ON QUASI-LOG STRUCTURES FOR COMPLEX ANALYTIC SPACES

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Abstract. We introduce the notion of quasi-log complex analytic spaces and establish various fundamental properties. Moreover, we prove that a semi-log canonical pair naturally has a quasi-log complex analytic space structure.

Contents

1. Introduction

This is the final part of the trilogy on the minimal model theory for projective morphisms between complex analytic spaces (see [Fu13] and [Fu15]).

The main purpose of this paper is to introduce the notion of *quasi-log complex analytic spaces* and establish various fundamental results. Let us see the definition of quasi-log complex analytic spaces, which may look artificial.

Definition 1.1 (Quasi-log complex analytic spaces, see Definition 4.1)**.** A *quasi-log complex analytic space*

$$\big(X, \omega, f\colon (Y, B_Y) \to X\big)$$

is a complex analytic space X endowed with an $\mathbb{R}$-line bundle (or a globally $\mathbb{R}$-Cartier divisor) ω on X, a closed analytic subspace $X_{-\infty} \subsetneq X$, and a finite collection $\{C\}$ of reduced and irreducible closed analytic subspaces of X such that there exists a projective

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morphism $f\colon (Y, B_Y) \to X$ from an analytic globally embedded simple normal crossing pair (Y, B_Y) satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism
$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$
where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$ on X.
(3) The collection of closed analytic subvarieties $\{C\}$ coincides with the f-images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

Since we treat $\mathbb{R}$-line bundles and globally $\mathbb{R}$-Cartier divisors on (not necessarily compact) complex analytic spaces, we need the following remark.

Remark 1.2 ($\mathbb{R}$-line bundles and globally $\mathbb{R}$-Cartier divisors)**.** Let X be a complex analytic space and let $\mathrm{Pic}(X)$ be the group of line bundles on X, that is, the *Picard group* of X. An element of $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ (resp. $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{Q}$) is called an $\mathbb{R}$-*line bundle* (resp. a $\mathbb{Q}$-*line bundle*) on X. In this paper, we write the group law of $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ additively for simplicity of notation. A *globally* $\mathbb{R}$-*Cartier* (resp. *globally* $\mathbb{Q}$-*Cartier*) *divisor* is a finite $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of Cartier divisors. If ω is a globally $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) divisor in Definition 1.1, then we can naturally see ω as an $\mathbb{R}$-line bundle (resp. a $\mathbb{Q}$-line bundle) on X. In Definition 1.1, we always assume that B_Y is a globally $\mathbb{R}$-Cartier divisor on Y implicitly. This assumption is harmless to applications because Y is usually a relatively compact open subset of a given complex analytic space. In that case, the support of B_Y has only finitely many irreducible components and then B_Y automatically becomes globally $\mathbb{R}$-Cartier. Under the assumption that B_Y is globally $\mathbb{R}$-Cartier, $K_Y + B_Y$ naturally defines an $\mathbb{R}$-line bundle on Y. The condition $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$ in Definition 1.1 (1) means that $f^*\omega = K_Y + B_Y$ holds in $\mathrm{Pic}(Y)\otimes_{\mathbb{Z}}\mathbb{R}$.

Note that the notion of quasi-log schemes was first introduced by Ambro in [A]. Definition 1.1 is an analytic counterpart of the notion of quasi-log schemes. For the details of the theory of quasi-log schemes, see [Fu7, Chapter 6] and [Fu11]. A gentle introduction to the theory of quasi-log schemes is [Fu2]. Although the definition of quasi-log complex analytic spaces looks complicated and artificial, we think that the following example shows that it is natural.

Example 1.3 (Normal pairs)**.** Let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces such that X is a normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. We sometimes call (X, Δ) a *normal pair*. We replace S with any relatively compact open subset of S. Then we can construct a projective bimeromorphic morphism $f\colon Y \to X$ with
$$K_Y + B_Y := f^*(K_X + \Delta)$$
such that Y is smooth and $\mathrm{Supp}\, B_Y$ is a simple normal crossing divisor on Y. Then
$$f^*(K_X + \Delta) \sim_{\mathbb{R}} K_Y + B_Y$$
obviously holds and
$$\mathcal{J}_{\mathrm{NLC}}(X, \Delta) := f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor)$$
is a well-defined coherent ideal sheaf on X which defines the non-lc locus $\mathrm{Nlc}(X, \Delta)$ of (X, Δ). By definition, C is a log canonical center of (X, Δ) if and only if C is not contained

in $\mathrm{Nlc}(X,\Delta)$ and is the f-image of some log canonical center of (Y,B_Y). Hence,

$$(X, K_X+\Delta, f\colon (Y,B_Y)\to X)$$

with $X_{-\infty}:=\mathrm{Nlc}(X,\Delta)$ satisfies the conditions in Definition 1.1, that is,

$$(X, K_X+\Delta, f\colon (Y,B_Y)\to X)$$

is a quasi-log complex analytic space. By construction, $X_{-\infty}=\emptyset$ if and only if (X,Δ) is log canonical.

In this paper, after we define quasi-log complex analytic spaces and prove the adjunction formula and vanishing theorems for them, we establish the basepoint-free theorem, the basepoint-freeness of Reid–Fukuda type, the effective freeness, the cone and contraction theorem, and so on, for quasi-log complex analytic spaces. We can use them for the study of normal pairs by Example 1.3. We note that the cone and contraction theorem for normal pairs, which is sufficient for the minimal model program for log canonical pairs, was already proved in the complex analytic setting in [Fu15]. We do not need the framework of quasi-log complex analytic spaces in [Fu15]. However, it seems to be difficult to prove the basepoint-free theorem of Reid–Fukuda type for normal pairs in the complex analytic setting without using the framework of quasi-log complex analytic spaces. By combining some results obtained in this paper with Example 1.3, we have:

Theorem 1.4 (Effective freeness of Reid–Fukuda type for log canonical pairs)**.** *Let $\pi\colon X\to S$ be a projective morphism of complex analytic spaces such that (X,Δ) is log canonical and that Δ is a $\mathbb{Q}$-divisor. Let $\mathcal{L}$ be a π-nef line bundle on X such that $a\mathcal{L}-(K_X+\Delta)$ is nef and log big over S with respect to (X,Δ) for some positive real number a. This means that $a\mathcal{L}-(K_X+\Delta)$ is nef and big over S and that $(a\mathcal{L}-(K_X+\Delta))|_C$ is big over $\pi(C)$ for every log canonical center C of (X,Δ). Then there exists a positive integer m_0, which depends only on $\dim X$ and a, such that $\mathcal{L}^{\otimes m}$ is π-generated for every $m\geq m_0$. Moreover, we may allow Δ to be an $\mathbb{R}$-divisor when $a\mathcal{L}-(K_X+\Delta)$ is π-ample over S in the above statement.*

Theorem 1.4 is a generalization of [Fu1, Theorem 2.2.4]. We note that we do not have to replace S with a relatively compact open subset of S in Theorem 1.4. The notion of quasi-log complex analytic spaces is very useful for the proof of Theorem 1.4. The author does not know how to prove Theorem 1.4 in the framework of [Fu15]. Precisely speaking, we first establish the basepoint-free theorem for quasi-log complex analytic spaces (see Theorem 6.1). Then, by using it, we prove the basepoint-free theorem of Reid–Fukuda type for quasi-log complex analytic spaces (see Theorem 7.1). Here, the framework of quasi-log complex analytic spaces plays an important role. Finally, we obtain the effective freeness for complex analytic quasi-log canonical pairs in Theorems 8.1 and 8.2. By combining it with Example 1.3, we have Theorem 1.4. Moreover, we think that we need the theory of quasi-log complex analytic spaces for the study of semi-log canonical pairs in the complex analytic setting by the following theorem.

Theorem 1.5 (Semi-log canonical pairs, see Theorem 10.1)**.** *Let $\pi\colon X\to S$ be a projective morphism of complex analytic spaces and let (X,Δ) be a semi-log canonical pair. Then, after replacing S with any relatively compact open subset of S, $[X, K_X+\Delta]$ naturally becomes a quasi-log complex analytic space such that $\mathrm{Nqlc}(X, K_X+\Delta)=\emptyset$ and that C is a qlc center of $[X,K_X+\Delta]$ if and only if C is a semi-log canonical center of (X,Δ).*

More precisely, we can construct a projective surjective morphism $f\colon (Y,B_Y)\to X$ from an analytic globally embedded simple normal crossing pair (Y,B_Y) such that the natural *map*

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$$

is an isomorphism and that C is the f-image of some stratum of (Y, B_Y) if and only if C is a semi-log canonical center of (X, Δ) or an irreducible component of X. Moreover, if every irreducible component of X has no self-intersection in codimension one, then we can make $f\colon Y \to X$ bimeromorphic.

Theorem 1.5 is obviously a complex analytic generalization of [Fu4, Theorem 1.2]. Example 1.6 may help us understand Theorem 1.5.

Example 1.6. We consider

$$X := \left(X_0X_2^2 - X_1^2(X_1 - 1) = 0\right) \subset \mathbb{P}^2.$$

Then $(X, 0)$ is a projective semi-log canonical curve. Let $\alpha\colon M \to \mathbb{P}^2$ be the blow-up at $[1 : 0 : 0] \in \mathbb{P}^2$. We put $Y := X' + E$, where X' is the strict transform of X on M and E is the α-exceptional curve. Then it is easy to see that Y is a simple normal crossing divisor on M,

$$\alpha^*(K_{\mathbb{P}^2} + X) = K_M + X' + E, \tag{1.1}$$

and $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$, where $f := \alpha|_Y$. By (1.1) and adjunction,

$$f^*K_X = K_Y$$

holds. Thus

$$(X, K_X, f\colon (Y, 0) \to X)$$

is a quasi-log complex analytic space with $X_{-\infty} = \emptyset$. We note that X is irreducible but Y is reducible. In particular, $f\colon Y \to X$ is not bimeromorphic.

By Theorem 1.5, we can use the results established for quasi-log complex analytic spaces in this paper to study semi-log canonical pairs. Of course, by combining Theorems 8.1 and 8.2 with Theorem 1.5, we see that Theorem 1.4 holds for complex analytic semi-log canonical pairs. More precisely, the basepoint-free theorem and its variants hold true for semi-log canonical pairs in the complex analytic setting. Although we do not state it explicitly here, the cone and contraction theorem holds in full generality for complex analytic semi-log canonical pairs. We note that this paper is not self-contained. We strongly recommend that the reader looks at [Fu15] before reading this paper. Roughly speaking, this paper explains how to use the strict support condition and the vanishing theorems established in [Fu13] systematically by introducing the framework of quasi-log complex analytic spaces.

Remark 1.7. We are mainly interested in projective morphisms between complex analytic spaces. Roughly speaking, in [Fu12], we translated [BCHM] and [HM] into the complex analytic setting. We note that the framework of the minimal model program established in [Na1] and [Na2] is almost sufficient for [Fu12]. We do not need [Fu13], which is an analytic generalization of [Fu7, Chapter 5], for [Fu12]. The ACC for log canonical thresholds in the complex analytic setting (see [Fu14]) is an easy consequence of [Fu12] and [HMX]. On the other hand, [Fu15] and this paper heavily depend on [Fu13]. We note that [Fu15] is a complex analytic generalization of [Fu3] based on [Fu13]. This paper explains how to generalize [Fu7, Chapter 6], [Fu4], [Fu5], and [Fu6] into the complex analytic setting.

We make a remark on [Fu4] for the reader's convenience.

Remark 1.8. Note that [Fu4, Definition A.20] has some subtle troubles. For the details, see [Fu7, Definition 2.1.25, Remark 2.1.16, and Lemma 2.1.18]. The proof of [Fu4, Theorem 1.12] is insufficient. For the details, see Theorem 10.4 and Remark 10.5 below.

We look at the organization of this paper. In Section 2, we collect some basic definitions and results necessary for this paper. In Section 3, we recall the strict support condition and the vanishing theorems for analytic simple normal crossing pairs established in [Fu13]. Note that we do not prove them in this paper. In Section 4, which is the main part of this paper, we introduce the notion of quasi-log complex analytic spaces and prove some basic properties. In Section 5, we prepare several useful lemmas. Although they may look complicated and artificial, they are very important. In Section 6, we prove the basepoint-free theorem for quasi-log complex analytic spaces. Then, in Section 7, we prove the basepoint-free theorem of Reid–Fukuda type for quasi-log complex analytic spaces. In Section 8, we establish the effective basepoint-freeness and effective very ampleness for quasi-log complex analytic spaces. The argument in this section is new and is slightly simpler than the known one. In Section 9, we discuss the cone and contraction theorem for quasi-log complex analytic spaces. In Subsection 9.1, we prove that any extremal ray is spanned by a rational curve. In Section 10, we treat complex analytic semi-log canonical pairs. Roughly speaking, we show that a semi-log canonical pair naturally becomes a quasi-log complex analytic space. In Subsection 10.1, we explain some vanishing theorems for the reader's convenience. In Subsection 10.2, we briefly discuss Shokurov's polytopes for semi-log canonical pairs for the sake of completeness.

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In this paper, we assume that every complex analytic space is *Hausdorff* and *second-countable*. An irreducible and reduced complex analytic space is called a *complex variety*. We will freely use the basic definitions and results on complex analytic geometry in [BS] and [Fi]. Nakayama's book [Na2] may be helpful. We will also freely use Serre's GAGA (see [Se]) throughout this paper. We strongly recommend that the reader looks at [Fu15] before reading this paper. This paper is a continuation of [Fu15] and is also a supplement to [Fu15].

2. Preliminaries

In this section, we will recall some basic definitions and properties of complex analytic spaces necessary for subsequent sections. For the details, see [Fu15, Section 2].

2.1 (Hybrids of $\mathbb{R}$-line bundles and globally $\mathbb{R}$-Cartier divisors)**.** As we already mentioned in Remark 1.2, we usually treat hybrids of $\mathbb{R}$-line bundles and globally $\mathbb{R}$-Cartier divisors on a complex analytic space X. Note that a *globally* $\mathbb{R}$*-Cartier divisor* is a finite positive $\mathbb{R}$-linear combination of Cartier divisors. We often write

$$\mathcal{L} + A \sim_{\mathbb{R}} \mathcal{M} + B,$$

where $\mathcal{L}, \mathcal{M} \in \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, and A and B are globally $\mathbb{R}$-Cartier divisors on X. This means that

$$\mathcal{L} + \mathcal{A} = \mathcal{M} + \mathcal{B}$$

holds in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{R}$-line bundles naturally associated to A and B, respectively. We note that we usually write the group law of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ additively for simplicity of notation.

