ON RELATIVE FANO PAIRS

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Abstract. We prove that a relative Fano pair is a relative Mori dream space in the complex analytic setting.

Contents

1. Relative Fano pairs

The main purpose of this paper is to show that Theorem 1.1 follows from results established in [F1] and [F2]. Theorem 1.1 asserts that a relative Fano pair is a relative Mori dream space in the complex analytic setting, although we do not explicitly define Mori dream spaces for complex analytic spaces. This result can be regarded as a complex analytic generalization of [BCHM, Corollary 1.3.2].

Theorem 1.1 (cf. [BCHM, Corollary 1.3.2]). *Let (X, Δ) be a kawamata log terminal pair. Let $\pi\colon X \to Y$ be a projective surjective morphism between normal complex varieties such that $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$, and let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Assume that X is $\mathbb{Q}$-factorial over W and that $-(K_X+\Delta)$ is π-ample over W. Then the following properties hold:*

(i) *The isomorphism*
$$\varinjlim_{W\subset U} \operatorname{Pic}\left(\pi^{-1}(U)/U\right) \simeq A^1(X/Y;W)$$
holds, where U runs through all open neighborhoods of W.

(ii) *The nef cone*
$$\operatorname{Nef}(X/Y;W) := \left\{\zeta \in N^1(X/Y;W) \mid \zeta \text{ is } \pi\text{-nef over } W\right\}$$
is spanned by finitely many π-semiample line bundles defined over some open neighborhood of W.

(iii) *The pseudo-effective cone*
$$\overline{\operatorname{Eff}}(X/Y;W) := \left\{\zeta \in N^1(X/Y;W) \mid \zeta \text{ is } \pi\text{-pseudo-effective over } W\right\}$$

Date: 2026/5/6, version 0.08.
2020 *Mathematics Subject Classification.* Primary 14E30; Secondary 32C15.
Key words and phrases. Mori dream spaces, minimal model program, complex analytic spaces, Fano pairs, movable cones, nef cones, pseudo-effective cones.

is a rational polyhedral cone in $N^1(X/Y;W)$*. Moreover, for every* $\zeta \in \overline{\mathrm{Eff}}(X/Y;W)$*, after possibly shrinking* Y *around* W*, there exists an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* D *on* X *such that* $\zeta = D$ *in* $\overline{\mathrm{Eff}}(X/Y;W)$*.*

(iv) *The movable cone*

$$\overline{\mathrm{Mov}}(X/Y;W)$$

in $N^1(X/Y;W)$ *is defined over the rationals.*

(v) *There exists a finite collection of small bimeromorphic modifications* $f_i\colon X \dashrightarrow X_i$ *over some open neighborhood* U_i *of* W *such that each* X_i *is* $\mathbb{Q}$*-factorial over* W *and satisfies* (ii)*, and the movable cone* $\overline{\mathrm{Mov}}(X/Y;W)$ *is the union of the cones* $f_i^*(\mathrm{Nef}(X_i/Y;W))$*.*

In this note, we do not use the theory of Mori dream spaces (see [HK] and [O]), since it has been developed only for algebraic varieties. Note that [HK] and [O] rely on Mumford's geometric invariant theory. We also note that a very special case of Theorem 1.1 was already used in [KS].

Acknowledgments. The author would like to thank Professors Travis Schedler, Shinnosuke Okawa, and Yoshinori Gongyo for valuable discussions and suggestions. The author was partially supported by JSPS KAKENHI Grant Numbers JP19H01787, JP20H00111, JP21H00974, JP21H04994, and JP23K20787.

Throughout this note, we freely use the results of [F1].

2. A brief review of the minimal model program with scaling

In this section, we recall some results from [F1]. We begin with the following lemma, which follows easily from [F1].

Lemma 2.1 (Minimal model program with scaling)**.** *Let* (X,Δ) *be a kawamata log terminal pair. Let* $\pi\colon X \to Y$ *be a projective morphism between complex varieties, and let* W *be a Stein compact subset of* Y *such that* $\Gamma(W,\mathcal{O}_Y)$ *is noetherian. Assume that* $-(K_X+\Delta)$ *is* π*-ample. Let* D *be an* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor on* X *such that* $D-(K_X+\Delta)$ *is* π*-ample over* W*. Then, after shrinking* Y *around* W *if necessary, we can take an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* B *on* X *such that* $B \sim_{\mathbb{R}} D-(K_X+\Delta)$ *and* $(X,\Delta+B)$ *is kawamata log terminal. Hence, we can run the* $(K_X+\Delta+B)$*-minimal model program with scaling over* Y *around* W*, which always terminates. Furthermore, if* D *and* $K_X+\Delta$ *are both* $\mathbb{Q}$*-Cartier, then we may choose* $B \sim_{\mathbb{Q}} D-(K_X+\Delta)$*.*

Lemma 2.1 is an immediate consequence of [F1, Theorem 1.7].

