BIG $\mathbb{R}$-DIVISORS

OSAMU FUJINO

0.1. **Big $\mathbb{R}$-divisors.** In this subsection, we introduce the notion of big $\mathbb{R}$-divisors on singular varieties. The basic references of big $\mathbb{R}$-divisors are [L, 2.2] and [N, II. §3 and §5]. Since we have to consider big $\mathbb{R}$-divisors on *non-normal* varieties, we give supplementary definitions and arguments to [L] and [N].

First, let us quickly recall the definition of big Cartier divisors on normal complete varieties. For details, see, for example, [KMM, §0-3].

def-big **Definition 0.1** (Big Cartier divisors)**.** Let X be a normal complete variety and D a Cartier divisor on X. Then D is *big* if one of the following equivalent conditions holds.

(1) $\max_{m\in\mathbb{N}}\{\dim \Phi_{|mD|}(X)\} = \dim X$, where $\Phi_{|mD|} : X \dashrightarrow \mathbb{P}^N$ is the rational map associated to the linear system $|mD|$ and $\Phi_{|mD|}(X)$ is the image of $\Phi_{|mD|}$.
(2) There exist a rational number α and a positive integer m_0 such that
$$\alpha m^{\dim X} \leq \dim H^0(X, \mathcal{O}_X(mm_0D)).$$

It is well known that we can take $m_0 = 1$ in the condition (2).

One of the most important properties of big Cartier divisors is known as Kodaira's lemma.

kod-lem **Lemma 0.2** (Kodaira's lemma)**.** *Let X be a normal complete variety and D a big Cartier divisor on X. Then, for an arbitrary Cartier divisor M, we have $H^0(X, \mathcal{O}_X(lD - M)) \neq 0$ for $l \gg 0$.*

Proof. By replacing X with its resolution, we can assume that X is smooth and projective. Then it is sufficient to show that for a very ample Cartier divisor A, $H^0(X, \mathcal{O}_X(lD - A)) \neq 0$ for $l \gg 0$. Since we have the exact sequence
$$0 \to \mathcal{O}_X(lD - A) \to \mathcal{O}_X(lD) \to \mathcal{O}_Y(lD) \to 0,$$

Date: 2009/8/3, Version 1.15.
This note will be contained in my book. I thank Professors Takeshi Abe, Atsushi Moriwaki, and Noboru Nakayama.

where Y is a general member of $|A|$, and since there exist positive rational numbers α, β such that $\alpha l^{\dim X} \leq \dim H^0(X, \mathcal{O}_X(lD))$ and $\dim H^0(Y, \mathcal{O}_Y(lD)) \leq \beta l^{\dim Y}$ for $l \gg 0$, we have $H^0(X, \mathcal{O}_X(lD - A)) \neq 0$ for $l \gg 0$. □

For non-normal varieties, we need the following definition.

def-big2 **Definition 0.3** (Big Cartier divisors on non-normal varieties)**.** Let X be a complete irreducible variety and D a Cartier divisor on X. Then D is *big* if $\nu^* D$ is big on X^ν, where $\nu : X^\nu \to X$ is the normalization.

Before we define big $\mathbb{R}$-divisors, let us recall the definition of big $\mathbb{Q}$-divisors.

defnQ **Definition 0.4** (Big $\mathbb{Q}$-divisors)**.** Let X be a complete irreducible variety and D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then D is *big* if mD is a big Cartier divisor for some positive integer m.

We note the following obvious lemma.

lem0555 **Lemma 0.5.** *Let $f : W \to V$ be a birational morphism between normal varieties and D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V. Then D is big if and only if so is $f^* D$.*

Next, let us start to consider big $\mathbb{R}$-divisors.

defnA **Definition 0.6** (Big $\mathbb{R}$-divisors on complete varieties)**.** An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on a complete irreducible variety X is *big* if it can be written in the form

$$D = \sum_i a_i D_i$$

where each D_i is a big Cartier divisor and a_i is a positive real number for every i.

Let us recall an easy but very important lemma.

lemABC **Lemma 0.7** (cf. nakayama2 [N, 2.11. Lemma])**.** *Let $f : Y \to X$ be a proper surjective morphism between normal varieties with connected fibers. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then we have a canonical isomorphism*

$$\mathcal{O}_X(\llcorner D \lrcorner) \simeq f_* \mathcal{O}_Y(\llcorner f^* D \lrcorner).$$

lem088 **Lemma 0.8.** *Let D be a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a* smooth *projective variety X. Then there exist a positive rational number α and a positive integer m_0 such that*

$$\alpha m^{\dim X} \leq \dim H^0(X, \mathcal{O}_X(\llcorner m m_0 D \lrcorner))$$

for $m \gg 0$.

