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Fundamental Properties of Basic Slc-trivial Fibrations I

Dedicated to Professor Shigefumi Mori on the occasion of his seventieth birthday

by

Osamu FUJINO

Abstract

We introduce the notion of basic slc-trivial fibrations. It is a generalization of that of Ambro's lc-trivial fibrations. Then we study fundamental properties of basic slc-trivial fibrations by using the theory of variations of mixed Hodge structure on cohomology with compact support. More precisely, we prove that the moduli part of a basic slc-trivial fibration is b-potentially nef. Note that the notion of basic slc-trivial fibrations is closely related to that of normal irreducible quasi-log canonical pairs. So the results obtained in this paper will play an important role in the theory of quasi-log schemes. Here we give a structure theorem for normal irreducible quasi-log canonical pairs as an application of the main theorem. This result makes the theory of quasi-log schemes more powerful and more flexible.

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Contents

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O. Fujino: Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan;
e-mail: fujino@math.sci.osaka-u.ac.jp

§1. Introduction

Let us introduce *basic slc-trivial fibrations* $f : (X, B) \to Y$. They consist of a projective surjective morphism $f : X \to Y$ from a simple normal crossing variety X to a normal irreducible variety Y such that (X, B) is a simple normal crossing pair and that $K_X + B$ is $\mathbb{Q}$-linearly trivial over Y. More precisely, we assume:

(1) Y is a normal irreducible variety,
(2) every stratum of X is dominant onto Y and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$,
(3) B is a $\mathbb{Q}$-divisor on X such that (X, B) is a simple normal crossing pair and that $B = B^{\leq 1}$ holds over the generic point of Y,
(4) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $K_X + B \sim_{\mathbb{Q}} f^*D$, and
(5) $\operatorname{rank} f_*\mathcal{O}_X(\lceil -(B^{<1})\rceil) = 1$.

We note that X is not necessarily irreducible in the above setup. It may be a *reducible* simple normal crossing variety. Of course, we are mainly interested in the case where X is reducible. The notion of basic slc-trivial fibrations is a natural generalization of that of *lc-trivial fibrations* (see [A4] and [FG2]) and will suit the theory of quasi-log schemes very well.

In the above setup, let $\sigma : Y' \to Y$ be a birational morphism from a normal irreducible variety Y'. Then we can construct the following commutative diagram of basic slc-trivial fibrations:

$$\begin{tikzcd}
(X', B_{X'}) \arrow[r, "\mu"] \arrow[d, "f'"'] & (X, B) \arrow[d, "f"] \\
Y' \arrow[r, "\sigma"'] & Y,
\end{tikzcd}$$

where $B_{X'}$ is defined by $K_{X'} + B_{X'} = \mu^*(K_X + B)$ and $f' : (X', B_{X'}) \to Y'$ is nothing but the base change of $f : (X, B) \to Y$ by $\sigma : Y' \to Y$ on a nonempty Zariski open set of Y'. We call $f' : (X', B_{X'}) \to Y'$ an *induced basic slc-trivial fibration* of $f : (X, B) \to Y$ by $\sigma : Y' \to Y$. As for lc-trivial fibrations, we can define a *discriminant* $\mathbb{Q}$*-b-divisor* $\mathbf{B}$ and a *moduli* $\mathbb{Q}$*-b-divisor* $\mathbf{M}$ on Y associated to $f : (X, B) \to Y$ (see 4.5).

Before we state the main theorem of this paper, we have to introduce the notion of *potentially nef* $\mathbb{Q}$-divisors.

Definition 1.1 (Potentially nef divisors, see Definition 2.5). Let X be a normal irreducible variety and let D be a divisor on X. If there exist a normal complete variety $\overline{X}$ which contains X as a dense Zariski open set and a nef divisor $\overline{D}$ on $\overline{X}$ such that $D = \overline{D}|_X$, then D is called a *potentially nef* divisor on X. A finite $\mathbb{R}_{>0}$-linear (resp. $\mathbb{Q}_{>0}$-linear) combination of potentially nef divisors is called a *potentially nef* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor).

Let us state the main theorem of this paper, which is a generalization of [A4, Theorem 0.2] (see also [FG2, Theorem 3.6]).

Theorem 1.2 (Main Theorem). *Let $f : (X, B) \to Y$ be a basic slc-trivial fibration and let $\mathbf{B}$ and $\mathbf{M}$ be the induced discriminant and moduli $\mathbb{Q}$-b-divisors of Y respectively. Then we have the following properties:*

(i) $\mathbf{K} + \mathbf{B}$ *is $\mathbb{Q}$-b-Cartier, and*

(ii) $\mathbf{M}$ *is b-potentially nef, that is, there exists a proper birational morphism $\sigma : Y' \to Y$ from a normal variety Y' such that $\mathbf{M}_{Y'}$ is a potentially nef $\mathbb{Q}$-divisor on Y' and that $\mathbf{M} = \overline{\mathbf{M}_{Y'}}$.*

We note that $\mathbf{K}$ in Theorem 1.2 is the *canonical b-divisor* of Y. For the precise definition of $\mathbb{Q}$-*b-divisors* and *b-potentially nef divisors*, see Definition 2.12 below.

Theorem 1.2 can be restated as follows without using b-divisors.

Theorem 1.3. *Let $f : (X, B) \to Y$ be a basic slc-trivial fibration. Then there is a proper birational morphism $\sigma : Y' \to Y$ from a normal variety Y' such that*

(i) *$K_{Y'} + B_{Y'}$ is $\mathbb{Q}$-Cartier and $\nu^*(K_{Y'} + B_{Y'}) = K_{Y''} + B_{Y''}$ for every proper birational morphism $\nu : Y'' \to Y'$ from a normal variety Y'', and*

(ii) *$M_{Y'}$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y' that is potentially nef and $\nu^* M_{Y'} = M_{Y''}$ for every proper birational morphism $\nu : Y'' \to Y'$ from a normal variety Y''.*

We note that $B_{Y'}$ (resp. $B_{Y''}$) is the discriminant $\mathbb{Q}$-divisor on Y' (resp. Y'') and that $M_{Y'}$ (resp. $M_{Y''}$) is the moduli $\mathbb{Q}$-divisor on Y' (resp. Y'') in Theorem 1.3.

In [A4], Florin Ambro established Theorem 1.2 under the assumption that (X, B) has only sub kawamata log terminal singularities over the generic point of Y. The case where (X, B) has only sub log canonical singularities over the generic point of Y was proved in [FG2]. Note that Ambro used the theory of variations of

Hodge structure in [A4] and Gongyo and the author used the theory of variations of mixed Hodge structure in [FG2].

On moduli $\mathbb{Q}$-b-divisors, we pose the following conjecture.

Conjecture 1.4 (b-semi-ampleness conjecture). *Let $f : (X, B) \to Y$ be a basic slc-trivial fibration. Then the moduli part $\mathbf{M}$ is b-semi-ample.*

By Lemma 4.12 below, we see that it is sufficient to prove Conjecture 1.4 under the extra assumption that Y is complete. Conjecture 1.4 is still widely open even when X is a smooth irreducible variety and $B = 0$. For some known cases and related topics, we recommend the reader to see [Ka2], [Fn2], [A5], [PrSh], [FG2], and so on. In a joint paper with Taro Fujisawa and Haidong Liu (see [FFL]), we will prove:

Theorem 1.5. *If Y is complete and $\mathbf{M}_{Y'}$ is numerically trivial in Theorem 1.2, then $\mathbf{M}_{Y'} \sim_{\mathbb{Q}} 0$ holds.*

As an easy consequence of Theorem 1.5, we have the following result.

Corollary 1.6. *Conjecture 1.4 holds true when Y is a curve.*

We note that Theorem 1.5 is a generalization of [A5, Theorem 3.5] and [Fl, Theorem 1.3] and that Corollary 1.6 is a generalization of [A4, Theorem 0.1]. For the details of Theorem 1.5 and Corollary 1.6, see [FFL].

As an application of Theorem 1.3, we will prove the following theorem. Theorem 1.7, which is one of the main motivations of this paper, will play an crucial role in the theory of quasi-log schemes.

Theorem 1.7 (Structure theorem for normal irreducible quasi-log canonical pairs). *Let $[X, \omega]$ be a quasi-log canonical pair such that X is a normal irreducible variety. Then there exists a projective birational morphism $p : X' \to X$ from a smooth quasi-projective variety X' such that*

$$K_{X'} + B_{X'} + M_{X'} = p^*\omega,$$

where $B_{X'}$ is a subboundary $\mathbb{R}$-divisor, that is, $B_{X'} = B_{X'}^{\leq 1}$, such that $\operatorname{Supp} B_{X'}$ *is a simple normal crossing divisor and that $B_{X'}^{<0}$ is p-exceptional, and $M_{X'}$ is a potentially nef $\mathbb{R}$-divisor on X'. Furthermore, we can make $B_{X'}$ satisfy $p(B_{X'}^{=1}) =$ $\mathrm{Nqklt}(X, \omega)$.*

We further assume that $[X, \omega]$ has a $\mathbb{Q}$-structure. Then we can make $B_{X'}$ and $M_{X'}$ $\mathbb{Q}$-divisors in the above statement.

We note that there are many examples of quasi-log canonical pairs in the theory of minimal models.

Example 1.8. (1) Let (X, Δ) be a quasi-projective semi-log canonical pair. Then $[X, \omega]$, where $\omega = K_X + \Delta$, is a quasi-log canonical pair such that W is a qlc stratum of $[X, \omega]$ if and only if W is an slc stratum of (X, Δ). For the details, see [Fn7, Theorem 1.2].

(2) Let W be a qlc stratum of a quasi-log canonical pair $[X, \omega]$. Then $[W, \omega|_W]$ is also a quasi-log canonical pair by adjunction (see [Fn10, Theorem 6.3.5]).

(3) Let $[X, \omega]$ be a quasi-log canonical pair such that X is irreducible. Let $\nu : X^\nu \to X$ be the normalization of X. Then we can prove that $[X^\nu, \nu^*\omega]$ is a quasi-log canonical pair. For the details, see [FLh1, Theorem 1.1].

(4) Let W be an slc stratum of a quasi-projective semi-log canonical pair (X, Δ). Then, by (1), (2), and (3) above, we see that $[W, \omega|_W]$ and $[W^\nu, \nu^*(\omega|_W)]$ are quasi-log canonical pairs, where $\omega = K_X + \Delta$ and $\nu : W^\nu \to W$ is the normalization of W.

Here we give an important remark on Theorem 1.7.

Remark 1.9 (Generalized polarized pairs). We put $B_X = p_*B_{X'}$ and $M_X = p_*M_{X'}$ in Theorem 1.7. Then B_X is a boundary $\mathbb{R}$-divisor on X, that is, an effective $\mathbb{R}$-divisor on X with $B_X = B_X^{\leq 1}$, since $B_{X'}^{<0}$ is p-exceptional. Of course, $K_X + B_X + M_X$ is $\mathbb{R}$-Cartier by construction. Let $X \to S$ be any projective morphism between quasi-projective varieties. Then, $(X, B_X + M_X)$ is a *generalized polarized pair* which comes with the data $X' \xrightarrow{p} X \longrightarrow S$ and $M_{X'}$ as in [BZ, Definition 1.4]. Moreover, we can easily check that $(X, B_X + M_X)$ is *generalized lc* in the sense of [BZ, Definition 4.1]. We note that $(X, B_X + M_X)$ is *generalized klt* in the sense of [BZ, Definition 4.1] when $\mathrm{Nqklt}(X, \omega) = \emptyset$. Since $M_{X'}$ is potentially nef $\mathbb{R}$-divisor, $M_{X'}$ is a finite $\mathbb{R}_{>0}$-linear combination of relatively nef Cartier divisors over S. Hence $(X, B_X + M_X)$ is an NQC g-pair in the sense of [HL, Definition 2.13]. For the details of generalized polarized pairs, we recommend the reader to see [BZ, Section 4] and [HL].

By Theorem 1.7, we can prove a kind of subadjunction formula for minimal qlc strata of quasi-log canonical pairs. Corollary 1.10 is a complete generalization of [Ka3, Theorem 1]. For a different generalization of [Ka3, Theorem 1], see [FG1, Theorem 1.2]. We also recommend the reader to see [Fn16] for a generalization of Corollary 1.10.

Corollary 1.10 (Subadjunction for minimal qlc strata). *Let $[X, \omega]$ be a quasi-log canonical pair and let W be a minimal qlc stratum of $[X, \omega]$. We assume that W*

is quasi-projective and H is any ample $\mathbb{R}$-divisor on W. Then we can construct an effective $\mathbb{R}$-divisor Δ_W on W such that (W, Δ_W) is kawamata log terminal with $K_W + \Delta_W \sim_{\mathbb{R}} \omega|_W + H$. We further assume that $[X, \omega]$ has a $\mathbb{Q}$-structure and H is an ample $\mathbb{Q}$-divisor on W. Then we can make Δ_W a $\mathbb{Q}$-divisor with $K_W + \Delta_W \sim_{\mathbb{Q}} \omega|_W + H$.

As an application of Theorem 1.7, we prove:

Corollary 1.11 ([FLh2]). *Every quasi-log canonical pair has only Du Bois singularities.*

Corollary 1.11 is a complete generalization of [Ko3, Corollary 6.32]. We discuss Corollary 1.11 and some related topics in a joint paper with Haidong Liu (see [FLh2]). We note that the arguments in [FLh2] and this paper are free from the minimal model program.

By using Theorem 1.7, we will also prove:

Corollary 1.12 (Simply connectedness and rationally chain connectedness of quasi-log canonical Fano pairs, [Fn16]) *Let $[X, \omega]$ be a connected projective quasi-log canonical pair. Assume that $-\omega$ is ample. Then X is simply connected and rationally chain connected.*

Corollary 1.13 (Lengths of extremal rational curves, [Fn16]). *Let $[X, \omega]$ be a quasi-log canonical pair and let $\pi : X \to S$ be a projective morphism onto a variety S. Then every ω-negative extremal ray R of the relative Kleiman–Mori cone $\overline{NE}(X/S)$ is spanned by a rational curve C with $0 < -\omega \cdot C \leq 2 \dim X$.*

We will discuss a generalization of Corollary 1.10, Corollaries 1.12 and 1.13, and some other applications in [Fn16].

Finally, as an application of Theorem 1.7, we prove the following Fujita-type freeness for quasi-log canonical surfaces in a joint paper with Haidong Liu (see [FLh3]).

Corollary 1.14 ([FLh3]). *Let $[X, \omega]$ be a projective quasi-log canonical pair of dimension two and let M be a Cartier divisor on X. We put $N = M - \omega$. Assume that $N^2 \cdot X_i > 4$ for every two-dimensional irreducible component X_i of X and that $N \cdot C \geq 2$ for every curve C on X. Then the complete linear system $|M|$ is basepoint-free.*

Corollary 1.14 is a generalization of the result for semi-log canonical surfaces obtained in [Fn11].

We strongly recommend the reader to see [FLh2], [FLh3], [FFL], and [Fn16] after reading this paper.

1.15 (Historical comments on related papers, and so on). One of the starting points of this paper is Mori's work in [M, Section 5, Part II]. It is a prototype of the so-called Fujino–Mori canonical bundle formula (see [FM]). We note that [FM] is an expanded version of Mori's unpublished preprint written and circulated around 1994. We also note that the moduli part is called the *semistable part* in [FM]. In [Ka3, Theorem 2], Kawamata essentially proved that the moduli part of a klt-trivial fibration is nef. After the author learned [Ka3, Theorem 2], he soon got some applications of Kawamata's result in [Fn1] and then obtained the so-called Fujino–Mori canonical bundle formula with Shigefumi Mori by combining Mori's unpublished preprint with [Ka3, Theorem 2]. Then the author discussed the semi-ampleness of semistable parts for certain algebraic fiber spaces in [Fn2] and also proved that the semistable part behaves very well under pull-back in [Fn2, Section 4]. In [Fn3, Section 4], he essentially proved that the moduli part of an lc-trivial fibration is nef. This result is a direct generalization of [Ka3, Theorem 2]. From the Hodge theoretic viewpoint, [Ka3] is *pure* and [Fn3, Section 4] is *mixed*. We note that [Fn3, Sections 4 and 5] was not published. If the author remembers correctly, he planned to divide [Fn3] into two papers following the editor's recommendation (see [Fn4, Remark 1.1]). On the other hand, Ambro started to study some applications of [Ka3, Theorem 2] in his thesis (see [A1]) independently. Then he formulated lc-trivial fibrations, which are now called klt-trivial fibrations in this paper, and proved that the moduli part is b-nef (see [A4]). His result recovers [Ka3, Theorem 2]. However, his proof is different from Kawamata's original one in [Ka3] and is essentially the same as the arguments in [M, Section 5, Part II] and [Fn2, Section 4]. Moreover, in [A5], Ambro proved that the moduli part of a klt-trivial fibration is *b-nef and abundant* under some mild assumptions. Note that this deep result was generalized for lc-trivial fibrations by [FG2]. More precisely, in [FG2], Gongyo and the author showed how to reduce some problems for lc-trivial fibrations to those for klt-trivial fibrations. On the semi-ampleness, Kawamata essentially proved that the moduli part of an lc-trivial fibration is semi-ample when the dimension of general fibers is one in [Ka2] (see also [PrSh]). As we mentioned before, the b-semi-ampleness conjecture (see Conjecture 1.4) is still widely open. We recommend the reader to see [Fn8], where the author discussed various topics around lc-trivial fibrations. Roughly speaking, in [Fn8], the author formulated lc-trivial fibrations for Kähler manifolds and proved the finite generation of canonical rings for compact Kähler manifolds. We also recommend the reader to see [Fn12] for a survey on some related topics. Finally, we note that Kollár surveys lc-trivial fibrations in [Ko2]. His treatment is slightly different from others.

From 2006 to 2007, the author wrote a preprint [Fn5], where he obtained some generalizations of Kollár's injectivity, vanishing, and torsion-free theorems by using the theory of mixed Hodge structures on cohomology with compact support. Note that a completely revised and expanded version of [Fn5] is now published as Chapter 5 of [Fn10] (see also [Fn9] and [Fn13]). The main motivation of [Fn5] is to establish some generalizations of Kollár's theorems for the theory of quasi-log schemes introduced by Florin Ambro (see [A3]). In 2009, he wrote a very preliminary version of [FF1] and started a joint work with Taro Fujisawa. One of his motivations of [FF1] is to formulate an ultimate generalization of the Fujita–Zucker–Kawamata semipositivity theorem and obtain some kind of canonical bundle formula for reducible varieties by using the theory of variations of mixed Hodge structure on cohomology with compact support. Soon after they released a preprint version of [FF1] in 2012, the author got the projectivity of the coarse moduli spaces of stable varieties in [Fn14] as an easy application of [FF1]. He thought that the paper [Fn14] was an important unexpected application of [FF1] because everyone thought that the projectivity of the coarse moduli spaces of stable varieties had been already proved in [Ko1]. Note that the main result of [Fn14] now can be proved without using the theory of variations of mixed Hodge structure (see [Fn15]). The proof in [Fn15] uses the Kollár–Ohsawa type vanishing theorem for simple normal crossing pairs.

In this paper, we discuss a kind of canonical bundle formula for reducible varieties, which we call a basic slc-trivial fibration, as an application of [FF1]. This paper relates the theory of variations of mixed Hodge structure on cohomology with compact support discussed in [FF1] to the theory of quasi-log schemes established in [Fn10, Chapter 6]. Therefore, the results in this paper will play a crucial role in the study of quasi-log schemes.

