition (d) implies [F9, Lemma 2.3] and that we can use the condition (e) in the proof of [F9, Lemma 2.4]. □

Remark 4.5. We note that $\mathrm{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner)$ is a birational invariant for $f : Y \to Z$ by Lemma 3.12.

Remark 4.6. If (X, B) is klt and bH is Cartier, then it is obvious that $\pi_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner + jb\overline{H}) \simeq \pi_*\mathcal{O}_X(jbH)$ for every positive integer j (see Remark 3.18).

Remark 4.7. We can easily generalize Theorem 4.4 to varieties in class $\mathcal{C}$ by suitable modifications. For details, see [F9, Section 4].

The following examples help us understand the condition (d).

Example 4.8. Let $X = E$ be an elliptic curve and let $P \in X$ be a closed point. Take a general member $P_1 + P_2 + P_3 \in |3P|$. We put $B = \frac{1}{3}(P_1 + P_2 + P_3) - P$. Then (X, B) is sub klt and $K_X + B \sim_{\mathbb{Q}} 0$. In this case, $\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner) \simeq \mathcal{O}_X(P)$ and $H^0(X, \mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner)) \simeq k$.

Example 4.9. Let $f : X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to Z = \mathbb{P}^1$ be the Hirzebruch surface and let C (resp. E) be the positive (resp. negative) section of f. We take a general member $B_0 \in |5C|$. Note that $|5C|$ is a free linear system on X. We put $B = -\frac{1}{2}E + \frac{1}{2}B_0$ and consider the pair (X, B). Then (X, B) is sub klt. We put $H = 0$. Then H is a nef Cartier divisor on X and $aH - (K_X + B) \sim_{\mathbb{Q}} \frac{1}{2}F$ for every rational number a, where F is a fiber of f. Therefore, $aH - (K_X + B)$ is nef and abundant for every rational number a. In this case, $\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner) \simeq \mathcal{O}_X(E)$. So, we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner + j\overline{H})) &\simeq H^0(X, \mathcal{O}_X(E)) \simeq k \\ &\simeq H^0(X, \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X(jH)) \end{aligned}$$

for every integer j. Therefore, $\pi : X \to \operatorname{Spec} k$, H, and (X, B) satisfy the conditions (a), (b), (c), and (e) in Theorem 4.4. However, (d) is not satisfied. In our case, it is easy to see that $f : X \to Z$ is the Iitaka fibration with respect to $H - (K_X + B)$. Since $f_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner) \simeq f_*\mathcal{O}_X(E)$, we have $\mathrm{rank} f_*\mathcal{O}_X(\ulcorner \mathbf{A}(X, B)\urcorner) = 2$.

Remark 4.10. In Theorem 4.4, the assumptions (a), (b), (c) are the same as in Theorem 4.2. The condition (e) is indispensable by Example 2.3 for sub klt pairs. By using the non-vanishing theorem for generalized normal crossing varieties in [K1, Theorem 5.1], which is the hardest part to prove in [K1], the semi-ampleness of H seems to follow from the conditions (a), (b), (c), and (e). However, we need (d) to apply Ambro's canonical bundle formula to the Iitaka fibration

$f : Y \to Z$. See, for example, [F9, Section 3]. Unfortunately, as we saw in Example 4.9, the condition (d) does not follow from the other assumptions. Anyway, the condition (d) is automatically satisfied if (X, B) is klt (see [F9, Lemma 2.3]).

4.11 (Examples)**.** The following two examples show that the effective version of Theorem 4.2 does not necessarily hold. The first one is an obvious example.

Example 4.12. Let $X = E$ be an elliptic curve and let m be an arbitrary positive integer. Then there is a Cartier divisor H on X such that $mH \sim 0$ and $lH \not\sim 0$ for $0 < l < m$. Therefore, the effective version of Theorem 4.2 does not necessarily hold.

The next one shows the reason why Theorem 2.4 does not imply the effective version of Theorem 4.2.

Example 4.13. Let E be an elliptic curve and $G = \mathbb{Z}/m\mathbb{Z} = \langle \zeta \rangle$, where ζ is a primitive m-th root of unity. We take an m-torsion point $a \in E$. The cyclic group G acts on $E \times \mathbb{P}^1$ as follows:

$$E \times \mathbb{P}^1 \ni (x, [X_0 : X_1]) \mapsto (x + a, [\zeta X_0 : X_1]) \in E \times \mathbb{P}^1.$$

We put $X = (E \times \mathbb{P}^1)/G$. Then X has a structure of elliptic surface $p : X \to \mathbb{P}^1$. In this setting,

$$K_X = p^* \left(K_{\mathbb{P}^1} + \frac{m-1}{m}[0] + \frac{m-1}{m}[\infty] \right).$$

We put $H = p^{-1}(0)_{\mathrm{red}}$. Then H is a Cartier divisor on X. It is easy to see that H is nef and $H - K_X$ is nef and abundant. Moreover, $\kappa(X, aH - K_X) = \nu(X, aH - K_X) = 1$ for every rational number $a > 0$. It is obvious that $|mH|$ is free. However, $|lH|$ is not free for $0 < l < m$. Thus, the effective version of Theorem 4.2 does not hold.