2.2 (Divisors)**.** Let X be a reduced equidimensional complex analytic space. A *prime divisor* on X is an irreducible and reduced closed analytic subspace of codimension one. An $\mathbb{R}$-*divisor* D on X is a formal sum

$$D = \sum_i a_i D_i,$$

where D_i is a prime divisor on X with $D_i \neq D_j$ for $i \neq j$, $a_i \in \mathbb{R}$ for every i, and the *support*

$$\mathrm{Supp}\, D := \bigcup_{a_i \neq 0} D_i$$

is a closed analytic subset of X. In other words, the formal sum $\sum_i a_i D_i$ is locally finite. If $a_i \in \mathbb{Z}$ (resp. $a_i \in \mathbb{Q}$) for every i, then D is called a *divisor* (resp. $\mathbb{Q}$-*divisor*) on X. Note that a divisor is sometimes called an *integral Weil divisor* in order to emphasize the condition that $a_i \in \mathbb{Z}$ for every i. If $0 \leq a_i \leq 1$ (resp. $a_i \leq 1$) holds for every i, then an $\mathbb{R}$-divisor D is called a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor.

Let $D = \sum_i a_i D_i$ be an $\mathbb{R}$-divisor on X such that D_i is a prime divisor for every i with $D_i \neq D_j$ for $i \neq j$. The *round-down* $\lfloor D \rfloor$ of D is defined to be the divisor

$$\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i,$$

where $\lfloor x \rfloor$ is the integer defined by $x - 1 < \lfloor x \rfloor \leq x$ for every real number x. The *round-up* and the *fractional part* of D are defined to be

$$\lceil D \rceil := -\lfloor -D \rfloor, \quad \text{and} \quad \{D\} := D - \lfloor D \rfloor,$$

respectively. We put

$$D^{=1} := \sum_{a_i = 1} D_i, \quad D^{<1} := \sum_{a_i < 1} a_i D_i, \quad \text{and} \quad D^{>1} := \sum_{a_i > 1} a_i D_i.$$

Let D be an $\mathbb{R}$-divisor on X and let x be a point of X. If D is written as a finite $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of Cartier divisors on some open neighborhood of x, then D is said to be $\mathbb{R}$-*Cartier at* x (resp. $\mathbb{Q}$-*Cartier at* x). If D is $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) at x for every $x \in X$, then D is said to be $\mathbb{R}$-*Cartier* (resp. $\mathbb{Q}$-*Cartier*). Note that a $\mathbb{Q}$-Cartier $\mathbb{R}$-divisor D is automatically a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor by definition. If D is a finite $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of Cartier divisors on X, then we say that D is a *globally* $\mathbb{R}$-*Cartier* $\mathbb{R}$-*divisor* (resp. *globally* $\mathbb{Q}$-*Cartier* $\mathbb{Q}$-*divisor*).

Two $\mathbb{R}$-divisors D_1 and D_2 are said to be *linearly equivalent* if $D_1 - D_2$ is a principal Cartier divisor. The linear equivalence is denoted by $D_1 \sim D_2$. Two $\mathbb{R}$-divisors D_1 and D_2 are said to be $\mathbb{R}$-*linearly equivalent* (resp. $\mathbb{Q}$-*linearly equivalent*) if $D_1 - D_2$ is a *finite* $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of principal Cartier divisors. When D_1 is $\mathbb{R}$-linearly (resp. $\mathbb{Q}$-linearly) equivalent to D_2, we write $D_1 \sim_{\mathbb{R}} D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$).

Remark 2.3. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and let U be any relatively compact open subset of X. Then it is easy to see that $D|_U$ is a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on U.

2.4 (Singularities of pairs)**.** We have already discussed the notion of singularities of pairs for complex analytic spaces in detail in [Fu15, 2.1]. Hence we omit the details here. We do not repeat the definitions of *log canonical pairs*, *kawamata log terminal pairs*, *log canonical centers*, and so on. Here we define *semi-log canonical pairs* for complex analytic spaces.

Let X be an equidimensional reduced complex analytic space that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let X^{nc} be the largest open subset of X consisting of smooth points and normal crossing points. Then we have an invertible dualizing sheaf $\omega_{X^{\mathrm{nc}}}$ on X^{nc}. We put $\omega_X := \iota_*\omega_{X^{\mathrm{nc}}}$, where $\iota\colon X^{\mathrm{nc}} \hookrightarrow X$, and call it the *canonical sheaf* of X. Since $\operatorname{codim}_X(X \setminus X^{\mathrm{nc}}) \geq 2$ and X satisfies Serre's S_2 condition, ω_X is a reflexive sheaf of rank one on X. Although we can not always define K_X globally with $\mathcal{O}_X(K_X) \simeq \omega_X$, we use the symbol K_X as a formal divisor class with an isomorphism $\mathcal{O}_X(K_X) \simeq \omega_X$ if there is no danger of confusion.

Definition 2.5 (Semi-log canonical pairs)**.** Let X be an equidimensional reduced complex analytic space that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor on X such that any irreducible component of $\operatorname{Supp}\Delta$ is not contained in the singular locus of X. In this situation, the pair (X, Δ) is called a *semi-log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and
(2) (X^ν, Θ) is log canonical, where $\nu\colon X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta := \nu^*(K_X + \Delta)$.

Let (X, Δ) be a semi-log canonical pair. A closed analytic subvariety C is called a *semi-log canonical center* (an *slc center*, for short) *of* (X, Δ) if C is the ν-image of some log canonical center of (X^ν, Θ). A closed subvariety S is sometimes called an *slc stratum* if S is an slc center of (X, Δ) or S is an irreducible component of X.

Let X be an equidimensional complex analytic space. A real vector space spanned by the prime divisors on X is denoted by $\mathrm{WDiv}_{\mathbb{R}}(X)$, which has a canonical basis given by the prime divisors. Let D be an element of $\mathrm{WDiv}_{\mathbb{R}}(X)$. Then the sup norm of D with respect to this basis is denoted by $\|D\|$. Let V be a finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals. Let L be a compact subset of X. We put

$$\mathcal{L}(V; L) := \{\Delta \in V \mid (X, \Delta) \text{ is semi-log canonical at } L\}.$$

Then we can check that $\mathcal{L}(V; L)$ is a rational polytope. For the details, see [Fu15, 2.10], where we treat the case where X is normal. We will use $\mathcal{L}(V; L)$ in Subsection 10.2.

2.6 (Kleiman–Mori cones)**.** Here we briefly discuss the basics about Kleiman–Mori cones in the complex analytic setting. For the details, see [Fu12, Section 4] and [Fu15, Section 11].

Let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces and let W be a compact subset of S. Let $Z_1(X/S; W)$ be the free abelian group generated by the projective integral curves C on X such that $\pi(C)$ is a point of W. Let U be any open neighborhood of W. Then we can consider the following intersection pairing

$$\cdot : \operatorname{Pic}\bigl(\pi^{-1}(U)\bigr) \times Z_1(X/S; W) \to \mathbb{Z}$$

given by $\mathcal{L} \cdot C \in \mathbb{Z}$ for $\mathcal{L} \in \operatorname{Pic}(\pi^{-1}(U))$ and $C \in Z_1(X/S; W)$. We say that $\mathcal{L}$ is *π-numerically trivial over* W when $\mathcal{L} \cdot C = 0$ for every $C \in Z_1(X/S; W)$. We take $\mathcal{L}_1, \mathcal{L}_2 \in \operatorname{Pic}(\pi^{-1}(U))$. If $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ is π-numerically trivial over W, then we write $\mathcal{L}_1 \equiv_W \mathcal{L}_2$ and say that $\mathcal{L}_1$ is *numerically equivalent* to $\mathcal{L}_2$ over W. We put

$$\widetilde{A}(U, W) := \operatorname{Pic}\bigl(\pi^{-1}(U)\bigr) /{\equiv_W}$$

and define

$$A^1(X/S;W) := \varinjlim_{W\subset U} \widetilde{A}(U,W),$$

where U runs through all the open neighborhoods of W.

From now on, we further assume that $A^1(X/S;W)$ is a finitely generated abelian group. Then we can define the *relative Picard number* $\rho(X/S;W)$ to be the rank of $A^1(X/S;W)$. We put

$$N^1(X/S;W) := A^1(X/S;W) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $A_1(X/S;W)$ be the image of

$$Z_1(X/S;W) \to \mathrm{Hom}_{\mathbb{Z}}\left(A^1(X/S;W), \mathbb{Z}\right)$$

given by the above intersection pairing. Then we set

$$N_1(X/S;W) := A_1(X/S;W) \otimes_{\mathbb{Z}} \mathbb{R}.$$

In this setting, we can define the *Kleiman–Mori cone*

$$\overline{\mathrm{NE}}(X/S;W)$$

of $\pi\colon X \to S$ over W, that is, $\overline{\mathrm{NE}}(X/S;W)$ is the closure of the convex cone in $N_1(X/S;W)$ spanned by the projective integral curves C on X such that $\pi(C)$ is a point of W. An element $\zeta \in N^1(X/S;W)$ is called *π-nef over* W or *nef over* W if $\zeta \geq 0$ on $\overline{\mathrm{NE}}(X/S;W)$, equivalently, $\zeta|_{\pi^{-1}(w)}$ is nef in the usual sense for every $w \in W$.

When $A^1(X/S;W)$ is finitely generated, equivalently, $\dim N^1(X/S;W)$ is finite, we can formulate Kleiman's ampleness criterion (see [Fu15, Theorem 11.5]) and discuss the cone and contraction theorem for projective morphisms between complex analytic spaces (see Section 9 below). We note that $A^1(X/S;W)$ is not always finitely generated.

Remark 2.7 (Nakayama's finiteness)**.** By Nakayama's finiteness (see [Fu12, Subsection 4.1] and [Fu15, Subsection 11.1]), it is known that $\dim N^1(X/S;W)$ is finite under the assumption that $W \cap Z$ has only finitely many connected components for any analytic subset Z defined over an open neighborhood of W. In particular, if W is a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian, then $\dim N^1(X/S;W)$ is finite.

2.8 (Big $\mathbb{R}$-line bundles)**.** Let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces such that X is irreducible and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. If $\mathcal{L}$ is a finite positive $\mathbb{R}$-linear combination of π-big line bundles on X, then $\mathcal{L}$ is said to be *big over* S.

We will use the following convention throughout this paper.

2.9. The expression '... for every $m \gg 0$' means that 'there exists a positive real number m_0 such that ... for every $m \geq m_0$.'

3. On vanishing theorems

In this section, we will briefly recall the the vanishing theorems and the strict support condition established in [Fu13], which is an analytic generalization of [Fu7, Chapter 5]. Let us start with the definition of *analytic simple normal crossing pairs*.

Definition 3.1 (Analytic simple normal crossing pairs)**.** Let X be a simple normal crossing divisor on a smooth complex analytic space M and let B be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(B + X)$ is a simple normal crossing divisor on M and that B and X have no common irreducible components. Then we put $D := B|_X$ and consider the pair (X, D).

We call (X, D) an *analytic globally embedded simple normal crossing pair* and M the *ambient space* of (X, D).

If the pair (X, D) is locally isomorphic to an analytic globally embedded simple normal crossing pair at any point of X and the irreducible components of X and D are all smooth, then (X, D) is called an *analytic simple normal crossing pair*.

As we explained in 2.4, we use the symbol K_X as a formal divisor class with an isomorphism $\mathcal{O}_X(K_X) \simeq \omega_X$ if there is no danger of confusion, where ω_X is the *dualizing sheaf* of X.

Remark 3.2. Let X be a smooth complex analytic space and let D be an $\mathbb{R}$-divisor on X such that $\operatorname{Supp} D$ is a simple normal crossing divisor on X. Then, by considering $M := X \times \mathbb{C}$, we can see (X, D) as an analytic globally embedded simple normal crossing pair.

The notion of *strata*, which is a generalization of that of log canonical centers, plays a crucial role.

Definition 3.3 (Strata)**.** Let (X, D) be an analytic simple normal crossing pair such that D is effective. Let $\nu\colon X^\nu \to X$ be the normalization. We put

$$K_{X^\nu} + \Theta := \nu^*(K_X + D).$$

This means that Θ is the union of $\nu_*^{-1}D$ and the inverse image of the singular locus of X. If S is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ), then S is called a *stratum* of (X, D). By definition, S is a stratum of (X, D) if and only if S is a stratum of $(X, D^{=1})$.

We recall Siu's theorem on coherent analytic sheaves, which is a special case of [Si, Theorem 4].

Theorem 3.4. *Let $\mathcal{F}$ be a coherent sheaf on a complex analytic space X. Then there exists a locally finite family $\{Y_i\}_{i\in I}$ of complex analytic subvarieties of X such that*

$$\operatorname{Ass}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = \{\mathfrak{p}_{x,1}, \dots, \mathfrak{p}_{x,r(x)}\}$$

holds for every point $x \in X$, where $\mathfrak{p}_{x,1}, \dots, \mathfrak{p}_{x,r(x)}$ are the prime ideals of $\mathcal{O}_{X,x}$ associated to the irreducible components of the germs $Y_{i,x}$ of Y_i at x with $x \in Y_i$. We note that each Y_i is called an associated subvariety *of $\mathcal{F}$.*

Now we are ready to state the main result of [Fu13].

Theorem 3.5 ([Fu13, Theorem 1.1])**.** *Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $f\colon X \to Y$ be a projective morphism to a complex analytic space Y and let $\mathcal{L}$ be a line bundle on X. Let q be an arbitrary non-negative integer. Then we have the following properties.*

(i) (*Strict support condition*)*. If $\mathcal{L} - (\omega_X + \Delta)$ is f-semi-ample, then every associated subvariety of $R^qf_*\mathcal{L}$ is the f-image of some stratum of (X, Δ).*

(ii) (*Vanishing theorem*)*. If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^*\mathcal{H}$ holds for some π-ample $\mathbb{R}$-line bundle $\mathcal{H}$ on Y, where $\pi\colon Y \to Z$ is a projective morphism to a complex analytic space Z, then we have*

$$R^p\pi_*R^qf_*\mathcal{L} = 0$$

for every $p > 0$.

We make a supplementary remark on Theorem 3.5.

Remark 3.6. In Theorem 3.5, we always assume that Δ is *globally* $\mathbb{R}$-*Cartier*, that is, Δ is a finite $\mathbb{R}$-linear combination of Cartier divisors. Under this assumption, we can obtain an $\mathbb{R}$-line bundle $\mathcal{N}$ on X naturally associated to $\mathcal{L} - (\omega_X + \Delta)$. The assumption in (i) means that $\mathcal{N}$ is a finite positive $\mathbb{R}$-linear combination of π-semi-ample line bundles on X. The assumption in (ii) says that $\mathcal{N} = f^*\mathcal{H}$ holds in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

We do not prove Theorem 3.5 here. For the details of the proof of Theorem 3.5, see [Fu13], which depends on Saito's theory of mixed Hodge modules (see [Sa1], [Sa2], [Sa3], [FFS], and [Sa4]) and Takegoshi's analytic generalization of Kollár's torsion-free and vanishing theorem (see [Ta]). We note that Theorem 3.5 is one of the main ingredients of this paper. Or, we can see this paper as an application of Theorem 3.5. In order to explain the vanishing theorem of Reid–Fukuda type, we prepare the notion of nef and log big $\mathbb{R}$-line bundles.