We briefly explain the organization of this paper. In Section 2, we fix the notation and recall various basic results for the reader's convenience. Here we introduce the notion of potentially nef divisors and explain some basic properties. Section 3 is a short section on the theory of variations of mixed Hodge structure on cohomology with compact support. We explain some results in [FF1]. Note that Theorem 3.1 is the main ingredient of this paper. Theorem 3.1 is a generalization of the Fujita–Zucker–Kawamata semipositivity theorem. In Section 4, we introduce the notion of (pre-)basic slc-trivial fibrations, define discriminant $\mathbb{Q}$-b-divisors and moduli $\mathbb{Q}$-b-divisors, and study some basic properties. The notion of basic slc-trivial fibrations is a generalization of that of Ambro's lc-trivial fibrations. In Section 5, we treat an inversion of adjunction for pre-basic slc-trivial fibrations under some assumptions. Although we do not need the result in Section 5 explicitly in this paper, the calculation in Section 5 may help the reader understand Theorem 1.7. In Section 6, we take a cyclic cover of the generic fiber of a given basic slc-trivial fibration to construct a new pre-basic slc-trivial fibration. Then we interpret the moduli part of a given basic slc-trivial fibration Hodge theoretically. In Section 7, we discuss various covering lemmas essentially due to Yujiro Kawamata. We will use them in the subsequent sections. In Section 8, we prove that the moduli part of a basic slc-trivial fibration behaves very well under pull-back by generically finite surjective morphisms with some mild assumptions. Section 9 is devoted to the proof of the main theorem: Theorem 1.2. In Section 10, we treat normal irreducible quasi-log canonical pairs. By the main result in Section 10, we see that a normal irreducible quasi-log canonical pair with $\mathbb{Q}$-structure can be seen as a basic slc-trivial fibration. This fact is one of the main motivations to introduce the notion of basic slc-trivial fibrations. In Section 11, we prove Theorem 1.7 as an application of Theorem 1.2. By this theorem, we see that normal irreducible quasi-log canonical pairs are similar to log canonical pairs. Section 12 is a short section on a remark about the basepoint-free theorem for quasi-log canonical pairs. In the final section: Section 13, we give some supplementary remarks on [FF1], which is one of the main ingredients of this paper, for the reader's convenience.

Conventions. We will work over $\mathbb{C}$, the complex number field, throughout this paper. We will freely use the basic notation of the minimal model program as in [Fn6] and [Fn10]. A *scheme* means a separated scheme of finite type over $\mathbb{C}$. A *variety* means a reduced scheme, that is, a reduced separated scheme of finite type over $\mathbb{C}$. In this paper, a variety may be reducible. However, we sometimes assume that a variety is irreducible without mentioning it explicitly if there is no danger of confusion. The set of integers (resp. rational numbers or real numbers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{Q}$ or $\mathbb{R}$). The set of nonnegative (resp. positive) rational numbers is denoted by $\mathbb{Q}_{\geq 0}$ (resp. $\mathbb{Q}_{>0}$). We use $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{>0}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$ similarly.

§2. Preliminaries

In this section, we fix the notation and recall some basic results for the reader's convenience.

2.1 (Divisors). Let X be a scheme with structure sheaf $\mathcal{O}_X$ and let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$. Let $\mathcal{K}_X^*$ denote the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$, and $\mathcal{O}_X^*$ the sheaf of invertible elements in $\mathcal{O}_X$. We note that $\mathcal{O}_X \subset \mathcal{K}_X$ and $\mathcal{O}_X^* \subset \mathcal{K}_X^*$ hold. A *Cartier divisor* D on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, D is an element of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. A $\mathbb{Q}$-*Cartier divisor* (resp. An $\mathbb{R}$-*Cartier divisor*) is an element of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)\otimes_{\mathbb{Z}}\mathbb{Q}$ (resp. $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)\otimes_{\mathbb{Z}}\mathbb{R}$).

Let D_1 and D_2 be two $\mathbb{R}$-Cartier divisors on X. Then D_1 is *linearly* (resp. $\mathbb{Q}$-*linearly*, or $\mathbb{R}$-*linearly*) *equivalent* to D_2, denoted by $D_1 \sim D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$, or $D_1 \sim_{\mathbb{R}} D_2$) if

$$D_1 = D_2 + \sum_{i=1}^{k} r_i(f_i)$$

such that $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{Z}$ (resp. $r_i \in \mathbb{Q}$, or $r_i \in \mathbb{R}$) for every i. We note that (f_i) is a *principal Cartier divisor* associated to f_i, that is, the image of f_i by

$$\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Let $f : X \to Y$ be a morphism between schemes. If there exists an $\mathbb{R}$-Cartier (resp. a $\mathbb{Q}$-Cartier) divisor B on Y such that $D_1 \sim_{\mathbb{R}} D_2 + f^*B$ (resp. $D_1 \sim_{\mathbb{Q}} D_2 + f^*B$), then D_1 is said to be *relatively* $\mathbb{R}$-*linearly* (*resp.* $\mathbb{Q}$-*linearly*) *equivalent* to D_2. It is denoted by $D_1 \sim_{\mathbb{R},f} D_2$ or $D_1 \sim_{\mathbb{R},Y} D_2$ (resp. $D_1 \sim_{\mathbb{Q},f} D_2$ or $D_1 \sim_{\mathbb{Q},Y} D_2$).

From now on, let X be an equidimensional scheme. We note that X is not necessarily regular in codimension one. A (*Weil*) *divisor* D on X is a finite formal sum

$$D = \sum_i d_i D_i,$$

where D_i is an irreducible reduced closed subscheme of X of pure codimension one and d_i is an integer for every i such that $D_i \neq D_j$ for every $i \neq j$. If $d_i \in \mathbb{Q}$ (resp. $d_i \in \mathbb{R}$) for every i, then D is called a $\mathbb{Q}$-*divisor* (resp. an $\mathbb{R}$-*divisor*). Let $D = \sum_i d_i D_i$ be an $\mathbb{R}$-divisor as above. We put

$$D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{=1} = \sum_{d_i = 1} D_i, \quad \text{and} \quad \lceil D \rceil = \sum_i \lceil d_i \rceil D_i,$$

where $\lceil d_i \rceil$ is the integer defined by $d_i \leq \lceil d_i \rceil < d_i + 1$. Moreover, we put $\lfloor D \rfloor = -\lceil -D \rceil$ and $\{D\} = D - \lfloor D \rfloor$. Let D be an $\mathbb{R}$-divisor. We call D a *subboundary* $\mathbb{R}$-divisor if $D = D^{\leq 1}$ holds. When D is effective and $D = D^{\leq 1}$ holds, we call D a *boundary* $\mathbb{R}$-divisor.

We further assume that $f : X \to Y$ is a surjective morphism onto an irreducible variety Y. Then we put

$$D^v = \sum_{f(D_i) \subsetneq Y} d_i D_i \quad \text{and} \quad D^h = D - D^v,$$

and call D^v the *vertical part* and D^h the *horizontal part* of D with respect to $f : X \to Y$, respectively.

2.2 (Singularities of pairs). A pair (X, Δ) consists of a normal variety X and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A pair (X, Δ) is called *sub kawamata log terminal* (resp. *sub log canonical*) if for any proper birational morphism $f : Y \to X$ from a normal variety Y, every coefficient of Δ_Y is < 1 (resp. ≤ 1) where

$$K_Y + \Delta_Y := f^*(K_X + \Delta).$$

A pair (X, Δ) is called *kawamata log terminal* (resp. *log canonical*) if (X, Δ) is sub kawamata log terminal (resp. sub log canonical) and Δ is effective.

Let (X, Δ) be a sub log canonical pair and let W be a closed subset of X. Then W is called a *log canonical center* of (X, Δ) if there exist a proper birational morphism $f : Y \to X$ from a normal variety Y and a prime divisor E on Y such that $\operatorname{mult}_E \Delta_Y = 1$ and $f(E) = W$.

We note that $-\operatorname{mult}_E \Delta_Y$ is denoted by $a(E, X, \Delta)$ for any prime divisor E on Y and is called the *discrepancy coefficient* of E with respect to (X, Δ).

Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Under this assumption, we can define the discrepancy coefficient $a(E, X, \Delta)$ for any prime divisor E *over* X by taking a suitable resolution of singularities. The *minimal log discrepancy* of (X, Δ) in a closed subset $Z \subsetneq X$ is

$$\operatorname{mld}_Z(X, \Delta) := \inf_{c_X(E) \subset Z} a(E, X, \Delta) + 1,$$

where E is a prime divisor over X and $c_X(E)$ is the center of E on X.

In this paper, we mainly treat reducible varieties. So we need the notion of (*sub*) *semi-log canonical singularities.*

Definition 2.3 (Semi-log canonical singularities). Let X be an equidimensional variety that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an $\mathbb{R}$-divisor on X such that no irreducible component of $\operatorname{Supp} \Delta$ is contained in the singular locus of X and that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We say that (X, Δ) has only *sub semi-log canonical* (*sub slc*, for short) singularities if (X^ν, Δ_{X^ν}) is sub log canonical, where $\nu : X^\nu \to X$ is the normalization of X and $K_{X^\nu} + \Delta_{X^\nu} = \nu^*(K_X + \Delta)$, that is, Δ_{X^ν} is the sum of the inverse images of Δ and the conductor of X. An *slc center* of (X, Δ) is the ν-image of an lc center of (X^ν, Δ_{X^ν}). An *slc stratum* of (X, Δ) means either an slc center of (X, Δ) or an irreducible component of X. If (X, Δ) has only sub semi-log canonical singularities and Δ is effective, then we say that (X, Δ) has only *semi-log canonical* (*slc*, for short) singularities.

If (X, Δ) is (sub) semi-log canonical and X is normal, then (X, Δ) is (sub) log canonical by definition.

For the details of semi-log canonical singularities, see [Fn7] and [Ko3].

2.4 (Potentially nef divisors). Let us introduce the notion of *potentially nef* divisors. It is indispensable for the main theorem of this paper: Theorem 1.2.

Definition 2.5 (Potentially nef divisors). Let X be a normal irreducible variety and let D be a divisor on X. If there exist a completion $\overline{X}$ of X, that is, $\overline{X}$ is a normal complete variety and contains X as a dense Zariski open set, and a nef divisor $\overline{D}$ on $\overline{X}$ such that $D = \overline{D}|_X$, then D is called a *potentially nef* divisor on X. A finite $\mathbb{R}_{>0}$-linear (resp. $\mathbb{Q}_{>0}$-linear) combination of potentially nef divisors is called a *potentially nef* $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor).

The following easy lemma is very important in some applications.

Lemma 2.6. *Let X be a normal irreducible quasi-projective variety, let D be a potentially nef divisor on X, and let H be an ample divisor on X. Then $D+H$ is ample.*

We give a detailed proof for the reader's convenience.

Proof of Lemma 2.6. It is sufficient to prove that the line bundle $\mathcal{O}_X(D+H)$ is ample. Therefore, by replacing D and H with mD and mH for some positive integer m, respectively, we may assume that H is very ample (see, for example, [H, Chapter II, Theorem 7.6]). Thus, there exists an embedding $i : X \hookrightarrow \mathbb{P}^N$ such that $\mathcal{O}_X(H) \simeq i^*\mathcal{O}_{\mathbb{P}^N}(1)$. Let $X^\dagger$ be the closure of X in $\mathbb{P}^N$. Let $\overline{X}$ be a completion of X on which there is a nef divisor $\overline{D}$ such that $D = \overline{D}|_X$. By [L, Lemma 2.2], which is an easy application of the flattening theorem (see [RG, Théorème (5.2.2)]), we can take an ideal sheaf $\mathcal{I}$ on $X^\dagger$ with $\operatorname{Supp} \mathcal{O}_{X^\dagger}/\mathcal{I} \subset X^\dagger \setminus X$ such that the blow-up of $X^\dagger$ along $\mathcal{I}$ eliminates the indeterminacy of $X^\dagger \dashrightarrow \overline{X}$. Therefore, by taking the normalization of the blow-up of $X^\dagger$ along $\mathcal{I}$, we get a projective birational morphism $\alpha : \widetilde{X} \to X^\dagger$, which is an isomorphism over X, from a normal variety $\widetilde{X}$ such that the induced birational map $\beta : \widetilde{X} \dashrightarrow \overline{X}$ is a morphism, and an effective divisor E on $\widetilde{X}$ such that $\operatorname{Supp} E \subset \widetilde{X} \setminus X$ and $-E$ is α-ample. Note that we can see X as a Zariski open set of $\widetilde{X}$.

$$\begin{tikzcd}
\overline{X} & & \\
X \arrow[u, hook] \arrow[r, hook] & X^\dagger \arrow[ul, dashed] & \widetilde{X} \arrow[l, "\alpha"] \arrow[ull, "\beta"']
\end{tikzcd}$$

Therefore, we can construct an ample line bundle $\mathcal{L}$ on $\widetilde{X}$ such that $\mathcal{L}|_X \simeq \mathcal{O}_X(lH)$ for some positive integer l. We consider the nef divisor $\beta^*\overline{D}$ on $\widetilde{X}$. Since $\widetilde{X}$ is projective and $\mathcal{L}$ is an ample line bundle on $\widetilde{X}$, $\mathcal{L} \otimes \mathcal{O}_{\widetilde{X}}(l\beta^*\overline{D})$ is ample. By restricting it to X, we obtain that $\mathcal{O}_X(lD + lH)$ is an ample line bundle on X. Thus, $\mathcal{O}_X(D + H)$ is ample. This is what we wanted. $\square$

We note that any Cartier divisor is potentially nef when X is affine.

Lemma 2.7. *Let X be a normal irreducible affine variety and let D be a* (*not necessarily effective*) *Weil divisor on X which is Cartier. Then there exist a normal irreducible projective variety $\overline{X}$ containing X as a dense Zariski open set and a Weil divisor $\overline{D}$ on $\overline{X}$ such that $D = \overline{D}|_X$ and that $\mathcal{O}_{\overline{X}}(\overline{D})$ is a very ample line bundle on $\overline{X}$. In particular, D is potentially nef.*

Proof. We fix a closed embedding $X \subset \mathbb{C}^N$. Then we take the closure X_1 of X in $\mathbb{P}^N$. Note that there exists a hyperplane H on $\mathbb{P}^N$ such that

$$\operatorname{Supp} H|_{X_1} = X_1 \setminus X. \tag{2.1}$$

Let X_2 be the normalization of X_1. In this situation, we can see X as a dense Zariski open set of X_2. Let D_2 be the closure of D in X_2. We take an ample Cartier divisor $H^\dagger$ on X_2 and a sufficiently large positive integer l. Then we can take an effective Weil divisor Γ which is linearly equivalent to $D_2 + lH^\dagger$, that is, $\Gamma - D_2 \sim lH^\dagger$. We take the normalization of the blow-up of X_2 along the ideal sheaf $\mathcal{O}_{X_2}(-\Gamma)$. Then we get a projective birational morphism $p : X_3 \to X_1$ from a normal variety X_3 and a Weil divisor D_3 on X_3 such that p is an isomorphism over X and that D_3 is a Cartier divisor satisfying $D = D_3|_X$. Note that we saw X as a dense Zariski open set of X_3. As in the proof of Lemma 2.6, we take the normalization of the blow-up of X_1 along a suitable ideal sheaf on X_1 to eliminate the indeterminacy of $p^{-1} : X_1 \dashrightarrow X_3$. Then we get the following commutative diagram

$$\begin{tikzcd} & \overline{X} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X_1 \arrow[rr, dashed, "p^{-1}"'] & & X_3, \end{tikzcd}$$

where $\alpha : \overline{X} \to X_1$ is a projective birational morphism from a normal irreducible variety $\overline{X}$ such that α is an isomorphism over X. By using (2.1), we can construct an ample divisor A on $\overline{X}$ with $\operatorname{Supp} A = \overline{X} \setminus X$. Of course, we saw X as a dense Zariski open set of $\overline{X}$. We put $\overline{D} := \beta^* D_3 + mA$ for some sufficiently large positive integer m. Then $\overline{D}$ is very ample and $\overline{D}|_X = D$ by construction. Therefore, we see that D is potentially nef. $\square$

We prepare one more easy lemma on ample divisors.

Lemma 2.8. *Let $f : X \to Y$ be a projective morphism between quasi-projective varieties. Let D be an f-ample Cartier (resp. $\mathbb{Q}$-Cartier or $\mathbb{R}$-Cartier) divisor on X and let H be an ample divisor on Y. Then $D + mf^*H$ is an ample divisor (resp. $\mathbb{Q}$-divisor or $\mathbb{R}$-divisor) for every sufficiently large positive integer m.*

Proof. If D is $\mathbb{Q}$-Cartier, then it is well known that $D + mf^*H$ is ample for every sufficiently large positive integer m. When D is an f-ample $\mathbb{R}$-divisor, we can write $D = \sum_{i=1}^{k} d_i D_i$ where $d_i \in \mathbb{R}_{>0}$ and D_i is an f-ample divisor for every i. Then

$$D + mf^*H = \sum_{i=1}^{k} d_i(D_i + m_i f^*H) + \left(m - \sum_{i=1}^{k} m_i d_i\right) f^*H,$$

where m_i is a positive integer such that $D_i + m_i f^*H$ is ample for every i. Therefore, it is sufficient to prove that $aA + bf^*H$ is ample, where a and b are positive real numbers and A is an ample divisor on X. We fix a positive integer l such that $A + lf^*H$ is ample. We take a positive real number c such that $0 < c \ll 1$ and $b - c \in \mathbb{Q}_{>0}$. Then we take a positive rational number d with $0 < d \ll 1$. In this situation, we can write

$$aA + bf^*H = \frac{c}{l}(A + lf^*H) + (dA + (b-c)f^*H) + \left(a - \frac{c}{l} - d\right) A.$$

This means that $aA + bf^*H$ is ample. Thus, we obtain that $D + mf^*H$ always can be written as a finite $\mathbb{R}_{>0}$-linear combination of ample divisors on X for every sufficiently large positive integer m. This is what we wanted. $\square$

We give some remarks on potentially nef divisors.

Remark 2.9. (1) Let X be a normal irreducible variety and let D be a potentially nef $\mathbb{R}$-divisor on X. Then $D \cdot C \geq 0$ for every complete integral curve C on X. In particular, D is π-nef for any proper morphism $\pi : X \to S$ onto a variety S.

(2) Let $\pi : X \to S$ be a projective morphism from a normal quasi-projective irreducible variety onto a quasi-projective variety S. Let D be a π-nef $\mathbb{R}$-divisor on X, let A be a π-ample $\mathbb{R}$-divisor on X, and let H be an ample divisor on S. Then, by Lemma 2.8, we can easily see that $D + A + m\pi^*H$ is an ample $\mathbb{R}$-divisor on X, that is, a finite $\mathbb{R}_{>0}$-linear combination of ample divisors on X, for every sufficiently large positive integer m. We note that $D + A$ is a π-ample $\mathbb{R}$-divisor on X.

2.10 (b-divisors). Let us quickly recall the notion of *b-divisors* introduced by Shokurov (see [Sh, Section 1]). We note that a b-divisor was originally called a *bi-divisor* in [Sh].

Let X be a normal variety and let $\mathrm{Div}X$ be the space of Weil divisors on X. A *b-divisor* on X is an element:

$$\mathbf{D} \in \mathbf{Div}X = \lim_{Y \to X} \mathrm{Div}Y,$$

where the (projective) limit is taken over all proper birational morphism $f : Y \to X$ from a normal variety Y under the pushforward homomorphism $f_* : \mathrm{Div}Y \to \mathrm{Div}X$. We can define $\mathbb{Q}$-*b-divisors* on X similarly. If $\mathbf{D} = \sum d_\Gamma \Gamma$ is a ($\mathbb{Q}$-)b-divisor on a normal variety X and $f : Y \to X$ is a proper birational morphism from a normal variety Y, then the *trace* of $\mathbf{D}$ on Y is the ($\mathbb{Q}$-)divisor

$$\mathbf{D}_Y := \sum_{\Gamma \text{ is a divisor on } Y} d_\Gamma \Gamma.$$

The $\mathbb{Q}$-*Cartier closure* of a $\mathbb{Q}$-Cartier ($\mathbb{Q}$-)divisor D on a normal variety X is the $\mathbb{Q}$-b-divisor $\overline{D}$ with trace

$$\overline{D}_Y = f^* D,$$

where $f : Y \to X$ is a proper birational morphism from a normal variety Y.