5. Base point free theorem of Reid–Fukuda type

The following theorem is a reformulation of the main theorem of [F2].

Theorem 5.1 (Base point free theorem of Reid–Fukuda type)**.** *Let X be a non-singular variety and let B be a $\mathbb{Q}$-divisor on X such that* $\mathrm{Supp} B$ *is a simple normal crossing divisor and (X, B) is sub lc. Let $\pi : X \to S$ be a proper morphism onto a variety S and let D be a π-nef Cartier divisor on X. Assume the following conditions*:

1. *$rD - (K_X + B)$ is nef and log big over S for some positive integer r, and*

(2) (*Saturation condition*) *there exists a positive integer* j_0 *such that* $\pi_*\mathcal{O}_X(\ulcorner \mathbf{A}^*(X,B)\urcorner + j\overline{D}) \subseteq \pi_*\mathcal{O}_X(jD)$ *for every integer* $j \geq j_0$.

Then mD *is* π*-generated for every* $m \gg 0$*, that is, there exists a positive integer* m_0 *such that for every* $m \geq m_0$ *the natural homomorphism* $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ *is surjective.*

Let us recall the definition of *nef and log big divisors* on sub lc pairs.

Definition 5.2. Let (X,B) be a sub lc pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Let $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is *nef and log big* over S if and only if $\mathcal{L}$ is π-nef and π-big and the restriction $\mathcal{L}|_W$ is big over $\pi(W)$ for every lc center W of the pair (X,B). A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on X is said to be nef and log big over S if and only if so is $\mathcal{O}_X(cH)$, where c is a positive integer such that cH is Cartier.

Proof of Theorem 5.1. We write $B = T + B_+ - B_-$ such that T, B_+, and B_- are effective divisors, they have no common irreducible components, $\llcorner B_+ \lrcorner = 0$, and $\llcorner T \lrcorner = T$. If $T = 0$, then (X,B) is sub klt. So, theorem follows from Theorem 2.1. Thus, we assume $T \neq 0$. Let T_0 be an irreducible component of T. If $m \geq r$, then

$$mD + \ulcorner B_- \urcorner - T_0 - (K_X + B + \ulcorner B_- \urcorner - T_0) = mD - (K_X + B)$$

is nef and log big over S for the pair $(X, B + \ulcorner B_- \urcorner - T_0)$. We note that $B + \ulcorner B_- \urcorner - T_0$ is effective. Therefore, $R^1\pi_*\mathcal{O}_X(\ulcorner B_- \urcorner - T_0 + mD) = 0$ for $m \geq r$ by the vanishing theorem: Lemma 5.3. Thus, we obtain the following commutative diagram for $m \geq \max\{r, j_0\}$:

$$\begin{array}{ccccc}
\pi_*\mathcal{O}_X(\ulcorner B_- \urcorner + mD) & \longrightarrow & \pi_*\mathcal{O}_{T_0}(\ulcorner B_-|_{T_0} \urcorner + mD|_{T_0}) & \longrightarrow & 0 \\
\uparrow \cong & & \uparrow \iota & & \\
\pi_*\mathcal{O}_X(mD) & \xrightarrow{\alpha} & \pi_*\mathcal{O}_{T_0}(mD|_{T_0}). & &
\end{array}$$

Here, we used

$$\begin{aligned}
\pi_*\mathcal{O}_X(mD) &\subseteq \pi_*\mathcal{O}_X(\ulcorner B_- \urcorner + mD) \\
&\simeq \pi_*\mathcal{O}_X(\ulcorner \mathbf{A}^*(X,B) \urcorner + m\overline{D}) \\
&\subseteq \pi_*\mathcal{O}_X(mD)
\end{aligned}$$

for $m \geq j_0$ (see Lemma 3.22). We put $K_{T_0} + B_{T_0} = (K_X + B)|_{T_0}$ and $D_{T_0} = D|_{T_0}$. Then (T_0, B_{T_0}) is sub lc and it is easy to see that $rD_{T_0} - (K_{T_0} + B_{T_0})$ is nef and log big over $\pi(T_0)$. It is obvious that T_0 is non-singular and $\mathrm{Supp} B_{T_0}$ is a simple normal crossing divisor. We note that $\pi_*\mathcal{O}_{T_0}(\ulcorner \mathbf{A}^*(T_0, B_{T_0}) \urcorner + j\overline{D_{T_0}}) \simeq \pi_*\mathcal{O}_{T_0}(jD_{T_0})$ for every

$j \geq \max\{r, j_0\}$ follows from the above diagram, that is, the natural inclusion ι is isomorphism for $m \geq \max\{r, j_0\}$. Thus, α is surjective for $m \geq \max\{r, j_0\}$. By induction, mD_{T_0} is π-generated for every $m \gg 0$. We can apply the same argument to every irreducible component of T. Therefore, the relative base locus of mD is disjoint from T for every $m \gg 0$ since the restriction map $\alpha : \pi_*\mathcal{O}_X(mD) \to \pi_*\mathcal{O}_{T_0}(mD_{T_0})$ is surjective for every irreducible component T_0 of T. The arguments in [Fk1, Proof of Theorem 3], which is a variant of the X-method, work without any changes (cf. Theorem 6.1). So, we obtain that mD is π-generated for every $m \gg 0$. $\square$

The following vanishing theorem was already used in the proof of Theorem 5.1. The proof is an easy exercise by induction on $\dim X$ and on the number of the irreducible components of $\llcorner \Delta \lrcorner$.