Definition 3.7. Let $f\colon X \to Y$ and $\pi\colon Y \to Z$ be projective morphisms between complex analytic spaces and let $\mathcal{H}$ be an $\mathbb{R}$-line bundle on Y. Let Δ be a boundary $\mathbb{R}$-divisor on X such that (X, Δ) is an analytic simple normal crossing pair. We say that $\mathcal{H}$ is *nef and log big over Z with respect to* $f\colon (X, \Delta) \to Y$ if $\mathcal{H}$ is nef over Z and $\mathcal{H}|_{f(S)}$ is big over $\pi \circ f(S)$ for every stratum S of (X, Δ).

We note that if $\mathcal{H}$ is π-ample then it is nef and log big over Z with respect to $f\colon (X, \Delta) \to Y$. Therefore, Theorem 3.8 is obviously a generalization of Theorem 3.5 (ii).

Theorem 3.8 (Vanishing theorem of Reid–Fukuda type, see [Fu13, Theorem 1.2])**.** *Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $f\colon X \to Y$ and $\pi\colon Y \to Z$ be projective morphisms between complex analytic spaces and let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^*\mathcal{H}$ holds such that $\mathcal{H}$ is an $\mathbb{R}$-line bundle, which is nef and log big over Z with respect to $f\colon (X, \Delta) \to Y$, on Y, then*

$$R^p\pi_*R^qf_*\mathcal{L} = 0$$

holds for every $p > 0$ and every q.

The reader can find the detailed proof of Theorem 3.8 in [Fu13], which is harder than that of Theorem 3.5 (ii). As an easy application of Theorem 3.8, we can establish the vanishing theorem of Reid–Fukuda type of log canonical pairs for projective morphisms between complex analytic spaces. Theorem 3.9 can be seen as a generalization of the Kawamata–Viehweg vanishing theorem for projective morphisms between complex analytic spaces.

Theorem 3.9 (Vanishing theorem of Reid–Fukuda type for log canonical pairs)**.** *Let (X, Δ) be a log canonical pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let L be a $\mathbb{Q}$-Cartier integral Weil divisor on X. Assume that $L-(K_X+\Delta)$ is nef and big over Y and that $(L-(K_X+\Delta))|_C$ is big over $\pi(C)$ for every log canonical center C of (X, Δ). Then*

$$R^q\pi_*\mathcal{O}_X(L) = 0$$

holds for every $q > 0$.

Proof. The proof of [Fu7, Theorem 5.7.6] works by Theorem 3.8. □

Theorem 3.9 will be generalized for semi-log canonical pairs in Theorem 10.2 by using Theorem 1.5.

4. Quasi-log structures for complex analytic spaces

This section is the main part of this paper. In this section, we will discuss *quasi-log structures* on complex analytic spaces. For the details of the theory of quasi-log schemes, see [Fu7, Chapter 6] and [Fu11].

Let us define *quasi-log complex analytic spaces.*

Definition 4.1 (Quasi-log complex analytic spaces)**.** A *quasi-log complex analytic space*

$$\big(X, \omega, f\colon (Y, B_Y) \to X\big)$$

is a complex analytic space X endowed with an $\mathbb{R}$-line bundle (or a globally $\mathbb{R}$-Cartier divisor) ω on X, a closed analytic subspace $X_{-\infty} \subsetneq X$, and a finite collection $\{C\}$ of reduced and irreducible closed analytic subspaces of X such that there exists a projective morphism $f\colon (Y, B_Y) \to X$ from an analytic globally embedded simple normal crossing pair (Y, B_Y) satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism

$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$

where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$ on X.
(3) The collection of closed analytic subvarieties $\{C\}$ coincides with the f-images of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We often simply write $[X, \omega]$ to denote the above data

$$\big(X, \omega, f\colon (Y, B_Y) \to X\big)$$

if there is no risk of confusion. The closed analytic subvarieties C are called the *qlc strata* of $[X, \omega]$. If a qlc stratum C is not an irreducible component of X, then it is called a *qlc center* of $[X, \omega]$. The closed analytic subspace $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$. We note that we sometimes use $\mathrm{Nqlc}(X, \omega)$ or $\mathrm{Nqlc}\,(X, \omega, f\colon (Y, B_Y) \to X)$ to denote $X_{-\infty}$. We usually call $f\colon (Y, B_Y) \to X$ a *quasi-log resolution* of $[X, \omega]$.

In the above definition, if ω is a $\mathbb{Q}$-line bundle (or a globally $\mathbb{Q}$-Cartier divisor), B_Y is a $\mathbb{Q}$-divisor, and $f^*\omega \sim_{\mathbb{Q}} K_Y + B_Y$ holds, then we say that

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

has a $\mathbb{Q}$-*structure.*

We make an important remark.

Remark 4.2. As in Remark 1.2, we can naturally see ω as an $\mathbb{R}$-line bundle on X in Definition 4.1. In Definition 4.1 (1), $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$ means that B_Y is globally $\mathbb{R}$-Cartier, that is, B_Y is a finite $\mathbb{R}$-linear combination of Cartier divisors, and that $f^*\omega = \omega_Y + \mathcal{B}_Y$ holds in $\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$, where ω_Y is the dualizing sheaf of Y and $\mathcal{B}_Y$ is an $\mathbb{R}$-line bundle on Y naturally associated to the globally $\mathbb{R}$-Cartier divisor B_Y. Similarly, $f^*\omega \sim_{\mathbb{Q}} K_Y + B_Y$ means that $f^*\omega = \omega_Y + \mathcal{B}_Y$ holds in $\mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The notion of *quasi-log canonical pairs* is useful.

Definition 4.3 (Quasi-log canonical pairs)**.** In Definition 4.1, if $X_{-\infty} = \emptyset$, then

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

is called a *quasi-log canonical pair.* We sometimes simply say that $[X, \omega]$ is a *qlc pair.*

The most important result on quasi-log complex analytic spaces is the following adjunction formula. It is an easy consequence of the strict support condition in Theorem 3.5 (i).

Theorem 4.4 (Adjunction formula for quasi-log complex analytic spaces)**.** *Let*

$$\big(X, \omega, f\colon (Y, B_Y) \to X\big)$$

be a quasi-log complex analytic space and let X' be the union of $X_{-\infty}$ with a union of some qlc strata of $[X, \omega]$. Then, after replacing X with any relatively compact open subset of X, we can construct a projective morphism $f'\colon (Y', B_{Y'}) \to X'$ from an analytic globally embedded simple normal crossing pair $(Y', B_{Y'})$ such that

$$\big(X', \omega', f'\colon (Y', B_{Y'}) \to X'\big)$$

is a quasi-log complex analytic space with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.

The proof of [Fu7, Theorem 6.3.5 (i)] works without any modifications.

Sketch of Proof of Theorem 4.4. We replace X with a relatively compact open subset of X. Let M be the ambient space of (Y, B_Y). By taking a suitable projective bimeromorphic modification of M (see [Fu7, Proposition 6.3.1]), we may assume that the union of all strata of (Y, B_Y) mapped to X', which is denoted by Y', is a union of some irreducible components of Y. We put $K_{Y'} + B_{Y'} := (K_Y + B_Y)|_{Y'}$ and $Y'' := Y - Y'$. We also put $A := \lceil -(B_Y^{<1}) \rceil$ and $N := \lfloor B_Y^{>1} \rfloor$, and consider the following short exact sequence:

$$0 \to \mathcal{O}_{Y''}(A - N - Y') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{Y'}(A - N) \to 0.$$

Then we have the following long exact sequence:

$$\begin{aligned} 0 &\longrightarrow f_*\mathcal{O}_{Y''}(A - N - Y') \longrightarrow f_*\mathcal{O}_Y(A - N) \longrightarrow f_*\mathcal{O}_{Y'}(A - N) \\ &\overset{\delta}{\longrightarrow} R^1 f_*\mathcal{O}_{Y''}(A - N - Y') \longrightarrow \cdots . \end{aligned} \tag{4.1}$$

By the strict support condition in Theorem 3.5 (i), every associated subvariety of

$$R^1 f_*\mathcal{O}_{Y''}(A - N - Y')$$

is the f-image of some stratum of $(Y'', \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''})$, where $K_{Y''} + B_{Y''} := (K_Y + B_Y)|_{Y''}$, since

$$\begin{aligned} (A - N - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''}) &= -(K_{Y''} + B_{Y''}) \\ &\sim_{\mathbb{R}} -(f^*\omega)|_{Y''}. \end{aligned}$$

On the other hand, the support of $f_*\mathcal{O}_{Y'}(A - N)$ is contained in $f(Y')$. Hence, the connecting homomorphism δ in (4.1) is zero. Thus we obtain the following short exact sequence

$$0 \to f_*\mathcal{O}_{Y''}(A - N - Y') \to \mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_{Y'}(A - N) \to 0. \tag{4.2}$$

We put $\mathcal{I}_{X'} := f_*\mathcal{O}_{Y''}(A - N - Y')$ and define a complex analytic space structure on X' by $\mathcal{I}_{X'}$. Then we can check that $f' := f|_{Y'}\colon (Y', B_{Y'}) \to X'$ and $\omega' := \omega|_{X'}$ satisfy all the desired properties. We note that the short exact sequence (4.2) is

$$0 \to \mathcal{I}_{X'} \to \mathcal{I}_{X_{-\infty}} \to \mathcal{I}_{X'_{-\infty}} \to 0. \tag{4.3}$$

For the details, see, for example, the proof of [Fu7, Theorem 6.3.5 (i)]. □

The following theorem is an important supplement to Theorem 4.4.

Theorem 4.5. *The complex analytic structure on X' of the quasi-log complex analytic space*

$$\big(X', \omega', f' \colon (Y', B_{Y'}) \to X'\big)$$

defined in Theorem 4.4 is independent of the construction of $f' \colon (Y', B_{Y'}) \to X'$. Therefore, the defining ideal sheaf $\mathcal{I}_{X'}$ of X' is a globally well-defined coherent ideal sheaf on X.

Sketch of Proof of Theorem 4.5. On any relatively compact open subset U of X, we defined $\mathcal{I}_{X'}$ in the proof of Theorem 4.4. By [Fu7, Proposition 6.3.6], we see that it is independent of the construction. Hence we get a globally well-defined coherent defining ideal sheaf $\mathcal{I}_{X'}$ of X'. This is what we wanted. For the details, see the proof of [Fu7, Proposition 6.3.6]. □

For various inductive treatments, the notion of $\mathrm{Nqklt}(X, \omega)$ is very useful.

Corollary 4.6. *Let*

$$(X, \omega, f \colon (Y, B_Y) \to X)$$

be a quasi-log complex analytic space. The union of $X_{-\infty}$ with all qlc centers of $[X, \omega]$ is denoted by $\mathrm{Nqklt}(X, \omega)$*, or, more precisely,*

$$\mathrm{Nqklt}\,(X, \omega, f \colon (Y, B_Y) \to X)\,.$$

If $\mathrm{Nqklt}(X, \omega) \neq X_{-\infty}$*, then, after replacing X with any relatively compact open subset of X,*

$$\left[\mathrm{Nqklt}(X, \omega), \omega|_{\mathrm{Nqklt}(X,\omega)}\right]$$

naturally becomes a quasi-log complex analytic space by adjunction.

Proof. This is a special case of Theorem 4.4. □

If we apply Corollary 4.6 to Example 1.3, then $\mathrm{Nqklt}(X, K_X + \Delta) = \mathrm{Nklt}(X, \Delta)$ holds, where $\mathrm{Nklt}(X, \Delta)$ denotes the *non-klt locus* of (X, Δ). Moreover, we have $\mathcal{I}_{\mathrm{Nqklt}(X, K_X+\Delta)} = \mathcal{J}(X, \Delta)$, where $\mathcal{J}(X, \Delta)$ is the usual *multiplier ideal sheaf* of (X, Δ) and $\mathcal{I}_{\mathrm{Nqklt}(X, K_X+\Delta)}$ is the defining ideal sheaf of $\mathrm{Nqklt}(X, K_X + \Delta)$ on X.

For geometric applications, we need vanishing theorems. Of course, they follow from the vanishing theorems for analytic simple normal crossing pairs (see Theorem 3.5 (ii) and Theorem 3.8).

Theorem 4.7 (Vanishing theorem I)**.** *Let*

$$\big(X, \omega, f \colon (Y, B_Y) \to X\big)$$

be a quasi-log complex analytic space and let $\pi \colon X \to S$ be a projective morphism between complex analytic spaces. Let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L} - \omega$ is nef and log big over S with respect to $[X, \omega]$, that is, $\mathcal{L} - \omega$ is nef over S and $(\mathcal{L} - \omega)|_C$ is big over $\pi(C)$ for every qlc stratum C of $[X, \omega]$. Then

$$R^i\pi_*(\mathcal{I}_{X_{-\infty}} \otimes \mathcal{L}) = 0$$

holds for every $i > 0$.

Sketch of Proof of Theorem 4.7. We take an arbitrary point $s \in S$. It is sufficient to prove $R^i\pi_*(\mathcal{I}_{X_{-\infty}} \otimes \mathcal{L}) = 0$ for every $i > 0$ on a relatively compact open neighborhood U_s of $s \in S$. We replace X and S with $\pi^{-1}(U_s)$ and U_s, respectively. From now on, we use the notation in the proof of Theorem 4.4. In the proof of Theorem 4.4, we put $X' := X_{-\infty}$.

Then Y' is the union of all strata of (Y, B_Y) mapped to $X_{-\infty}$. In this situation, we can check that the following natural inclusion

$$f_*\mathcal{O}_{Y''}(A-N-Y') \hookrightarrow f_*\mathcal{O}_Y(A-N)$$

is an isomorphism. Since

$$(f^*\mathcal{L}+(A-N-Y'))\,|_{Y''}-(K_{Y''}+\{B_{Y''}\}+B^{=1}_{Y''}-Y'|_{Y''}) \sim_{\mathbb{R}} (f^*(\mathcal{L}-\omega))|_{Y''},$$

we obtain

$$\begin{aligned} R^i\pi_*(\mathcal{L}\otimes\mathcal{I}_{X_{-\infty}}) &\simeq R^i\pi_*\left(\mathcal{L}\otimes f_*\mathcal{O}_Y(A-N)\right) \\ &= R^i\pi_*\left(\mathcal{L}\otimes f_*\mathcal{O}_{Y''}(A-N-Y')\right) \\ &= R^i\pi_*\left(f_*(f^*\mathcal{L}\otimes\mathcal{O}_{Y''}(A-N-Y'))\right)=0 \end{aligned}$$

for every $i>0$ by Theorem 3.8. □

The following vanishing theorem and Theorem 4.4 will play a crucial role in the theory of quasi-log complex analytic spaces. We can see it as a generalization of the Kawamata–Viehweg–Nadel vanishing theorem for projective morphisms between complex analytic spaces.