Proof of Lemma 2.1. By Bertini's theorem, we may choose an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B as in the statement. We choose a π-ample effective $\mathbb{Q}$-divisor A on X such that $(X,\Delta+B+A)$ is kawamata log terminal and $K_X+\Delta+B+A$ is π-nef over W, after shrinking Y around W if necessary. Then, by [F1, Theorem 1.7], we can run the $(K_X+\Delta+B)$-minimal model program with scaling over Y around W. Since B is π-ample over W, this minimal model program always terminates. □

The following lemma plays a crucial role in this note.

Lemma 2.2. *Let*

$$(X,\Delta+B) =: (X_0,\Delta_0+B_0) \overset{\varphi_0}{\dashrightarrow} (X_1,\Delta_1+B_1) \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{m-1}}{\dashrightarrow} (X_m,\Delta_m+B_m)$$

be the $(K_X+\Delta+B)$-minimal model program with scaling over Y around W constructed in Lemma 2.1. Then D is π-pseudo-effective if and only if (X_m,Δ_m+B_m) is a good minimal model of $(X,\Delta+B)$ over Y around W.

Moreover, D is in $\overline{\mathrm{Mov}}(X/Y;W)$ if and only if each φ_i is a flip, i.e., the induced map $X \dashrightarrow X_m$ is small, and (X_m,Δ_m+B_m) is a good minimal model of $(X,\Delta+B)$ over Y around W.

We note that each step φ_i exists only after suitably shrinking Y around W. Hence we need to replace Y with a sufficiently small Stein open neighborhood of W at each step.

Proof of Lemma 2.2. We set $D_0 := D$ and define $D_{i+1} := \varphi_{i*}D_i$ for every i. Then D is π-pseudo-effective if and only if D_i is π_i-pseudo-effective for every i, where $\pi_i\colon X_i \to Y$ is the structure morphism. Hence, if D is π-pseudo-effective, it follows from the construction of the minimal model program with scaling that $K_{X_m}+\Delta_m+B_m$ is π_m-nef over W. Note that (X_m,Δ_m+B_m) is a kawamata log terminal pair and that B_m is π_m-big. Therefore, $K_{X_m}+\Delta_m+B_m$ is π_m-semiample over an open neighborhood of W by the basepoint-free theorem. It follows from the construction of the minimal model program that φ_i is a flip for every i when D is in $\overline{\mathrm{Mov}}(X/Y;W)$. The converse direction is immediate from the definitions. This completes the proof. □

Remark 2.3. In Lemma 2.2, using Lemma 2.4 below, we can construct an effective $\mathbb{Q}$-divisor Θ_i on X_i such that (X_i,Θ_i) is kawamata log terminal and that $-(K_{X_i}+\Theta_i)$ is π_i-ample over W for every i.

Lemma 2.4 is well known in the algebraic case.

Lemma 2.4. *Let $\pi\colon X\to Y$ and $\pi_Z\colon Z\to Y$ be projective morphisms of normal complex varieties and let W be a Stein compact subset of Y. Let $\varphi\colon X\to Z$ be a projective bimeromorphic contraction over Y such that $\pi=\pi_Z\circ\varphi$.*

(i) *If (X,Δ) is kawamata log terminal and $-(K_X+\Delta)$ is π-ample over W, then, after suitably shrinking Y around W, we can find Δ_Z such that (Z,Δ_Z) is kawamata log terminal and $-(K_Z+\Delta_Z)$ is π_Z-ample.*
(ii) *If (Z,Δ_Z) is kawamata log terminal, $-(K_Z+\Delta_Z)$ is π_Z-ample, and φ is small, then, after suitably shrinking Y around W, we can take Δ such that (X,Δ) is kawamata log terminal and $-(K_X+\Delta)$ is π-ample over W.*

We prove Lemma 2.4 for the sake of completeness.

Proof of Lemma 2.4. Throughout the proof, we freely shrink Y around W without explicit mention.

Let H be a π_Z-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z chosen so that $-(K_X+\Delta)-\varphi^*H$ is π-ample. By Bertini's theorem, we may choose an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on X such that $B\sim_{\mathbb{R}} -(K_X+\Delta)-\varphi^*H$ and $(X,\Delta+B)$ is kawamata log terminal. We set $\Delta_Z := \varphi_*(\Delta+B)$. Since $K_X+\Delta+B\sim_{\mathbb{R}} -\varphi^*H$, it follows that (Z,Δ_Z) is kawamata log terminal and that $K_Z+\Delta_Z\sim_{\mathbb{R}} -H$. Hence we obtain (i).

