ON RELATIVE FANO PAIRS

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Abstract. Roughly speaking, we show that a relative Fano pair is a relative Mori dream space for projective morphisms of complex analytic spaces.

Contents

1. Introduction

In this short note, we will show that Theorem 1.1 follows from [F1]. Roughly speaking, Theorem 1.1 says that a relative Fano pair is a relative Mori dream space. Obviously, it is a complex analytic generalization of [BCHM, Corollary 1.3.2].

Theorem 1.1 (cf. [BCHM, Corollary 1.3.2]). *Let (X, Δ) be a kawamata log terminal pair. Let $\pi\colon X \to Y$ be a projective surjective morphism between normal complex varieties with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$ and let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. We assume that X is $\mathbb{Q}$-factorial over W and that $-(K_X + \Delta)$ is π-ample over W. Then we have the following properties:*

(i) *the isomorphism*
$$\varinjlim_{W \subset U} \operatorname{Pic}\left(\pi^{-1}(U)/U\right) \simeq A^1(X/Y; W)$$
holds, where U runs through all the open neighborhoods of W,

(ii) *the nef cone*
$$\operatorname{Nef}(X/Y; W) := \left\{\zeta \in N^1(X/Y; W) \mid \zeta \text{ is } \pi\text{-nef over } W\right\}$$
is spanned by finitely many π-semiample line bundles which are defined over some open neighborhood of W,

(iii) *the pseudo-effective cone*
$$\overline{\operatorname{Eff}}(X/Y; W) := \left\{\zeta \in N^1(X/Y; W) \mid \zeta \text{ is } \pi\text{-pseudo-effective over } W\right\}$$
is a rational polyhedral cone in $N^1(X/Y; W)$, and for every $\zeta \in \overline{\operatorname{Eff}}(X/Y; W)$ we can take an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X after shrinking Y around W suitably such that $\zeta = D$ holds in $\overline{\operatorname{Eff}}(X/Y; W)$,

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(iv) *the movable cone*

$$\overline{\mathrm{Mov}}(X/Y;W)$$

in $N^1(X/Y;W)$ *is defined over the rationals, and*

(v) *there exists a finite collection of small bimeromorphic modifications* $f_i\colon X \dashrightarrow X_i$ *over some open neighborhood* U_i *of* W *such that each* X_i *is* $\mathbb{Q}$*-factorial over* W *and satisfies* (ii)*, and the movable cone* $\overline{\mathrm{Mov}}(X/Y;W)$ *is the union of the* $f_i^*(\mathrm{Nef}(X_i/Y;W))$.

In this note, we do not use the theory of Mori dream spaces (see [HK] and [O]) since it is established only for algebraic varieties. Note that [HK] and [O] depend on Mumford's geometric invariant theory. We also note that a very special case of Theorem 1.1 has already been used in [KS].

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Throughout this note, we will freely use [F1].

2. Minimal model program with scaling

In this section, we will explain some results established in [F1]. Let us start with the following lemma, which easily follows from [F1].

Lemma 2.1 (Minimal model program with scaling)**.** *Let* (X,Δ) *be a kawamata log terminal pair. Let* $\pi\colon X\to Y$ *be a projective morphism between complex varieties and let* W *be a Stein compact subset of* Y *such that* $\Gamma(W,\mathcal{O}_Y)$ *is noetherian. Assume that* $-(K_X+\Delta)$ *is* π*-ample. Let* D *be an* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor on* X *such that* $D-(K_X+\Delta)$ *is* π*-ample over* W*. Then, after shrinking* Y *around* W *suitably, we can take an effective* $\mathbb{R}$*-Cartier* $\mathbb{R}$*-divisor* B *on* X *such that* $B\sim_{\mathbb{R}} D-(K_X+\Delta)$ *and* $(X,\Delta+B)$ *is kawamata log terminal. Hence we can run the* $(K_X+\Delta+B)$*-minimal model program with scaling over* Y *around* W*. Note that it always terminates. Furthermore, if* D *and* $K_X+\Delta$ *are both* $\mathbb{Q}$*-Cartier, then we can make* $B\sim_{\mathbb{Q}} D-(K_X+\Delta)$ *in the above statement.*

Lemma 2.1 directly follows from [F1, Theorem 1.7].

Proof of Lemma 2.1. By Bertini's theorem, we can take a desired effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B. We take a π-ample effective $\mathbb{Q}$-divisor A on X such that $(X,\Delta+B+A)$ is kawamata log terminal and $K_X+\Delta+B+A$ is π-nef over W after shrinking Y around W suitably. Then, by [F1, Theorem 1.7], we can run the $(K_X+\Delta+B)$-minimal model program with scaling over Y around W. Since B is π-ample over W, this minimal model program always terminates. □

The following easy lemma plays a crucial role in this note.