Lemma 5.3. *Let $\pi : X \to S$ be a proper morphism from a non-singular variety X. Let $\Delta = \sum d_i \Delta_i$ be a sum of distinct prime divisors such that* SuppΔ *is a simple normal crossing divisor and d_i is a rational number with $0 \leq d_i \leq 1$ for every i. Let D be a Cartier divisor on X. Assume that $D - (K_X + \Delta)$ is nef and log big over S for the pair (X, Δ). Then $R^i\pi_*\mathcal{O}_X(D) = 0$ for every $i > 0$.*

As in Theorem 2.4, effective freeness holds under the same assumption as in Theorem 5.1.

Theorem 5.4 (Effective freeness)**.** *We use the same notation and assumption as in* Theorem 5.1*. Then there exists a positive integer l, which depends only on* $\dim X$ *and* $\max\{r, j_0\}$*, such that lD is π-generated, that is, $\pi^*\pi_*\mathcal{O}_X(lD) \to \mathcal{O}_X(lD)$ is surjective.*

Sketch of the proof. If (X, B) is sub klt, then this theorem is nothing but Theorem 2.4. So, we can assume that (X, B) is not sub klt. In this case, the arguments in [Fk1, §4] work with only minor modifications. From now on, we use the notation in [Fk1, §4]. By minor modifications, the proof in [Fk1, §4] works under the following weaker assumptions: X is non-singular and Δ is a $\mathbb{Q}$-divisor on X such that SuppΔ is a simple normal crossing divisor and (X, Δ) is sub lc. In [Fk1, Claim 5], E_i is f-exceptional. In our setting, it is not true. However, $0 \leq \sum_{cb_i - e_i + p_i < 0} \ulcorner -(cb_i - e_i + p_i) \urcorner E_i \leq \ulcorner \mathbf{A}^*(X, \Delta)_Y \urcorner$, which always holds even when Δ is not effective, is sufficient for us. It is because we can use the saturation condition (2) in Theorem 5.1. We leave the details as an exercise for the reader since all we have to do is to repeat the arguments in [Ko1, Section 2] and [Fk1, §4]. $\square$

The final statement in this section is the (effective) base point free theorem of Reid–Fukuda type for dlt pairs.

Corollary 5.5. *Let (X, B) be a dlt pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Let D be a π-nef Cartier divisor on X. Assume that $rD-(K_X+B)$ is nef and log big over S for some positive integer r. Then there exists a positive integer m_0 such that mD is π-generated for every $m \geq m_0$ and we can find a positive integer l, which depends only on* $\dim X$ *and r, such that lD is π-generated.*

Proof. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp} f_*^{-1}B$ are simple normal crossing divisors, $K_Y+B_Y = f^*(K_X+B)$, and f is an isomorphism over all the generic points of lc centers of the pair (X, B). Then (Y, B_Y) is sub lc, and $rD_Y - (K_Y + B_Y)$ is nef and log big over S, where $D_Y = f^*D$. Since $\ulcorner \mathbf{A}^*(X, B)\urcorner$ is effective and exceptional over X, $p_*\mathcal{O}_Y(\ulcorner \mathbf{A}^*(Y, B_Y)\urcorner + j\overline{D_Y}) \simeq p_*\mathcal{O}_Y(jD_Y)$ for every j, where $p = \pi \circ f$. So, we can apply Theorems 5.1 and 5.4 to D_Y and (Y, B_Y). We finish the proof. □

For the (effective) base point freeness for lc pairs, see [F6], [F10, Theorem 1.2], [F11, Theorem 13.1], and [F12, 3.3.1 Base Point Free Theorem].

6. Variants of base point free theorems due to Fukuda

The starting point of this section is a slight generalization of Theorem 2.1. It is essentially the same as [Fk1, Theorem 3].

Theorem 6.1. *Let X be a non-singular variety and let B be a $\mathbb{Q}$-divisor on X such that (X, B) is sub lc and* $\mathrm{Supp} B$ *is a simple normal crossing divisor. Let $\pi : X \to S$ be a proper morphism onto a variety S and let H be a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Assume the following conditions*:

1. *$H - (K_X + B)$ is nef and big over S, and*
2. *(Saturation condition) there exist positive integers b and j_0 such that $\pi_*\mathcal{O}_X(\ulcorner \mathbf{A}^*(X, B)\urcorner + jb\overline{H}) \subseteq \pi_*\mathcal{O}_X(jbH)$ for every integer $j \geq j_0$, and*
3. *there is a positive integer c such that cH is Cartier and*
$$\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$$
is π-generated, where $T = -\mathbf{N}(X, B)_X$.

Then H is π-semi-ample.