Theorem 4.8 (Vanishing theorem II)**.** *Let*

$$\big(X, \omega, f\colon (Y, B_Y)\to X\big)$$

be a quasi-log complex analytic space and let X' be the union of $X_{-\infty}$ with a union of some qlc strata of $[X,\omega]$. Let $\pi\colon X\to S$ be a projective morphism between complex analytic spaces and let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L}-\omega$ is nef over S and $(\mathcal{L}-\omega)|_C$ is big over $\pi(C)$ for every qlc stratum C of $[X,\omega]$ which is not contained in X'. Then

$$R^i\pi_*(\mathcal{I}_{X'}\otimes\mathcal{L})=0$$

holds for every $i>0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X. In particular, if $\mathcal{L}-\omega$ is ample over S, then

$$R^i\pi_*(\mathcal{I}_{X'}\otimes\mathcal{L})=0$$

holds for every $i>0$.

Sketch of Proof of Theorem 4.8. We take an arbitrary point $s\in S$. It is sufficient to prove $R^i\pi_*(\mathcal{I}_{X'}\otimes\mathcal{L})=0$ for every $i>0$ on a relatively compact open neighborhood U_s of $s\in S$. Therefore, we replace S and X with U_s and $\pi^{-1}(U_s)$, respectively. From now on, we use the same notation as in the proof of Theorem 4.4. We note that

$$(f^*\mathcal{L}+(A-N-Y'))\,|_{Y''}-(K_{Y''}+\{B_{Y''}\}+B^{=1}_{Y''}-Y'|_{Y''}) \sim_{\mathbb{R}} (f^*(\mathcal{L}-\omega))|_{Y''}$$

holds. By Theorem 3.8, we obtain

$$\begin{aligned} R^i\pi_*(\mathcal{L}\otimes\mathcal{I}_{X'}) &= R^i\pi_*\left(\mathcal{L}\otimes f_*\mathcal{O}_{Y''}(A-N-Y')\right) \\ &= R^i\pi_*\left(f_*(f^*\mathcal{L}\otimes\mathcal{O}_{Y''}(A-N-Y'))\right)=0 \end{aligned}$$

for every $i>0$. We finish the proof. □

As in the algebraic case, the following important property holds.

Lemma 4.9. *Let $[X,\omega]$ be a quasi-log complex analytic space with $X_{-\infty}=\emptyset$, that is, $[X,\omega]$ is a quasi-log canonical pair. We assume that every qlc stratum of $[X,\omega]$ is an irreducible component of X, equivalently,* $\mathrm{Nqklt}(X,\omega)=\emptyset$. *Then X is normal.*

Sketch of Proof of Lemma 4.9. Let $f\colon (Y, B_Y) \to X$ be a quasi-log resolution. By the assumption that $X_{-\infty}$ is empty, we see that the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$$

is an isomorphism. This implies that $\mathcal{O}_X \simeq f_*\mathcal{O}_Y$ holds. Hence f has connected fibers. Therefore, every connected component of X is an irreducible component of X by the assumption that every stratum of $[X, \omega]$ is an irreducible component of X. Thus, we may assume that X is irreducible and every stratum of Y is mapped onto X. In this case, it is well known and is easy to prove that X is normal. For the details, see, for example, the proof of [Fu15, Theorem 7.1]. □

By Theorem 4.4 and Lemma 4.9, we can prove:

Theorem 4.10 (Basic properties of qlc strata)**.** *Let $[X, \omega]$ be a quasi-log complex analytic space with $X_{-\infty} = \emptyset$, that is, $[X, \omega]$ is a quasi-log canonical pair. Then we have the following properties.*

(i) *The intersection of two qlc strata is a union of some qlc strata.*
(ii) *For any point $x \in X$, the set of all qlc strata passing through x has a unique* (*with respect to the inclusion*) *element C_x. Moreover, C_x is normal at x.*

Sketch of Proof of Theorem 4.10. Let C_1 and C_2 be two qlc strata of $[X, \omega]$. We may assume that $C_1 \neq C_2$ with $C_1 \cap C_2 \neq \emptyset$. We take $P \in C_1 \cap C_2$. It is sufficient to find a qlc stratum C such that $P \in C \subset C_1 \cap C_2$ for the proof of (i). We put $X' := C_1 \cup C_2$ and $\omega' := \omega|_{X'}$. Then, after shrinking X around P suitably, $[X', \omega']$ becomes a quasi-log complex analytic space by adjunction (see Theorem 4.4). Note that X' is reducible at P. Therefore, by Lemma 4.9 above, there exists a qlc center $C^\dagger$ of $[X, \omega]$ such that $P \in C^\dagger \subset X'$. By this fact, we can easily prove (i). For the details, see the proof of [Fu7, Theorem 6.3.11 (i)]. The uniqueness of the minimal (with respect to the inclusion) qlc stratum follows from (i) and the normality of the minimal qlc stratum follows from Lemma 4.9. So we finish the proof of (ii). □

Theorem 4.10 will play a crucial role in the theory of quasi-log complex analytic spaces.

5. Some basic operations on quasi-log structures

In this section, we will discuss some basic operations on quasi-log structures in the complex analytic setting. Almost all of them are well known for quasi-log schemes (see [Fu5, Section 3], [Fu7, Chapter 6], [Fu9, Subsection 4.3], and so on). They will play an important role in the subsequent sections.

Lemma 5.1. *Let*

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

be a quasi-log complex analytic space and let $P \in X$ be a point. Let $D_1, \ldots, D_k$ be effective Cartier divisors on X such that $P \in \operatorname{Supp} D_i$ for every i. We assume that no irreducible component of Y is mapped into $\bigcup_{i=1}^k \operatorname{Supp} D_i$. Then, after replacing X with any relatively compact open neighborhood of P, $\left[X, \omega + \sum_{i=1}^k D_i\right]$ naturally becomes a quasi-log complex analytic space. Moreover, if $\operatorname{Nqlc}\left(X, \omega + \sum_{i=1}^k D_i\right) = \emptyset$, then $k \leq \dim_P X$ holds. More precisely, $k \leq \dim_P C_P$ holds, where C_P is the minimal qlc stratum of $[X, \omega]$ passing through P.

Sketch of Proof of Lemma 5.1. After replacing X with a relatively compact open neighborhood of P, we may assume that

$$\left(Y, \sum_{i=1}^{k} f^*D_i + \operatorname{Supp} B_Y\right)$$

is an analytic globally embedded simple normal crossing pair by [Fu7, Proposition 6.3.1] and [BM1]. Then we see that

$$f\colon \left(Y, B_Y + \sum_{i=1}^{k} f^*D_i\right) \to X$$

naturally gives a quasi-log structure on $\left[X, \omega + \sum_{i=1}^{k} D_i\right]$. When we prove $k \leq \dim_P X$, we can freely replace X with a relatively compact open neighborhood of P. Hence, we can easily see that the proof of [Fu7, Lemma 6.3.13] works with only some minor modifications. □

Lemma 5.2 is a very basic result in the theory of quasi-log complex analytic spaces.

Lemma 5.2. *Let $[X, \omega]$ be a quasi-log complex analytic space and let D be an effective $\mathbb{R}$-Cartier divisor on X. This means that D is a finite positive $\mathbb{R}$-linear combination of Cartier divisors. Then, after replacing X with any relatively compact open subset of X, $[X, \omega + D]$ naturally becomes a quasi-log complex analytic space.*

Sketch of Proof of Lemma 5.2. Let $f\colon (Y, B_Y) \to X$ be a quasi-log resolution. By replacing X with any relatively compact open subset of X and taking a suitable projective bimeromorphic modification of M, the ambient space of (Y, B_Y), we may assume that the union of all strata of (Y, B_Y) mapped to $\operatorname{Supp} D \cup \operatorname{Nqlc}(X, \omega)$ by f, which is denoted by Y', is a union of some irreducible components of Y (see [Fu7, Proposition 6.3.1] and [BM1]). We put $Y'' := Y - Y'$, $K_{Y''} + B_{Y''} := (K_Y + B_Y)|_{Y''}$, and $f'' := f|_{Y''}$. We may further assume that $(Y'', B_{Y''} + (f'')^*D)$ is an analytic globally embedded simple normal crossing pair. Then

$$(X, \omega + D, f''\colon (Y'', B_{Y''}) \to X)$$

naturally becomes a quasi-log complex analytic space. For the details, see the proof of [Fu9, Lemma 4.20]. We note that the quasi-log structure of $(X, \omega + D, f''\colon (Y'', B_{Y''}) \to X)$ constructed above coincides with that of $(X, \omega, f\colon (Y, B_Y) \to X)$ outside $\operatorname{Supp} D$. □

The following lemma is useful since we can reduce various problems to the case where X is irreducible (see also [Fu9, Lemmas 4.18 and 4.19]).

Lemma 5.3 (see [Fu5, Lemmas 3.12 and 3.14])**.** *Let $[X, \omega]$ be a quasi-log complex analytic space and let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces. We put $X^\dagger := \overline{X \setminus X_{-\infty}}$, the closure of $X \setminus X_{-\infty}$ in X, with the reduced structure. Then, after replacing S with any relatively compact open subset of S, $[X^\dagger, \omega^\dagger := \omega|_{X^\dagger}]$ naturally becomes a quasi-log complex analytic space with the following properties.*

(1) *C is a qlc stratum of $[X, \omega]$ if and only if C is a qlc stratum of $[X^\dagger, \omega^\dagger]$.*
(2) *$\mathcal{I}_{\mathrm{Nqlc}(X^\dagger, \omega^\dagger)} = \mathcal{I}_{\mathrm{Nqlc}(X, \omega)}$ holds.*

Proof of Lemma 5.3. In Step 1, we will construct a quasi-log resolution

$$f'\colon (Y', B_{Y'}) \to X$$

such that every irreducible component of Y' is mapped to $X^\dagger$ by f'. Then, in Step 2, we will construct the desired quasi-log structure on $[X^\dagger, \omega^\dagger]$.

Step 1. Let M be the ambient space of (Y, B_Y). After replacing S with any relatively compact open subset of S, by taking some projective bimeromorphic modification of M, we may assume that the union of all strata of (Y, B_Y) that are not mapped to $\overline{X \setminus X_{-\infty}}$, which is denoted by Y'', is a union of some irreducible components of Y (see [Fu7, Proposition 6.3.1] and [BM1]). We may further assume that the union of all strata of (Y, B_Y) mapped to $\overline{X \setminus X_{-\infty}} \cap X_{-\infty}$ is a union of some irreducible components of Y. We put $Y' := Y - Y''$ and $K_{Y''} + B_{Y''} := (K_Y + B_Y)|_{Y''}$. We consider the short exact sequence

$$0 \to \mathcal{O}_{Y''}(-Y') \to \mathcal{O}_Y \to \mathcal{O}_{Y'} \to 0.$$

As usual, we put $A := \lceil -(B_Y^{<1}) \rceil$ and $N := \lfloor B_Y^{>1} \rfloor$. By applying $\otimes \mathcal{O}_Y(A - N)$, we have

$$0 \to \mathcal{O}_{Y''}(A - N - Y') \to \mathcal{O}_Y(A - N) \to \mathcal{O}_{Y'}(A - N) \to 0.$$

By taking $R^i f_*$, we obtain

$$\begin{aligned} 0 &\to f_*\mathcal{O}_{Y''}(A - N - Y') \to f_*\mathcal{O}_Y(A - N) \to f_*\mathcal{O}_{Y'}(A - N) \\ &\to R^1 f_*\mathcal{O}_{Y''}(A - N - Y') \to \cdots . \end{aligned}$$

By the strict support condition (see Theorem 3.5 (i)), no associated subvariety of

$$R^1 f_*\mathcal{O}_{Y''}(A - N - Y')$$

is contained in $f(Y') \cap X_{-\infty}$. Note that

$$\begin{aligned} (A - N - Y')|_{Y''} - (K_{Y''} + \{B_{Y''}\} + B_{Y''}^{=1} - Y'|_{Y''}) &= -(K_{Y''} + B_{Y''}) \\ &\sim_{\mathbb{R}} -(f^*\omega)|_{Y''}. \end{aligned}$$

Therefore, the connecting homomorphism

$$\delta : f_*\mathcal{O}_{Y'}(A - N) \to R^1 f_*\mathcal{O}_{Y''}(A - N - Y')$$

is zero. This implies that

$$0 \to f_*\mathcal{O}_{Y''}(A - N - Y') \to \mathcal{I}_{X_{-\infty}} \to f_*\mathcal{O}_{Y'}(A - N) \to 0$$

is exact. The ideal sheaf $\mathcal{J} = f_*\mathcal{O}_{Y''}(A - N - Y')$ is zero when it is restricted to $X_{-\infty}$ because $\mathcal{J} \subset \mathcal{I}_{X_{-\infty}} = \mathcal{I}_{\mathrm{Nqlc}(X,\omega)}$. On the other hand, $\mathcal{J}$ is zero on $X \setminus X_{-\infty}$ because $f(Y'') \subset X_{-\infty}$. Therefore, we obtain $\mathcal{J} = 0$. Thus we have $\mathcal{I}_{X_{-\infty}} = f_*\mathcal{O}_{Y'}(A - N)$. So $f' = f|_{Y'} \colon (Y', B_{Y'}) \to X$, where $K_{Y'} + B_{Y'} := (K_Y + B_Y)|_{Y'}$, gives the same quasi-log structure as one given by $f \colon (Y, B_Y) \to X$.