Proof. By using Lemma 0.2, we can find an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E on X such that $D - E$ is ample. Therefore, there exists a positive integer m_0 such that $A = \llcorner m_0 D - m_0 E \lrcorner$ is ample. We note that $m_0 D = A + \{m_0 D - m_0 E\} + m_0 E$. This implies that $mA \leq m m_0 D$ for any positive integer m. Therefore,

$$\dim H^0(X, \mathcal{O}_X(mA)) \leq \dim H^0(X, \mathcal{O}_X(\llcorner m m_0 D \lrcorner)).$$

So, we can find a positive rational number α such that

$$\alpha m^{\dim X} \leq \dim H^0(X, \mathcal{O}_X(\llcorner m m_0 D \lrcorner)).$$

It is the desired inequality. □

rem099 **Remark 0.9.** By Lemma 0.8, Definition 0.6 is compatible with Definition 0.4.

wkodaira **Lemma 0.10** (Weak Kodaira's lemma)**.** *Let X be a projective irreducible variety and D a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then we can write*

$$D \sim_{\mathbb{R}} A + E,$$

where A is an ample $\mathbb{Q}$-divisor on X and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X.

Proof. Let B be a big Cartier divisor on X and H a general very ample Cartier divisor on X. We consider the short exact sequence

$$0 \to \mathcal{O}_X(lB - H) \to \mathcal{O}_X(lB) \to \mathcal{O}_H(lB) \to 0$$

for any l. It is easy to see that $\dim H^0(X, \mathcal{O}_X(lB)) \geq \alpha l^{\dim X}$ and $\dim H^0(H, \mathcal{O}_H(lB)) \leq \beta l^{\dim H}$ for some positive rational numbers α, β, and for $l \gg 0$. Therefore, $H^0(X, \mathcal{O}_X(lB - H)) \neq 0$ for some large l. This means that $lB \sim H + G$ for some effective Cartier divisor G. By Definition 0.6, we can write $D = \sum_i a_i D_i$ where a_i is a positive real number and D_i is a big Cartier divisor for every i. By applying the above argument to each D_i, we can easily obtain the desired decomposition $D \sim_{\mathbb{R}} A + E$. □

We prepare an important lemma.

lemN **Lemma 0.11.** *Let X be a complete irreducible variety and N a numerically trivial $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then N can be written in the form*

$$N = \sum_i r_i N_i$$

where each N_i is a numerically trivial Cartier divisor and r_i is a real number for every i.

Proof. Let Z_j be an integral 1-cycle on X for $1 \le j \le \rho = \rho(X)$ such that $\{[Z_1], \cdots, [Z_\rho]\}$ is a basis of the vector space $N_1(X)$. The condition that an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $B = \sum_i b_i B_i$, where b_i is a real number and B_i is Cartier for every i, is numerically trivial is given by the integer linear equations

$$\sum_i b_i (B_i \cdot Z_j) = 0$$

on b_i for $1 \le j \le \rho$. Any real solution to these equations is an $\mathbb{R}$-linear combination of integral ones. Thus, we obtain the desired expression $N = \sum_i r_i N_i$. □

The following proposition seems to be very important.

lemD **Proposition 0.12.** *Let X be a complete irreducible variety. Let D and D' be $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X. If $D \equiv D'$, then D is big if and only if so is D'.*

Proof. We put $N = D' - D$. Then N is a numerically trivial $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. By Lemma 0.11, we can write $N = \sum_i r_i N_i$, where r_i is a real number and N_i is a numerically trivial Cartier divisor for every i. By Definition 0.6, we are reduced to showing that if B is a big Cartier divisor and G is a numerically trivial Cartier divisor, then $B + rG$ is big for any real number r. If r is not a rational number, we can write

$$B + rG = t(B + r_1 G) + (1 - t)(B + r_2 G)$$

where r_1 and r_2 are rational, $r_1 < r < r_2$, and t is a real number with $0 < t < 1$. Therefore, we can assume that r is rational. Let $f : Y \to X$ be a resolution. Then it is sufficient to check that $f^*B + rf^*G$ is big by Lemma 0.5. So, we can assume that X is smooth and projective. By Kodaira's lemma (cf. Lemma 0.2), we can write $lB \sim A + E$, where A is an ample Cartier divisor, E is an effective Carteir divisor, and l is a positive integer. Thus, $l(B + rG) \sim (A + lrG) + E$. We note that $A + lrG$ is an ample $\mathbb{Q}$-divisor. This implies that $B + rG$ is a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We finish the proof. □