Proof. If (X, B) is sub klt, then this follows from Theorem 2.1. By replacing H by a multiple, we can assume that $b = 1$, $j_0 = 1$, and

$c = 1$. Since $lH + \ulcorner \mathbf{A}^*_X \urcorner - T - (K_X + \{B\}) = lH - (K_X + B)$ is nef and big over S for every positive integer l, we have the following commutative diagram by the Kawamata–Viehweg vanishing theorem:

$$\begin{array}{ccccc}
\pi_*\mathcal{O}_X(lH + \ulcorner \mathbf{A}^*_X \urcorner) & \longrightarrow & \pi_*(\mathcal{O}_T(lH) \otimes \mathcal{O}_T(\ulcorner \mathbf{A}^*_X|_T \urcorner)) & \longrightarrow & 0 \\
\uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\iota} & & \\
\pi_*\mathcal{O}_X(lH) & \underset{\alpha}{\longrightarrow} & \pi_*\mathcal{O}_T(lH). & &
\end{array}$$

Thus, the natural inclusion ι is an isomorphism and α is surjective for every $l \geq 1$. In particular, $\pi_*\mathcal{O}_X(lH) \neq 0$ for every $l \geq 1$. The same arguments as in [Fk1, Proof of Theorem 3] show that H is π-semi-ample. $\square$

The main purpose of this section is to prove Theorem 6.2 below, which is a generalization of Theorem 4.4 and Theorem 6.1. The basic strategy of the proof is the same as that of Theorem 4.4. That is, by using Ambro's canonical bundle formula, we reduce it to the case when $H - (K_X + B)$ is nef and big. This is nothing but Theorem 6.1.

Theorem 6.2. *Let X be a non-singular variety and let B be a $\mathbb{Q}$-divisor on X such that (X, B) is sub lc and* Supp$B$ *is a simple normal crossing divisor. Let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(a) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X,*
(b) *$H - (K_X + B)$ is π-nef and π-abundant,*
(c) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S,*
(d) *let $f : Y \to Z$ be the Iitaka fibration with respect to $H - (K_X + B)$ over S. We assume that there exists a proper birational morphism $\mu : Y \to X$ and put $K_Y + B_Y = \mu^*(K_X + B)$. In this setting, we assume* $\mathrm{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}^*(Y, B_Y) \urcorner) = 1$,
(e) (*Saturation condition*) *there exist positive integers b and j_0 such that bH is Cartier and $\pi_*\mathcal{O}_X(\ulcorner \mathbf{A}^*(X, B) \urcorner + jb\overline{H}) \subseteq \pi_*\mathcal{O}_X(jbH)$ for every positive integer $j \geq j_0$, and*
(f) *there is a positive integer c such that cH is Cartier and*
$$\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$$
is π-generated, where $T = -\mathbf{N}(X, B)_X$.

Then H is π-semi-ample.

Proof. If $H - (K_X + B)$ is big, then this follows from Theorem 6.1. So, we can assume that $H - (K_X + B)$ is not big. Form now on, we use the notation in the proof of Theorem 4.2 which is given in [F9, Section 2]. We just explain how to modify that proof. Let us recall the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[\pi]{} & S \end{array}$$

in the proof of [F9, Theorem 1.1], where $f : Y \to Z$ is the Iitaka fibration with respect to $H - (K_X + B)$ over S. For the details, see [F9, Section 2]. We note that $\mu^* H = H_Y$ and $H_Y \sim f^* D$. Here, we replaced H with a multiple and assumed that H and D are Cartier (see [F9, p.307]). We can also assume that $b = j_0 = 1$ in (e) and $c = 1$ in (f) by replacing H with a multiple. We start with the following obvious lemma.

Lemma 6.3. *We put $T' = -\mathbf{N}(X, B)_Y$. Then $\mu(T') \subset T$. Therefore, $\mathcal{O}_{T'}(H_Y) := \mathcal{O}_Y(H_Y)|_{T'}$ is p-generated, where $p = \pi \circ \mu$.*

Lemma 6.4. *If $f(T') = Z$, then H_Y is p-semi-ample. In particular, H is π-semi-ample.*

Proof. There exists an irreducible component T'_0 of T' such that $f(T'_0) = Z$. Since $(H_Y)|_{T'_0} \sim (f^* D)|_{T'_0}$ is p-semi-ample, D is φ-semi-ample. This implies that H_Y is p-semi-ample and H is π-semi-ample. □

Therefore, we can assume that T' is not dominant onto Z. Thus $\mathbf{A}(Y, B_Y) = \mathbf{A}^*(Y, B_Y)$ over the generic point of Z. Equivalently, (Y, B_Y) is sub klt over the generic point of Z. As in [F9, Proof of Theorem 1.1], we have the properties:

(1) $K_Y + B_Y \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M)$, where B_Z is the *discriminant* $\mathbb{Q}$-divisor of (Y, B_Y) on Z and M is the *moduli* $\mathbb{Q}$-divisor on Z,
(2′) (Z, B_Z) is sub lc,
(3) M is a φ-nef $\mathbb{Q}$-divisor on Z,
(4′) $\varphi_* \mathcal{O}_Z(\ulcorner \mathbf{A}^*(Z, B_Z) \urcorner + j\overline{D}) \subseteq \varphi_* \mathcal{O}_Z(jD)$ for every positive integer j,
(5) $D - (K_Z + B_Z)$ is φ-nef and φ-big,
(6) Y and Z are non-singular and $\mathrm{Supp} B_Y$ and $\mathrm{Supp} B_Z$ are simple normal crossing divisors, and
(7) $\mathcal{O}_{T''}(D) := \mathcal{O}_Z(D)|_{T''}$ is φ-generated where $T'' = -\mathbf{N}(Z, B_Z)_Z$.