Definition 2.11 (Canonical b-divisor). Let X be a normal variety and let ω be a top rational differential form of X. Then (ω) defines a b-divisor $\mathbf{K}$. We call $\mathbf{K}$ the *canonical b-divisor* of X.

We need the following definition for Theorem 1.2 and Conjecture 1.4.

Definition 2.12 (b-potentially nef and b-semi-ample $\mathbb{Q}$-b-divisors, and $\mathbb{Q}$-b-Cartier divisors). Let X be a normal variety. A $\mathbb{Q}$-b-divisor $\mathbf{D}$ of X is *b-potentially nef* (resp. *b-semi-ample*) if there exists a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ and $\mathbf{D}_{X'}$ is potentially nef (resp. semi-ample). A $\mathbb{Q}$-b-divisor $\mathbf{D}$ of X is $\mathbb{Q}$-*b-Cartier* if there is a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

Lemma 2.13. *Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be $\mathbb{Q}$-b-divisors on a normal variety X. Assume that*

$$\mathbf{D}_1 = \mathbf{D}_2 + r\overline{(g)}$$

holds, where $\overline{(g)}$ is a $\mathbb{Q}$-Cartier closure of a principal Cartier divisor (g) associated to $g \in \Gamma(X, \mathcal{K}_X^*)$ and r is a rational number. In this situation, if $\mathbf{D}_1$ is $\mathbb{Q}$-b-Cartier, that is, $\mathbf{D}_1 = \overline{\mathbf{D}_{1X'}}$ for some proper birational morphism $X' \to X$ from a normal variety X', then $\mathbf{D}_2 = \overline{\mathbf{D}_{2X'}}$ holds.

Proof. It is obvious by definition. □

For more details on b-divisors, see, for example, [C, 2.3.2 b-divisors].

2.14 (Simple normal crossing pairs). In this paper, we will mainly treat simple normal crossing pairs.

Definition 2.15. We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y, where Y has a regular system of parameters $(x_1, \ldots, x_p, y_1, \ldots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X. If $(X, 0)$ is a simple normal crossing pair, then X is called a *simple normal crossing variety*. If (X, D) is a simple normal crossing pair and D is reduced, then D is called a *simple normal crossing divisor* on X. Let (X, D) be a simple normal crossing pair such that $D = D^{\leq 1}$ holds. Then it is easy to see that (X, D) is sub slc in the sense of Definition 2.3. In this situation, we simply say that W is a *stratum* of (X, D) if W is an slc stratum of (X, D) in the sense of Definition 2.3. We note that a stratum of a simple normal crossing variety X means a stratum of a simple normal crossing pair $(X, 0)$.

Let X be a simple normal crossing variety and let Δ be an $\mathbb{R}$-divisor on X such that no irreducible component of $\operatorname{Supp} \Delta$ is contained in the singular locus of X and that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let Z be a closed subset $Z \subsetneq X$ such that Z contains no stratum of X. Then we put

$$\operatorname{mld}_Z(X, \Delta) := \operatorname{mld}_{\nu^{-1}(Z)}(X^\nu, \Theta),$$

where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$, that is, Θ is the sum of the inverse images of Δ and the singular locus of X. We call $\operatorname{mld}_Z(X, \Delta)$ the *minimal log discrepancy* of (X, Δ) in a closed subset Z. We will use it in Theorem 5.1 below.

We close this section with a useful lemma. We will often use it in the subsequent sections without mentioning it explicitly. We note that the classical topology means the Euclidean topology in Lemma 2.16.

Lemma 2.16. *Let* (X, D) *be a simple normal crossing pair with* $\dim X = n$ *and let* $f : X \to Z$ *be a morphism onto an* m*-dimensional smooth irreducible variety* Z*. Assume that every stratum of* $(X, \operatorname{Supp} D)$ *is smooth over* Z*. Let* $a \in X$ *be any* *closed point. Then we have the following local analytic description of* $f : (X, D) \to Z$ *in a neighborhood of* $a \in X$.

(i) U *and* V *are open neighborhoods of* $a \in X$ *and* $f(a) \in Z$ *in the classical topology, respectively.*
(ii) W *is an open set of* $\mathbb{C}^{n+1}$ *in the classical topology.*
(iii) $(z_1, \ldots, z_m)$ *and* $(z_1, \ldots, z_{n+1})$ *are systems of local analytic coordinates of* V *and* W, *respectively.*
(iv) $p : W \to V$ *is the projection given by* $(z_1, \ldots, z_{n+1}) \mapsto (z_1, \ldots, z_m)$.
(v) U *is defined by a monomial equation* $z_{m+1} \cdots z_{m+p} = 0$ *in* W *and* $a = (0, \ldots, 0) \in W$.
(vi) $D|_U = \sum_{i=1}^{r} \alpha_i (z_{m+p+i} = 0)|_U$ *with* $\alpha_i \in \mathbb{R}$.
(vii) $f|_U = p \circ \iota$, *where* ι *is the natural closed embedding* $U \hookrightarrow W$.

$$
\begin{tikzcd}
U \arrow[r, hook, "\iota"] \arrow[dr, "f|_U"'] & W \arrow[d, "p"] \\
& V
\end{tikzcd}
$$

Let $\rho : Z' \to Z$ *be a morphism from a smooth irreducible variety* Z'. *We put* $X' = X \times_Z Z'$ *and consider the following commutative diagram.*

$$
\begin{tikzcd}
X \arrow[d, "f"'] & X' \arrow[l, "\rho'"'] \arrow[d, "f'"] \\
Z & Z' \arrow[l, "\rho"]
\end{tikzcd}
$$

Let D' *be the pull-back of* D *on* X' *by* ρ'. *Then we can easily see that* (X', D') *is a simple normal crossing pair and every stratum of* $(X', \operatorname{Supp} D')$ *is smooth over* Z' *by the above local analytic description of* $f : (X, D) \to Z$.

Proof. By definition, X is Zariski locally a simple normal crossing divisor on a smooth variety Y in a neighborhood of $a \in X$ (see Definition 2.15). By taking a small open set W of Y containing a in the classical topology, $f|_U : U \to Z$, where $U := X \cap W$, extends to a holomorphic map $W \to Z$ (see, for example, [Fi, 0.22. Corollary 2]). Since every stratum of $(X, \operatorname{Supp} D)$ is smooth over Z by assumption, we obtain the desired local analytic description by shrinking W suitably around a and taking a small open neighborhood V of $f(a)$ in Z in the classical topology. By this local analytic description, we can easily see that $f : (X, D) \to Z$ behaves well under base change. $\square$

§3. Variations of mixed Hodge structure

In this section, let us quickly recall the main result of [FF1] (see also [FFS]). We note that Theorem 3.1 is the main ingredient of Theorem 1.2. Theorem 3.1 follows from the theory of variations of mixed Hodge structure on cohomology with compact support.

Theorem 3.1 ([FF1, Theorems 7.1 and 7.3]). *Let (X, D) be a simple normal crossing pair such that D is reduced and let $f : X \to Y$ be a projective surjective morphism onto a smooth variety Y such that every stratum of (X, D) is dominant onto Y. Assume that there exists a simple normal crossing divisor Σ_Y on Y such that every stratum of (X, D) is smooth over $Y^* = Y \setminus \Sigma_Y$. Then we have*

(i) *$f_*\omega_{X/Y}(D)$ is a locally free sheaf on Y.*

We further assume that all the local monodromies on the local system

$$R^d(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*}$$

around Σ_Y are unipotent, where $d = \dim X - \dim Y$, $X^* = f^{-1}(Y^*)$, $D^* = D|_{X^*}$, and $\iota : X^* \setminus D^* \hookrightarrow X^*$. Then we have the following properties.

(ii) *$\big(f_*\omega_{X/Y}(D)\big)|_V$ is a nef locally free sheaf on V, where V is any complete subvariety of Y.*

(iii) *Let $\rho : Y' \to Y$ be a morphism from a smooth variety Y' such that $\rho^{-1}(\Sigma_Y)$ is a simple normal crossing divisor on Y'. Let (X', D') be a simple normal crossing pair and let $f' : X' \to Y'$ be a projective surjective morphism onto Y' such that $f' : (X', D') \to Y'$ is nothing but the base change of $f : (X, D) \to Y$ by $\rho : Y' \to Y$ over $Y \setminus \Sigma_Y$ and that every stratum of (X', D') is dominant onto Y'. Then there exists a natural isomorphism $\rho^*(f_*\omega_{X/Y}(D)) \simeq f'_*\omega_{X'/Y'}(D')$ of locally free sheaves which extends the base change isomorphism over $Y \setminus \Sigma_Y$.*

We sketch the proof of Theorem 3.1 for the reader's convenience. The details are contained in [FF1] (see also Section 13 for some supplementary remarks).

Sketch of proof. By [FF1, Theorem 4.15], the local system $R^d(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*}$ underlies a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure on Y^*. Moreover, it is admissible (see, for example, [FF1, Definition 3.11]). We put

$$\mathcal{V}^d_{Y^*} = R^d(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*} \otimes \mathcal{O}_{Y^*}.$$

Let

$$\cdots \subset F^{p+1}(\mathcal{V}^d_{Y^*}) \subset F^p(\mathcal{V}^d_{Y^*}) \subset F^{p-1}(\mathcal{V}^d_{Y^*}) \subset \cdots$$

be the Hodge filtration. By [FF1, Theorem 7.3 (b)], we obtain that $f_*\omega_{X/Y}(D)$ is isomorphic to the upper canonical extension of

$$\left(\mathrm{Gr}^0_F(\mathcal{V}^d_{Y^*})\right)^* = \mathcal{H}om_{\mathcal{O}_{Y^*}}\left(\mathrm{Gr}^0_F(\mathcal{V}^d_{Y^*}), \mathcal{O}_{Y^*}\right).$$

In particular, $f_*\omega_{X/Y}(D)$ is a locally free sheaf on Y. For the details of the (upper) canonical extensions of Hodge bundles, see [FF1, Remark 7.4]. Hence, we get (i). When all the local monodromies on the local system $R^d(f|_{X^*})_*\iota_!\mathbb{Q}_{X^*\setminus D^*}$ around Σ_Y are unipotent, $f_*\omega_{X/Y}(D)$ is the canonical extension of

$$\left(\mathrm{Gr}^0_F(\mathcal{V}^d_{Y^*})\right)^* = \mathcal{H}om_{\mathcal{O}_{Y^*}}\left(\mathrm{Gr}^0_F(\mathcal{V}^d_{Y^*}), \mathcal{O}_{Y^*}\right).$$

Therefore $f_*\omega_{X/Y}(D) \simeq \left(\mathrm{Gr}^0_F(\mathcal{V}^d_{Y})\right)^*$, where $\mathrm{Gr}^0_F(\mathcal{V}^d_{Y})$ is the canonical extension of

$$\mathrm{Gr}^0_F(\mathcal{V}^d_{Y^*}) = F^0(\mathcal{V}^d_{Y^*})/F^1(\mathcal{V}^d_{Y^*}).$$

Note that $\mathrm{Gr}^0_F(\mathcal{V}^d_{Y})$ is isomorphic to $R^df_*\mathcal{O}_X(-D)$ by [FF1, Theorem 7.1 (2)]. Thus we obtain that $\left(f_*\omega_{X/Y}(D)\right)|_V$ is a nef locally free sheaf on V for any complete subvariety V of Y (see [FF1, Remark 5.22, Corollary 5.23, and Theorem 7.1 (4)]). So we get (ii). As we saw above, $f_*\omega_{X/Y}(D)$ can be characterized by using canonical extensions of Hodge bundles. We note that canonical extensions of Hodge bundles behave well under pull-back by $\rho : Y' \to Y$ such that $\rho^{-1}(\Sigma_Y)$ is a simple normal crossing divisor on Y'. More precisely, we see that the pull-back of $\left(\mathrm{Gr}^0_F(\mathcal{V}^d_{Y})\right)^*$ is isomorphic to the canonical extension of the pull-back of $\left(\mathrm{Gr}^0_F(\mathcal{V}^d_{Y^*})\right)^*$. Therefore, we get a natural isomorphism $\rho^*\left(f_*\omega_{X/Y}(D)\right) \simeq f'_*\omega_{X'/Y'}(D')$, which is nothing but (iii). □

Remark 3.2. In Theorem 3.1, the same results hold for $R^if_*\omega_{X/Y}(D)$ for every i. We only treat the case where $i = 0$ since it is sufficient for our purposes in this paper. For the details of the cases where $i \neq 0$, see [FF1].

We recommend the interested reader to see [FF1] for the details of Theorem 3.1.

§4. Basic slc-trivial fibrations

In this section, we introduce the notion of *(pre-)basic slc-trivial fibrations* and define *discriminant* $\mathbb{Q}$-*b-divisors* and *moduli* $\mathbb{Q}$-*b-divisors* for (pre-)basic slc-trivial fibrations.

Let us start with the definition of (pre-)basic slc-trivial fibrations.

Definition 4.1 (Basic slc-trivial fibration). A *pre-basic slc-trivial fibration* $f : (X, B) \to Y$ consists of a projective surjective morphism $f : X \to Y$ and a simple normal crossing pair (X, B) satisfying the following properties:

(1) Y is a normal irreducible variety,

(2) every stratum of X is dominant onto Y and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$,

(3) B is a $\mathbb{Q}$-divisor such that $B = B^{\leq 1}$ holds over the generic point of Y, and

(4) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that

$$K_X + B \sim_{\mathbb{Q}} f^*D.$$

If a pre-basic slc-trivial fibration $f : (X, B) \to Y$ also satisfies

(5) $\operatorname{rank} f_*\mathcal{O}_X(\lceil -B^{<1} \rceil) = 1$,

then it is called a *basic slc-trivial fibration*.

Before we study basic slc-trivial fibrations, we make a remark on lc-trivial fibrations and klt-trivial fibrations for the reader's convenience.

Remark 4.2 (Lc-trivial fibrations and klt-trivial fibrations). Let $f : (X, B) \to Y$ be a basic slc-trivial fibration. Roughly speaking, if X is irreducible and (X, B) is sub log canonical (resp. sub kawamata log terminal) over the generic point of Y, then $f : (X, B) \to Y$ is called an *lc-trivial fibration* (resp. a *klt-trivial fibration*). We note that a klt-trivial fibration is called an lc-trivial fibration in [A4] (see [A4, Definition 2.1]). For the details, see [FG2, Definitions 3.1 and 3.2].

The notion of basic slc-trivial fibrations is a generalization of that of lc-trivial fibrations.

4.3 (Induced (pre-)basic slc-tirival fibrations). Let $f : (X, B) \to Y$ be a (pre-)basic slc-trivial fibration and let $\sigma : Y' \to Y$ be a generically finite surjective morphism from a normal irreducible variety Y'. Then we have an *induced* (*pre-*)*basic slc-trivial fibration* $f' : (X', B_{X'}) \to Y'$, where $B_{X'}$ is defined by $\mu^*(K_X + B) = K_{X'} + B_{X'}$, with the following commutative diagram:

$$\begin{tikzcd} (X', B_{X'}) \arrow[r, "\mu"] \arrow[d, "f'"'] & (X, B) \arrow[d, "f"] \\ Y' \arrow[r, "\sigma"'] & Y, \end{tikzcd}$$

where X' coincides with $X \times_Y Y'$ over a nonempty Zariski open set of Y'. More precisely, X' is a simple normal crossing variety with a morphism $X' \to X \times_Y Y'$

that is an isomorphism over a nonempty Zariski open set of Y' such that X' is projective over Y' and that every stratum of X' is dominant onto Y'.

Lemma 4.4. *Let $f'_i : (X'_i, B_{X'_i}) \to Y'$ be an induced (pre-)basic slc-trivial fibration for $i = 1, 2$. Then there exist an induced (pre-)basic slc-trivial fibration $f'_3 : (X'_3, B_{X'_3}) \to Y'$ and a commutative diagram*

$$\begin{tikzcd} & X'_3 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X'_1 \arrow[dr, "f'_1"'] & & X'_2 \arrow[dl, "f'_2"] \\ & Y' & \end{tikzcd}$$

such that p_i induces a birational correspondence between each stratum of X'_3 and X'_i and that $K_{X'_3} + B_{X'_3} = p_i^*(K_{X'_i} + B_{X'_i})$ holds for $i = 1, 2$.

Proof. By definition, there exists a nonempty Zariski open set U of Y' such that X'_1 and X'_2 coincide with $X \times_Y Y'$ over U. By [BVP, Theorem 1.4], we can take a common partial resolution X'_3 of X'_1 and X'_2, which coincides with $X \times_Y Y'$ over U, with the desired properties. □

4.5 (Discriminant and moduli $\mathbb{Q}$-b-divisors). Let $f : (X, B) \to Y$ be a (pre-)basic slc-trivial fibration as in Definition 4.1. Let P be a prime divisor on Y. By shrinking Y around the generic point of P, we assume that P is Cartier. We set

$$b_P := \max \left\{ t \in \mathbb{Q} \;\middle|\; \begin{array}{l} (X, B + tf^*P) \text{ is sub slc over} \\ \text{the generic point of } P \end{array} \right\}$$

and set

$$B_Y = \sum_P (1 - b_P)P,$$

where P runs over prime divisors on Y. Equivalently, we have

$$b_P = \max \left\{ t \in \mathbb{Q} \;\middle|\; \begin{array}{l} (X^\nu, \Theta + t\nu^* f^*P) \text{ is sub log canonical} \\ \text{over the generic point of } P \end{array} \right\},$$

where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + B)$, that is, Θ is the sum of the inverse images of B and the singular locus of X. Then it is easy to see that B_Y is a well-defined $\mathbb{Q}$-divisor on Y and is called the *discriminant $\mathbb{Q}$-divisor* of $f : (X, B) \to Y$. We set

$$M_Y = D - K_Y - B_Y$$

and call M_Y the *moduli* $\mathbb{Q}$*-divisor* of $f:(X,B)\to Y$. By definition, we have

$$K_X+B\sim_{\mathbb{Q}} f^*(K_Y+B_Y+M_Y).$$

Let $\sigma:Y'\to Y$ be a proper birational morphism from a normal variety Y' and let $f':(X',B_{X'})\to Y'$ be an induced (pre-)basic slc-trivial fibration by $\sigma:Y'\to Y$. We can define $B_{Y'}$, $K_{Y'}$ and $M_{Y'}$ such that $\sigma^*D=K_{Y'}+B_{Y'}+M_{Y'}$, $\sigma_*B_{Y'}=B_Y$, $\sigma_*K_{Y'}=K_Y$ and $\sigma_*M_{Y'}=M_Y$. We note that $B_{Y'}$ is independent of the choice of $(X',B_{X'})$, that is, $B_{Y'}$ is well defined, by Lemma 4.4 above and Lemma 4.6 below. Hence there exist a unique $\mathbb{Q}$-b-divisor $\mathbf{B}$ such that $\mathbf{B}_{Y'}=B_{Y'}$ for every $\sigma:Y'\to Y$ and a unique $\mathbb{Q}$-b-divisor $\mathbf{M}$ such that $\mathbf{M}_{Y'}=M_{Y'}$ for every $\sigma:Y'\to Y$. Note that $\mathbf{B}$ is called the *discriminant* $\mathbb{Q}$*-b-divisor* and that $\mathbf{M}$ is called the *moduli* $\mathbb{Q}$*-b-divisor* associated to $f:(X,B)\to Y$. We sometimes simply say that $\mathbf{M}$ is the *moduli part* of $f:(X,B)\to Y$.

The following lemma has already been used in the definition of discriminant $\mathbb{Q}$-b-divisors in 4.5.