We define $K_X+\Delta' := \varphi^*(K_Z+\Delta_Z)$. Then (X,Δ') is kawamata log terminal and $-(K_X+\Delta')$ is π-semiample and π-big. By Kodaira's lemma, we may choose an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E on X such that $(X,\Delta'+E)$ is kawamata log terminal and $-(K_X+\Delta'+E)$ is π-ample. Setting $\Delta := \Delta'+E$, we obtain the required properties. Thus we get (ii). □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The proof given here is based on [F1] and [F2]. We do not use the theory of Mori dream spaces (see [HK] and [O]).

Proof of Theorem 1.1. We divide the proof into several steps.

Step 1. We prove (i).

Let $\mathcal{L}$ be a line bundle defined over an open neighborhood of W such that $\mathcal{L} \equiv_W 0$, that is, $\mathcal{L} \cdot C = 0$ for every projective curve C on X with $\pi(C)$ a point of W. Then $\mathcal{L}$ is π-nef over W, and $\mathcal{L} - (K_X + \Delta)$ is π-ample over W. We note that (X, Δ) is kawamata log terminal. Therefore, after possibly shrinking Y around W, the basepoint-free theorem implies that $\mathcal{L} \simeq \pi^*\mathcal{M}$ for some line bundle $\mathcal{M}$ on Y, since $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$. In particular, we obtain

$$\varinjlim_{W \subset U} \operatorname{Pic}\left(\pi^{-1}(U)/U\right) \simeq A^1(X/Y; W),$$

which proves (i).

Step 2 (Basepoint-free theorem)**.** Let $\mathcal{L}$ be a line bundle defined over an open neighborhood of W. Assume that $\mathcal{L}$ is π-nef over W. Then, by the basepoint-free theorem, $\mathcal{L}$ is π-semiample over some open neighborhood of W, since $\mathcal{L} - (K_X + \Delta)$ is π-ample over W and (X, Δ) is kawamata log terminal. Therefore, $\mathcal{L}$ is π-nef over W if and only if it is π-semiample over some open neighborhood of W.

Step 3. We prove (ii).

By the cone theorem, the Kleiman–Mori cone $\overline{\mathrm{NE}}(X/Y; W)$ is spanned by finitely many $(K_X + \Delta)$-negative extremal rays, since $-(K_X + \Delta)$ is π-ample over W and (X, Δ) is kawamata log terminal. Hence the nef cone $\operatorname{Nef}(X/Y; W)$, which is the dual cone of $\overline{\mathrm{NE}}(X/Y; W)$ by definition, is spanned by finitely many line bundles which are π-nef over W, after shrinking Y around W suitably. Therefore, $\operatorname{Nef}(X/Y; W)$ is spanned by finitely many π-semiample line bundles defined over an open neighborhood of W (see Step 2). This proves (ii).

Step 4. Let $\mathcal{L}$ be a line bundle defined over an open neighborhood U of W. By replacing Y with a relatively compact Stein open neighborhood of W contained in U, we may assume that there exists a Cartier divisor D on X such that $\mathcal{L} \simeq \mathcal{O}_X(D)$ and that $\operatorname{Supp} D$ has only finitely many irreducible components.

More generally, let $\mathcal{L}$ be an $\mathbb{R}$-line bundle (resp. a $\mathbb{Q}$-line bundle). Then, after possibly shrinking Y around W, we can find a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (resp. a globally $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor) D on X such that $\mathcal{L} = D$ in $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and that $\operatorname{Supp} D$ has only finitely many irreducible components.

Step 5. We prove (iii).

Throughout this step, we freely shrink Y around W without explicit mention. We put $\rho := \rho(X/Y; W)$. We take line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_\rho$ on X such that $\{\mathcal{L}_1, \ldots, \mathcal{L}_\rho\}$ forms a basis of $A^1(X/Y; W) \otimes_{\mathbb{Z}} \mathbb{Q}$. For each i, we take a Cartier divisor D_i such that $\mathcal{L}_i \simeq \mathcal{O}_X(D_i)$. Without loss of generality, we may assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier by replacing Δ suitably.