By Proposition 0.12, we can discuss the bigness of $L - \omega$ in Theorem ?? below, where ω is the quasi-log canonical class of the quasi-log pair $[X, \omega]$. We note that ω is defined up to $\mathbb{R}$-linear equivalence class (see Remark ??).

Proposition 0.13 seems to be missing in the literature.

lemCD **Proposition 0.13.** *Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on a* normal *complete variety X. Then the following conditions are equivalent.*

(1) *D is big.*
(2) *There exist a positive rational number α and a positive integer m_0 such that*

$$\alpha m^{\dim X} \leq \dim H^0(X, \mathcal{O}_X(\llcorner m m_0 D \lrcorner))$$

for $m \gg 0$.

Proof. First, we assume (2). Let $f : Y \to X$ be a resolution. By Lemma 0.7, we have $\alpha m^{\dim X} \leq \dim H^0(X, \mathcal{O}_X(\llcorner m m_0 f^*D \lrcorner))$. By the usual argument as in the proof of Kodaira's lemma (cf. Lemma 0.2), we can write $f^*D \equiv A + E$, where A is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and E is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. By using Lemma 0.14 and Lemma 0.15 below, we can write $A + E \equiv \sum a_i G_i$ where a_i is a positive real number and G_i is a big Cartier divisor for every i. By Proposition 0.12, f^*D is a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. Let D' be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X whose coefficients are very close to those of D. Then $A + f^*D' - f^*D$ is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y. Therefore, $f^*D' \equiv (A + f^*D' - f^*D) + E$ is also a big $\mathbb{Q}$-divisor on Y. By Lemma 0.5, D' is a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. This means that there exists a big Cartier divisor M on X (see Example 0.16 below). By the assumption, we can write $lD \sim M + E'$, where E' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (see, for example, the usual proof of Kodaira's lemma: Lemma 0.2). By using Lemma 0.14 and Lemma 0.15 below, we can write $M + E' \equiv \sum a'_i G'_i$, where a'_i is a positive real number and G'_i is a big Cartier divisor for every i. By Proposition 0.12, D is a big $\mathbb{R}$-divisor on X.

Next, we assume (1). Let $f : Y \to X$ be a resolution. Then f^*D is big by Definition 0.6 and Lemma 0.5. By Lemma 0.7 and Lemma 0.8, we obtain the desired estimate in (2). $\square$

We have already used the following lemmas in the proof of Proposition 0.13.

lemCDD **Lemma 0.14.** *Let X be a normal variety and B an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then B can be written in the form*

$$B = \sum_i b_i B_i$$

where each B_i is an effective Cartier divisor and b_i is a positive real number for every i.

Proof. We can write $B = \sum_{j=1}^{l} d_j D_j$, where d_j is a real number and D_j is Cartier for every j. We put $E = \cup_j \mathrm{Supp} D_j$. Let $E = \sum_{k=1}^{m} E_k$ be the irreducible decomposition. We can write $D_j = \sum_{k=1}^{m} a_k^j E_k$ for every j. Note that a_k^j is integer for every j and k. We can also write $B = \sum_{k=1}^{m} c_k E_k$ with $c_k \geq 0$ for every k. We consider

$$\mathcal{E} = \{(r_1, \cdots, r_l) \in \mathbb{R}^l \mid \sum_{j=1}^{l} r_j a_k^j \geq 0 \text{ for every } k\} \subset \mathbb{R}^l.$$

Then $\mathcal{E}$ is a rational convex polyhedral cone and $(d_1, \cdots, d_l) \in \mathcal{E}$. Therefore, we can find effective Cartier divisors B_i and positive real numbers b_i such that $B = \sum_i b_i B_i$. □

lemCDDD **Lemma 0.15.** *Let B be a big Cartier divisor on a normal variety X and G an effective Cartier divisor on X. Then $B + rG$ is big for any positive real number r.*