Lemma 4.6. *Let $f_i:(X_i,B_i)\to Y$ be a pre-basic slc-trivial fibration for $i=1,2$. Assume that there exists a morphism $p:X_2\to X_1$ over Y which induces a birational correspondence between each irreducible component of X_1 and X_2 such that $K_{X_2}+B_2=p^*(K_{X_1}+B_1)$ holds. Then $f_1:(X_1,B_1)\to Y$ and $f_2:(X_2,B_2)\to Y$ induce the same discriminant $\mathbb{Q}$-divisor on Y.*

Proof. Let P be a prime divisor on Y. We may assume that P is Cartier by shrinking Y around P as above. Since $(X_1,B_1+tf_1^*P)$ is sub slc over the generic point of P if and only if $(X_2,B_2+tf_2^*P)$ is sub slc over the generic point of P for every $t\in\mathbb{Q}$. Therefore, $f_1:(X_1,B_1)\to Y$ and $f_2:(X_2,B_2)\to Y$ induce the same discriminant $\mathbb{Q}$-divisor on Y by the definition of discriminant $\mathbb{Q}$-divisors. □

When $(X,\operatorname{Supp}B+\operatorname{Supp}f^*P)$ is a simple normal crossing pair, we can explicitly write down b_P.

Remark 4.7 ([Ka3, Theorem 2] and [A1, Remark 3.1]). Let $f:(X,B)\to Y$ be a pre-basic slc-trivial fibration and let P be a prime divisor on Y. By shrinking Y around the generic point of P, we assume that P is Cartier. If $(X,\operatorname{Supp}B+\operatorname{Supp}f^*P)$ is a simple normal crossing pair and the irreducible decomposition $f^*P=\sum_j w_jQ_j$ satisfies $f(Q_j)=P$ for every j, then we can explicitly write

$$b_P=\min_j\frac{1-d_j}{w_j}, \tag{4.1}$$

where $d_j = \mathrm{mult}_{Q_j} B$ for every j, by direct calculations. Equivalently, we have

$$\mathrm{mult}_P B_Y = 1 - b_P = \max_j \frac{d_j + w_j - 1}{w_j}. \tag{4.2}$$

Note that (4.2) plays a crucial role when we compare the minimal log discrepancy of (X, B) with that of (Y, B_Y). See, for example, the proof of Theorem 5.1 below.

We give a small remark on the definition of discriminant $\mathbb{Q}$-divisors.

Remark 4.8. Let $f : (X, B) \to Y$ be a pre-basic slc-trivial fibration. We do not need condition (4) in Definition 4.1 in order to define the discriminant $\mathbb{Q}$-divisor B_Y.

We will use condition (5) in Definition 4.1 to relate the moduli $\mathbb{Q}$-divisor M_Y with some Hodge bundles (see Proposition 6.3 below).

Remark 4.9. Let $f : (X, B) \to Y$ be a pre-basic slc-trivial fibration. We start with $K_X + B \sim_{\mathbb{Q}} f^*D^\dagger$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D^\dagger$ on Y. Note that $D \sim_{\mathbb{Q}} D^\dagger$ holds since $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$. In this setting, we put

$$M_Y^\dagger := D^\dagger - K_Y - B_Y$$

and obtain the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}^\dagger$ associated to $f : (X, B) \to Y$ as in 4.5 above. We note that K_Y is well defined modulo linear equivalence and the discriminant $\mathbb{Q}$-b-divisor $\mathbf{B}$ is independent of D and $D^\dagger$ (see Remark 4.8). Therefore, we have $g \in \Gamma(Y, \mathcal{K}_Y^*)$ and a rational number r such that

$$\mathbf{M} = \mathbf{M}^\dagger + r\overline{(g)}$$

holds. Therefore, if $\mathbf{M}^\dagger = \overline{\mathbf{M}^\dagger_{Y'}}$ holds for some proper birational morphism $\sigma : Y' \to Y$ from a normal variety Y', then $\mathbf{M} = \overline{\mathbf{M}_{Y'}}$ holds true by Lemma 2.13.

We prepare an elementary finite base change formula, which will be used in Sections 8 and 9.

Lemma 4.10 ([A1, Theorem 3.2]). *Let us consider a commutative diagram:*

$$\begin{tikzcd} (X', B_{X'}) \arrow[r, "\mu"] \arrow[d, "f'"'] & (X, B) \arrow[d, "f"] \\ Y' \arrow[r, "\sigma"'] & Y, \end{tikzcd}$$

where $f : (X, B) \to Y$ is a pre-basic slc-trivial fibration, $\sigma : Y' \to Y$ is a finite surjective morphism of normal irreducible varieties, and $f' : (X', B_{X'}) \to Y'$

is an induced pre-basic slc-trivial fibration. Then $\sigma^*(K_Y + B_Y) = K_{Y'} + B_{Y'}$ *holds, where* B_Y *(resp.* $B_{Y'}$*) is the discriminant* $\mathbb{Q}$*-divisor of* $f : (X, B) \to Y$ *(resp.* $f' : (X', B_{X'}) \to Y'$*).*

Remark 4.11. In Lemma 4.10, $K_Y + B_Y$ is not necessarily $\mathbb{Q}$-Cartier. However, we can define $\sigma^*(K_Y + B_Y)$ since σ is a finite surjective morphism between normal varieties.

Proof of Lemma 4.10. Without loss of generality, we may assume that Y and Y' are both smooth by shrinking Y suitably. Let P' be a prime divisor on Y'. We put $P = \sigma(P')$ and $w = \mathrm{mult}_{P'} \sigma^* P$. Then it is sufficient to see $wb_P = b_{P'}$ because $\sigma^*(K_Y + P) = K_{Y'} + P'$ holds in a neighborhood of the generic point of P'. By the definition of discriminant $\mathbb{Q}$-divisors, we may assume that X is smooth by replacing (X, B) with (X^ν, Θ), where $\nu : X^\nu \to X$ is the normalization with $K_{X^\nu} + \Theta = \nu^*(K_X + B)$ as usual. Hence X' is also smooth.

We take any $c \leq b_P$. Then $K_X + B + cf^*P$ is sub log canonical over the generic point of P. Therefore, $K_{X'} + B_{X'} + c(f \circ \mu)^*P = K_{X'} + B_{X'} + c(f')^*\sigma^*P$ is sub log canonical over the generic point of P'. Since $\sigma^*P = wP'$, $K_{X'} + B_{X'} + cw(f')^*P'$ is sub log canonical over the generic point of P'. This implies that $cw \leq b_{P'}$. Thus we get $b_{P'} \geq wb_P$.

We take any $c \geq b_P$. By taking a suitable birational modification of X, we may assume that there exists a prime divisor E on X such that $a(E, X, B + cf^*P) \leq -1$ and $f(E) = P$. Since X' is a resolution of $X \times_Y Y'$, we can find a prime divisor E' on X' such that $\mu(E') = E$, $f'(E') = P'$, and $a(E', X', B_{X'} + cw(f')^*P') = a(E', X', B_{X'} + c(f \circ \sigma)^*P) \leq -1$. Therefore, we get $cw \geq b_{P'}$. This implies $wb_P \geq b_{P'}$.

Thus we obtain $wb_P = b_{P'}$. This is what we wanted, that is, $\sigma^*(K_Y + B_Y) = K_{Y'} + B_{Y'}$. □

We close this section with the following easy lemma.

Lemma 4.12. *Let* $f : (X, B) \to Y$ *be a (pre-)basic slc-trivial fibration. Then there exists a (pre-)basic slc-trivial fibration* $\overline{f} : (\overline{X}, \overline{B}) \to \overline{Y}$ *such that*

(i) $\overline{Y}$ *is a normal complete variety which contains* Y *as a dense Zariski open set, and*
(ii) *the restriction of* $\overline{f} : (\overline{X}, \overline{B}) \to \overline{Y}$ *to* Y *coincides with* $f : (X, B) \to Y$*.*

Proof. We can write $K_X + B + r(\varphi) = f^*D$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y, $r \in \mathbb{Q}$, and $\varphi \in \Gamma(X, \mathcal{K}_X^*)$. We take a normal complete irreducible variety $\overline{Y}$ which contains Y as a dense Zariski open set. By taking a suitable birational modification (see [RG, Théorème (5.2.2)]), we may assume that there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\overline{D}$ on $\overline{Y}$ with $\overline{D}|_Y = D$. By using [BVP, Theorem 1.4], we can construct a complete simple normal crossing variety $\overline{X}$ which contains X as a dense Zariski open set and a projective morphism $\overline{f} : \overline{X} \to \overline{Y}$ which is an extension of $f : X \to Y$. By [BVP, Theorem 1.4], we may assume that every stratum of $\overline{X}$ is dominant onto $\overline{Y}$ and that $\Sigma := \overline{X} \setminus X$ is a simple normal crossing divisor on $\overline{X}$. We consider the Stein factorization

$$\overline{f} : \overline{X} \longrightarrow \overline{Z} := \operatorname{Spec}_{\overline{Y}} \overline{f}_* \mathcal{O}_{\overline{X}} \xrightarrow{\alpha} \overline{Y}$$

of $\overline{f} : \overline{X} \to \overline{Y}$. Note that α is an isomorphism over Y by construction. Since every irreducible component of X is dominant onto $\overline{Y}$, $\overline{Z}$ is an irreducible variety. Therefore, by Zarisiki's main theorem, α is an isomorphism. This means that $\overline{f}_* \mathcal{O}_{\overline{X}} \simeq \mathcal{O}_{\overline{Y}}$ holds. We may further assume that $(\overline{X}, \Sigma + \operatorname{Supp} B')$ is a simple normal crossing pair, where B' is the closure of B on $\overline{X}$. We put $\overline{B} := \overline{f}^* \overline{D} - K_{\overline{X}} - r(\varphi)$. Note that we can see φ as an element of $\Gamma(\overline{X}, \mathcal{K}^*_{\overline{X}})$. Then $\overline{f} : (\overline{X}, \overline{B}) \to \overline{Y}$ satisfies the desired properties. $\square$

Lemma 4.12 is indispensable for the proof of Theorem 1.2 (ii).

§5. Inversion of adjunction

In this section, we prove the following theorem, which is essentially the same as [A4, Theorem 3.1]. Although we do not use Theorem 5.1 explicitly in this paper, the arguments in the proof of Theorem 5.1 below may help the reader understand the proof of Theorem 1.7 in Section 11.

Theorem 5.1 (Inversion of adjunction). *Let $f : (X, B) \to Y$ be a pre-basic slc-trivial fibration such that $\mathbf{K} + \mathbf{B} = \overline{K_Y + B_Y}$, where $\mathbf{K}$ is the canonical b-divisor of Y and $\mathbf{B}$ is the discriminant $\mathbb{Q}$-b-divisor of $f : (X, B) \to Y$. Then there is a positive integer N such that*

$$\frac{1}{N} \operatorname{mld}_{f^{-1}(Z)}(X, B) \leq \operatorname{mld}_Z(Y, B_Y) \leq \operatorname{mld}_{f^{-1}(Z)}(X, B)$$

for every closed subset $Z \subsetneq Y$.

Proof. We take a proper birational morphism $\sigma : Y' \to Y$ from a smooth variety Y' such that $\sigma^{-1}(Z)$ is a divisor on Y' and that $\operatorname{Supp} \sigma^{-1}(Z) \cup \operatorname{Supp} \mathbf{B}_{Y'}$ is included in a simple normal crossing divisor $\Sigma_{Y'}$. Let $f' : (X', B_{X'}) \to Y'$ be an induced pre-basic slc-trivial fibration with

$$\begin{tikzcd} (X, B) \arrow[d, "f"'] & (X', B_{X'}) \arrow[l, "\mu"'] \arrow[d, "f'"] \\ Y & Y'. \arrow[l, "\sigma"] \end{tikzcd}$$

We may further assume that $\operatorname{Supp} B_{X'} \cup \operatorname{Supp}(f')^*\Sigma_{Y'}$ is included in a simple normal crossing divisor $\Sigma_{X'}$. Let $\Sigma_{Y'} = \sum_l P_l$ (resp. $\Sigma_{X'} = \sum_j Q_j$) be the irreducible decomposition of $\Sigma_{Y'}$ (resp. $\Sigma_{X'}$). We may assume that there exists j_0 such that $Q_{j_0} \subset (\sigma \circ f')^{-1}(Z)$ and $a(Q_{j_0}, X, B) + 1 = \operatorname{mld}_{f^{-1}(Z)}(X, B)$ when $\operatorname{mld}_{f^{-1}(Z)}(X, B) \geq 0$. When $\operatorname{mld}_{f^{-1}(Z)}(X, B) = -\infty$, we assume that $a(Q_{j_0}, X, B)+ 1 < 0$ holds. If we need, we take more blow-ups of Y' and may assume that $f'(Q_{j_0}) = P_{l_0}$ for some l_0 with the aid of the flattening theorem (see [RG, Théorème (5.2.2)]). By (4.1), we obtain

$$a(P_{l_0}, Y, B_Y) + 1 = a(P_{l_0}, Y', \mathbf{B}_{Y'}) + 1 \leq a(Q_{j_0}, X', B_{X'}) + 1 = a(Q_{j_0}, X, B) + 1.$$

Therefore, if $\operatorname{mld}_{f^{-1}(Z)}(X, B) \geq 0$, then $a(P_{l_0}, Y, B_Y) + 1 \leq \operatorname{mld}_{f^{-1}(Z)}(X, B)$. When $\operatorname{mld}_{f^{-1}(Z)}(X, B) = -\infty$, we get $a(P_{l_0}, Y, B_Y) + 1 < 0$. Hence, we obtain that

$$\operatorname{mld}_Z(Y, B_Y) \leq \operatorname{mld}_{f^{-1}(Z)}(X, B)$$

always holds.

If $\operatorname{mld}_{f^{-1}(Z)}(X, B) = -\infty$, then

$$\frac{1}{N} \operatorname{mld}_{f^{-1}(Z)}(X, B) \leq \operatorname{mld}_Z(Y, B_Y)$$

obviously holds for any positive integer N. Therefore, from now on, we may assume that $\operatorname{mld}_{f^{-1}(Z)}(X, B) \geq 0$. Let P_l be any prime divisor contained in $\sigma^{-1}(Z)$. Then

$$\begin{aligned} a(P_l, Y, B_Y) + 1 &= a(P_l, Y', \mathbf{B}_{Y'}) + 1 \\ &\geq \frac{1}{N_l} \left(\min_{f'(Q_j) = P_l} a(Q_j, X', B_{X'}) + 1 \right) \\ &\geq \frac{1}{N_l} \operatorname{mld}_{f^{-1}(Z)}(X, B) \end{aligned}$$

for some positive integer N_l by (4.1). By [A2, Theorem 2.3], we can check that

$$\left\{ \operatorname{mld}_{f^{-1}(Z)}(X, B) \,|\, Z \subsetneq Y \right\}$$

is a finite subset of $\mathbb{Q}_{\geq 0} \cup \{-\infty\}$. Therefore, we can take a positive integer N satisfying the desired properties. $\square$

§6. Cyclic cover of the generic fiber

The main purpose of this section is to interpret moduli parts of basic slc-trivial fibrations Hodge theoretically. We closely follow the formulation in [A4, Section 5]. The approach in [A4, Section 5] is essentially the same as those in [M, Section 5, Part II] and [Fn2, Section 4].

Let $f : (X, B) \to Y$ be a basic slc-trivial fibration such that Y is quasi-projective. Let F be a general fiber of $f : X \to Y$. We put

$$b(F, B_F) := \min\{m \in \mathbb{Z}_{>0} \,|\, m(K_F + B_F) \sim 0\}$$

where $K_F + B_F = (K_X + B)|_F$. Since Y is quasi-projective, we can take a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y and $\varphi \in \Gamma(X, \mathcal{K}_X^*)$ such that

$$K_X + B + \frac{1}{b}(\varphi) = f^*D$$

holds, where $b = b(F, B_F)$. Therefore, we have

$$K_X + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y), \tag{6.1}$$

as in 4.5.

6.1 (Cyclic cover of the generic fiber under the assumption that Y is smooth). In 6.1, we assume that Y is smooth. In particular, K_Y is Cartier. By taking some suitable blow-ups, we may assume that $\operatorname{Supp}(B - f^*(B_Y + M_Y))$ is a simple normal crossing divisor on X, $(B^h)^{=1}$ is Cartier, and every stratum of $(X, (B^h)^{=1})$ is dominant onto Y (see, for example, [BVP, Section 8] and [Fn14, Lemma 2.11]). Let $\pi : \widetilde{X} \to X$ be the b-fold cyclic cover associated to (6.1). Then we have the following commutative diagram.

$$\begin{tikzcd}
(X, B) \arrow[d, "f"'] & \widetilde{X} \arrow[l, "\pi"'] \arrow[dl, "\widetilde{f}"] \\
Y &
\end{tikzcd}$$

More explicitly, we put

$$\Delta = K_{X/Y} + B - f^*(B_Y + M_Y),$$

where $K_{X/Y} = K_X - f^*K_Y$. Then $b\Delta = -(\varphi) \sim 0$ holds by definition. We note that the support of $\{\Delta\}$ is a simple normal crossing divisor on X. We can define an $\mathcal{O}_X$-algebra structure of $\bigoplus_{i=0}^{b-1} \mathcal{O}_X(\lfloor i\Delta \rfloor)$ by $b\Delta = (\varphi^{-1}) \sim 0$. We note that

$$\mathcal{O}_X(\lfloor i\Delta \rfloor) \times \mathcal{O}_X(\lfloor j\Delta \rfloor) \to \mathcal{O}_X(\lfloor (i+j)\Delta \rfloor)$$

is well defined for $0 \leq i, j \leq b-1$ by $\lfloor i\Delta \rfloor + \lfloor j\Delta \rfloor \leq \lfloor (i+j)\Delta \rfloor$ and that

$$\mathcal{O}_X(\lfloor (i+j)\Delta \rfloor) \simeq \mathcal{O}_X(\lfloor (i+j-b)\Delta \rfloor)$$

for $i+j \geq b$ by $b\Delta = (\varphi^{-1}) \sim 0$. In this situation, we have the following description of $\widetilde{X}$:

$$\widetilde{X} = \mathrm{Spec}_X \bigoplus_{i=0}^{b-1} \mathcal{O}_X(\lfloor i\Delta \rfloor). \tag{6.2}$$

Let ζ be a fixed primitive b-th root of unity and let $G = \langle \rho \rangle$ be the cyclic group $\mathbb{Z}/b\mathbb{Z}$. Then G acts on $\bigoplus_{i=0}^{b-1} \mathcal{O}_X(\lfloor i\Delta \rfloor)$ by $\mathcal{O}_X$-algebra homomorphisms defined by:

$$\rho(l) = \zeta^i l$$

for a local section l of $\mathcal{O}_X(\lfloor i\Delta \rfloor)$.

Here, we give an alternative description of $\widetilde{X}$ for the reader's convenience, which is more familiar than (6.2). We put $\mathcal{L} = \mathcal{O}_X(-\lfloor \Delta \rfloor)$. Then we see that $b\{\Delta\} = (\varphi^{-1}) - b\lfloor \Delta \rfloor \in |\mathcal{L}^b|$. In this notation, we have

$$\widetilde{X} = \mathrm{Spec}_X \bigoplus_{i=0}^{b-1} \mathcal{L}^{-i}\left(\left\lfloor \frac{ib\{\Delta\}}{b} \right\rfloor\right) = \mathrm{Spec}_X \bigoplus_{i=0}^{b-1} \mathcal{L}^{-i}(\lfloor i\{\Delta\} \rfloor). \tag{6.3}$$

We note that

$$\mathcal{L}^{-i}(\lfloor i\{\Delta\} \rfloor) = \mathcal{O}_X(i\lfloor \Delta \rfloor + \lfloor i\{\Delta\} \rfloor) = \mathcal{O}_X(\lfloor i\Delta \rfloor).$$

Thus this usual description of the b-fold cyclic cover (6.3) coincides with the above description (6.2). We note that [Ko3, 2.3 Ramified covers] may be helpful.