Step 2. Let $\mathcal{I}_{X^\dagger}$ be the defining ideal sheaf of $X^\dagger$ on X. Let $f' : (Y', B_{Y'}) \to X$ be the quasi-log resolution constructed in Step 1. Note that

$$\begin{aligned} \mathcal{I}_{X_{-\infty}} &= f'_*\mathcal{O}_{Y'}(A - N) \\ &= f'_*\mathcal{O}_{Y'}(-N) \end{aligned}$$

and that

$$f'(N) = X_{-\infty} \cap f'(Y') = X_{-\infty} \cap X^\dagger$$

set theoretically, where $A = \lceil -(B_Y^{<1}) \rceil$ and $N = \lfloor B_Y^{>1} \rfloor$. We note that $A|_{Y'} = \lceil -(B_{Y'}^{<1}) \rceil$ and $N|_{Y'} = \lfloor B_{Y'}^{>1} \rfloor$ hold by definition. Moreover, we obtain

$$\mathcal{I}_{X^\dagger} \cap \mathcal{I}_{X_{-\infty}} = \mathcal{I}_{X^\dagger} \cap f_*\mathcal{O}_Y(A - N) \subset \mathcal{J} = f_*\mathcal{O}_{Y''}(A - N - Y') = \{0\}.$$

Thus we can construct the following big commutative diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{I}_{X_{-\infty}} & = & \mathcal{I}_{X^{\dagger}_{-\infty}} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{X^{\dagger}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X^{\dagger}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{X^{\dagger}} & \longrightarrow & \mathcal{O}_{X_{-\infty}} & \longrightarrow & \mathcal{O}_{X^{\dagger}_{-\infty}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

By construction, f' factors through $X^{\dagger}$. We put $f^{\dagger}\colon (Y', B_{Y'}) \to X^{\dagger}$. Then it is easy to see that $f^{\dagger}\colon (Y', B_{Y'}) \to X^{\dagger}$ gives the desired quasi-log structure on $[X^{\dagger}, \omega^{\dagger}]$.

We finish the proof. □

Lemma 5.4 will play a crucial role in the proof of the basepoint-free theorem of Reid–Fukuda type (see Theorem 7.1).

Lemma 5.4 (see [Fu5, Lemma 3.15])**.** *Let $[X, \omega]$ be a quasi-log complex analytic space and let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces. Let E be an effective $\mathbb{R}$-Cartier divisor on X. This means that E is a finite positive $\mathbb{R}$-linear combination of Cartier divisors. We put*

$$\widetilde{\omega} := \omega + \varepsilon E$$

with $0 < \varepsilon \ll 1$. Then, after replacing S with any relatively compact open subset of S, $[X, \widetilde{\omega}]$ naturally becomes a quasi-log complex analytic space with the following properties.

(1) *Let $\{C_i\}_{i\in I}$ be the set of all qlc centers of $[X, \omega]$ contained in* $\operatorname{Supp} E$. *We put*

$$X^{\star} := \left(\bigcup_{i\in I} C_i\right) \cup \operatorname{Nqlc}(X, \omega).$$

Then, by adjunction,

$$[X^{\star}, \omega^{\star} := \omega|_{X^{\star}}]$$

is a quasi-log complex analytic space and

$$X^{\star} = \operatorname{Nqlc}(X, \widetilde{\omega})$$

holds. More precisely,

$$\mathcal{I}_{X^{\star}} = \mathcal{I}_{\operatorname{Nqlc}(X, \widetilde{\omega})}$$

holds, where $\mathcal{I}_{X^{\star}}$ is the defining ideal sheaf of $X^{\star}$ on X.

(2) *C is a qlc center of $[X, \widetilde{\omega}]$ if and only if C is a qlc center of $[X, \omega]$ with $C \not\subset$* $\operatorname{Supp} E$.

Proof of Lemma 5.4. Let $f\colon (Y, B_Y) \to X$ be a quasi-log resolution. We replace S with any relatively compact open subset of S and take a suitable projective bimeromorphic modification of M, where M is the ambient space of (Y, B_Y). Then we may assume that the union of all strata of (Y, B_Y) mapped to $X^{\star}$, which is denoted by Y'', is a union of some irreducible components of Y (see [Fu7, Proposition 6.3.1] and [BM1]).

We put $Y' := Y - Y''$ and $K_{Y'} + B_{Y'} := (K_Y + B_Y)|_{Y'}$. We may further assume that $(Y', f^*E + \operatorname{Supp} B_{Y'})$ is an analytic globally embedded simple normal crossing pair. We consider

$$f\colon (Y', B_{Y'} + \varepsilon f^*E) \to X$$

with $0 < \varepsilon \ll 1$. We put $A := \lceil -(B_Y^{<1}) \rceil$ and $N := \lfloor B_Y^{>1} \rfloor$. Then $X^\star$ is defined by the ideal sheaf $f_*\mathcal{O}_{Y'}(A - N - Y'')$ by the proof of adjunction (see Theorem 4.4). We note that

$$\begin{aligned}(A - N - Y'')|_{Y'} &= -\lfloor B_{Y'} + \varepsilon f^*E \rfloor + (B_{Y'} + \varepsilon f^*E)^{=1} \\ &= \lceil -(B_{Y'} + \varepsilon f^*E)^{<1} \rceil - \lfloor (B_{Y'} + \varepsilon f^*E)^{>1} \rfloor.\end{aligned}$$

Therefore, if we define $\operatorname{Nqlc}(X, \widetilde{\omega})$ by the ideal sheaf

$$f_*\mathcal{O}_{Y'}(\lceil -(B_{Y'} + \varepsilon f^*E)^{<1} \rceil - \lfloor (B_{Y'} + \varepsilon f^*E)^{>1} \rfloor) = f_*\mathcal{O}_{Y'}(A - N - Y''),$$

then $f\colon (Y', B_{Y'} + \varepsilon f^*E) \to X$ gives the desired quasi-log structure on $[X, \widetilde{\omega}]$. □

By Lemma 5.1, we can prove:

Lemma 5.5. *Let $\varphi\colon X \to Z$ be a projective surjective morphism between complex analytic spaces such that $[X, \omega]$ is a quasi-log complex analytic space and that X is irreducible. Let P be an arbitrary point of Z. Let E be any positive-dimensional irreducible component of $\varphi^{-1}(P)$ with $E \not\subset \operatorname{Nqlc}(X, \omega)$. Then, after shrinking Z around P suitably, we can take an effective $\mathbb{R}$-Cartier divisor G on Z such that $[X, \omega + \varphi^*G]$ naturally becomes a quasi-log complex analytic space and E is a qlc stratum of $[X, \omega + \varphi^*G]$.*

We will use Lemma 5.5 when we prove the existence of ω-negative extremal rational curves (see Theorem 9.6).

Sketch of Proof of Lemma 5.5. If E is a qlc stratum of $[X, \omega]$, then it is sufficient to put $G = 0$. From now on, we assume that E is not a qlc stratum of $[X, \omega]$. We shrink Z around P suitably and take general effective Cartier divisors $D_1, \ldots, D_{n+1}$ with $P \in \operatorname{Supp} D_i$ for every i, where $n = \dim X$. By Lemma 5.2, $\left[X, \omega + \sum_{i=1}^{n+1} \varphi^*D_i\right]$ naturally becomes a quasi-log complex analytic space. We take a general point $Q \in E$. By Lemma 5.1, we see that $\left[X, \omega + \sum_{i=1}^{n+1} \varphi^*D_i\right]$ is not quasi-log canonical at Q. Therefore, we can find $0 < c < 1$ such that $G := c\sum_{i=1}^{n+1} D_i$ and E is a qlc center of $[X, \omega + \varphi^*G]$. This is what we wanted. □

Similarly, we also have:

Lemma 5.6. *Let $\varphi\colon X \to Z$ be a projective surjective morphism between complex analytic spaces with $\dim Z > 0$ such that $[X, \omega]$ is a quasi-log complex analytic space, X is irreducible, and $\operatorname{Nqlc}(X, \omega) = \emptyset$. Let P be an arbitrary point of Z with $\dim \varphi^{-1}(P) > 0$. Then, after shrinking Z around P suitably, there exists an effective $\mathbb{R}$-Cartier divisor G' on Z such that $[X, \omega + \varphi^*G']$ naturally becomes a quasi-log complex analytic space, there exists a positive-dimensional qlc center C of $[X, \omega + \varphi^*G']$ with $\varphi(C) = P$, $\dim \operatorname{Nqlc}(X, \omega + \varphi^*G') \leq 0$, and $\operatorname{Nqlc}(X, \omega + \varphi^*G') = \emptyset$ outside $\varphi^{-1}(P)$.*

We will use Lemma 5.6 in the proof of Theorem 9.4.

Sketch of Proof of Lemma 5.6. If there exists a positive-dimensional qlc center C of $[X, \omega]$ with $\varphi(C) = P$, then it is sufficient to put $G' = 0$. So we may assume that there are no positive-dimensional qlc centers in $\varphi^{-1}(P)$. From now on, we will use the same notation as in the proof of Lemma 5.5. It is not difficult to see that we can take $0 < c' < 1$ such that $[X, \omega + \varphi^*G']$, where $G' := c'\sum_{i=1}^{n+1} D_i$, satisfies all the desired properties. □

We close this section with the following easy lemma.

Lemma 5.7. *Let $(X, \omega, f\colon (Y, B_Y) \to X)$ be a quasi-log complex analytic space. We assume that the support of B_Y has only finitely many irreducible components. Then we obtain a $\mathbb{Q}$-divisor D_i on Y, a $\mathbb{Q}$-line bundle ω_i on X, and a positive real number r_i for $1 \leq i \leq k$ such that*

(i) $\sum_{i=1}^{k} r_i = 1$,
(ii) $\operatorname{Supp} D_i = \operatorname{Supp} B_Y$, $D_i^{=1} = B_Y^{=1}$, $\lfloor D_i^{>1} \rfloor = \lfloor B_Y^{>1} \rfloor$, *and* $\lceil -(D_i^{<1}) \rceil = \lceil -(B_Y^{<1}) \rceil$ *for every* i,
(iii) $\omega = \sum_{i=1}^{k} r_i \omega_i$ *holds in* $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ *and* $B_Y = \sum_{i=1}^{k} r_i D_i$, *and*
(iv) $(X, \omega_i, f\colon (Y, D_i) \to X)$ *is a quasi-log complex analytic space with* $K_Y + D_i \sim_{\mathbb{Q}} f^*\omega_i$ *for every* i.

We note that

$$\mathcal{I}_{\mathrm{Nqlc}(X, \omega_i)} = \mathcal{I}_{\mathrm{Nqlc}(X, \omega)}$$

holds for every i. In particular, if $\mathrm{Nqlc}(X, \omega) = \emptyset$, *then* $\mathrm{Nqlc}(X, \omega_i) = \emptyset$ *for every i. We also note that W is a qlc stratum of $[X, \omega]$ if and only if W is a qlc stratum of $[X, \omega_i]$ for every i.*

Proof of Lemma 5.7. The proof of [Fu9, Lemma 4.22] works without any changes. □

6. Basepoint-free theorem

In this section, we will prove the following basepoint-free theorem for quasi-log complex analytic spaces, which is a generalization of [Fu7, Theorem 6.5.1] and [Fu15, Theorem 9.1].

Theorem 6.1 (Basepoint-free theorem for quasi-log complex analytic spaces)**.** *Let*

$$(X, \omega, f\colon (Y, B_Y) \to X)$$

be a quasi-log complex analytic space and let $\pi\colon X \to S$ be a projective morphism between complex analytic spaces. Let W be a compact subset of S and let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L}$ is π-nef over W. We assume that

(i) *$q\mathcal{L} - \omega$ is π-ample over W for some real number $q > 0$, and*
(ii) *$\mathcal{L}^{\otimes m}|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-generated over some open neighborhood of W for every $m \gg 0$.*

Then there exists a relatively compact open neighborhood U of W such that $\mathcal{L}^{\otimes m}$ is π-generated over U for every $m \gg 0$.

Proof. By shrinking S around W suitably, we may assume that $\mathcal{L}^{\otimes m}|_{X_{-\infty}}$ is $\pi|_{X_{-\infty}}$-generated for every $m \gg 0$ by (ii). We take an arbitrary point $w \in W$. Then it is sufficient to prove that $\mathcal{L}^{\otimes m}$ is π-generated for every $m \gg 0$ over some relatively compact open neighborhood of w since W is compact. Hence, we may assume that $W = \{w\}$, S is Stein, and π is surjective. We will sometimes shrink S around W suitably without mentioning it explicitly throughout this proof. We use induction on the dimension of $X \setminus X_{-\infty}$. We note that Theorem 6.1 obviously holds true when $\dim X \setminus X_{-\infty} = 0$.

Step 1. In this step, we will prove that for every $m \gg 0$ there exists an open neighborhood U_m of w such that $\mathcal{L}^{\otimes m}$ is π-generated around $\mathrm{Nqklt}(X, \omega)$ over U_m.

We put $X' := \mathrm{Nqklt}(X, \omega)$. Then, after shrinking S around w suitably, $[X', \omega']$, where $\omega' := \omega|_{X'}$, is a quasi-log complex analytic space by adjunction when $X' \neq X_{-\infty}$ (see Theorem 4.4). If $X' = X_{-\infty}$, then $\mathcal{L}^{\otimes m}|_{X'}$ is π-generated for every $m \gg 0$ by assumption. If $X' \neq X_{-\infty}$, then $\mathcal{L}^{\otimes m}|_{X'}$ is π-generated for every $m \gg 0$ by induction on the dimension of $X \setminus X_{-\infty}$ after shrinking S around w suitably. We can take an open neighborhood

U_m of w such that $m\mathcal{L}-\omega$ is π-ample over U_m. Then $R^1\pi_*\left(\mathcal{I}_{X'}\otimes\mathcal{L}^{\otimes m}\right)=0$ on U_m (see Theorems 4.7 and 4.8). Thus, the restriction map

$$\pi_*\mathcal{L}^{\otimes m}\to \pi_*\left(\mathcal{L}^{\otimes m}|_{X'}\right)$$

is surjective on U_m. This implies that $\mathcal{L}^{\otimes m}$ is π-generated around $\mathrm{Nqklt}(X,\omega)$ over U_m. This is what we wanted.

Step 2. In this step, we will prove that $\pi_*\left(\mathcal{L}^{\otimes m}|_{X'}\right)\neq 0$ for every $m\gg 0$ when $X'\cap \mathrm{Nqklt}(X,\omega)$ is empty, where X' is any connected component of X with $w\in\pi(X')$.

Throughout this step, we can freely shrink S around w. Without loss of generality, we may assume that X itself is connected (see [Fu7, Lemma 6.3.12]). Then, by Lemma 4.9, X is a normal complex variety. We apply [Fu15, Lemma 8.2] to

$$(X,\omega,f\colon (Y,B_Y)\to X).$$

Let s be an analytically sufficiently general point of S. Then

$$(X_s,\omega|_{X_s},f_s\colon (Y_s,B_{Y_s})\to X_s)$$

is a projective quasi-log canonical pair, where $X_s:=\pi^{-1}(s)$, $Y_s:=(\pi\circ f)^{-1}(s)$, $f_s:=f|_{X_s}$, and $B_{Y_s}:=B_Y|_{Y_s}$. We may assume that $\mathcal{L}|_{X_s}$ is nef (see [Fu15, Lemma 3.5]) and $q\mathcal{L}|_{X_s}-\omega|_{X_s}$ is ample. By the basepoint-free theorem for quasi-log schemes (see [Fu7, Theorem 6.5.1]), we obtain that $\mathcal{L}^{\otimes m}|_{X_s}$ is basepoint-free for every $m\gg 0$. In particular, $|\mathcal{L}^{\otimes m}|_{X_s}|\neq\emptyset$ for every $m\gg 0$. This implies that $\pi_*\mathcal{L}^{\otimes m}\neq\emptyset$ for every $m\gg 0$. This is what we wanted.