We note that if d is a sufficiently large positive integer, then

$$\pm\frac{1}{d}D_i = K_X + \Delta + \left(\pm\frac{1}{d}D_i - (K_X + \Delta)\right)$$

where

$$\pm\frac{1}{d}D_i-(K_X+\Delta)$$

is π-ample for every i. Thus, by [F2], we can check that

$$R\left(X/Y,\frac{1}{d}D_1,-\frac{1}{d}D_1,\cdots,\frac{1}{d}D_\rho,-\frac{1}{d}D_\rho\right)$$

is a locally finitely generated $\mathbb{N}^{2\rho}$-graded $\mathcal{O}_Y$-algebra. Hence, it follows that

$$R(X/Y,D_1,-D_1,\cdots,D_\rho,-D_\rho)$$

is a locally finitely generated $\mathbb{N}^{2\rho}$-graded $\mathcal{O}_Y$-algebra. This implies that the pseudo-effective cone $\overline{\mathrm{Eff}}(X/Y;W)$ is a rational polyhedral cone in $N^1(X/Y;W)$.[1]

Let $\mathcal{L}$ be a π-pseudo-effective $\mathbb{R}$-line (resp. $\mathbb{Q}$-line) bundle on X. We can take an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor) D' on X such that $\mathcal{L}=D'$ holds in $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ (see Step 4). By applying Lemma 2.1 to $\frac{1}{m}D'$ for some $m\gg 0$, after finitely many flips and divisorial contractions, we can make D' semiample over Y. In particular, we can find an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (resp. an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor) D on X such that $D\sim_{\mathbb{R}}D'$ (resp. $D\sim_{\mathbb{Q}}D'$). Hence, $\mathcal{L}=D$ in $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$. This proves (iii).

Step 6. (iv) follows from (v) since $\mathrm{Nef}(X_i/Y;W)$ is a rational polyhedral cone. Thus, it suffices to prove (v).

Step 7. In this final step, we prove (v).

Throughout this step, we freely shrink Y around W without explicit mention. Without loss of generality, we may assume that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. We take a general effective π-ample $\mathbb{Q}$-divisor A on X such that $A\sim_{\mathbb{Q}}-(K_X+\Delta)$ and that $\mathrm{Supp}\,A$ has only finitely many irreducible components. It is obvious from the definition that

$$\overline{\mathrm{Mov}}(X/Y;W)\subset\overline{\mathrm{Eff}}(X/Y;W).$$

By (iii), $\overline{\mathrm{Eff}}(X/Y;W)$ is a rational polyhedral cone spanned by finitely many effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $D_1,\dots,D_k$ on X (see Step 5). By replacing D_j with $\frac{1}{m}D_j$ for some sufficiently large positive integer m, we may further assume that $(X,\Delta+D_j)$ is kawamata log terminal for every j. We note that

$$\begin{aligned}D_j&=K_X+\Delta+(D_j-(K_X+\Delta))\\&=K_X+\Delta+D_j+(-(K_X+\Delta))\\&\sim_{\mathbb{Q}}K_X+\Delta+D_j+A.\end{aligned}$$

Let V be the finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by $\Delta+D_1,\dots,\Delta+D_k$. Let $\mathcal{C}\subset\mathcal{L}_A(V;\pi^{-1}(W))$ be the rational polytope spanned by $\Delta+D_1+A,\dots,\Delta+D_k+A$. By the natural map

$$p\colon K_X+\mathcal{L}_A(V;\pi^{-1}(W))\to N^1(X/Y;W)$$

defined over the rationals, $\overline{\mathrm{Eff}}(X/Y;W)$ is the cone spanned by $p(K_X+\mathcal{C})$ in $N^1(X/Y;W)$ by construction. We apply [F1, Theorem E] (see also [F1, Lemma 18.3]) to $\mathcal{C}$. Then there exist finitely many bimeromorphic maps $f_i\colon X\dashrightarrow X_i$ over Y for $1\le i\le l$ with the following property. If $\psi\colon X\dashrightarrow Z$ is a good minimal model of (X,Θ) over Y for some $\Theta\in\mathcal{C}$, then there exist an index $1\le i\le l$ and an isomorphism $\xi\colon X_i\to Z$ such that

[1] A brief explanation may be added later.

$\psi = \xi \circ f_i$. We note that $K_X + \Theta \in \overline{\mathrm{Mov}}(X/Y; W)$ if and only if $\psi\colon X \dashrightarrow Z$ is small. We put

$$\mathcal{C}' := \bigcup_{f_i\colon \text{small}} \mathcal{W}_{f_i,A,\pi}(V; W) \subset \mathcal{C}.$$

By [F1, Corollary 11.19], $\mathcal{W}_{f_i,A,\pi}(V; W)$ is a rational polytope. Hence $\mathcal{C}'$ is also defined over the rationals. By construction, we see that the movable cone $\overline{\mathrm{Mov}}(X/Y; W)$ is the cone spanned by $p(K_X + \mathcal{C}')$ in $N^1(X/Y; W)$, and $f_i^*(\mathrm{Nef}(X_i/Y; W))$ is the cone spanned by $p(K_X + \mathcal{W}_{f_i,A,\pi}(V; W))$ in $N^1(X/Y; W)$. Therefore, we obtain (v).

We complete the proof of Theorem 1.1. □

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