By construction, $\pi : \widetilde{X} \to X$ is étale outside $\mathrm{Supp}\{\Delta\}$. We note that, over a neighborhood of the generic point of every irreducible component of $\mathrm{Supp}\{\Delta\}$, $\widetilde{X}$ is normal and $\pi : \widetilde{X} \to X$ is a well-known b-fold cyclic cover of X associated to $b\{\Delta\} \in |\mathcal{L}^b|$. By construction again, there exists $\widetilde{\varphi} \in \Gamma(\widetilde{X}, \mathcal{K}^*_{\widetilde{X}})$ such that $\pi^*\varphi = \widetilde{\varphi}^b$ in $\Gamma(\widetilde{X}, \mathcal{K}^*_{\widetilde{X}})$. We note that $\widetilde{X}$ is connected by the definition of $b = b(F, B_F)$. We also note that $G = \langle \rho \rangle$ acts on $\widetilde{\varphi}$ by $\rho(\widetilde{\varphi}) = \zeta^{-1}\widetilde{\varphi}$.

We define $B_{\widetilde{X}}$ by the formula $K_{\widetilde{X}} + B_{\widetilde{X}} = \pi^*(K_X + B)$. We can easily see that $(B^h_{\widetilde{X}})^{=1} = \pi^*((B^h)^{=1})$ holds. We can also check that $(\widetilde{X}, (B^h_{\widetilde{X}})^{=1})$ is semi-log canonical and that every slc stratum of $(\widetilde{X}, (B^h_{\widetilde{X}})^{=1})$ is dominant onto Y. Let $d : V \to \widetilde{X}$ be a projective birational morphism from a simple normal crossing variety V. Then we have the following commutative diagram.

$$
\begin{tikzcd}
(X, B) \arrow[d, "f"'] & \widetilde{X} \arrow[l, "\pi"'] \arrow[dl, "\widetilde{f}"'] & (V, B_V) \arrow[l, "d"'] \arrow[dll, "h"] \\
Y & &
\end{tikzcd}
\tag{6.4}
$$

We assume that d is an isomorphism over the generic point of every slc stratum of $(\widetilde{X}, (B^h_{\widetilde{X}})^{=1})$. We put $g := \pi \circ d : V \to X$ and

$$K_V + B_V = d^*(K_{\widetilde{X}} + B_{\widetilde{X}}) = g^*(K_X + B).$$

By taking some more blow-ups if necessary, we may assume that $(B^h_V)^{=1}$ is Cartier (see [BVP, Section 8] and [Fn14, Lemma 2.11]). We put $\psi = d^*\widetilde{\varphi} \in \Gamma(V, \mathcal{K}^*_V)$. Thus we have $g^*\varphi = \psi^b \in \Gamma(V, \mathcal{K}^*_V)$. Therefore,

$$K_V + B_V + (\psi) = h^*(K_Y + B_Y + M_Y) \tag{6.5}$$

holds. We further assume that (V, B_V) is a simple normal crossing pair. By construction,

$$\pi_*\omega_{\widetilde{X}/Y}((B^h_{\widetilde{X}})^{=1}) = \bigoplus_{i=0}^{b-1} \omega_{X/Y}((B^h)^{=1}) \otimes \mathcal{O}_X(\lceil -i\Delta \rceil).$$

We note that $(B^h_{\widetilde{X}})^{=1} = \pi^*((B^h)^{=1})$ holds and that G acts on $\pi_*\omega_{\widetilde{X}/Y}((B^h_{\widetilde{X}})^{=1})$ naturally. Since $K_V + (B^h_V)^{=1} = d^*(K_{\widetilde{X}} + (B^h_{\widetilde{X}})^{=1}) + E$, where E is a d-exceptional $\mathbb{Q}$-divisor such that $\lceil E \rceil \geq 0$, $d_*\omega_{V/Y}((B^h_V)^{=1}) = \omega_{\widetilde{X}/Y}((B^h_{\widetilde{X}})^{=1})$ holds. Therefore, the following eigensheaf decomposition holds:

$$
\begin{aligned}
&h_*\omega_{V/Y}((B^h_V)^{=1}) \\
&= \widetilde{f}_*\omega_{\widetilde{X}/Y}((B^h_{\widetilde{X}})^{=1}) \\
&= \bigoplus_{i=0}^{b-1} f_*\mathcal{O}_X(\lceil (1-i)K_{X/Y} - iB + if^*B_Y + if^*M_Y \rceil + (B^h)^{=1})
\end{aligned}
\tag{6.6}
$$

since $\Delta = K_{X/Y} + B - f^*(B_Y + M_Y)$. We note that

$$\operatorname{rank} f_*\mathcal{O}_X(\lceil -B + f^*B_Y + f^*M_Y \rceil + (B^h)^{=1}) = \operatorname{rank} f_*\mathcal{O}_X(\lceil -(B^{<1}) \rceil) = 1$$

by Definition 4.1 (5).

6.2 (Cyclic cover of the generic fiber when Y may be singular). Let Y_0 be a nonempty Zariski open set of Y such that Y_0 is smooth. By restricting (6.1) to Y_0, we have

$$K_{X_0} + B_0 + \frac{1}{b}(\varphi) = f_0^*(K_{Y_0} + B_{Y_0} + M_{Y_0}) \tag{6.7}$$

and can construct the following commutative diagram

(6.8)
$$\begin{tikzcd}
(X_0, B_0) \arrow[d, "f_0"'] & \widetilde{X}_0 \arrow[l, "\pi_0"'] \arrow[dl, "\widetilde{f}_0"] & (V_0, B_{V_0}) \arrow[l, "d_0"'] \arrow[dll, "h_0"] \\
Y_0 & &
\end{tikzcd}$$

similar to (6.4) since Y_0 is smooth. We take a projective birational modification $\sigma : Y' \to Y$ and construct an induced basic slc-trivial fibration $f' : (X', B_{X'}) \to Y'$ of $f : (X, B) \to Y$ by $\sigma : Y' \to Y$, and so on. Then we get the following commutative diagram similar to the diagram (6.4).

(6.9)
$$\begin{tikzcd}
(X', B_{X'}) \arrow[d, "f'"'] & \widetilde{X}' \arrow[l, "\pi'"'] \arrow[dl, "\widetilde{f}'"] & (V', B_{V'}) \arrow[l, "d'"'] \arrow[dll, "h'"] \\
Y' & &
\end{tikzcd}$$

We can assume that this new diagram (6.9) coincides with the diagram (6.8) over some nonempty Zariski open set of Y' by [BVP, Theorem 1.4]. By replacing $f : (X, B) \to Y$ and (V, B_V) with $f' : (X', B_{X'}) \to Y'$ and $(V', B_{V'})$ respectively, we can assume that Y is smooth and, furthermore, the following properties hold for (6.4).

(a) Y is a smooth quasi-projective irreducible variety, and X and V are quasi-projective simple normal crossing varieties.
(b) there exist simple normal crossing divisors Σ_X, Σ_V, and Σ_Y on X, V, and Y, respectively.
(c) f and h are projective surjective morphisms.
(d) the supports of B, B_V, and B_Y, M_Y are contained in Σ_X, Σ_V, and Σ_Y, respectively.
(e) every stratum of (X, Σ_X^h) and (V, Σ_V^h) is smooth over $Y \setminus \Sigma_Y$.
(f) $f^{-1}(\Sigma_Y) \subset \Sigma_X$, $f(\Sigma_X^v) \subset \Sigma_Y$, and $h^{-1}(\Sigma_Y) \subset \Sigma_V$, $h(\Sigma_V^v) \subset \Sigma_Y$.
(g) $(B^h)^{=1}$ and $(B_V^h)^{=1}$ are Cartier.

By definition and construction, we can easily check the following basic properties of $h : (V, B_V) \xrightarrow{g} (X, B) \xrightarrow{f} Y$. Proposition 6.3 is the main result of this section.

Proposition 6.3. *We have the following properties.*

(i) $\pi : \widetilde{X} \to X$ *is a Galois cover and its Galois group* G *is* $\mathbb{Z}/b\mathbb{Z}$.

(ii) *$h : (V, B_V) \to Y$ is a pre-basic slc-trivial fibration.*

(iii) *$f : (X, B) \to Y$ and $h : (V, B_V) \to Y$ induce the same discriminant and moduli part on Y.*

(iv) *Assume that, for any irreducible component P of $\operatorname{Supp}\Sigma_Y$, there exists a prime divisor Q on V such that $\operatorname{mult}_Q(-B_V + h^*B_Y) = 0$, $h(Q) = P$, and $\operatorname{mult}_Q h^*P = 1$. Then M_Y is an integral divisor and $\mathcal{O}_Y(M_Y)$ is a direct summand of $h_*\mathcal{O}_V(K_{V/Y} + (B^h_V)^{=1})$.*

(v) *In* (iv)*, we further assume that all the local monodromies on the local system*

$$R^{\dim V - \dim Y}(h|_{V^*})_*\iota_!\mathbb{Q}_{V^*\setminus (B^h_{V^*})^{=1}}$$

around Σ_Y are unipotent, where $Y^* = Y\setminus\Sigma_Y$, $V^* = h^{-1}(Y^*)$, $B_{V^*} = (B_V)|_{V^*}$, and $\iota : V^* \setminus (B^h_{V^*})^{=1} \hookrightarrow V^*$ is the natural open immersion. Then

$$\left(h_*\mathcal{O}_V(K_{V/Y} + (B^h_V)^{=1})\right)|_W$$

is a nef locally free sheaf on W, where W is any complete subvariety of Y. In particular, $(M_Y)|_W$ is a nef Cartier divisor on W.

Proof of Proposition 6.3. In the above construction, we have already described the action of $G = \mathbb{Z}/b\mathbb{Z}$ on $\widetilde{X}$ explicitly. Therefore, (i) is obvious. By the definition of $b = b(F, B_F)$, the general fiber of $h : V \to Y$ is connected. We consider the Stein factorization

$$h : V \longrightarrow Z := \operatorname{Spec}_Y h_*\mathcal{O}_V \xrightarrow{\alpha} Y$$

of $h : V \to Y$. Note that α is an isomorphism over a nonempty Zariski open set of Y since the general fibers of h are connected. Since every irreducible component of V is dominant onto Y, Z is an irreducible variety. Therefore, by Zariski's main theorem, α is an isomorphism. Hence we see that the natural map $\mathcal{O}_Y \to h_*\mathcal{O}_V$ is an isomorphism. By construction, $K_V + B_V \sim_{\mathbb{Q},f} 0$ and $B_V = B_V^{\leq 1}$ holds over the generic point of Y. Therefore, $h : (V, B_V) \to Y$ is a pre-basic slc-trivial fibration. This is (ii). By construction again, we see that $(X, B + tf^*P)$ is sub slc over the generic point of P if and only if $(V, B_V + th^*P)$ is sub slc over the generic point of P. Therefore, $f : (X, B) \to Y$ and $h : (V, B_V) \to Y$ induce the same discriminant $\mathbb{Q}$-divisor B_Y. This implies (iii) since $M_Y = D - K_Y - B_Y$.

From now on, we will prove (iv). We note that

$$K_X + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y)$$

and

$$K_V + B_V + (\psi) = h^*(K_Y + B_Y + M_Y). \tag{6.10}$$

By (6.10) and $\mathrm{mult}_Q(-B_V + h^*B_Y) = 0$ in (iv), we obtain $\mathrm{mult}_Q\, h^*M_Y \in \mathbb{Z}$. Since $\mathrm{mult}_Q\, h^*P = 1$ by the assumption in (iv), M_Y is an integral divisor on Y. By Theorem 3.1, $h_*\mathcal{O}_V(K_{V/Y} + (B_V^h)^{=1})$ is a locally free sheaf. As we saw above, by construction and assumption, we have the following eigensheaf decomposition

$$\begin{aligned}
&h_*\omega_{V/Y}((B_V^h)^{=1})\\
&= \widetilde{f}_*\omega_{\widetilde{X}/Y}((B_{\widetilde{X}}^h)^{=1})\\
&= \bigoplus_{i=0}^{b-1} f_*\mathcal{O}_X(\lceil (1-i)K_{X/Y} - iB + if^*B_Y + if^*M_Y\rceil + (B^h)^{=1}).
\end{aligned} \tag{6.11}$$

We note that the eigensheaf corresponding to the eigenvalue ζ^{-1} is

$$\mathcal{N} = f_*\mathcal{O}_X(\lceil -B + f^*B_Y + f^*M_Y\rceil + (B^h)^{=1}),$$

which is an invertible sheaf on Y. From now on, we will prove that $\mathcal{O}_Y(M_Y) = \mathcal{N}$ holds. Since $\mathcal{O}_Y(M_Y)$ and $\mathcal{N}$ are both invertible, we can freely replace Y with its Zariski open set Y^0 such that $\mathrm{codim}_Y(Y \setminus Y^0) \geq 2$. Therefore, we can assume that $\lceil -B + f^*B_Y\rceil + (B^h)^{=1}$ and $-B_V + h^*B_Y + (B_V^h)^{=1}$ are both effective. We have already seen that M_Y is an integral divisor. Since $\lceil -B + f^*B_Y\rceil + (B^h)^{=1}$ is effective, there exists a natural inclusion:

$$\mathcal{O}_Y(M_Y) \subset \mathcal{N} = \mathcal{O}_Y(M_Y) \otimes f_*\mathcal{O}_X(\lceil -B + f^*B_Y\rceil + (B^h)^{=1}).$$

More precisely, we have:

Claim. *The following equality*

$$\mathcal{O}_Y(M_Y) = \mathcal{N}$$

holds under the assumptions in (iv).

Proof of Claim. By (6.10), we see that

$$K_{V/Y} + (B_V^h)^{=1} + (\psi)$$

is effective over some nonempty Zariski open set of Y. Therefore, it defines a holomorphic section τ of $h_*\omega_{V/Y}((B_V^h)^{=1}) = \widetilde{f}_*\omega_{\widetilde{X}/Y}((B_{\widetilde{X}}^h)^{=1})$ on some nonempty Zariski open set of Y. Since $\psi = d^*\widetilde{\varphi}$, $G = \langle\rho\rangle$ acts on $\widetilde{\varphi}$ by $\rho(\widetilde{\varphi}) = \zeta^{-1}\widetilde{\varphi}$, and the eigensheaf corresponding to the eigenvalue ζ^{-1} is $\mathcal{N}$, we can consider τ a rational section of $\mathcal{N}$. Let a be an element of $\mathbb{C}(Y)$, that is, a is a rational function on Y. Then, by the above description, a rational section $a\cdot\tau$ of $\mathcal{N}$ corresponds to $h^*a\cdot\psi$. Let U be any nonempty Zariski open set of Y.

Assume that $a \cdot \tau \in \Gamma(U, \mathcal{N})$. Then

$$\big((h^*a \cdot \psi) + K_{V/Y} + (B_V^h)^{=1}\big)\,|_{h^{-1}(U)} \geq 0. \tag{6.12}$$

holds. On the other hand, we see that

$$(h^*a \cdot \psi) + K_{V/Y} + (B_V^h)^{=1} = h^*((a) + M_Y) + (-B_V + h^*B_Y + (B_V^h)^{=1}) \tag{6.13}$$

holds by (6.10). Since $-B_V + h^*B_Y$ contains no fibers over any codimension one points of Y by the assumption in (iv), (6.12) and (6.13) imply that

$$((a) + M_Y)\,|_U \geq 0.$$

Assume that $((a) + M_Y)\,|_U \geq 0$ holds. Then we obtain

$$\big((h^*a \cdot \psi) + K_{V/Y} + (B_V^h)^{=1}\big)\,|_{h^{-1}(U)} \geq 0 \tag{6.14}$$

by (6.13) because $-B_V + h^*B_Y + (B_V^h)^{=1}$ is effective. This means that $a \cdot \tau \in \Gamma(U, \mathcal{N})$.

Therefore, we obtain that the desired equality $\mathcal{O}_Y(M_Y) = \mathcal{N}$ holds. □

Hence, we obtain (iv). (v) is a direct consequence of (iv) and Theorem 3.1. □

We close this section with a remark on the assumptions in (iv) and (v) in Proposition 6.3. We will implicitly use it in Sections 7 and 8.

Remark 6.4. Let $h : (V, B_V) \to Y$ be a pre-basic slc-trivial fibration satisfying the assumptions in (iv) and (v) in Proposition 6.3. We consider the following commutative diagram of pre-basic slc-trivial fibrations:

$$\begin{tikzcd}
(V^\dagger, B_{V^\dagger}) \arrow[rr, "\alpha"] \arrow[dr, "h^\dagger"'] & & (V, B_V) \arrow[dl, "h"] \\
& Y &
\end{tikzcd}$$

where $h^\dagger : (V^\dagger, B_{V^\dagger}) \to Y$ is a pre-basic slc-trivial fibration, α is an isomorphism over $Y^* = Y \setminus \Sigma_Y$, and $K_{V^\dagger} + B_{V^\dagger} = \alpha^*(K_V + B_V)$. Then it is almost obvious that $h^\dagger : (V^\dagger, B_{V^\dagger}) \to Y$ also satisfies the assumptions in (iv) and (v) in Proposition 6.3.

§7. Covering lemmas revisited

In this section, we explain some covering lemmas, which are essentially due to Yujiro Kawamata (see [Ka1]). We will use Lemma 7.3, which is the main result of this section, in Sections 8 and 9.

Let us start with a well-known covering lemma in [Ka1].

Lemma 7.1 ([Ka1, Theorem 17]). *Let X be a smooth quasi-projective variety and let D be a simple normal crossing divisor on X such that $D = \sum_{j=1}^{r} D_j$ is the irreducible decomposition. Let N_j be a positive integer for $1 \leq j \leq r$. Then we can construct a finite ramified cover $\tau : Z \to X$ satisfying the following properties.*

(i) *Z is a smooth quasi-projective variety and there is a simple normal crossing divisor Σ_X on X such that $D \leq \Sigma_X$, τ is étale over $X \setminus \Sigma_X$, $\tau^{-1}(\Sigma_X)$ is a simple normal crossing divisor on Z.*

(ii) *We have $\tau^* D_j = N_j \tau^{-1}(D_j)$ for every $1 \leq j \leq r$.*

Since it is very important to understand how to construct $\tau : Z \to X$, we sketch the proof of Lemma 7.1 for the reader's convenience.

Sketch of proof. Here, we closely follow the presentation in [EV, 3.19. Lemma] and [V, Lemma 2.5]. We take an ample line bundle $\mathcal{A}$ on X such that $\mathcal{A}^{N_j} \otimes \mathcal{O}_X(-D_j)$ is generated by global sections for $1 \leq j \leq r$. We put $n = \dim X$. We take general members $H_1^{(j)}, \ldots, H_n^{(j)}$ of $\left|\mathcal{A}^{N_j} \otimes \mathcal{O}_X(-D_j)\right|$ for $1 \leq j \leq r$ such that $D + \sum_{i,j} H_i^{(j)}$ is a simple normal crossing divisor on X. Let $Z_i^{(j)}$ be the cyclic cover obtained by taking the N_j-th root out of $D_j + H_i^{(j)}$ (see [EV, 3.5. Cyclic covers] and [V, Lemma 2.3]). More explicitly, let $s_i^{(j)} \in \Gamma(X, \mathcal{A}^{N_j})$ be a section whose zero divisor is $D_j + H_i^{(j)}$. The dual of $s_i^{(j)} : \mathcal{O}_X \to \mathcal{A}^{N_j}$, that is, $\left(s_i^{(j)}\right)^{\vee} : \mathcal{A}^{-N_j} \to \mathcal{O}_X$, defines an $\mathcal{O}_X$-algebra structure on $\bigoplus_{l=0}^{N_j-1} \mathcal{A}^{-l}$. Then we can write $Z_i^{(j)} = \mathrm{Spec}_X \bigoplus_{l=0}^{N_j-1} \mathcal{A}^{-l}$. In this situation, we can check that the normalization of

$$\left(Z_1^{(1)} \times_X \cdots \times_X Z_n^{(1)}\right) \times_X \cdots \times_X \left(Z_1^{(r)} \times_X \cdots \times_X Z_n^{(r)}\right),$$

which is denoted by Z, has the desired properties. For the details, we recommend the reader to see [EV, 3.19. Lemma] and [V, Lemma 2.5]. We note that we can take $\Sigma_X = D + \sum_{i,j} H_i^{(j)}$ by construction. We will use the above description of Z in the proof of Lemma 7.3 below. $\square$

The following slight generalization of Lemma 7.1 is very important for our applications.