Proof. If r is rational, then this lemma is obvious by the definition of big $\mathbb{Q}$-divisors. If r is not rational, then we can write

$$B + rG = t(B + r_1 G) + (1 - t)(B + r_2 G)$$

where r_1 and r_2 are rational, $0 < r_1 < r < r_2$, and t is a real number with $0 < t < 1$. By Definition defnA 0.6, $B + rG$ is a big $\mathbb{R}$-divisor. □

Example exAB 0.16 implies that a normal complete variety does not always have big Cartier divisors. For the details of Example exAB 0.16, see fujino-km [?, Section 4].

exAB **Example 0.16.** Let Δ be the fan in $\mathbb{R}^3$ whose rays are generated by $v_1 = (1, 0, 1)$, $v_2 = (0, 1, 1)$, $v_3 = (-1, -2, 1)$, $v_4 = (1, 0, -1)$, $v_5 = (0, 1, -1)$, $v_6 = (-1, -1, -1)$ and whose maximal cones are

$$\langle v_1, v_2, v_4, v_5\rangle, \langle v_2, v_3, v_5, v_6\rangle, \langle v_1, v_3, v_4, v_6\rangle, \langle v_1, v_2, v_3\rangle, \langle v_4, v_5, v_6\rangle.$$

Then the associated toric threefold X is complete with $\rho(X) = 0$. More precisely, every Cartier divisor on X is linearly equivalent to zero.

Let $f : Y \to X$ be the blow-up along $v_7 = (0, 0, -1)$ and E the f-exceptional divisor on Y. Then we can check that $\rho(Y) = 1$ and that $\mathcal{O}_Y(E)$ is a generator of $\mathrm{Pic}(Y)$. Therefore, there are no big Cartier divisors on Y.

The next lemma is almost obvious.

lemC **Lemma 0.17.** *Let V be a complete irreducible variety and D a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on V. Let $g : W \to V$ be an arbitrary proper birational morphism from an irreducible variety W. Then g^*D is big.*

Proof. By Definition 0.6, we can assume that D is Cartier. We obtain the following commutative diagram.

$$\begin{array}{ccc} W & \xleftarrow{\mu} & W^{\nu} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ V & \xleftarrow[\nu]{} & V^{\nu} \end{array}$$

Here, $\mu : W^{\nu} \to W$ and $\nu : V^{\nu} \to V$ are the normalizations. Since $\nu^* D$ is big, $h^*\nu^* D = \mu^* g^* D$ is also big. We note that h is a birational morphism between normal varieties. Thus, $g^* D$ is big by Definition 0.3. □

Kodaira's lemma for big $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on normal varieties is also obvious by Proposition 0.13.

lemE **Lemma 0.18** (Kodaira's lemma for big $\mathbb{R}$-divisors on normal varieties)**.** *Let X be a complete irreducible* normal *variety and D a big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let M be an arbitrary Cartier divisor on X. Then there exist a positive integer l and an effective $\mathbb{R}$-divisor E on X such that $lD - M \sim E$.*

Finally, we discuss relatively big $\mathbb{R}$-divisors.

defnF **Definition 0.19** (Relatively big $\mathbb{R}$-divisors)**.** Let $\pi : X \to S$ be a proper morphism onto a variety S and D an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then D is called *π-big* if D_{η} is big on X_{η}, where η is the generic point of S.

We need the following lemma for the proofs of the Kawamata–Viehweg vanishing theorems (cf. Theorem ?? and Theorem ??).

lemGH **Lemma 0.20** (cf. [KMM, Corollary 0-3-6])**.** *Let $\pi : X \to S$ be a proper surjective morphism and D a π-nef and π-big $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then there exist a proper birational morphism $\mu : Y \to X$ from a smooth variety Y projective over S and divisors F_{α}'s on Y such that* $\mathrm{Supp}\mu^* D \cup (\cup F_{\alpha})$ *is a simple normal crossing divisor and such that $\mu^* D - \sum_{\alpha} \delta_{\alpha} F_{\alpha}$ is $\pi \circ \mu$-ample for some δ_{α} with $0 < \delta_{\alpha} \ll 1$.*

We can check Lemma 0.20 by Lemma 0.18 and Hironaka's resolution theorem.

References

kmm [KMM]
lposi [L]
nakayama2 [N]

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: fujino@math.kyoto-u.ac.jp