Once the above conditions were satisfied, D is φ-semi-ample by Theorem 6.1. Therefore, H is π-semi-ample. So, all we have to do is to check the above conditions. The conditions (1), (2′), (3), (5), (6) are satisfied by Ambro's result (see [F9, Proof of Theorem 1.1]). We note that $\mathrm{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}(Y, B_Y)\urcorner) = \mathrm{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}^*(Y, B_Y)\urcorner) = 1$. By the same computation as in [A5, Lemma 9.2.2 and Proposition 9.2.3], we have the following lemma.

Lemma 6.5. *$\mathcal{O}_Z(\ulcorner \mathbf{A}^*(Z, B_Z)\urcorner + j\overline{D}) \subseteq f_*\mathcal{O}_Y(\ulcorner \mathbf{A}^*(Y, B_Y)\urcorner + j\overline{H_Y})$ for every integer j.*

Thus, we have (4′) by the saturation condition (e) (for the details, see [F9, Proof of Theorem 1.1], and Lemma 3.24). By definition, we have

$$lH_Y + \ulcorner \mathbf{A}^*_Y \urcorner - T' - (K_Y + \{B_Y\}) \sim_{\mathbb{Q}} f^*((l-1)D + M_0),$$

where

$$H_Y - (K_Y + B_Y) = \mu^*(H - (K_X + B)) \sim_{\mathbb{Q}} f^*M_0.$$

Note that $(l-1)D + M_0$ is φ-nef and φ-big for $l \geq 1$. By the Kollár type injectivity theorem,

$$R^1p_*\mathcal{O}_Y(lH_Y + \ulcorner \mathbf{A}^*_Y \urcorner - T') \to R^1p_*\mathcal{O}_Y(lH_Y + \ulcorner \mathbf{A}^*_Y \urcorner)$$

is injective for $l \geq 1$. Note that the above injectivity can be checked easily by [F13, Theorem 1.1]. Here, we used the fact that $f(T') \subsetneq Z$. So, we have the following commutative diagram:

$$\begin{array}{ccccc}
p_*\mathcal{O}_Y(lH_Y + \ulcorner \mathbf{A}^*_Y \urcorner) & \longrightarrow & p_*(\mathcal{O}_{T'}(lH_Y) \otimes \mathcal{O}_{T'}(\ulcorner \mathbf{A}^*_Y|_{T'} \urcorner)) & \longrightarrow & 0 \\
\Big\uparrow \cong & & \Big\uparrow \iota & & \\
p_*\mathcal{O}_Y(lH_Y) & \xrightarrow[\alpha]{} & p_*\mathcal{O}_{T'}(lH_Y). & &
\end{array}$$

The isomorphism of the left vertical arrow follows from the saturation condition (e). Thus, the natural inclusion ι is an isomorphism and α is surjective for $l \geq 1$. In particular, the relative base locus of lH_Y is disjoint from T' since $\mathcal{O}_{T'}(lH_Y)$ is p-generated (cf. Lemma 6.3). On the other hand, $H_Y \sim f^*D$. Therefore, $\mathcal{O}_{T''}(D)$ is φ-generated since $T'' \subset f(T')$. So, we obtain the condition (7). We complete the proof of Theorem 6.2. □

As a corollary of Theorem 6.2, we obtain a slight generalization of Fukuda's result (cf. [Fk2, Proposition 3.3]). Before we explain the corollary, let us recall the definition of *non-klt loci*.

Definition 6.6 (Non-klt locus)**.** Let (X, B) be an lc pair. We consider the closed subset

$$\mathrm{Nklt}(X, B) = \{x \in X \mid (X, B) \text{ is not klt at } x\}$$

of X. We call $\mathrm{Nklt}(X, B)$ the *non-klt locus* of (X, B).

Corollary 6.7. *Let (X, B) be an lc pair and let $\pi : X \to S$ be a proper morphism onto a variety S. Assume the following conditions*:

(a) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X,*
(b) *$H - (K_X + B)$ is π-nef and π-abundant,*
(c) *$\kappa(X_\eta, (aH - (K_X + B))_\eta) \geq 0$ and $\nu(X_\eta, (aH - (K_X + B))_\eta) = \nu(X_\eta, (H - (K_X + B))_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S,*
(f) *there is a positive integer c such that cH is Cartier and*

$$\mathcal{O}_T(cH) := \mathcal{O}_X(cH)|_T$$

is π-generated, where $T = \mathrm{Nklt}(X, B)$ is the non-klt locus of (X, B).

Then H is π-semi-ample.

The readers can find applications of this corollary in [Fk2], [F7], and [FG2].