Step 3. Let p be a prime number and let k be a large positive integer. By Steps 1 and 2, after shrinking S around w suitably, we obtain that $\pi_*\mathcal{L}^{\otimes p^k}\neq 0$ and that $\mathcal{L}^{\otimes p^k}$ is π-generated around $\mathrm{Nqklt}(X,\omega)$. In this step, we will prove that if the relative base locus $\mathrm{Bs}_\pi|\mathcal{L}^{\otimes p^k}|$ with the reduced structure is not empty over w then, after shrinking S around w suitably again, there exists a positive integer $l>k$ such that $\mathrm{Bs}_\pi|\mathcal{L}^{\otimes p^l}|$ is strictly smaller than $\mathrm{Bs}_\pi|\mathcal{L}^{\otimes p^k}|$.

From now on, we will sometimes shrink S around w suitably without mentioning it explicitly. Let $f\colon (Y,B_Y)\to X$ be a quasi-log resolution. We take a general member $D\in|\mathcal{L}^{\otimes p^k}|$. Then we may assume that f^*D intersects any strata of $(Y,\mathrm{Supp}\,B_Y)$ transversally over $X\setminus\mathrm{Bs}_\pi|\mathcal{L}^{\otimes p^k}|$ by Bertini's theorem and that f^*D contains no strata of (Y,B_Y). By taking a suitable projective bimeromorphic modification of M, the ambient space of (Y,B_Y), we may assume that $(Y,f^*D+\mathrm{Supp}\,B_Y)$ is an analytic globally embedded simple normal crossing pair (see [BM1] and [Fu7, Proposition 6.3.1]). After shrinking S around w suitably, we take the maximal positive real number c such that B_Y+cf^*D is a subboundary over $X\setminus X_{-\infty}$. We note that $c\leq 1$ holds. Here, we used the fact that the natural map $\mathcal{O}_X\to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ is an isomorphism over $X\setminus X_{-\infty}$ (see [Fu10, Claim 3.5]). Then

$$f\colon (Y,B_Y+cf^*D)\to X$$

gives a natural quasi-log structure on the pair $[X,\omega':=\omega+cD]$ (see Lemma 5.2, [Fu10, Proposition 3.4], and so on). We note that $\mathrm{Nqlc}(X,\omega)=\mathrm{Nqlc}(X,\omega')$ holds by construction. We note that we may assume that $[X,\omega']$ has a qlc center C that intersects $\mathrm{Bs}_\pi|\mathcal{L}^{\otimes p^k}|\cap\pi^{-1}(w)$. Since $(q+cp^k)\mathcal{L}-\omega'$ is π-ample over W, for every $m\gg 0$, there exists some open neighborhood U'_m of w such that $\mathcal{L}^{\otimes m}$ is π-generated around $\mathrm{Nqklt}(X,\omega')$ over U'_m by Step 1. In particular, $\mathcal{L}^{\otimes m}$ is π-generated around C over U'_m. Thus, $\mathrm{Bs}_\pi|\mathcal{L}^{\otimes p^l}|$ is strictly smaller than $\mathrm{Bs}_\pi|\mathcal{L}^{\otimes p^k}|$ for some positive integer $l>k$. This is what we wanted.

Step 4. In this step, we will complete the proof.

By using Step 3 finitely many times, we obtain an open neighborhood V_p of w and a large positive integer n such that $\mathcal{L}^{\otimes p^n}$ is π-generated over V_p. We take another prime number p'. By the same argument, we obtain an open neighborhood $V_{p'}$ of w and a large positive integer n' such that $\mathcal{L}^{\otimes p'^{n'}}$ is π-generated over $V_{p'}$. Hence, we can take an open neighborhood U_w of w and a positive integer m_0 such that $\mathcal{L}^{\otimes m}$ is π-generated over U_w for every $m \geq m_0$ (see Lemma 8.6 below).

As we mentioned above, we can obtain a desired open neighborhood U of W since W is compact. We finish the proof. □

By combining Theorem 6.1 with Example 1.3, we can recover the basepoint-free theorem for normal pairs (see [Fu15, Theorem 9.1]).

Remark 6.2. For normal pairs, we first established the non-vanishing theorem (see [Fu15, Theorem 8.1]) and then proved the basepoint-free theorem (see [Fu15, Theorem 9.1]). On the other hand, in this section, we can directly prove the basepoint-free theorem (see Theorem 6.1) because quasi-log structures behave well for inductive treatments.

7. Basepoint-free theorem of Reid–Fukuda type

In this section, we will prove the basepoint-free theorem of Reid–Fukuda type. If we apply this theorem to kawamata log terminal pairs, then we can recover the Kawamata–Shokurov basepoint-free theorem for projective morphisms between complex analytic spaces.

Theorem 7.1 (Basepoint-free theorem of Reid–Fukuda type for quasi-log complex analytic spaces)**.** *Let $[X, \omega]$ be a quasi-log complex analytic space and let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces. Let $\mathcal{L}$ be a π-nef line bundle on X such that $q\mathcal{L} - \omega$ is nef and log big over S with respect to $[X, \omega]$ for some positive real number q. This means that $q\mathcal{L} - \omega$ is nef over S and that $(q\mathcal{L} - \omega)|_C$ is big over $\pi(C)$ for every qlc stratum C of $[X, \omega]$. We assume that $\mathcal{L}^{\otimes m}|_{X_{-\infty}}$ is π-generated for every $m \gg 0$. Then, after replacing S with any relatively compact open subset of S, $\mathcal{L}^{\otimes m}$ is π-generated for every $m \gg 0$.*

The following proof is essentially the same as the one for [Fu5, Theorem 1.1].

Proof of Theorem 7.1. In Step 1, we will reduce the problem to the case where $X \setminus X_{-\infty}$ is irreducible and the relative base locus of $\mathcal{L}^{\otimes m}$ is disjoint from $\mathrm{Nqklt}(X, \omega)$ for every $m \gg 0$.

Step 1. We use induction on $\dim(X \setminus X_{-\infty})$. It is obvious that the statement holds when $\dim(X \setminus X_{-\infty}) = 0$. We take an arbitrary point $P \in S$. It is sufficient to prove the statement over some open neighborhood of P. Hence we will freely shrink S around P throughout this proof. In particular, we may assume that S is Stein. Let C be any qlc stratum of $[X, \omega]$. We put $X' := C \cup \mathrm{Nqlc}(X, \omega)$. By adjunction (see Theorem 4.4), after replacing S with any relatively compact open neighborhood of P, $[X', \omega' := \omega|_{X'}]$ is a quasi-log complex analytic space. By the vanishing theorem (see Theorem 4.8), we have $R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{L}^{\otimes m}) = 0$ for every $m \geq q$. Therefore, the natural restriction map

$$\pi_*\mathcal{L}^{\otimes m} \to \pi_*\left(\mathcal{L}^{\otimes m}|_{X'}\right)$$

is surjective for every $m \geq q$. This implies that we may assume that $X \setminus X_{-\infty}$ is irreducible by replacing X with X'. If $\mathrm{Nqklt}(X, \omega) = \mathrm{Nqlc}(X, \omega)$, then $\mathcal{L}^{\otimes m}|_{\mathrm{Nqklt}(X,\omega)}$ is π-generated for every $m \gg 0$ by assumption. If $\mathrm{Nqklt}(X,\omega) \neq \mathrm{Nqlc}(X,\omega)$, then we know that $\mathcal{L}^{\otimes m}|_{\mathrm{Nqklt}(X,\omega)}$ is π-generated for every $m \gg 0$ by induction on $\dim(X \setminus X_{-\infty})$. By the vanishing theorem again (see Theorems 4.7 and 4.8), we have $R^1\pi_* \left(\mathcal{I}_{\mathrm{Nqklt}(X,\omega)} \otimes \mathcal{L}^{\otimes m}\right) = 0$ for every $m \geq q$. Thus, the restriction map

$$\pi_*\mathcal{L}^{\otimes m} \to \pi_* \left(\mathcal{L}^{\otimes m}|_{\mathrm{Nqklt}(X,\omega)}\right)$$

is surjective for every $m \geq q$. Hence, the relative base locus $\mathrm{Bs}_\pi |\mathcal{L}^{\otimes m}|$ of $\mathcal{L}^{\otimes m}$ is disjoint from $\mathrm{Nqklt}(X,\omega)$ for every $m \gg 0$.

Step 2. In this step, we will prove the basepoint-freeness under the extra assumption that X is the disjoint union of $X_{-\infty} = \mathrm{Nqlc}(X,\omega)$ and a qlc stratum C of $[X,\omega]$ such that C is the unique qlc stratum of $[X,\omega]$.

In the above setting, we may assume that $X_{-\infty} = \emptyset$ (see [Fu7, Lemma 6.3.12]). By Kodaira's lemma, after replacing S with any relatively compact open neighborhood of P, we can write

$$q\mathcal{L} - \omega \sim_{\mathbb{R}} A + E$$

on X such that A is a π-ample $\mathbb{Q}$-divisor on X and E is an effective $\mathbb{R}$-Cartier divisor on X. We put $\widetilde{\omega} = \omega + \varepsilon E$ with $0 < \varepsilon \ll 1$. Then $[X,\widetilde{\omega}]$ is a quasi-log complex analytic space with $\mathrm{Nqlc}(X,\widetilde{\omega}) = \emptyset$ by Lemma 5.4. We note that

$$q\mathcal{L} - \widetilde{\omega} \sim_{\mathbb{R}} (1-\varepsilon)(q\mathcal{L} - \omega) + \varepsilon A$$

is π-ample. Therefore, by the basepoint-free theorem (see Theorem 6.1), we obtain that $\mathcal{L}^{\otimes m}$ is π-generated for every $m \gg 0$ over some open neighborhood of P.

Step 3. By Step 2, we may assume that X is connected and $\mathrm{Nqklt}(X,\omega) \neq \emptyset$. Let p be any prime number. Then, by Step 1, the relative base locus $\mathrm{Bs}_\pi |p^l\mathcal{L}|$ of $p^l\mathcal{L}$ is strictly smaller than X for some large positive integer l. In this step, we will prove the following claim.

Claim. *If the relative base locus* $\mathrm{Bs}_\pi |p^l\mathcal{L}|$ *with the reduced structure is not empty over* P*, then there is a positive integer* k *with* $k > l$ *such that* $\mathrm{Bs}_\pi |p^k\mathcal{L}|$ *is strictly smaller than* $\mathrm{Bs}_\pi |p^l\mathcal{L}|$ *after shrinking* S *around* P *suitably.*

Proof of Claim. Note that the inclusion $\mathrm{Bs}_\pi |p^k\mathcal{L}| \subseteq \mathrm{Bs}_\pi |p^l\mathcal{L}|$ obviously holds for every positive integer $k > l$. Let us consider $[X^\dagger, \omega^\dagger]$ as in Lemma 5.3. Since $(q\mathcal{L} - \omega)|_{X^\dagger}$ is nef and big over S, after replacing S with any relatively compact open neighborhood of P, we can write

$$q\mathcal{L}|_{X^\dagger} - \omega^\dagger \sim_{\mathbb{R}} A + E$$

on $X^\dagger$ by Kodaira's lemma, where A is a π-ample $\mathbb{Q}$-divisor on $X^\dagger$ and E is an effective $\mathbb{R}$-Cartier divisor on $X^\dagger$. By Lemma 5.4, after replacing S with any relatively compact Stein open neighborhood of P, we have a new quasi-log structure on $[X^\dagger, \widetilde{\omega}]$, where $\widetilde{\omega} = \omega^\dagger + \varepsilon E$ with $0 < \varepsilon \ll 1$, such that

$$\mathrm{Nqlc}(X^\dagger, \widetilde{\omega}) = \left(\bigcup_{i \in I} C_i\right) \cup \mathrm{Nqlc}(X^\dagger, \omega^\dagger), \tag{7.1}$$

where $\{C_i\}_{i \in I}$ is the set of qlc centers of $[X^\dagger, \omega^\dagger]$ contained in $\mathrm{Supp}\, E$. We put $n := \dim X^\dagger$. Let $D_1, \ldots, D_{n+1}$ be general members of $|p^l\mathcal{L}|$. Let $f\colon (Y, B_Y) \to X^\dagger$ be a quasi-log resolution of $[X^\dagger, \widetilde{\omega}]$. We consider

$$f\colon \left(Y, B_Y + \sum_{i=1}^{n+1} f^*D_i\right) \to X^\dagger.$$

Without loss of generality, we may assume that

$$\left(Y, \sum_{i=1}^{n+1} f^*D_i + \operatorname{Supp} B_Y\right)$$

is an analytic globally embedded simple normal crossing pair by taking a suitable projective bimeromorphic modification of the ambient space of (Y, B_Y) (see [BM1] and [Fu7, Proposition 6.3.1]). Then, after shrinking S around P suitably, we can take $0 < c < 1$ such that

$$f\colon \left(Y, B_Y + c\sum_{i=1}^{n+1} f^*D_i\right) \to X^\dagger$$

gives a quasi-log structure on $\left[X^\dagger, \widetilde{\omega} + c\sum_{i=1}^{n+1} D_i\right]$ such that $\left[X^\dagger, \widetilde{\omega} + c\sum_{i=1}^{n+1} D_i\right]$ has only quasi-log canonical singularities on $X^\dagger \setminus \operatorname{Nqlc}(X^\dagger, \widetilde{\omega})$ and that there exists a qlc center C_0 of $\left[X^\dagger, \widetilde{\omega} + c\sum_{i=1}^{n+1} D_i\right]$ contained in $\operatorname{Bs}_\pi |p^l\mathcal{L}|$ with $C_0 \cap \pi^{-1}(P) \neq \emptyset$ (see Lemmas 5.1 and 5.2). We put $\widetilde{\omega} + c\sum_{i=1}^{n+1} D_i = \overline{\omega}$. Then, by construction,

$$C_0 \cap \operatorname{Nqlc}(X^\dagger, \overline{\omega}) = \emptyset$$

holds because

$$\operatorname{Bs}_\pi |p^l\mathcal{L}| \cap \operatorname{Nqklt}(X, \omega) = \emptyset.$$