Lemma 2.2. *Let*

$$(X,\Delta+B)=:(X_0,\Delta_0+B_0)\overset{\varphi_0}{\dashrightarrow}(X_1,\Delta_1+B_1)\overset{\varphi_1}{\dashrightarrow}\cdots\overset{\varphi_{m-1}}{\dashrightarrow}(X_m,\Delta_m+B_m)$$

be the $(K_X+\Delta+B)$*-minimal model program with scaling over* Y *around* W *in Lemma 2.1. Then* D *is* π*-pseudo-effective if and only if* (X_m,Δ_m+B_m) *is a good minimal model of* $(X,\Delta+B)$ *over* Y *around* W*. Moreover,* D *is in* $\overline{\mathrm{Mov}}(X/Y;W)$ *if and only if* φ_i *is a flip for every* i*, that is,* $X\dashrightarrow X_m$ *is small, and* (X_m,Δ_m+B_m) *is a good minimal model* *of* $(X, \Delta + B)$ *over* Y *around* W*. We note that each step* φ_i *exists only after shrinking* Y *around* W *suitably. Hence we have to replace* Y *with a small Stein open neighborhood of* W *repeatedly in the above process.*

Proof of Lemma 2.2. We put $D_0 := D$ and $D_{i+1} := \varphi_{i*}D_i$ for every i. Then D is π-pseudo-effective if and only if D_i is π_i-pseudo-effective for every i, where $\pi_i\colon X_i \to Y$ is the structure morphism. Hence if D is π-pseudo-effective then $K_{X_m} + \Delta_m + B_m$ is π_m-nef over W. Note that $(X_m, \Delta_m + B_m)$ is a kawamata log terminal pair and that B_m is π_m-big. Therefore, $K_{X_m} + \Delta_m + B_m$ is π_m-semiample over some open neighborhood of W by the basepoint-free theorem. It is easy to see that φ_i is a flip for every i when D is in $\overline{\mathrm{Mov}}(X/Y; W)$. The opposite direction is obvious. We finish the proof. □

We make an important remark on Lemmas 2.1 and 2.2.

Remark 2.3. In Lemma 2.2, by Lemma 2.4 below, we can take an effective $\mathbb{Q}$-divisor Θ_i on X_i such that (X_i, Θ_i) is kawamata log terminal and that $-(K_{X_i} + \Theta_i)$ is π_i-ample over W for every i.

Lemma 2.4 is well known in the algebraic case.

Lemma 2.4. *Let* $\pi\colon X \to Y$ *and* $\pi_Z\colon Z \to Y$ *be projective morphisms of normal complex varieties and let* W *be a Stein compact subset of* Y*. Let* $\varphi\colon X \to Z$ *be a projective bimeromorphic contraction over* Y *such that* $\pi = \pi_Z \circ \varphi$*.*

(i) *If* (X, Δ) *is kawamata log terminal and* $-(K_X + \Delta)$ *is* π*-ample over* W*, then, after shrinking* Y *around* W *suitably, we can find* Δ_Z *such that* (Z, Δ_Z) *is kawamata log terminal and* $-(K_Z + \Delta_Z)$ *is* π_Z*-ample.*

(ii) *If* (Z, Δ_Z) *is kawamata log terminal,* $-(K_Z + \Delta_Z)$ *is* π_Z*-ample, and* φ *is small, then, after shrinking* Y *around* W *suitably, we can take* Δ *such that* (X, Δ) *is kawamata log terminal and* $-(K_X + \Delta)$ *is* π*-ample over* W*.*

We prove Lemma 2.4 for the sake of completeness.

Proof of Lemma 2.4. Throughout this proof, we will freely shrink Y around W without mentioning it explicitly.

Let H be a π_Z-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z such that $-(K_Z + \Delta_Z) - \varphi^*H$ is still π-ample. By Bertini's theorem, we can take an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on X such that $B \sim_{\mathbb{R}} -(K_X + \Delta) - \varphi^*H$ and that $(X, \Delta + B)$ is kawamata log terminal. We put $\Delta_Z := \varphi_*(\Delta + B)$. Since $K_X + \Delta + B \sim_{\mathbb{R}} -\varphi^*H$, we obtain that (Z, Δ_Z) is kawamata log terminal with $K_Z + \Delta_Z \sim_{\mathbb{R}} -H$. Hence we obtain (i).