Proof. Let $h : X' \to X$ be a resolution such that $\mathrm{Exc}(h) \cup \mathrm{Supp} h_*^{-1}B$ is a simple normal crossing divisor and $K_{X'} + B_{X'} = h^*(K_X + B)$. Then $H_{X'} = h^*H$, $(X', B_{X'})$, and $\pi' = \pi \circ h : X' \to S$ satisfy the assumptions (a), (b), and (c) in Theorem 6.2. By the same argument as in the proof of [F9, Lemma 2.3], we obtain $\mathrm{rank} f_*\mathcal{O}_Y(\ulcorner \mathbf{A}^*(Y, B_Y)\urcorner) = 1$, where $f : Y \to Z$ is the Iitaka fibration as in (d) in Theorem 6.2. Note that $\ulcorner \mathbf{A}^*(Y, B_Y)\urcorner$ is effective and exceptional over X. Since B is effective, $\ulcorner \mathbf{A}^*(X, B)\urcorner$ is effective and exceptional over X,

$$\pi'_*\mathcal{O}_{X'}(\ulcorner \mathbf{A}^*(X', B_{X'})\urcorner + jb\overline{H_{X'}}) \subseteq \pi'_*\mathcal{O}_{X'}(jbH_{X'})$$

for every integer j, where b is a positive integer such that bH is Cartier. So, the saturation condition (e) in Theorem 6.2 is satisfied. Finally, $\mathcal{O}_{T'}(cH_{X'}) := \mathcal{O}_{X'}(cH_{X'})|_{T'}$ is π'-generated, where $T' = -\mathbf{N}(X, B)_{X'}$, by the assumption (f) and the fact that $h(T') \subset T$. So, the condition (f) in Theorem 6.2 for $H_{X'}$ and $(X', B_{X'})$ is satisfied. Therefore, $H_{X'}$ is π'-semi-ample by Theorem 6.2. Thus, H is π-semi-ample. □

Remark 6.8. (i) It is obvious that $\mathrm{Supp}(-\mathbf{N}(X, B)_X) \subseteq \mathrm{Nklt}(X, B)$. In general, $\mathrm{Supp}(-\mathbf{N}(X, B)_X) \subsetneq \mathrm{Nklt}(X, B)$. In particular, $\mathrm{Nklt}(X, B)$ is not necessarily of pure codimension one in X.

(ii) If (X, B) is dlt, then $\mathrm{Nklt}(X, B) = \mathrm{Supp}(-\mathbf{N}(X, B)_X) = \llcorner B \lrcorner$. Therefore, if (X, B) is dlt and S is a point, then Corollary 6.7 is nothing but Fukuda's result [Fk2, Proposition 3.3].

By combining Corollary 6.7 with [G, Theorem 1.5], we obtain the following result.

Corollary 6.9. *Let (X, B) be a projective dlt pair such that $\nu(K_X + B) = \kappa(K_X + B)$ and that $(K_X + B)|_{\llcorner B \lrcorner}$ is numerically trivial. Then $K_X + B$ is semi-ample.*

We close this section with a remark.

Remark 6.10. We can easily generalize Theorem 6.2 and Corollary 6.7 to varieties in class $\mathcal{C}$ by suitable modifications. We omit details here. See [F9, Section 4].

7. Base point free theorems for pseudo-klt pairs

In this section, we generalize the Kawamata–Shokurov base point free theorem and Kawamata's theorem: Theorem 4.2 for *klt pairs* to *pseudo-klt pairs*. We think that our formulation is useful when we study lc centers (see Proposition 7.8). First, we introduce the notion of *pseudo-klt pairs*.

Definition 7.1 (Pseudo-klt pair)**.** Let W be a normal variety. Assume the following conditions:

(1) there exist a sub klt pair (V, B) and a proper surjective morphism $f : V \to W$ with connected fibers,
(2) $f_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner) \simeq \mathcal{O}_W$, and
(3) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{K}$ on W such that $K_V + B \sim_{\mathbb{Q}} f^*\mathcal{K}$.

Then the pair $[W, \mathcal{K}]$ is called a *pseudo-klt pair*.

Although it is the first time that we use the name of *pseudo-klt pair*, the notion of pseudo-klt pair appeared in [F1], where we proved the cone and contraction theorem for *pseudo-klt* pairs (cf. [F1, Section 4]). We note that all the fundamental theorems for the log minimal model program for pseudo-klt pairs can be proved by the theory of quasi-log varieties (cf. [A1], [F8], and [F12]).

Remark 7.2. In Definition 7.1, we assume that W is normal. However, the normality of W follows from the condition (2) and the normality of V. Note that $\ulcorner \mathbf{A}(V, B)\urcorner$ is effective.

Remark 7.3. In the definition of pseudo-klt pairs, if (V, B) is klt, then $f_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner) \simeq \mathcal{O}_W$ is automatically satisfied. It is because $\ulcorner \mathbf{A}(V, B)\urcorner$ is effective and exceptional over V.

We note that a pseudo-klt pair is a very special example of Ambro's quasi-log varieties (see [A1, Definition 4.1]). More precisely, if $[V, \mathcal{K}]$ is a pseudo-klt pair, then we can easily check that $[V, \mathcal{K}]$ is a *qlc pair*. See, for example, [F8, Definition 3.1]. For the details of the theory of quasi-log varieties, see [F12].

Theorem 7.4. *Let $[W, \mathcal{K}]$ be a pseudo-klt pair. Assume that (V, B) is klt and W is projective or that W is affine. Then we can find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and that $\mathcal{K} \sim_{\mathbb{Q}} K_W + B_W$.*

Proof. When (X, B) is klt and W is projective, we can find B_W by [A3, Theorem 4.1]. When W is affine, this theorem follows from [F1, Theorem 1.2]. $\square$

It is conjectured that we can always find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and $\mathcal{K} \sim_{\mathbb{Q}} K_W + B_W$.