Lemma 7.2 (see [Ka3, Corollary 19] and [A4, Remark 4.2]). *Let X, D, and N_1, $\ldots$, N_r be as in Lemma 7.1. Let $\rho : X' \to X$ be a projective surjective morphism from a smooth quasi-projective variety X' such that $\rho^{-1}(D)$ is a simple normal crossing divisor on X'. Then we may assume that $\tau : Z \to X$ in Lemma 7.1 fits* *into a commutative diagram*

$$\begin{tikzcd} Z \arrow[d, "\tau"'] & Z' \arrow[l, "\rho'"'] \arrow[d, "\tau'"] \\ X & X' \arrow[l, "\rho"] \end{tikzcd}$$

satisfying the following properties.

(i) *τ' is a finite cover and ρ' is a projective morphism.*

(ii) *Z' is a smooth quasi-projective variety.*

(iii) *There is a simple normal crossing divisor $\Sigma_{X'}$ on X' such that τ' is étale over $X' \setminus \Sigma_{X'}$, $(\tau')^{-1}(\Sigma_{X'})$ is a simple normal crossing divisor, and $\rho^{-1}(\Sigma_X) \subset \Sigma_{X'}$, where Σ_X is the simple normal crossing divisor on X in Lemma 7.1.*

Although the proof of Lemma 7.2 is more or less well known to the experts, we give a detailed proof for the reader's convenience.

Proof of Lemma 7.2. We closely follow the presentation in [V, Corollary 2.6]. In the proof of Lemma 7.1, we can choose the divisors $H_i^{(j)}$ on X such that $D' := \rho^{-1}\left(D + \sum_{i,j} H_i^{(j)}\right)$ is a simple normal crossing divisor on X'. Let $Z^\dagger$ be the normalization of an irreducible component of $Z \times_X X'$. Then we get the following commutative diagram:

$$\begin{tikzcd} Z \arrow[d, "\tau"'] & Z^\dagger \arrow[l, "\rho^\dagger"'] \arrow[d, "\tau^\dagger"] \\ X & X'. \arrow[l, "\rho"] \end{tikzcd}$$

By construction, $\tau^\dagger$ is étale over $X' \setminus D'$. Let $D' = \sum_k D'_k$ be the irreducible decomposition. We put

$$N'_k := \operatorname*{lcm}_l \{ e(\Delta_k^l) \mid \Delta_k^l \text{ is an irreducible component of } (\tau^\dagger)^{-1}(D'_k)\},$$

where $e(\Delta_k^l)$ denotes the ramification index of Δ_k^l over D'_k. Let $\widetilde{\tau} : \widetilde{Z} \to X'$ be the finite cover constructed in Lemma 7.1 for X', D', and N'_k. Let Z' be the normalization of an irreducible component of $\widetilde{Z} \times_{X'} Z^\dagger$. Thus we get the following commutative diagram:

$$\begin{tikzcd} Z \arrow[d, "\tau"'] & Z^\dagger \arrow[l] \arrow[d, "\tau^\dagger"'] & Z' \arrow[l, "\beta"'] \arrow[dl, "\tau'"'] \arrow[d, "\alpha"] \\ X & X' \arrow[l, "\rho"] & \widetilde{Z}. \arrow[l, "\widetilde{\tau}"] \end{tikzcd}$$

Since $\widetilde{\tau} : \widetilde{Z} \to X'$ is constructed as a chain of finite cyclic covers, the same holds true for $\beta : Z' \to Z^{\dagger}$. The ramification index of a component of $\beta^{-1}(\Delta_k^l)$ over Δ_k^l is $N_k'/e(\Delta_k^l)$ by construction. Therefore, the ramification index of an irreducible component of $(\tau')^{-1}(D_k')$ over D_k' is given by

$$\frac{N_k'}{e(\Delta_k^l)} \cdot e(\Delta_k^l) = N_k'.$$

By the construction of $\widetilde{Z}$, this is nothing but the ramification index of an irreducible component of $(\widetilde{\tau})^{-1}(D_k')$ over D_k'. Therefore, $\alpha : Z' \to \widetilde{Z}$ is unramified in codimension one. Since $\widetilde{Z}$ is smooth, α is étale. Thus, we can easily check that $\tau' : Z' \to X'$ satisfies the desired properties. $\square$

The following lemma is the main result of this section. This somewhat technical covering lemma will play an important role in Sections 8 and 9.

Lemma 7.3 (Unipotent reduction for pre-basic slc-trivial fibrations). *Let $h : (V, B_V) \to Y$ be a pre-basic slc-trivial fibration such that Y is a smooth quasi-projective variety. Assume that there are simple normal crossing divisors Σ_V and Σ_Y on V and Y respectively such that $h^{-1}(\Sigma_Y) \subset \Sigma_V$, $h(\Sigma_V^v) \subset \Sigma_Y$, $\operatorname{Supp} B_V \subset \Sigma_V$, and every stratum of (V, Σ_V^h) is smooth over $Y \setminus \Sigma_Y$. Then there exist a finite cover $\gamma : Y' \to Y$ from a smooth quasi-projective variety Y' such that $\Sigma_{Y'} := \gamma^{-1}(\Sigma_Y)$ is a simple normal crossing divisor on Y' and a commutative diagram*

$$\begin{tikzcd}
V \arrow[d, "h"'] & V \times_Y Y' \arrow[l, "q"'] \arrow[d] & V' \arrow[l, "p"'] \arrow[dl, "h'"] \\
Y & Y' \arrow[l, "\gamma"]
\end{tikzcd}$$

with the following properties.

(i) *p is a projective birational morphism from a simple normal crossing variety V' which is an isomorphism over $Y' \setminus \Sigma_{Y'}$.*

(ii) *$h' : (V', B_{V'}) \to Y'$ is a pre-basic slc-trivial fibration, where $\gamma' := q \circ p : V' \to V$ and $K_{V'} + B_{V'} = (\gamma')^*(K_V + B_V)$.*

(iii) *There exists a simple normal crossing divisor $\Sigma_{V'}$ on V' such that $(\gamma')^{-1}(\Sigma_V) \subset \Sigma_{V'}$, $\operatorname{Supp} B_{V'} \subset \Sigma_{V'}$, $h'(\Sigma_{V'}^v) \subset \Sigma_{Y'}$, $(h')^{-1}(\Sigma_{Y'}) \subset \Sigma_{V'}$, and every stratum of $(V', \Sigma_{V'}^h)$ is smooth over $Y' \setminus \Sigma_{Y'}$.*

(iv) *$h' : (V', B_{V'}) \to Y'$ satisfies the assumptions in* (iv) *and* (v) *in Proposition 6.3.*

Proof. Let $\Sigma_Y = \sum_{j=1}^r P_j$ and $\Sigma_V = \sum_l Q_l$ be the irreducible decomposition of Σ_Y and Σ_V respectively. In this case, we can write $h^*P_j = \sum_l w_j^l Q_l$ with $w_j^l \in \mathbb{Z}_{\geq 0}$

for every j. Let M_j be the monodromy matrix on the local system

$$R^{\dim V-\dim Y}(h|_{V^*})_*\iota_!\mathbb{Q}_{V^*\setminus (B^h_{V^*})^{=1}}$$

around P_j, where $Y^*:=Y\setminus \Sigma_Y$, $V^*:=h^{-1}(Y^*)$, $B_{V^*}:=B_V|_{V^*}$, and $\iota: V^*\setminus (B^h_{V^*})^{=1}\hookrightarrow V^*$. By [FF1, Theorem 4.15], the local system

$$R^{\dim V-\dim Y}(h|_{V^*})_*\iota_!\mathbb{Q}_{V^*\setminus (B^h_{V^*})^{=1}}$$

underlies a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure on Y^*. In particular, M_j is quasi-unipotent. We put

$$m_j:=\min\{m\in \mathbb{Z}_{>0}\,|\, M_j^m \text{ is unipotent}\}$$

and

$$w_j:=\operatorname*{lcm}_l\{w^l_j\,|\, h(Q_l)=P_j\}$$

Then we set

$$N_j:=\operatorname{lcm}\{m_j, w_j\}.$$

By applying Lemma 7.1 to Y, Σ_Y, and N_j, we can construct a finite cover $\gamma: Y'\to Y$. More precisely, let $\mathcal{A}$ be an ample line bundle on Y such that $\mathcal{A}^{N_j}\otimes \mathcal{O}_Y(-P_j)$ is generated by global sections for $1\leq j\leq r$. We put $n=\dim Y$. We take general members $H^{(j)}_1, \ldots, H^{(j)}_n$ of $|\mathcal{A}^{N_j}\otimes \mathcal{O}_Y(-P_j)|$ for $1\leq j\leq r$ such that $\Sigma_Y+\sum_{i,j}H^{(j)}_i$ is a simple normal crossing divisor on Y. By the above data $\mathcal{A}$, N_j, and $\Sigma_Y+\sum_{i,j}H^{(j)}_i$, we can construct a finite cover $\gamma: Y'\to Y$, which is a chain of cyclic covers (see the proof of Lemma 7.1). Let $s\in \Gamma(Y, \mathcal{A}^{N_j})$ be a section whose zero divisor is $P_j+H^{(j)}_1$. The dual of $s:\mathcal{O}_Y\to \mathcal{A}^{N_j}$, that is, $s^\vee: \mathcal{A}^{-N_j}\to \mathcal{O}_Y$, defines an $\mathcal{O}_Y$-algebra structure on $\bigoplus_{i=0}^{N_j-1}\mathcal{A}^{-i}$. From now on, we will look at $\gamma: Y'\to Y$ in a neighborhood of the generic point of P_j. Therefore, by shrinking Y suitably, we assume that $(s=0)=P_j$. By construction, we can easily see that $\gamma: Y'\to Y$ can be decomposed as follows:

$$\gamma: Y'\overset{\beta}{\longrightarrow}\widetilde{Y}\overset{\alpha}{\longrightarrow} Y,$$

where $\widetilde{Y}=\operatorname{Spec}_Y\bigoplus_{i=0}^{N_j-1}\mathcal{A}^{-i}$, $\alpha:\widetilde{Y}\to Y$ is the cyclic cover obtained by taking the N_j-th root out of P_j, and $\beta: Y'\to \widetilde{Y}$ is a finite étale morphism. Let us consider $V\times_Y\widetilde{Y}=\operatorname{Spec}_V\bigoplus_{i=0}^{N_j-1}(h^*\mathcal{A})^{-i}$. Note that the $\mathcal{O}_V$-algebra structure on $\bigoplus_{i=0}^{N_j-1}(h^*\mathcal{A})^{-i}$ is defined by the dual of $h^*s\in \Gamma(V, (h^*\mathcal{A})^{N_j})$, that is, $(h^*s)^\vee$: $(h^*\mathcal{A})^{-N_j}\to \mathcal{O}_V$. We put

$$\widetilde{Z}:=\operatorname{Spec}_V\bigoplus_{i=0}^{N_j-1}(h^*\mathcal{A})^{-i}\otimes \mathcal{O}_V\left(\left\lfloor\frac{ih^*P_j}{N_j}\right\rfloor\right).$$

Of course, the $\mathcal{O}_V$-algebra structure on

$$\bigoplus_{i=0}^{N_j-1}(h^*\mathcal{A})^{-i}\otimes\mathcal{O}_V\left(\left\lfloor\frac{ih^*P_j}{N_j}\right\rfloor\right)$$

is defined by the isomorphism

$$(h^*\mathcal{A})^{-N_j}\otimes\mathcal{O}_V\left(\left\lfloor\frac{N_jh^*P_j}{N_j}\right\rfloor\right)\xrightarrow{\sim}\mathcal{O}_V,$$

which is induced by $h^*s\in\Gamma(V,(h^*\mathcal{A})^{N_j})$. Then we get a morphism $\widetilde{Z}\to V\times_Y\widetilde{Y}$, which is an isomorphism over $\widetilde{Y}\setminus\alpha^{-1}(P_j)$, induced by the natural map of $\mathcal{O}_V$-algebras

$$\bigoplus_{i=0}^{N_j-1}(h^*\mathcal{A})^{-i}\to\bigoplus_{i=0}^{N_j-1}(h^*\mathcal{A})^{-i}\otimes\mathcal{O}_V\left(\left\lfloor\frac{ih^*P_j}{N_j}\right\rfloor\right).$$

We put $Z':=\widetilde{Z}\times_{\widetilde{Y}}Y'$ and take a suitable birational modification $a:V'\to Z'$. Then we get the following big commutative diagram:

(7.1)
$$\begin{tikzcd}
 & \widetilde{Z} \arrow[dl, "b"'] \arrow[d] & Z' \arrow[l, "\beta_2"'] \arrow[d] & V' \arrow[l, "a"'] \arrow[dl, "p"] \arrow[ddl, "h'"] \\
V \arrow[d, "h"'] & V\times_Y\widetilde{Y} \arrow[l] \arrow[d] & V\times_Y Y' \arrow[l, "\beta_1"] \arrow[d] & \\
Y & \widetilde{Y} \arrow[l, "\alpha"] & Y' \arrow[l, "\beta"] &
\end{tikzcd}$$

where β,β_1, and β_2 are finite étale morphisms. We put $\widetilde{P}_j=\alpha^{-1}(P_j)$. We define $B_{\widetilde{Z}}$ by $K_{\widetilde{Z}}+B_{\widetilde{Z}}=b^*(K_V+B_V)$. Similarly, we put $\beta_2^*(K_{\widetilde{Z}}+B_{\widetilde{Z}})=K_{Z'}+B_{Z'}$ and $a^*(K_{Z'}+B_{Z'})=K_{V'}+B_{V'}$. Without loss of generality, by shrinking Y suitably, we may assume that $h(Q_l)=P_j$ holds if $h(Q_l)\subset P_j$. By the construction of $\alpha:\widetilde{Y}\to Y$ and the definition of N_j, $c^*\widetilde{P}_j$ is reduced (see [KM, Proposition 7.23]) and $(\widetilde{Z},c^*\widetilde{P}_j)$ is semi-log canonical, where $c:\widetilde{Z}\to\widetilde{Y}$. We note that $Z'\to Y'$ in (7.1) is the base change of $c:\widetilde{Z}\to\widetilde{Y}$ by an étale morphism $\beta:Y'\to\widetilde{Y}$. Therefore, we can take a birational modification $a:V'\to Z'$ which is an isomorphism over $Y'\setminus\gamma^{-1}(P_j)$ such that $h':(V',B_{V'})\to Y'$ is a pre-basic slc-trivial fibration satisfying the desired properties. Although we constructed $h':(V',B_{V'})\to Y'$ after shrinking Y around the generic point of P_j, we can construct a desired pre-basic slc-trivial fibration $h':(V',B_{V'})\to Y'$ globally without shrinking Y by the above local description and [BVP, Theorem 1.4]. $\square$

§8. Pull-back of the moduli parts

In this section, we see that the moduli parts behave well under pull-back by generically finite surjective morphisms with some mild assumptions.

Let

$$K_X + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y)$$

and $h : (V, B_V) \xrightarrow{g} (X, B) \xrightarrow{f} Y$ be as in Section 6 which satisfies conditions (a)–(g) in Section 6. Let $\gamma : Y' \to Y$ be a generically finite surjective morphism from a smooth quasi-projective variety Y'. Assume that there is a simple normal crossing divisor $\Sigma_{Y'}$ which contains $\gamma^{-1}(\Sigma_Y)$. By base change, we have a commutative diagram:

(8.1)

$$\begin{tikzcd}
& V \arrow[dl, "g"'] \arrow[dd] & & V' \arrow[ll, "\nu"'] \arrow[dl, "g'"'] \arrow[dd] \\
X \arrow[dr, "f"'] & & X' \arrow[ll, "\sigma"', crossing over] \arrow[dr, "f'"] & \\
& Y & & Y' \arrow[ll, "\gamma"]
\end{tikzcd}$$

where $h' : (V', B_{V'}) \xrightarrow{g'} (X', B_{X'}) \xrightarrow{f'} Y'$ satisfies the same properties, that is, (a)–(g) in Section 6, and it is nothing but the base change of $h : (V, B_V) \xrightarrow{g} (X, B) \xrightarrow{f} Y$ by $\gamma : Y' \to Y$ over $Y \setminus \Sigma_Y$. We note that $B_{X'}$ and $B_{V'}$ are induced by crepant pull-back, that is, $K_{X'} + B_{X'} = \sigma^*(K_X + B_X)$ and $K_{V'} + B_{V'} = \nu^*(K_V + B_V)$, $\Sigma_{X'} \supset \sigma^{-1}(\Sigma_X)$, $\Sigma_{V'} \supset \nu^{-1}(\Sigma_V)$, and $\varphi' = \sigma^*\varphi$. In this situation, we say that the setup $h' : (V', B_{V'}) \xrightarrow{g'} (X', B_{X'}) \xrightarrow{f'} Y'$ is induced from $h : (V, B_V) \xrightarrow{g} (X, B_X) \xrightarrow{f} Y$ by the base change $\gamma : Y' \to Y$.

In the above setup, we have the following theorem, which is a generalization of [A4, Proposition 5.5]. Note that [A4, Proposition 5.5] is a generalization of [Fn2, Proposition 4.2].

Theorem 8.1. *Let $h : (V, B_V) \xrightarrow{g} (X, B) \xrightarrow{f} Y$ be a setup as in Section 6 which satisfies conditions* (a)–(g) *in Section 6. Let $\gamma : Y' \to Y$ be a generically finite projective surjective morphism from a smooth quasi-projective variety Y'. Assume that there exists a simple normal crossing divisor $\Sigma_{Y'}$ on Y' which contains $\gamma^{-1}(\Sigma_Y)$. We consider an induced setup $h' : (V', B_{V'}) \xrightarrow{g'} (X', B_{X'}) \xrightarrow{f'} Y'$ as in* (8.1). *Let $M_{Y'}$ be the moduli part of the induced setup $h' : (V', B_{V'}) \xrightarrow{g'} (X', B_{X'}) \xrightarrow{f'} Y'$. Then we obtain $\gamma^* M_Y = M_{Y'}$.*

Proof. We divide the proof into the following two steps.

Step 1. In this step, we further assume that $h : (V, B_V) \to Y$ and $h' : (V', B_{V'}) \to Y'$ satisfy the assumptions in (iv) and (v) in Proposition 6.3. Then the moduli parts M_Y and $M_{Y'}$ are both integral divisors. By Theorem 3.1, there exists a natural isomorphism

$$\gamma^* \left(h_* \mathcal{O}_V(K_{V/Y} + (B_V^h)^{=1}\right) \simeq h'_* \mathcal{O}_{V'}(K_{V'/Y'} + (B_{V'}^h)^{=1}),$$

which is compatible with the action of the Galois group $G = \mathbb{Z}/b\mathbb{Z}$ (see Proposition 6.3). Therefore, we have an induced isomorphism of eigensheaves corresponding to the eigenvalue ζ^{-1}. Thus we obtain the isomorphism $\gamma^*\mathcal{O}_Y(M_Y) \simeq \mathcal{O}_{Y'}(M_{Y'})$. This means that $\gamma^* M_Y - M_{Y'}$ is linearly trivial. If γ is finite, then we know that $\gamma^* M_Y = M_{Y'}$ holds by Lemma 4.10. Therefore, $\gamma^* M_Y - M_{Y'}$ is exceptional over Y. More precisely, $\operatorname{codim}_Y \gamma(E) \geq 2$ holds for $E = \gamma^* M_Y - M_{Y'}$. Thus we get $\gamma^* M_Y = M_{Y'}$ in this special case since $\gamma^* M_Y - M_{Y'}$ is linearly trivial.