Note that $\operatorname{Nqlc}(X^\dagger, \overline{\omega}) = \operatorname{Nqlc}(X^\dagger, \widetilde{\omega})$ by construction. We also note that

$$(q + c(n+1)p^l)\mathcal{L}|_{X^\dagger} - \overline{\omega} \sim_{\mathbb{R}} (1-\varepsilon)(q\mathcal{L}|_{X^\dagger} - \omega^\dagger) + \varepsilon A$$

is ample over S. Therefore,

$$\pi_*\left(\mathcal{L}^{\otimes m}|_{X^\dagger}\right) \to \pi_*\left(\mathcal{L}^{\otimes m}|_{C_0}\right) \oplus \pi_*\left(\mathcal{L}^{\otimes m}|_{\operatorname{Nqlc}(X^\dagger, \overline{\omega})}\right) \tag{7.2}$$

is surjective for every $m \geq q + c(n+1)p^l$ by Theorem 4.8. Moreover, $\mathcal{L}^{\otimes m}|_{C_0}$ is π-generated for every $m \gg 0$ by the basepoint-free theorem (see Theorem 6.1). Note that $[C_0, \overline{\omega}|_{C_0}]$ is a quasi-log complex analytic space with only quasi-log canonical singularities (see [Fu7, Lemma 6.3.12]). Therefore, we can construct a section s of $\mathcal{L}^{\otimes p^k}|_{X^\dagger}$ for some positive integer $k > l$ such that $s|_{C_0}$ is not zero and s is zero on $\operatorname{Nqlc}(X^\dagger, \overline{\omega})$ by (7.2). Thus s is zero on

$$\operatorname{Nqlc}(X^\dagger, \overline{\omega}) = \operatorname{Nqlc}(X^\dagger, \widetilde{\omega}) = \left(\bigcup_{i \in I} C_i\right) \cup \operatorname{Nqlc}(X^\dagger, \omega^\dagger)$$

by (7.1). In particular, s is zero on $\operatorname{Nqlc}(X^\dagger, \omega^\dagger)$. Hence, s can be seen as a section of $\mathcal{L}^{\otimes p^k}$ because $\mathcal{I}_{\operatorname{Nqlc}(X^\dagger, \omega^\dagger)} = \mathcal{I}_{\operatorname{Nqlc}(X, \omega)}$ by construction (see Lemma 5.3). More precisely, we can see

$$s \in \pi_*\left(\mathcal{I}_{\operatorname{Nqlc}(X^\dagger, \overline{\omega})} \otimes \mathcal{L}^{\otimes p^k}\right)$$

by construction. Since

$$\mathcal{I}_{\operatorname{Nqlc}(X^\dagger, \overline{\omega})} \subset \mathcal{I}_{\operatorname{Nqlc}(X^\dagger, \omega^\dagger)} = \mathcal{I}_{\operatorname{Nqlc}(X, \omega)},$$

we have

$$s \in \pi_*\left(\mathcal{I}_{\operatorname{Nqlc}(X^\dagger, \overline{\omega})} \otimes \mathcal{L}^{\otimes p^k}\right) \subset \pi_*\left(\mathcal{I}_{\operatorname{Nqlc}(X, \omega)} \otimes \mathcal{L}^{\otimes p^k}\right) \subset \pi_*\left(\mathcal{L}^{\otimes p^k}\right).$$

Therefore, $\operatorname{Bs}_\pi |p^k\mathcal{L}|$ is strictly smaller than $\operatorname{Bs}_\pi |p^l\mathcal{L}|$ over P. We complete the proof of Claim. □

Step 4. By the noetherian induction, after shrinking S around P suitably, $p^l\mathcal{L}$ and $p'^{l'}\mathcal{L}$ are both π-generated for large positive integers l and l', where p and p' are distinct prime numbers. Hence there exists a positive integer m_0 such that $\mathcal{L}^{\otimes m}$ is π-generated for every $m \geq m_0$ (see Lemma 8.6 below).

We finish the proof. □

As we saw above, since $\pi\colon X \to S$ is projective in Theorem 7.1, the proof of [Fu5, Theorem 1.1] works even when $\pi\colon X \to S$ is not algebraic. When $\pi\colon X \to S$ is algebraic but is only *proper*, the proof of Theorem 7.1 is unexpectedly difficult. For the details, see the proof of [Fu11, Theorem 1.1].

8. Effective freeness

In this section, we will prove the following effective freeness and effective very ampleness. This type of effective freeness was originally due to Kollár (see [Ko1]). Note that his method was already generalized for quasi-log schemes in [Fu6]. Here, we give a slightly simpler proof for quasi-log complex analytic spaces.

Theorem 8.1 (Effective freeness for quasi-log complex analytic spaces)**.** *Let $[X, \omega]$ be a quasi-log complex analytic space with $X_{-\infty} = \emptyset$ and let $\pi\colon X \to S$ be a projective morphism between complex analytic spaces. Let $\mathcal{L}$ be a π-nef line bundle on X such that $a\mathcal{L} - \omega$ is π-ample over S for some non-negative integer a. Then there exists a positive integer $m = m(\dim X, a)$, which only depends on $\dim X$ and a, such that $\mathcal{L}^{\otimes m}$ is π-generated. Moreover, there exists a positive integer $m_0 = m_0(\dim X, a)$ depending only on $\dim X$ and a such that $\mathcal{L}^{\otimes l}$ is π-generated for every $l \geq m_0$.*

If ω is a $\mathbb{Q}$-line bundle in Theorem 8.1, then we have:

Theorem 8.2. *In Theorem 8.1, if ω is a $\mathbb{Q}$-line bundle (or a globally $\mathbb{Q}$-Cartier divisor), then we may replace the assumption that $a\mathcal{L} - \omega$ is π-ample over S with a weaker one that $a\mathcal{L} - \omega$ is nef and log big over S with respect to $[X, \omega]$.*

If $\mathcal{L}$ is π-ample in Theorem 8.1, then we have the following effective very ampleness.

Theorem 8.3 (Effective very ampleness for quasi-log complex analytic spaces)**.** *Let $[X, \omega]$ be a quasi-log complex analytic space with $X_{-\infty} = \emptyset$ and let $\pi\colon X \to S$ be a projective morphism between complex analytic spaces. Let $\mathcal{L}$ be a π-ample line bundle on X such that $a\mathcal{L} - \omega$ is π-nef over S for some non-negative integer a. Then there exists a positive integer $m' = m'(\dim X, a)$ depending only on $\dim X$ and a such that $\mathcal{L}^{\otimes m'}$ is π-very ample. Moreover, there exists a positive integer $m'_0 = m'_0(\dim X, a)$ depending only on $\dim X$ and a such that $\mathcal{L}^{\otimes l'}$ is π-very ample for every $l' \geq m'_0$.*

We will use the following easy lemmas in the proof of Theorem 8.1.

Lemma 8.4. *Let $P(x)$ be a polynomial and let a and n be positive integers. Assume that, with at most n exceptions, $P(a + j) \neq 0$ holds for every non-negative integer j. Then, for every positive integer $m \geq 2(a + n)$, there exists a non-negative integer j_0 with $0 \leq j_0 \leq n$ such that $P(a + j_0) \neq 0$ and $P(m - a - j_0) \neq 0$.*

Proof. We note that $m - a - j = a + (m - 2a) - j$ and $m - 2a \geq 2n$. Therefore, we can easily find some non-negative integer j_0 with $0 \leq j_0 \leq n$ such that $P(a + j_0) \neq 0$ and $P(m - a - j_0) \neq 0$. □

Lemma 8.5. *Let n_0 and n_1 be positive integers such that* $\gcd(n_0, n_1) = 1$. *We put* $n_2 := (kn_0 + 1)n_1$, *where k is any positive integer. Then* $\gcd(n_0, n_2) = 1$.

Proof. It is obvious. □

Lemma 8.6. *Let a and b be positive integers with $1 < a < b$ such that* $\gcd(a, b) = 1$. *Then, for any positive integer l with $l \geq a\left(b - \lceil \frac{b}{a} \rceil\right)$, there exist non-negative integers u and v such that $l = ua + vb$.*

Proof. It is an easy exercise. For the details, see, for example, the proof of [Fu11, Lemma 4.3]. □

Let us prove Theorem 8.1. The proof is new and is slightly simpler than the one given in [Fu6] for quasi-log schemes.

Proof of Theorem 8.1. We take an arbitrary point $s \in S$. It is sufficient to prove the existence of $m(\dim X, a)$ and $m_0(\dim X, a)$ over some open neighborhood of s. Therefore, we replace S with a relatively compact Stein open neighborhood of s.

Step 1. Let X' be an irreducible component of X. Then X' is a qlc stratum of $[X, \omega]$. Hence, by adjunction (see Theorem 4.4), we see that $[X', \omega' := \omega|_{X'}]$ is a quasi-log complex analytic space with $X'_{-\infty} = \emptyset$. By the vanishing theorem (see Theorem 4.8), we have

$$R^1\pi_*(\mathcal{I}_{X'} \otimes \mathcal{L}^{\otimes j}) = 0$$

for every $j \geq a$. Thus the natural restriction map

$$\pi_*\mathcal{L}^{\otimes j} \to \pi_*(\mathcal{L}^{\otimes j}|_{X'})$$

is surjective for every $j \geq a$. Therefore, we replace X with X' and may assume that X is irreducible.

Step 2. In this step, we will prove the following claims.

Claim 1 (see [Fu6, Lemma 3.3])**.** *For every positive integer $m_1 \geq 2(a + \dim X)$, there exists an effective Cartier divisor D_1 on X such that $D_1 \in |\mathcal{L}^{\otimes m_1}|$ and that* $\operatorname{Supp} D_1$ *contains no qlc strata of $[X, \omega]$.*

Proof of Claim 1. Let C be any qlc stratum of $[X, \omega]$. We consider the following short exact sequence:

$$0 \to \mathcal{I}_C \otimes \mathcal{L}^{\otimes j} \to \mathcal{L}^{\otimes j} \to \mathcal{L}^{\otimes j}|_C \to 0,$$

where $\mathcal{I}_C$ is the defining ideal sheaf of C on X. By the vanishing theorem (see Theorem 4.8),

$$R^i\pi_*\left(\mathcal{I}_C \otimes \mathcal{L}^{\otimes j}\right) = R^i\pi_*\mathcal{L}^{\otimes j} = R^i\pi_*\left(\mathcal{L}^{\otimes j}|_C\right) = 0$$

for every $i \geq 1$ and $j \geq a$. Therefore,

$$\chi(C_s, \mathcal{L}^{\otimes j}|_{C_s}) = \dim H^0(C_s, \mathcal{L}^{\otimes j}|_{C_s})$$

holds for $j \geq a$, where C_s is an analytically sufficiently general fiber of $C \to \pi(C)$. Note that $\chi(C_s, \mathcal{L}^{\otimes j}|_{C_s})$ is a non-zero polynomial in j since $\mathcal{L}^{\otimes m}$ is π-generated for every $m \gg 0$ by the basepoint-free theorem (see Theorem 6.1). We also note that the restriction map

$$\pi_*\mathcal{L}^{\otimes j} \to \pi_*\left(\mathcal{L}^{\otimes j}|_C\right) \tag{8.1}$$

is surjective for every $j \geq a$. Thus, with at most $\dim C_s$ exceptions, we have

$$\dim H^0(C_s, \mathcal{L}^{\otimes (a+j)}|_{C_s}) \neq 0$$

for $j \geq 0$. By Lemma 8.4, we see that

$$\dim H^0(C_s, \mathcal{L}^{\otimes m_1}|_{C_s}) \neq 0$$

holds for $m_1 \geq 2(a + \dim X)$. Therefore, we have $C \not\subset \mathrm{Bs}_\pi |\mathcal{L}^{\otimes m_1}|$ for $m_1 \geq 2(a + \dim X)$ by (8.1). By this observation, we can take a desired effective Cartier divisor $D_1 \in |\mathcal{L}^{\otimes m_1}|$ for every $m_1 \geq 2(a + \dim X)$. □

By the basepoint-free theorem (see Theorem 6.1), we have the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad p \quad} & Z \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

such that $\mathcal{L} \simeq p^*\mathcal{L}_Z$ for some q-ample line bundle $\mathcal{L}_Z$ on Z with $p_*\mathcal{O}_X \simeq \mathcal{O}_Z$.

Claim 2. *If* $\mathrm{Bs}_\pi |\mathcal{L}^{\otimes m_1}|$ *contains no qlc strata of* $[X, \omega]$*, then*

$$\dim \mathrm{Bs}_q |\mathcal{L}_Z^{\otimes m_2}| < \dim \mathrm{Bs}_q |\mathcal{L}_Z^{\otimes m_1}|$$

holds for every positive integer $m_2 \geq 2\left(a + (\dim X + 1)m_1 + \dim X\right)$ *with* $m_1 | m_2$.

Proof of Claim 2. We take general members $D_1, \ldots, D_{\dim X+1} \in |\mathcal{L}^{\otimes m_1}| = |p^*\mathcal{L}_Z^{\otimes m_1}|$. Let $\mathcal{B}$ be any irreducible component of $\mathrm{Bs}_q |\mathcal{L}_Z^{\otimes m_1}|$. Then, by Lemmas 5.1 and 5.2, we can take $0 < c < 1$ such that $[X, \omega + cD]$, where $D := \sum_{i=1}^{\dim X+1} D_i$, has a natural quasi-log structure with the following properties:

- there exists a qlc center V of $[X, \omega + cD]$ such that $p(V) = \mathcal{B}$, and
- $p(\mathrm{Nqlc}(X, \omega + cD))$ does not contain $\mathcal{B}$.