7.5 (Examples)**.** We collect basic examples of pseudo-klt pairs.

Example 7.6. A klt pair is a pseudo-klt pair.

Example 7.7. Let $f : X \to W$ be a Mori fiber space. Then we can find a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{K}$ on W such that $[W, \mathcal{K}]$ is a pseudo-klt pair. It is because we can find an effective $\mathbb{Q}$-divisor B on X such that $K_X + B \sim_{\mathbb{Q}, f} 0$ and (X, B) is klt.

Proposition 7.8. *An exceptional lc center W of an lc pair (X, B) is a pseudo-klt pair for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{K}$ on W.*

Proof. We take a resolution $g : Y \to X$ such that $\mathrm{Exc}(g) \cup g_*^{-1}B$ has a simple normal crossing support. We put $K_Y + B_Y = g^*(K_X + B)$. Then $-B_Y = \mathbf{A}(X, B)_Y = \mathbf{A}_Y = \mathbf{A}_Y^* + \mathbf{N}_Y$, where $\mathbf{N}_Y = -\sum_{i=0}^k E_i$. Without loss of generality, we can assume that $f(E) = W$ and $E = E_0$. By shrinking X around W, we can assume that $\mathbf{N}_Y = -E$. Note that $R^1 g_* \mathcal{O}_Y(\ulcorner \mathbf{A}_Y^* \urcorner - E) = 0$ by the Kawamata–Viehweg vanishing theorem since $\ulcorner \mathbf{A}_Y^* \urcorner - E = K_Y + \{-\mathbf{A}_Y^*\} - g^*(K_X + B)$. Therefore, $g_*\mathcal{O}_Y(\ulcorner \mathbf{A}_Y^* \urcorner) \simeq \mathcal{O}_X \to g_*\mathcal{O}_E(\ulcorner \mathbf{A}_Y^*|_E \urcorner)$ is surjective. This implies that $g_*\mathcal{O}_E(\ulcorner \mathbf{A}_Y^*|_E \urcorner) \simeq \mathcal{O}_W$. In particular, W is normal. If we put $K_E + B_E = (K_Y + B_Y)|_E$, then (E, B_E) is sub klt and $\mathbf{A}_Y^*|_E = \mathbf{A}(E, B_E)_E = -B_E$. So, $g_*\mathcal{O}_E(\ulcorner \mathbf{A}(E, B_E) \urcorner) = g_*\mathcal{O}_E(\ulcorner -B_E \urcorner) \simeq \mathcal{O}_W$. Since $K_E + B_E = (K_Y + B_Y)|_E$ and $K_Y + B_Y = g^*(K_X + B)$, we can find a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{K}$ on W such that $K_E + B_E \sim_{\mathbb{Q}} g^*\mathcal{K}$. Therefore, W is a pseudo-klt pair. $\square$

We give an important remark on *minimal* lc centers.

Remark 7.9 (Subadjunction for minimal lc center)**.** Let (X, B) be a projective or affine lc pair and let W be a minimal lc center of the pair (X, B). Then we can find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and $K_W + B_W \sim_{\mathbb{Q}} (K_X + B)|_W$. For the details, see [FG1, Theorems 4.1, 7.1].

The following theorem is the Kawamata–Shokurov base point free theorem for pseudo-klt pairs. We give a simple proof depending on Kawamata's positivity theorem. Although Theorem 7.10 seems to be contained in [A1, Theorem 7.2], there are no proofs of [A1, Theorem 7.2] in [A1].

Theorem 7.10. *Let $[W, \mathcal{K}]$ be a pseudo-klt pair, let $\pi : W \to S$ be a proper morphism onto a variety S and let D be a π-nef Cartier divisor on W. Assume that $rD - \mathcal{K}$ is π-nef and π-big for some positive integer r. Then mD is π-generated for every $m \gg 0$.*

Proof. Without loss of generality, we can assume that S is affine. By the usual technique (cf. [K2, Theorem 1] and [F1, Theorem 1.2]), we have

$$\mathcal{K} + \varepsilon(rD - \mathcal{K}) \sim_{\mathbb{Q}} K_W + \Delta_W$$

such that (W, Δ_W) is klt for some sufficiently small rational number $0 < \varepsilon \ll 1$ (see also [Ko2, Theorem 8.6.1]). Then $rD - (K_W + \Delta_W) \sim_{\mathbb{Q}} (1-\varepsilon)(rD - \mathcal{K})$, which is π-nef and π-big. Therefore, mD is π-generated for every $m \gg 0$ by the usual Kawamata–Shokurov base point free theorem. □

The next theorem is the main theorem of this section. It is a generalization of Kawamata's theorem in [K1] (cf. Theorem 4.2) for pseudo-klt pairs.

Theorem 7.11. *Let $[W, \mathcal{K}]$ be a pseudo-klt pair and let $\pi : W \to S$ be a proper morphism onto a variety S. Assume the following conditions:*

(i) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on W,*
(ii) *$H - \mathcal{K}$ is π-nef and π-abundant, and*
(iii) *$\kappa(W_\eta, (aH - \mathcal{K})_\eta) \geq 0$ and $\nu(W_\eta, (aH - \mathcal{K})_\eta) = \nu(W_\eta, (H - \mathcal{K})_\eta)$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.*

Then H is π-semi-ample.