Step 2. In this step we treat the general case. By Lemma 7.3, we can construct a finite cover $\tau : \overline{Y} \to Y$ such that an induced setup $\overline{h} : (\overline{V}, B_{\overline{V}}) \xrightarrow{\overline{g}} (\overline{X}, B_{\overline{X}}) \xrightarrow{\overline{f}} \overline{Y}$ as in (8.1) satisfies the assumptions in (iv) and (v) in Proposition 6.3. By construction, we may assume that there is a simple normal crossing divisor Σ_1 on Y such that $\Sigma_Y \subset \Sigma_1$, τ is étale over $Y \setminus \Sigma_1$, and $\gamma^{-1}(\Sigma_1)$ is a simple normal crossing divisor on Y'. We may further assume that $\gamma^{-1}(\Sigma_1) \cup \Sigma_{Y'}$ is contained in a simple normal crossing divisor. By Lemma 7.2, we can construct the following commutative diagram:

$$\begin{tikzcd}
\overline{Y} \arrow[d, "\tau"'] & \widetilde{Y} \arrow[l, "\widetilde{\gamma}"'] \arrow[d, "\widetilde{\tau}"] \\
Y & Y' \arrow[l, "\gamma"]
\end{tikzcd}$$

where $\widetilde{\tau} : \widetilde{Y} \to Y'$ is a finite cover from a smooth quasi-projective variety $\widetilde{Y}$, and there is a simple normal crossing divisor Σ_2 on Y' such that $\gamma^{-1}(\Sigma_1) \cup \Sigma_{Y'} \subset \Sigma_2$, $\widetilde{\tau}$ is étale over $Y' \setminus \Sigma_2$, and $(\widetilde{\tau})^{-1}(\Sigma_2)$ is a simple normal crossing divisor on $\widetilde{Y}$. Then we apply Lemma 7.3 again. We get a finite cover $\overline{Y}' \to \widetilde{Y}$ from a smooth quasi-projective variety $\overline{Y}'$ and the following commutative diagram:

$$\begin{tikzcd}
 & & \overline{Y}' \arrow[dll, "\gamma'"'] \arrow[dl] \arrow[ddl, "\tau'"] \\
\overline{Y} \arrow[d, "\tau"'] & \widetilde{Y} \arrow[l, "\widetilde{\gamma}"] \arrow[d, "\widetilde{\tau}"'] & \\
Y & Y' \arrow[l, "\gamma"] &
\end{tikzcd}$$

such that an induced setup $\overline{h}' : (\overline{V}', B_{\overline{V}'}) \xrightarrow{\overline{g}'} (\overline{X}', B_{\overline{X}'}) \xrightarrow{\overline{f}'} \overline{Y}'$ satisfies the assumptions in (iv) and (v) in Proposition 6.3. Hence, $\overline{h} : (\overline{V}, B_{\overline{V}}) \to \overline{Y}$ and $\overline{h}' : (\overline{V}', B_{\overline{V}'}) \to \overline{Y}'$ satisfy the assumptions in Step 1. Therefore, we get $M_{\overline{Y}'} = (\gamma')^* M_{\overline{Y}}$, We note that $\tau^* M_Y = M_{\overline{Y}}$ and $(\tau')^* M_{Y'} = M_{\overline{Y}'}$ hold by Lemma 4.10 because τ and τ' are both finite. Thus we get $(\tau')^*(M_{Y'} - \gamma^* M_Y) = 0$. This implies that $M_{Y'} = \gamma^* M_Y$ holds.

Thus, we obtain $\gamma^* M_Y = M_{Y'}$. $\square$

§9. Proof of Theorem 1.2

In this section, we prove Theorem 1.2, which is the main theorem of this paper. Theorem 1.2 obviously generalizes [A4, Theorem 0.2] and [FG2, Theorem 3.6]. Since we have already checked that the moduli part of a given basic slc-trivial fibration behaves well under pull-back by generically finite surjective morphisms with some mild assumptions in Theorem 8.1, there are no difficulties to prove Theorem 1.2.

Let us prove Theorem 1.2.

Proof of Theorem 1.2. Let $f : (X, B) \to Y$ be a basic slc-trivial fibration. As in 4.5, we can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y).$$

Without loss of generality, by taking a projective birational modification $\sigma : Y' \to Y$ from a smooth quasi-projective variety Y' and considering an induced basic slc-trivial fibration $f' : (X', B_{X'}) \to Y'$ of $f : (X, B) \to Y$, we may assume that Y is a smooth quasi-projective variety. By Remark 4.9, we may further assume that

$$K_X + B + \frac{1}{b}(\varphi) = f^*(K_Y + B_Y + M_Y)$$

holds, where $b = b(F, B_F)$ and $\varphi \in \Gamma(X, \mathcal{K}_X^*)$. It is sufficient to prove that the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-potentially nef in the above setup. By taking a birational modification of X which is an isomorphism over the generic point of every stratum of X, we may assume that $\mathrm{Supp}(B - f^*(B_Y + M_Y))$ is a simple normal crossing divisor on X, $(B^h)^{=1}$ is a Cartier divisor on X, and every stratum of $(X, (B^h)^{=1})$ is dominant onto Y (see, for example, [BVP, Theorem 1.4 and Section 8] and [Fn14, Lemma 2.11]). As in Section 6, by taking the b-fold cyclic cover $\pi : (\widetilde{X}, B_{\widetilde{X}}) \to (X, B)$ associated to

$$K_{X/Y} + B - f^*(B_Y + M_Y) = \frac{1}{b}(\varphi^{-1})$$

and a suitable birational modification $d : (V, B_V) \to (\widetilde{X}, B_{\widetilde{X}})$, we get

$$h : (V, B_V) \xrightarrow{g} (X, B) \xrightarrow{f} Y.$$

Then we take a projective birational morphism $\sigma : Y' \to Y$ from a smooth quasi-projective variety Y' and obtain an induced setup $h' : (V', B_{V'}) \xrightarrow{g'} (X', B_{X'}) \xrightarrow{f'} Y'$ which satisfies conditions (a)–(g) in Section 6.

From now on, we will prove that $\nu^*(\mathbf{M}_{Y'}) = \mathbf{M}_{Y''}$ and $\nu^*(K_{Y'} + \mathbf{B}_{Y'}) = K_{Y''} + \mathbf{B}_{Y''}$ hold for every proper birational morphism $\nu : Y'' \to Y'$ from a normal variety Y''. We take a common resolution

$$\begin{tikzcd} & Y''' \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y' & & Y'' \arrow[ll, "\nu"] \end{tikzcd}$$

such that Y''' is a smooth quasi-projective variety and that $p^{-1}(\Sigma_{Y'})$ is a simple normal crossing divisor on Y'''. We consider an induced setup $h''' : (V''', B_{V'''}) \xrightarrow{g'''} (X''', B_{X'''}) \xrightarrow{f'''} Y'''$ as in Section 8. By Theorem 8.1, we get $p^*\mathbf{M}_{Y'} = \mathbf{M}_{Y'''}$. Thus we obtain $p^*(K_{Y'} + \mathbf{B}_{Y'}) = K_{Y'''} + \mathbf{B}_{Y'''}$. Since $q : Y''' \to Y''$ is birational, $\nu^*\mathbf{M}_{Y'} = \mathbf{M}_{Y''}$ and $\nu^*(K_{Y'} + \mathbf{B}_{Y'}) = K_{Y''} + \mathbf{B}_{Y''}$ follow from the above relations by taking q_*.

Finally, we will prove that $\mathbf{M}_{Y'}$ is potentially nef. By Lemma 4.12, we can compactify $f : (X, B) \to Y$ and may assume that X and Y are both complete varieties. Therefore, it is sufficient to prove that $\mathbf{M}_{Y'}$ is nef. Let $\tau : \overline{Y}' \to Y'$ be a suitable finite cover from a smooth projective variety $\overline{Y}'$ as in Lemma 7.3. More precisely, $\overline{h}' : (\overline{V}', B_{\overline{V}'}) \to \overline{Y}'$ satisfies the assumptions in (iv) and (v) in Proposition 6.3, where $\overline{h}' : (\overline{V}', B_{\overline{V}'}) \xrightarrow{\overline{g}'} (\overline{X}', B_{\overline{X}'}) \xrightarrow{\overline{f}'} \overline{Y}'$ is an induced setup from $h' : (V', B_{V'}) \xrightarrow{g'} (X', B_{X'}) \xrightarrow{f'} Y'$ by $\tau : \overline{Y}' \to Y'$. Then $\tau^*\mathbf{M}_{Y'} = \mathbf{M}_{\overline{Y}'}$ holds by Lemma 4.10 since τ is finite. By Proposition 6.3, $\mathbf{M}_{\overline{Y}'}$ is a nef Cartier divisor. Therefore, $\mathbf{M}_{Y'}$ is nef. Hence, we obtain that $\mathbf{M}$ is b-potentially nef. $\square$

§10. Quasi-log canonical pairs

In this section, let us recall the basic definitions of quasi-log canonical pairs and prove a result on normal irreducible quasi-log canonical pairs, which will play a crucial role in the proof of Theorem 1.7. For the details of the theory of quasi-log schemes, see [Fn10, Chapter 6]. We note that our formulation in [Fn10, Chapter 6] is slightly different from Ambro's original one (see [A3]).

Let us start with the definition of globally embedded simple normal crossing pairs. We will soon use it for the definition of quasi-log canonical pairs (see Definition 10.2).

Definition 10.1 (Globally embedded simple normal crossing pairs). Let Y be a simple normal crossing divisor on a smooth variety M and let B be an $\mathbb{R}$-divisor on M such that Y and B have no common irreducible components and that the support of $Y + B$ is a simple normal crossing divisor on M. In this situation, the pair (Y, B_Y), where $B_Y := B|_Y$, is called a *globally embedded simple normal crossing pair*.

Of course, a globally embedded simple normal crossing pair is a simple normal crossing pair in the sense of Definition 2.15. We note that a simple normal crossing variety can not always be embedded as a simple normal crossing divisor on a smooth variety (see [Fn10, Example 5.2.7]). Therefore, a simple normal crossing pair is not necessarily a globally embedded simple normal crossing pair.

Let us quickly look at the definition of quasi-log canonical pairs.

Definition 10.2 (Quasi-log canonical pairs). Let X be a scheme and let ω be an $\mathbb{R}$-Cartier divisor (or an $\mathbb{R}$-line bundle) on X. Let $f : Y \to X$ be a proper morphism from a globally embedded simple normal crossing pair (Y, B_Y). If the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$$

is an isomorphism, B_Y is a subboundary $\mathbb{R}$-divisor, and

$$f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$$

holds, then $(X, \omega, f : (Y, B_Y) \to X)$ or simply $[X, \omega]$ is called a *quasi-log canonical pair* (*qlc pair*, for short).

We say that $(X, \omega, f : (Y, B_Y) \to X)$ or $[X, \omega]$ has a $\mathbb{Q}$-*structure* if B_Y is a $\mathbb{Q}$-divisor, ω is a $\mathbb{Q}$-Cartier divisor (or a $\mathbb{Q}$-line bundle), and $f^*\omega \sim_{\mathbb{Q}} K_Y + B_Y$ holds in the above definition.

Let $(X, \omega, f : (Y, B_Y) \to X)$ be a quasi-log canonical pair as in Definition 10.2. Let $\nu : Y^\nu \to Y$ be the normalization. We put $K_{Y^\nu} + \Theta = \nu^*(K_Y + B_Y)$, that is, Θ

is the sum of the inverse images of B_Y and the singular locus of Y. Then (Y^ν, Θ) is sub log canonical in the usual sense (see 2.2). Let W be a log canonical center of (Y^ν, Θ) or an irreducible component of Y^ν. Then $f\circ\nu(W)$ is called a *qlc stratum* of $(X, \omega, f : (Y, B_Y) \to X)$. If there is no danger of confusion, we simply call it a qlc stratum of $[X, \omega]$. If C is a qlc stratum of $[X, \omega]$ but is not an irreducible component of X, then C is called a *qlc center* of $(X, \omega, f : (Y, B_Y) \to X)$ or simply of $[X, \omega]$. The union of all qlc centers of $[X, \omega]$ is denoted by $\mathrm{Nqklt}(X, \omega, f : (Y, B_Y) \to X)$ or simply by $\mathrm{Nqklt}(X, \omega)$. It is important that by adjunction (see [Fn10, Theorem 6.3.5 (i)]) $[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X,\omega)}]$ has a natural quasi-log canonical structure induced by $(X, \omega, f : (Y, B_Y) \to X)$.

The following theorem is the main result of this section. Although this is a special case of [FLh1, Theorem 1.1], we give a detailed proof for the reader's convenience.

Theorem 10.3 (see [FLh1, Theorem 1.1]). *Let $(X, \omega, f : (Y, B_Y) \to X)$ be a quasi-log canonical pair. Assume that X is a normal irreducible variety. Then we can construct a projective surjective morphism $f' : Y' \to X$ with the following properties:*

(i) *$(Y', B_{Y'})$ is a globally embedded simple normal crossing pair, Y' is quasi-projective, $B_{Y'}$ is a subboundary $\mathbb{R}$-divisor, and $K_{Y'} + B_{Y'} \sim_{\mathbb{R}} (f')^*\omega$,*
(ii) *the natural map $\mathcal{O}_X \to f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil)$ is an isomorphism, and*
(iii) *every stratum of Y' is dominant onto X.*

Therefore, $(X, \omega, f' : (Y', B_{Y'}) \to X)$ is also a quasi-log canonical pair. Moreover, we have:

(iv) *if C is a qlc stratum of $(X, \omega, f' : (Y', B_{Y'}) \to X)$ then C is a qlc stratum of $(X, \omega, f : (Y, B_Y) \to X)$, and*
(v) $\mathrm{Nqklt}(X, \omega, f' : (Y', B_{Y'}) \to X) = \mathrm{Nqklt}(X, \omega, f : (Y, B_Y) \to X)$.

Furthermore, if $K_Y + B_Y \sim_{\mathbb{Q}} f^*\omega$, then $K_{Y'} + B_{Y'} \sim_{\mathbb{Q}} (f')^*\omega$ holds by construction. We note that if $(X, \omega, f' : (Y', B_{Y'}) \to X)$ has a $\mathbb{Q}$-structure then $f : (Y', B_{Y'}) \to X$ is a basic slc-trivial fibration in the sense of Definition 4.1.

Proof. By [Fn10, Proposition 6.3.1], we may assume that Y is quasi-projective and that the union of all strata of (Y, B_Y) mapped to $\mathrm{Nqklt}(X, \omega, f : (Y, B_Y) \to X)$, which is denoted by Y'', is a union of some irreducible components of Y by taking some suitable blow-ups of the ambient space M of Y. We put $Y' = Y - Y''$ and

$K_{Y'} + B_{Y'} = (K_Y + B_Y)|_{Y'}$. Then we obtain the following commutative diagram:

$$\begin{tikzcd} Y' \arrow[r, hook, "\iota"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ V \arrow[r, "p"'] & X \end{tikzcd}$$

where $\iota : Y' \hookrightarrow Y$ is a natural closed immersion and

$$Y' \xrightarrow{f'} V \xrightarrow{p} X$$

is the Stein factorization of $f \circ \iota : Y' \to X$. By construction, the natural map $\mathcal{O}_V \to f'_*\mathcal{O}_{Y'}$ is an isomorphism and every stratum of Y' is dominant onto V. By construction again, $\iota : Y' \hookrightarrow Y$ is an isomorphism over the generic point of X. Therefore, p is birational. Thus, $p : V \to X$ is an isomorphism by Zariski's main theorem since X is normal and p is a finite birational morphism. So we have the following commutative diagram.

$$\begin{tikzcd} Y' \arrow[r, hook, "\iota"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

By construction, it is obvious that $B_{Y'}$ is a subboundary $\mathbb{R}$-divisor and that $K_{Y'} + B_{Y'} \sim_{\mathbb{R}} (f')^*\omega$ holds. Of course, if $K_Y + B_Y \sim_{\mathbb{Q}} f^*\omega$, then $K_{Y'} + B_{Y'} \sim_{\mathbb{Q}} (f')^*\omega$.

Claim. *The natural map*

$$\alpha : \mathcal{O}_X \to f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil)$$

is an isomorphism.

Proof of Claim. Since X is normal and $f'_*\mathcal{O}_{Y'}(\lceil -(B^{<1}_{Y'})\rceil)$ is torsion-free, it is sufficient to see that α is an isomorphism in codimension one. Let P be a prime divisor on X such that $P \subset \mathrm{Nqklt}(X, \omega, f : (Y, B_Y) \to X)$. We note that every fiber of f is connected by $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$. Thus, by construction, there exists an irreducible component of $B^{=1}_{Y'}$ which maps onto P. Therefore, the effective divisor $\lceil -(B^{<1}_{Y'})\rceil$ does not contain the whole fiber of f' over the generic point of P. Thus, α is an isomorphism at the generic point of P. This implies that the natural map α is an isomorphism. $\square$

By Claim, $(X, \omega, f' : (Y', B_{Y'}) \to X)$ is a quasi-log canonical pair. By construction, if C is a qlc stratum of $(X, \omega, f' : (Y', B_{Y'}) \to X)$ then C is a qlc stratum of $(X, \omega, f : (Y, B_Y) \to X)$. By construction again, it is easy to see that

$$\mathcal{I}_{\mathrm{Nqklt}(X,\omega,f:(Y,B_Y)\to X)} = f'_*\mathcal{O}_{Y'}(\lceil -(B_{Y'}^{<1})\rceil - Y''|_{Y'}) = \mathcal{I}_{\mathrm{Nqklt}(X,\omega,f':(Y',B_{Y'})\to X)}$$

(see the proof of [Fn10, Theorem 6.3.5 (i)]). Therefore, this new quasi-log canonical pair $(X, \omega, f' : (Y', B_{Y'}) \to X)$ is the desired one. We note that $f' : (Y', B_{Y'}) \to X$ is a basic slc-trivial fibration in the sense of Definition 4.1 when the quasi-log canonical pair $(X, \omega, f' : (Y', B_{Y'}) \to X)$ has a $\mathbb{Q}$-structure. □

Theorem 10.3 is one of the main motivations to introduce the notion of basic slc-trivial fibrations.

We close this section with an important remark on *embedded* qlc centers.

Remark 10.4. In Theorem 10.3, let C be an embedded qlc center of $(X, \omega, f : (Y, B_Y) \to X)$, that is, C is a qlc center of $(X, \omega, f : (Y, B_Y) \to X)$ that is not an irreducible component of $\mathrm{Nqklt}(X, \omega, f : (Y, B_Y) \to X)$. Then it is not clear whether C is also a qlc center of $(X, \omega, f' : (Y', B_{Y'}) \to X)$ or not by the above construction of $f' : (Y', B_{Y'}) \to X$. In Theorem 10.3, we just claim that the equality

$$\mathrm{Nqklt}(X, \omega, f' : (Y', B_{Y'}) \to X) = \mathrm{Nqklt}(X, \omega, f : (Y, B_Y) \to X)$$

holds.

§11. Structure theorem for normal qlc pairs

In this section, we prove Theorem 1.7. We believe that Theorem 1.7 will make the theory of quasi-log schemes more powerful and flexible. We treat various nontrivial applications of Theorem 1.7 in [FLh2], [FLh3], and [Fn16].

Let us start with the following elementary lemma.