We put $K_X + \Delta' := \varphi^*(K_Z + \Delta_Z)$. Then (X, Δ') is kawamata log terminal and $-(K_X + \Delta')$ is π-semiample and π-big. By Kodaira's lemma, we can take an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E on X such that $(X, \Delta' + E)$ is kawamata log terminal and $-(K_X + \Delta' + E)$ is π-ample. Hence $\Delta := \Delta' + E$ satisfies all the desired properties. Thus we get (ii). □

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. The proof of Theorem 1.1 given here is an application of [F1] and [F2]. We do not use the theory of Mori dream spaces (see [HK] and [O]).

Proof of Theorem 1.1. We divide the proof into several small steps.

Step 1. In this step, we will prove (i).

Let $\mathcal{L}$ be a line bundle defined over some open neighborhood of W with $\mathcal{L} \equiv_W 0$, that is, $\mathcal{L} \cdot C = 0$ holds for every projective curve C on X such that $\pi(C)$ is a point of W. In this case, we have that $\mathcal{L}$ is π-nef over W and that $\mathcal{L} - (K_X + \Delta)$ is π-ample over W. We note that (X, Δ) is kawamata log terminal. Therefore, after shrinking Y around W suitably, we can write $\mathcal{L} \simeq \pi^*\mathcal{M}$ for some line bundle $\mathcal{M}$ on Y by the basepoint-free theorem since $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$. This implies that

$$\varinjlim_{W \subset U} \operatorname{Pic}\left(\pi^{-1}(U)/U\right) \simeq A^1(X/Y; W)$$

holds, that is, (i) holds true.

Step 2 (Basepoint-free theorem)**.** Let $\mathcal{L}$ be a line bundle defined over some open neighborhood of W. Assume that $\mathcal{L}$ is π-nef over W. Then, by the basepoint-free theorem, $\mathcal{L}$ is π-semiample over some open neighborhood of W since $\mathcal{L} - (K_X + \Delta)$ is π-ample over W and (X, Δ) is kawamata log terminal. Therefore, $\mathcal{L}$ is π-nef over W if and only if $\mathcal{L}$ is π-semiample over some open neighborhood of W.

Step 3. In this step, we will prove (ii).

By the cone theorem, the Kleiman–Mori cone $\overline{\mathrm{NE}}(X/Y; W)$ is spanned by a finitely many $(K_X + \Delta)$-negative extremal rays since $-(K_X + \Delta)$ is π-ample over W and (X, Δ) is kawamata log terminal. Hence the nef cone $\mathrm{Nef}(X/Y; W)$, which is the dual cone of $\overline{\mathrm{NE}}(X/Y; W)$ by definition, is spanned by finitely many line bundles which are π-nef over W after shrinking Y around W suitably. Therefore, $\mathrm{Nef}(X/Y; W)$ is spanned by finitely many π-semiample line bundles defined over some open neighborhood of W (see Step 2). This means that (ii) holds true.

Step 4. Let $\mathcal{L}$ be a line bundle defined over some open neighborhood U of W. By replacing Y with a relatively compact Stein open neighborhood of W contained in U, we can find a Cartier divisor D on X such that $\mathcal{L} \simeq \mathcal{O}_X(D)$ and that $\operatorname{Supp} D$ has only finitely many irreducible components. More generally, let $\mathcal{L}$ be an $\mathbb{R}$-line (resp. a $\mathbb{Q}$-line) bundle. Then we can find a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor) D on X after shrinking Y around W suitably such that $\mathcal{L} = D$ holds in $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and that $\operatorname{Supp} D$ has only finitely many irreducible components.

Step 5. In this step, we will prove (iii).

Throughout this step, we will freely shrink Y around W suitably without mentioning it explicitly. We put $\rho := \rho(X/Y; W)$. We take line bundles $\mathcal{L}_1, \cdots, \mathcal{L}_\rho$ on X such that $\{\mathcal{L}_1, \cdots, \mathcal{L}_\rho\}$ is a basis of $A^1(X/Y; W) \otimes_{\mathbb{Z}} \mathbb{Q}$. We take a Cartier divisor D_i with $\mathcal{L}_i \simeq \mathcal{O}_X(D_i)$ for every i. Without loss of generality, we may assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier by replacing Δ suitably. We note that if d is a sufficiently large positive integer then

$$\pm\frac{1}{d}D_i = K_X + \Delta + \left(\pm\frac{1}{d}D_i - (K_X + \Delta)\right)$$

such that

$$\pm\frac{1}{d}D_i - (K_X + \Delta)$$

is π-ample for every i. Thus, by [F2], we can easily check that

$$R\left(X/Y, \frac{1}{d}D_1, -\frac{1}{d}D_1, \cdots, \frac{1}{d}D_\rho, -\frac{1}{d}D_\rho\right)$$

is a locally finitely generated $\mathbb{N}^{2\rho}$-graded $\mathcal{O}_Y$-algebra. Hence, we obtain that

$$R(X/Y, D_1, -D_1, \cdots, D_\rho, -D_\rho)$$

is a locally finitely generated $\mathbb{N}^{2\rho}$-graded $\mathcal{O}_Y$-algebra. This implies that the pseudo-effective cone $\overline{\mathrm{Eff}}(X/Y;W)$ is a rational polyhedral cone in $N^1(X/Y;W)$.