Proof. By definition, there exists a proper surjective morphism $f : V \to W$ from a sub klt pair (V, B). Without loss of generality, we can assume that V is non-singular and $\mathrm{Supp} B$ is a simple normal crossing divisor. By definition, $f_*\mathcal{O}_V(\ulcorner -B \urcorner) \simeq \mathcal{O}_W$. From now on, we assume that H is Cartier by replacing it with a multiple. Then $f_*\mathcal{O}_V(\ulcorner -B \urcorner + jH_V) \simeq \mathcal{O}_W(jH)$ by the projection formula for every integer j, where $H_V = f^*H$. Pushing forward by π, we have

$$\begin{aligned} p_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B) \urcorner + j\overline{H_V}) &= p_*\mathcal{O}_V(\ulcorner -B \urcorner + jH_V) \\ &\simeq \pi_*\mathcal{O}_W(jH) \\ &\simeq p_*\mathcal{O}_V(jH_V) \end{aligned}$$

for every integer j, where $p = \pi \circ f$. This is nothing but the saturation condition: Assumption (e) in Theorem 4.4. We put $L = H - \mathcal{K}$. We consider the Iitaka fibration with respect to L over S as in [F9, Proof of Theorem 1.1]. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} V & = & V \\ {\scriptstyle f}\downarrow & & \downarrow \\ W & \xleftarrow{\mu} & U \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xleftarrow[\varphi]{} & Z \end{array}$$

where $g : U \to Z$ is the Iitaka fibration over S and $\mu : U \to W$ is a birational morphism. Note that we can assume that $f : V \to W$ factors through U by blowing up V.

Lemma 7.12. $\mathrm{rank} h_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner) = 1$, *where* $h : V \to U \to Z$.

Proof of Lemma 7.12. This proof is essentially the same as that of [F9, Lemma 2.3]. First, we can assume that S is affine. Let A be an ample divisor on Z such that $h_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner) \otimes \mathcal{O}_Z(A)$ is φ-generated. We note that we can assume that $\mu^*L \sim_{\mathbb{Q}} g^*M$ since L is π-nef and π-abundant, where M is a φ-nef and φ-big $\mathbb{Q}$-divisor on Z. If we choose a large and divisible integer m, then $\mathcal{O}_Z(A) \subset \mathcal{O}_Z(mM)$. Thus

$$\begin{aligned} & \varphi_*(h_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner) \otimes \mathcal{O}_Z(A)) \\ \subseteq\ & \varphi_*(h_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner) \otimes \mathcal{O}_Z(mM)) \\ \simeq\ & p_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner + m\overline{f^*L}) \\ \simeq\ & \pi_*\mathcal{O}_W(mL) \\ \simeq\ & \varphi_*\mathcal{O}_Z(mM). \end{aligned}$$

Therefore, we have $\mathrm{rank} h_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner) \leq 1$. Since $\mathcal{O}_Z \subset h_*\mathcal{O}_V \subset h_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner)$, we obtain $\mathrm{rank} h_*\mathcal{O}_V(\ulcorner \mathbf{A}(V, B)\urcorner) = 1$ □

Note that $h : V \to Z$ is the Iitaka fibration with respect to f^*L over S. The assumption (c) in Theorem 4.4 easily follows from (iii). Thus, by Theorem 4.4, we have that H_V is p-semi-ample. Equivalently, H is π-semi-ample. □

The final theorem of this paper is a base point free theorem for minimal lc centers.

Theorem 7.13. *Let* (X, B) *be an lc pair and let* W *be a minimal lc center of* (X, B). *Let* $\pi : W \to S$ *be a proper morphism onto a variety* S. *Assume the following conditions*:

(i) *H is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on W,*
(ii) *$H - (K_X + B)|_W$ is π-nef and π-abundant, and*
(iii) *$\kappa(W_\eta, (aH - (K_X + B))|_{W_\eta}) \geq 0$ and $\nu(W_\eta, (aH - (K_X + B))|_{W_\eta}) = \nu(W_\eta, (H - (K_X + B))|_{W_\eta})$ for some $a \in \mathbb{Q}$ with $a > 1$, where η is the generic point of S.*

Then H is π-semi-ample.

Proof. Let $f : Y \to X$ be a dlt blow-up such that $K_Y + B_Y = f^*(K_X + B)$ (see, for example, [F11, Theorem 10.4] or [F14, Section 4]). Then we can take a minimal lc center Z of (Y, B_Y) such that $f(Z) = W$. Note that $K_Z + B_Z = (K_Y + B_Y)|_Z$ is klt. We also note that W is normal (see, for example, [F10, Theorem 2.4 (4)] or [F11, Theorem 9.1 (4)]). Let

$$f : Z \xrightarrow{\ g\ } V \xrightarrow{\ h\ } W$$

be the Stein factorization of $f : Z \to W$. Then $[V, h^*((K_X + B)|_W)]$ is a pseudo-klt pair by $g : (Z, B_Z) \to V$. We note that H is π-semi-ample if and only if h^*H is $\pi \circ h$-semi-ample. By Theorem 7.11, h^*H is semi-ample over S. We finish the proof. $\square$

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