Lemma 11.1. *Let $(X, \omega, f : (Y, B_Y) \to X)$ be a quasi-log canonical pair such that X is a normal irreducible variety and that every stratum of Y is dominant onto X. Then we obtain a $\mathbb{Q}$-divisor D_i on Y, a $\mathbb{Q}$-Cartier divisor ω_i on X, and a positive real number r_i for $1 \le i \le k$ such that*

(i) $\sum_{i=1}^k r_i = 1$,

(ii) $D_i = D_i^{\le 1}$, $\mathrm{Supp}\, D_i = \mathrm{Supp}\, B_Y$, $D_i^{=1} = B_Y^{=1}$, *and* $\lceil -(D_i^{<1})\rceil = \lceil -(B_Y^{<1})\rceil$ *for every* i,

(iii) $\omega = \sum_{i=1}^k r_i\omega_i$ *and* $B_Y = \sum_{i=1}^k r_i D_i$, *and*

(iv) $(X, \omega_i, f : (Y, D_i) \to X)$ *is a quasi-log canonical pair with* $K_Y + D_i \sim_{\mathbb{Q}} f^*\omega_i$ *for every* i.

Proof. We put $B_Y=\sum_j b_jB_j$, where B_j is a simple normal crossing divisor on Y for every j, $b_{j_1}\neq b_{j_2}$ for $j_1\neq j_2$, and $\operatorname{Supp} B_{j_1}$ and $\operatorname{Supp} B_{j_2}$ have no common irreducible components for $j_1\neq j_2$. We may assume that $b_j\in\mathbb{R}\setminus\mathbb{Q}$ for $1\leq j\leq l$ and $b_j\in\mathbb{Q}$ for $j\geq l+1$. We put $\omega=\sum_{p=1}^m a_p\omega_p$, where $a_p\in\mathbb{R}$ and ω_p is a Cartier divisor on X for every p. We can write

$$K_Y+B_Y+\sum_{q=1}^n c_q(\varphi_q)=\sum_{p=1}^m a_pf^*\omega_p$$

where $c_q\in\mathbb{R}$ and $\varphi_q\in\Gamma(Y,\mathcal{K}_Y^*)$ for every q. We consider the following linear map

$$\psi:\mathbb{R}^{l+m+n}\longrightarrow\Gamma(Y,\mathcal{K}_Y^*/\mathcal{O}_Y^*)\otimes_{\mathbb{Z}}\mathbb{R}$$

defined by

$$\psi(x_1,\dots,x_{l+m+n})=\sum_{\alpha=1}^m x_\alpha f^*\omega_\alpha-\sum_{\beta=1}^n x_{m+\beta}(\varphi_\beta)-\sum_{\gamma=1}^l x_{m+n+\gamma}B_\gamma.$$

We note that ψ is defined over $\mathbb{Q}$. By construction,

$$\mathcal{A}:=\psi^{-1}\left(K_Y+\sum_{j\geq l+1}b_jB_j\right)$$

is a nonempty affine subspace of $\mathbb{R}^{l+m+n}$ defined over $\mathbb{Q}$. We put

$$P:=(a_1,\dots,a_m,c_1,\dots,c_n,b_1,\dots,b_l)\in\mathcal{A}.$$

We can take $P_1,\dots,P_k\in\mathcal{A}\cap\mathbb{Q}^{l+m+n}$ and $r_1,\dots,r_k\in\mathbb{R}_{>0}$ such that $\sum_{i=1}^k r_i=1$ and $\sum_{i=1}^k r_iP_i=P$ in $\mathcal{A}$. Note that we can make P_i arbitrary close to P for every i. So we may assume that P_i is sufficiently close to P for every i. For each P_i, we obtain

$$K_Y+D_i\sim_{\mathbb{Q}}f^*\omega_i \tag{11.1}$$

which satisfies (ii) by using ψ. By construction, (i) and (iii) hold. By (11.1) and (ii),

$$(X,\omega_i,f:(Y,D_i)\to X)$$

is a quasi-log canonical pair for every i. Therefore, we get (iv). $\square$

We prepare one more lemma for the proof of Theorem 1.7, which is essentially contained in [Fn10, Chapter 6].

Lemma 11.2. *Let $(X, \omega, f : (Y, B_Y) \to X)$ be a quasi-log canonical pair such that X is a normal irreducible variety. We assume that every stratum of Y is dominant onto X. Let P be a prime divisor on X which is Cartier. We put*

$$b_P := \max \left\{ t \in \mathbb{R} \;\middle|\; \begin{array}{l} (Y, B_Y + tf^*P) \textit{ is sub slc over} \\ \textit{the generic point of } P \end{array} \right\}.$$

Then $b_P \leq 1$ holds.

Proof. If P is a qlc center of $[X, \omega]$, then $b_P = 0$. Therefore, from now on, we assume that P is not a qlc center of $[X, \omega]$. By shrinking X around the generic point of P, we may assume that X is quasi-projective and that $(Y, B_Y + b_P f^*P)$ is sub slc. By taking a suitable birational modification of Y (see [BVP, Theorem 1.4]), we may further assume that $(Y, \operatorname{Supp} B_Y + \operatorname{Supp} f^*P)$ is a simple normal crossing pair. In this situation, $(X, \omega + b_P P, f : (Y, B_Y + b_P f^*P) \to X)$ has a natural quasi-log canonical structure. In order to prove $b_P \leq 1$, we may further assume that X is a smooth curve and P is a point of X by taking general hyperplanes of X and by using adjunction. If $b_P > 1$, then $((B_Y + f^*P)^v)^{<1} = (B_Y + f^*P)^v$ holds over P. This implies that $f^*P \leq \lceil -(B_Y^{<1}) \rceil$. Thus we get

$$\mathcal{O}_X \subsetneq \mathcal{O}_X(P) \subset f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$$

in a neighborhood of P. This is a contradiction because the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1}) \rceil)$$

is an isomorphism. Therefore, we obtain $b_P \leq 1$. □

Let us start the proof of Theorem 1.7.

Proof of Theorem 1.7. By Theorem 10.3, we may assume that there exists a projective surjective morphism $f : (Y, B_Y) \to X$ from a simple normal crossing pair (Y, B_Y) such that every stratum of Y is dominant onto X and that $(X, \omega, f : (Y, B_Y) \to X)$ is a quasi-log canonical pair. By taking some more blow-ups, we may further assume that $(B_Y^h)^{=1}$ is Cartier and that every stratum of $(Y, (B_Y^h)^{=1})$ is dominant onto X (see, for example, [BVP, Theorem 1.4 and Section 8] and [Fn14, Lemma 2.11]).

Step 1. In this step, we treat the case where $[X, \omega]$ has a $\mathbb{Q}$-structure. In this situation, $f : (Y, B_Y) \to X$ is a basic slc-trivial fibration (see Theorem 10.3). Let $\mathbf{B}$ be the discriminant $\mathbb{Q}$-b-divisor and let $\mathbf{M}$ be the moduli $\mathbb{Q}$-b-divisor associated to $f : (Y, B_Y) \to X$. Since (Y, B_Y) is sub slc, $\mathbf{B}_X$ is a subboundary $\mathbb{Q}$-divisor on

X, that is, $\mathbf{B}_X = (\mathbf{B}_X)^{\leq 1}$. By Lemma 11.2, we obtain that $\mathbf{B}_X$ is an effective $\mathbb{Q}$-divisor on X. By the definition of qlc centers, we have $f((B_Y^v)^{=1}) = \mathrm{Nqklt}(X, \omega)$. We take a projective birational morphism $p : X' \to X$ from a smooth quasi-projective variety X'. Let $f' : (Y', B_{Y'}) \to X'$ be an induced basic slc-trivial fibration with the following commutative diagram.

$$
\begin{tikzcd}
(Y, B_Y) \arrow[d, "f"'] & (Y', B_{Y'}) \arrow[l, "q"'] \arrow[d, "f'"] \\
X & X' \arrow[l, "p"]
\end{tikzcd}
$$

By Theorem 1.2, we may assume that there exists a simple normal crossing divisor $\Sigma_{X'}$ on X' such that $\mathbf{M} = \overline{\mathbf{M}_{X'}}$, $\mathrm{Supp}\, \mathbf{M}_{X'}$ and $\mathrm{Supp}\, \mathbf{B}_{X'}$ are contained in $\Sigma_{X'}$, and that every stratum of $(Y', \mathrm{Supp}\, B^h_{Y'})$ is smooth over $X' \setminus \Sigma_{X'}$. Of course, we may assume that $M_{X'} := \mathbf{M}_{X'}$ is potentially nef by Theorem 1.2. We may further assume that every irreducible component of $q_*^{-1}\left((B_Y^v)^{=1}\right)$ is mapped onto a prime divisor in $\Sigma_{X'}$ with the aid of the flattening theorem (see [RG, Théorème (5.2.2)]). We put $B_{X'} := \mathbf{B}_{X'}$. Note that $B_{X'}$ is a subboundary $\mathbb{Q}$-divisor on X' since $(Y', B_{Y'})$ is sub slc. In the above setup, $f'(q_*^{-1}(B_Y^v)^{=1}) \subset B_{X'}^{=1}$ by the definition of $\mathbf{B}$. Thus, we get $\mathrm{Nqklt}(X, \omega) \subset p(B_{X'}^{=1})$. On the other hand, we can easily see that $p(B_{X'}^{=1}) \subset \mathrm{Nqklt}(X, \omega)$ by definition. Therefore, $p(B_{X'}^{=1}) = \mathrm{Nqklt}(X, \omega)$ holds. Since $p_* B_{X'} = \mathbf{B}_X$ and $\mathbf{B}_X$ is effective, $B_{X'}^{<0}$ is p-exceptional. Hence, $B_{X'}$ and $M_{X'}$ satisfy the desired properties. We note that $B_{X'}$ and $M_{X'}$ are obviously $\mathbb{Q}$-divisors by construction.

Step 2. In this step, we treat the general case. We first use Lemma 11.1 and get a positive real number r_i and $(X, \omega_i, f : (Y, D_i) \to X)$ for $1 \leq i \leq k$ with the properties in Lemma 11.1. Then we apply the argument in Step 1 to

$$(X, \omega_i, f : (Y, D_i) \to X)$$

for every i. By Theorem 1.2, we can take a projective birational morphism $p : X' \to X$ from a smooth quasi-projective variety X' which works for

$$(X, \omega_i, f : (Y, D_i) \to X)$$

for every i. By summing them up with weight r_i, we get $\mathbb{R}$-divisors $B_{X'}$ and $M_{X'}$ with the desired properties. In this case, we do not claim that $B_{X'}$ is the discriminant of $f' : (Y', B_{Y'}) \to X'$.

Therefore, we get $p : X' \to X$, $B_{X'}$, and $M_{X'}$ with the desired properties. $\square$

As we mentioned in Remark 1.9, $(X, B_X + M_X)$, where $B_X := p_*B_{X'}$ and $M_X := p_*M_{X'}$, is generalized lc in the sense of [BZ, Definition 4.1]. Moreover, if $\mathrm{Nqklt}(X, \omega) = \emptyset$, then $(X, B_X + M_X)$ is generalized klt in the sense of [BZ, Definition 4.1]. We note that $M_{X'}$ is a finite $\mathbb{R}_{>0}$-linear combination of relatively nef Cartier divisors. Hence $(X, B_X + M_X)$ is an NQC g-pair in the sense of [HL, Definition 2.13].

Finally, we prove Corollary 1.10.

Proof of Corollary 1.10. By adjunction (see [Fn10, Theorem 6.3.5]), $[W, \omega|_W]$ is a quasi-log canonical pair. Since W is a minimal qlc stratum of $[X, \omega]$, W is a normal irreducible variety and $\mathrm{Nqklt}(W, \omega|_W) = \emptyset$ holds (see [Fn10, Theorem 6.3.5 and Lemma 6.3.9]). By Theorem 1.7, we can take a projective birational morphism $p : W' \to W$ from a smooth quasi-projective variety W', a subboundary $\mathbb{R}$-divisor $B_{W'}$ whose support is a simple normal crossing divisor on W', a potentially nef $\mathbb{R}$-divisor $M_{W'}$ on W' such that $p^*(\omega|_W) = K_{W'} + B_{W'} + M_{W'}$. Since $\mathrm{Nqklt}(W, \omega|_W) = \emptyset$ holds, we may assume that $B_{W'} = B_{W'}^{<1}$. By taking some more blow-ups, if necessary, we may further assume that there exists an effective p-exceptional Cartier divisor E on W' such that $\mathrm{Supp}\, B_{W'} \cup \mathrm{Supp}\, E$ is contained in a simple normal crossing divisor and that $-E$ is p-ample. We note that $-\varepsilon E + p^*H + M_{W'}$ is semi-ample for any $0 < \varepsilon \ll 1$. Therefore, we can take a general effective $\mathbb{R}$-divisor $G \sim_{\mathbb{R}} -\varepsilon E + p^*H + M_{W'}$ such that $\mathrm{Supp}(B_{W'} + \varepsilon E + G)$ is a simple normal crossing divisor on W' and $\lfloor B_{W'} + \varepsilon E + G \rfloor \leq 0$. By construction, $K_{W'} + B_{W'} + M_{W'} + p^*H \sim_{\mathbb{R}} K_{W'} + B_{W'} + \varepsilon E + G$ holds. We put $\Delta_W = p_*(B_{W'} + \varepsilon E + G)$. Then Δ_W satisfies the desired properties.

When $[X, \omega]$ has a $\mathbb{Q}$-structure and H is an ample $\mathbb{Q}$-divisor, it is easy to see that we can make Δ_W a $\mathbb{Q}$-divisor with $K_W + \Delta_W \sim_{\mathbb{Q}} \omega|_W + H$ by the above construction of Δ_W. □

§12. On the basepoint-freeness

In this section, we give a small remark on the basepoint-free theorem for quasi-log canonical pairs.

The following theorem is a special case of the basepoint-free theorem for quasi-log schemes (see [Fn10, Theorem 6.5.1]). We can quickly reduce Theorem 12.1 to the usual Kawamata–Shokurov basepoint-free theorem for kawamata log terminal pairs by Corollary 1.10. Note that the general basepoint-free theorem for quasi-log schemes (see [Fn10, Theorem 6.5.1]) easily follows from Theorem 12.1. For the details, see Claims 1, 3, and 4 in the proof of [Fn10, Theorem 6.5.1].

Theorem 12.1 (Basepoint-free theorem, see [Fn10, Theorem 6.5.1]). *Let $[X, \omega]$ be a quasi-log canonical pair with* $\mathrm{Nqklt}(X, \omega) = \emptyset$ *and let $\pi : X \to S$ be a projective morphism between schemes. Let L be a π-nef Cartier divisor on X. Assume that $qL - \omega$ is π-ample for some real number $q > 0$. Then there exists a positive number m_0 such that $\mathcal{O}_X(mL)$ is π-generated for every integer $m \geq m_0$.*

Proof. Without loss of generality, we may assume that S is quasi-projective. Then X is also quasi-projective. Therefore, we can take an ample $\mathbb{Q}$-divisor H on X such that $qL - (\omega + H)$ is still π-ample. By Corollary 1.10, we can take an effective $\mathbb{R}$-divisor Δ_X on X such that $\omega + H \sim_{\mathbb{R}} K_X + \Delta_X$ and that (X, Δ_X) is kawamata log terminal. Therefore, by the usual Kawamata–Shokurov basepoint-free theorem for kawamata log terminal pairs, we obtain a positive number m_0 such that $\mathcal{O}_X(mL)$ is π-generated for every integer $m \geq m_0$. □

§13. Supplements to [FF1]

In this section, we give some supplementary remarks on [FF1] for the reader's convenience. We believe that there are no serious troubles in [FF1]. However, we found that it contains some minor mistakes and ambiguities. So we fix them here. For a completely different approach to the results in [FF1] based on Saito's theory of mixed Hodge modules, see [FFS].

13.1 (Base change theorem). We note that the statement of [FF1, Lemma 3.4 (iv)] is correct. However, the proof of [FF1, Lemma 3.4 (iv)] is somewhat misleading. Therefore, we recommend the interested reader to see [Fs1, Lemma 2.20] and its proof. We think that [FF1, Lemma 3.4] is an easy exercise.

13.2 (Semipositivity theorem). In [FF1, Section 5], we discussed a generalization of the Fujita–Zucker–Kawamata semipositivity theorem (see [FF1, Theorem 5.21]), which plays a crucial role in this paper. We used [FF1, Corollary 5.23], which is an easy consequence of [FF1, Theorem 5.21], in Theorem 3.1 (ii). Unfortunately, there are some ambiguities in the arguments in [FF1, Section 5]. In [FF1, 5.8], we defined the condition (mMH). It was not precise enough because the real structure was not mentioned explicitly. In [Fs2, Section 2], Taro Fujisawa, who is one of the authors of [FF1], removed the ambiguities from [FF1, Section 5]. We recommend the reader to see [Fs2]. We also recommend the interested reader to see [FFS, Theorem 3] and [FF3]. In [FF3], we give an analytic generalization of the Fujita–Zucker–Kawamata semipositivity theorem whose proof is completely different from the arguments in [FF1, Section 5].

13.3 (Lemma on two filtrations). In Section 4 of [FF1], the lemma on two filtrations [D, Propositions (7.2.5) and (7.2.8)] (see also [PeSt, Theorem 3.12]) was used several times (explicitly stated at p. 608, the proof of Lemma 4.5, p. 610, Remark 4.6, p. 618, Step 1 of the proof of Lemma 4.10 and p. 623, the proof of Lemma 4.12, and implicitly used at p. 611, the proof of Lemma 4.8). However, there are missing points in the arguments.

Let K be a complex, W a finite increasing filtration on K and F a finite decreasing filtration on K. In order to apply the lemma on two filtrations for the spectral sequence

$$(E_r^{p,q}(K,W), F_{\mathrm{rec}}),$$

it is necessary to discuss about the E_0-terms. More precisely, it has to be checked that the strictness of the filtration F on the complex $\mathrm{Gr}_m^W K$ holds true for all m. Here we will explain how to check this strictness for the case of Lemma 4.10 of [FF1] mentioned above. For the other cases, the similar arguments are valid.

In Step 1 of the proof of Lemma 4.10, the bifiltered complex

$$(Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet), L, F)$$

is studied. Thus the strictness of the filtration F on the complex

$$\mathrm{Gr}_m^L Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet)$$

has to be checked for all m. Under the canonical isomorphism

$$\begin{aligned}\mathrm{Gr}_m^L Rf_*\Omega_{X_\bullet/\Delta}(\log E_\bullet) &\simeq Rf_* \mathrm{Gr}_m^L \Omega_{X_\bullet/\Delta}(\log E_\bullet)\\ &\simeq Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m})[m],\end{aligned}$$

the filtration F coincides with the filtration induced from the stupid filtration, which is denoted by F again, on the complex $\Omega_{X_{-m}/\Delta}(\log E_{-m})$. Therefore it suffices to prove the strictness of the filtration F on $Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m})$ that is induced by the stupid filtration F on $\Omega_{X_{-m}/\Delta}(\log E_{-m})$. The strictness of F on $Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m})$ is equivalent to the E_1-degeneracy of the spectral sequence $E_r^{p,q}(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F)$. We note that the morphism of E_r-terms

$$\begin{aligned}d_r : E_r^{p,q}&(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F)\\ &\longrightarrow E_r^{p+r,q-r+1}(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F)\end{aligned}$$

is zero on Δ^* for all p, q and for all $r \geq 1$ because $X_{-m} \longrightarrow \Delta$ is smooth and projective over Δ^*. On the other hand,

$$\begin{aligned}E_1^{p,q}(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F) &= R^{p+q}f_{-m*}\mathrm{Gr}_F^p \Omega_{X_{-m}/\Delta}(\log E_{-m})\\ &= R^q f_{-m*}\Omega^p_{X_{-m}/\Delta}(\log E_{-m}),\end{aligned}$$

is a locally free $\mathcal{O}_\Delta$-module of finite rank by [St, (2.11) Theorem]. Therefore the morphism of E_1-terms d_1 is zero on the whole Δ for all p, q. Inductively on r, we obtain that $E_r^{p,q}(Rf_{-m*}\Omega_{X_{-m}/\Delta}(\log E_{-m}), F)$ is a locally free $\mathcal{O}_\Delta$-module of finite rank and that d_r is zero on the whole Δ for all p, q and for all $r \geq 1$. Thus the E_1-degeneracy is proved.

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