Let $\mathcal{L}$ be a π-pseudo-effective $\mathbb{R}$-line (resp. $\mathbb{Q}$-line) bundle on X. We can take an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (resp. a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor) D' on X such that $D' = \mathcal{L}$ holds in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (see Step 4). By applying Lemma 2.1 to $\frac{1}{m}D'$ for some $m \gg 0$, after finitely many flips and divisorial contractions, we can make D' semiample over Y. In particular, we find an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor) D on X such that $D \sim_{\mathbb{R}} D'$ (resp. $D \sim_{\mathbb{Q}} D'$). Hence, $\mathcal{L} = D$ in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. This means that (iii) holds.

Step 6. (iv) follows from (v) since $\mathrm{Nef}(X_i/Y;W)$ is a rational polyhedral cone. Hence it is sufficient to prove (v).

Step 7. In this final step, we will prove (v).

Throughout this step, we will freely shrink Y around W suitably without mentioning it explicitly. Without loss of generality, we may assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We take an effective general π-ample $\mathbb{Q}$-divisor A on X such that $A \sim_{\mathbb{Q}} -(K_X + \Delta)$ and that the number of the irreducible components of $\mathrm{Supp}\, A$ is finite. It is obvious that

$$\overline{\mathrm{Mov}}(X/Y;W) \subset \overline{\mathrm{Eff}}(X/Y;W)$$

holds by definition. By (iii), $\overline{\mathrm{Eff}}(X/Y;W)$ is a rational polyhedral cone and is spanned by finitely many effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $D_1, \cdots, D_k$ on X (see Step 5). By replacing D_j with $\frac{1}{m}D_j$ for some large positive integer m, we may further assume that $(X, \Delta + D_j)$ is kawamata log terminal for every j. We note that

$$\begin{aligned} D_j &= K_X + \Delta + (D_j - (K_X + \Delta)) \\ &= K_X + \Delta + D_j + (-(K_X + \Delta)) \\ &\sim_{\mathbb{Q}} K_X + \Delta + D_j + A. \end{aligned}$$

Let V be the finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by $\Delta + D_1, \cdots, \Delta + D_k$. Let $\mathcal{C} \subset \mathcal{L}_A(V;\pi^{-1}(W))$ be the rational polytope spanned by $\Delta + D_1 + A, \cdots, \Delta + D_k + A$. By the following natural map

$$p\colon K_X + \mathcal{L}_A(V;\pi^{-1}(W)) \to N^1(X/Y;W)$$

defined over the rationals, $\overline{\mathrm{Eff}}(X/Y;W)$ is the cone spanned by $p(K_X + \mathcal{C})$ in $N^1(X/Y;W)$ by construction. We apply [F1, Theorem E] (see also [F1, Lemma 18.3]) to $\mathcal{C}$. Then there exist finitely many bimeromorphic maps $f_i\colon X \dashrightarrow X_i$ over Y for $1 \leq i \leq l$ with the following properties. If $\psi\colon X \dashrightarrow Z$ is a good minimal model of (X, Θ) over Y for some $\Theta \in \mathcal{C}$, then there exists an index $1 \leq i \leq l$ and an isomorphism $\xi\colon X_i \to Z$ such that $\psi = \xi \circ f_i$ holds. We note that $K_X + \Theta \in \overline{\mathrm{Mov}}(X/Y;W)$ if and only if $\psi\colon X \dashrightarrow Z$ is small. We put

$$\mathcal{C}' := \bigcup_{f_i\colon \mathrm{small}} \mathcal{W}_{f_i, A, \pi}(V;W) \subset \mathcal{C}.$$

By [F1, Corollary 11.19], $\mathcal{W}_{f_i,A,\pi}(V;W)$ is a rational polytope. Hence $\mathcal{C}'$ is also defined over the rationals. By construction, we see that the movable cone $\overline{\mathrm{Mov}}(X/Y;W)$ is the cone spanned by $p(K_X + \mathcal{C}')$ in $N^1(X/Y;W)$ and $f_i^*\,(\mathrm{Nef}(X/Y;W))$ is the cone spanned by $p\,(K_X + \mathcal{W}_{f_i,A,\pi}(V;W))$ in $N^1(X/Y;W)$. Therefore, we obtain (v).

We finish the proof of Theorem 1.1. □

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