We put $V' := V \cup \mathrm{Nqlc}(X, \omega + cD)$. Then, by the proof of adjunction (see (4.3) in Theorem 4.4), we have the following short exact sequence:

$$0 \to \mathcal{I}_{V'} \to \mathcal{I}_{\mathrm{Nqlc}(X, \omega+cD)} \to \mathcal{I}_{\mathrm{Nqlc}(V', (\omega+cD)|_{V'})} \to 0.$$

We note that $\mathcal{L}^{\otimes j} - (\omega + cD)$ is π-ample for every $j \geq a + (\dim X + 1)m_1$. Therefore,

$$R^i\pi_*\left(\mathcal{I}_{V'} \otimes \mathcal{L}^{\otimes j}\right) = R^i\pi_*\left(\mathcal{I}_{\mathrm{Nqlc}(X,\omega+cD)} \otimes \mathcal{L}^{\otimes j}\right) = R^i\pi_*\left(\mathcal{I}_{\mathrm{Nqlc}(V',(\omega+cD)|_{V'})} \otimes \mathcal{L}^{\otimes j}\right) = 0$$

for every $i > 0$ and $j \geq a + (\dim X + 1)m_1$. In particular, we have the following short exact sequence:

$$\begin{aligned} 0 \to \pi_*\left(\mathcal{I}_{V'} \otimes \mathcal{L}^{\otimes j}\right) &\to \pi_*\left(\mathcal{I}_{\mathrm{Nqlc}(X,\omega+cD)} \otimes \mathcal{L}^{\otimes j}\right) \\ &\to \pi_*\left(\mathcal{I}_{\mathrm{Nqlc}(V',(\omega+cD)|_{V'})} \otimes \mathcal{L}^{\otimes j}\right) \to 0 \end{aligned} \tag{8.2}$$

for $j \geq a + (\dim X + 1)m_1$. Let $V' \xrightarrow{p'} V'' \to p(V')$ be the Stein factorization of $V' \to p(V')$. By construction, we see that $p'_*\left(\mathcal{I}_{\mathrm{Nqlc}(V',(\omega+cD)|_{V'})}\right)$ is a non-zero ideal sheaf on V''. Therefore,

$$\pi_*\left(\mathcal{I}_{\mathrm{Nqlc}(V',(\omega+cD)|_{V'})} \otimes \mathcal{L}^{\otimes k}\right) \neq 0$$

for every $k \gg 0$ since $\mathcal{L} \simeq p^*\mathcal{L}_Z$ and $\mathcal{L}_Z$ is q-ample. Let s be an analytically sufficiently general point of $\pi(V)$. We put $V'_s := V'|_{\pi^{-1}(s)}$. Thus, with at most $\dim V'_s$ exceptions,

$$\pi_*\left(\mathcal{I}_{\mathrm{Nqlc}(V',(\omega+cD)|_{V'})} \otimes \mathcal{L}^{\otimes(a+(\dim X+1)m_1+j)}\right) \neq 0$$

for $j \geq 0$. Hence,

$$\pi_*\left(\mathcal{I}_{\mathrm{Nqlc}(V',(\omega+cD)|_{V'})} \otimes \mathcal{L}^{\otimes m_2}\right) \neq 0$$

for $m_2 \geq 2\,(a + (\dim X + 1)m_1 + \dim X)$ by Lemma 8.4. Thus, by (8.2), we obtain that $V \not\subset \mathrm{Bs}_\pi\,|\mathcal{L}^{\otimes m_2}|$ for every $m_2 \geq 2\,(a + (\dim X + 1)m_1 + \dim X)$. This implies that $\mathcal{B} = p(V) \not\subset \mathrm{Bs}_q\,|\mathcal{L}_Z^{\otimes m_2}|$ for every $m_2 \geq 2\,(a + (\dim X + 1)m_1 + \dim X)$. Therefore,

$$\dim \mathrm{Bs}_q\,|\mathcal{L}_Z^{\otimes m_2}| < \dim \mathrm{Bs}_q\,|\mathcal{L}_Z^{\otimes m_1}|$$

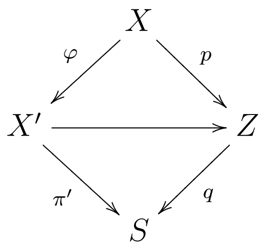
holds for $m_2 \geq 2\,(a + (\dim X + 1)m_1 + \dim X)$ with $m_1 | m_2$. This is what we wanted. We note that $\mathcal{B}$ is any irreducible component of $\mathrm{Bs}_q\,|\mathcal{L}_Z^{\otimes m_1}|$. □

Step 3. In this step, we will complete the proof.

By Claim 1, we see that $\mathrm{Bs}_\pi\,|\mathcal{L}^{\otimes(2(a+\dim X))}| \subsetneq X$ and $\mathrm{Bs}_\pi\,|\mathcal{L}^{\otimes(2(a+\dim X))}|$ contains no qlc strata of $[X, \omega]$. Then we use Claim 2 finitely many times. We finally obtain $m = m(\dim X, a)$, which only depends on $\dim X$ and a, such that $\mathcal{L}^{\otimes m(\dim X, a)}$ is π-generated. By Claims 1, 2, and Lemma 8.5, we can also take $m^\dagger$, which only depends on $\dim X$ and a, such that $\mathcal{L}^{\otimes m^\dagger}$ is π-generated with $\gcd(m(\dim X, a), m^\dagger) = 1$. Therefore, by Lemma 8.6, we can find a positive integer $m_0 = m_0(\dim X, a)$ depending only on $\dim X$ and a such that $\mathcal{L}^{\otimes l}$ is π-generated for every $l \geq m_0(\dim X, a)$.

We finish the proof. □

Proof of Theorem 8.2. We take a positive integer b with $b \geq a$ such that $b\omega$ is a line bundle on X. We put $\mathcal{M} := a(b+1)\mathcal{L} - b\omega$. Then $\mathcal{M}$ is a π-nef line bundle such that $\mathcal{M}$ and $\mathcal{M} - \omega = (b+1)(a\mathcal{L} - \omega)$ are nef and log big over S with respect to $[X, \omega]$. After replacing S with any relatively compact open subset of S, we have a contraction morphism $\varphi\colon X \to X'$ over S associated to $\mathcal{M}$ and a contraction morphism $p\colon X \to Z$ over S associated to $\mathcal{L}$ by the basepoint-free theorem of Reid–Fukuda type (see Theorem 7.1). Then we have the following commutative diagram:



such that $\mathcal{L} \simeq \varphi^*\mathcal{L}'$ for some line bundle $\mathcal{L}'$ on X' and $\omega \sim_{\mathbb{Q}} \varphi^*\omega'$ for some $\mathbb{Q}$-line bundle ω' on X'. Let $f\colon (Y, B_Y) \to X$ be a quasi-log resolution of $[X, \omega]$. Then

$$(X', \omega', \varphi \circ f\colon (Y, B_Y) \to X')$$

naturally becomes a quasi-log complex analytic space with $X'_{-\infty} = \emptyset$. By construction, $\mathcal{L}'$ is π'-nef over S and $(a+1)\mathcal{L}' - \omega'$ is π'-ample over S since $a + 1 \geq \frac{a(b+1)}{b}$. We note that

$$(a+1)\mathcal{L}' - \omega' \sim_{\mathbb{Q}} \frac{1}{b}\mathcal{M}' + \left((a+1) - \frac{a(b+1)}{b}\right)\mathcal{L}',$$

where $\mathcal{M} \simeq \varphi^*\mathcal{M}'$. Thus, by Theorem 8.1, we obtain that $\mathcal{L}'^{\otimes m(\dim X, a+1)}$ is π'-generated and $\mathcal{L}'^{\otimes l}$ is π'-generated for every $l \geq m_0(\dim X, a+1)$. This implies the desired effective freeness for $\mathcal{L} \simeq \varphi^*\mathcal{L}'$. □

Let us prove Theorem 8.3.

Sketch of Proof of Theorem 8.3. By assumption, $(a+1)\mathcal{L} - \omega$ is π-ample over S. We put

$$m' = m'(\dim X, a) := (\dim X + 1) \times m(\dim X, a+1),$$

where $m(\dim X, a+1)$ is a positive integer obtained in Theorem 8.1. Then, we can check that $\mathcal{L}^{\otimes m'}$ is π-very ample (see, for example, [Fu6, Lemma 4.1] and [Fu8, Lemma 7.1]). We put

$$m'_0 = m'_0(\dim X, a) := m_0(\dim X, a+1) + m'(\dim X, a),$$

where $m_0(\dim X, a+1)$ is a positive integer obtained in Theorem 8.1. Then, it is easy to see that $\mathcal{L}^{\otimes l'}$ is π-very ample for every $l' \geq m'_0$. □

Finally, we prove Theorem 1.4.

Proof of Theorem 1.4. We replace S with any relatively compact open subset of S. Then $[X, K_X+\Delta]$ naturally becomes a quasi-log complex analytic space with $\mathrm{Nqlc}(X, K_X+\Delta) = \emptyset$. By Theorems 8.1 and 8.2, there exists a positive integer m_0 depending only on $\dim X$ and a such that $\mathcal{L}^{\otimes m}$ is π-generated for every $m \geq m_0$. We note that m_0 is independent of S. Hence we see that $\mathcal{L}^{\otimes m}$ is π-generated for every $m \geq m_0$ without replacing S with a relatively compact open subset of S. This is what we wanted. □

9. Cone theorem

In this section, we will briefly see that the cone and contraction theorem holds for quasi-log complex analytic spaces. We note that we have already established it for normal pairs in [Fu15, Theorem 1.2]. Let us start with the rationality theorem for quasi-log complex analytic spaces.

Theorem 9.1 (Rationality theorem for quasi-log complex analytic spaces)**.** *Let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces and let W be a compact subset of S. Let $[X, \omega]$ be a quasi-log complex analytic space such that ω is a $\mathbb{Q}$-line bundle. Let H be a π-ample line bundle on X. Assume that ω is not π-nef over W and that r is a positive real number such that*

(i) *$H + r\omega$ is π-nef over W but is not π-ample over W, and*
(ii) *$(H + r\omega)|_{\mathrm{Nqlc}(X,\omega)}$ is $\pi|_{\mathrm{Nqlc}(X,\omega)}$-ample over W.*

Then r is a rational number, and in reduced form, it has denominator at most $a(d+1)$, where $d := \max_{w \in W} \dim \pi^{-1}(w)$ and a is a positive integer such that $a\omega$ is a line bundle in a neighborhood of $\pi^{-1}(W)$.

There are no difficulties to adapt the proof of [Fu15, Theorem 10.1] for Theorem 9.1. So we omit the proof of Theorem 9.1 here. By using the basepoint-free theorem (see Theorem 6.1) and the rationality theorem (see Theorem 9.1), we can establish the cone and contraction theorem for quasi-log complex analytic spaces.

Theorem 9.2 (Cone and contraction theorem for quasi-log complex analytic spaces)**.** *Let $[X, \omega]$ be a quasi-log complex analytic space and let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces. Let W be a compact subset of S. We assume that the dimension of $N^1(X/S; W)$ is finite. Then we have*

$$\overline{\mathrm{NE}}(X/S; W) = \overline{\mathrm{NE}}(X/S; W)_{\omega \geq 0} + \overline{\mathrm{NE}}(X/S; W)_{\mathrm{Nqlc}(X,\omega)} + \sum_j R_j$$

with the following properties.

(1) *$\mathrm{Nqlc}(X, \omega)$ is the non-qlc locus of $[X, \omega]$ and $\overline{\mathrm{NE}}(X/S; W)_{\mathrm{Nqlc}(X,\omega)}$ is the subcone of $\overline{\mathrm{NE}}(X/S; W)$ which is the closure of the convex cone spanned by the projective integral curves C on $\mathrm{Nqlc}(X, \omega)$ such that $\pi(C)$ is a point of W.*

(2) R_j *is an* ω*-negative extremal ray of* $\overline{\mathrm{NE}}(X/S;W)$ *which satisfies*

$$R_j \cap \overline{\mathrm{NE}}(X/S;W)_{\mathrm{Nqlc}(X,\omega)} = \{0\}$$

for every j.

(3) *Let* $\mathcal{A}$ *be a* π*-ample* $\mathbb{R}$*-line bundle on* X. *Then there are only finitely many* R_j*'s included in* $\overline{\mathrm{NE}}(X/S;W)_{(\omega+\mathcal{A})<0}$. *In particular, the* R_j*'s are discrete in the half-space* $\overline{\mathrm{NE}}(X/S;W)_{\omega<0}$.

(4) *Let* F *be any face of* $\overline{\mathrm{NE}}(X/S;W)$ *such that*

$$F \cap \left(\overline{\mathrm{NE}}(X/S;W)_{\omega\geq 0} + \overline{\mathrm{NE}}(X/S;W)_{\mathrm{Nqlc}(X,\omega)}\right) = \{0\}.$$

Then, after shrinking S *around* W *suitably, there exists a contraction morphism* $\varphi_F\colon X \to Z$ *over* S *satisfying the following properties.*

(i) *Let* C *be a projective integral curve on* X *such that* $\pi(C)$ *is a point of* W. *Then* $\varphi_F(C)$ *is a point if and only if the numerical equivalence class* $[C]$ *of* C *is in* F.

(ii) *The natural map* $\mathcal{O}_Z \to (\varphi_F)_*\mathcal{O}_X$ *is an isomorphism.*

(iii) *Let* $\mathcal{L}$ *be a line bundle on* X *such that* $\mathcal{L}\cdot C = 0$ *for every curve* C *with* $[C] \in F$. *Then, after shrinking* S *around* W *suitably again, there exists a line bundle* $\mathcal{L}_Z$ *on* Z *such that* $\mathcal{L} \simeq \varphi_F^*\mathcal{L}_Z$ *holds.*

Sketch of Proof of Theorem 9.2. The proof of the cone theorem for normal pairs in the complex analytic setting, which is described in [Fu15, Section 12], works with only some minor modifications since we have already established the basepoint-free theorem (see Theorem 6.1) and the rationality theorem (see Theorem 9.1) for quasi-log complex analytic spaces. Note that we can use Lemma 5.7 to reduce problems to the case where ω is a $\mathbb{Q}$-line bundle. □

As an immediate application of Theorem 9.2, we have:

Theorem 9.3 (Basepoint-freeness for $\mathbb{R}$-line bundles)**.** *Let* $\pi\colon X \to S$ *be a projective morphism between complex analytic spaces and let* W *be a compact subset of* S *such that the dimension of* $N^1(X/S;W)$ *is finite. Let* $[X,\omega]$ *be a quasi-log complex analytic space with* $X_{-\infty} = \emptyset$ *and let* $\mathcal{L}$ *be an* $\mathbb{R}$*-line bundle defined on some open neighborhood of* $\pi^{-1}(W)$ *such that* $\mathcal{L}$ *is* π*-nef over* W. *We assume that* $a\mathcal{L}-\omega$ *is* π*-ample over* W *for some positive real number* a. *Then there exists an open neighborhood* U *of* W *such that* $\mathcal{L}$ *is* π*-semi-ample over* U.

Sketch of Proof of Theorem 9.3. Without loss of generality, we may assume that $a = 1$ by replacing $\mathcal{L}$ with $a\mathcal{L}$. As in the proof of [Fu15, Theorem 15.1], we can write $\mathcal{L} = \sum_{i=1}^m r_i\mathcal{L}_i$ such that

- $\mathcal{L}_i$ is a $\mathbb{Q}$-line bundle for every i,
- r_i is a positive real number for every i with $\sum_{i=1}^m r_i = 1$, and
- $\mathcal{L}_i - \omega$ is π-ample over W for every i,

with the aid of the cone theorem (see Theorem 9.2). By the usual basepoint-free theorem (see Theorem 6.1), $\mathcal{L}_i$ is π-semi-ample over some open neighborhood of W for every i. This implies that $\mathcal{L}$ is a finite positive $\mathbb{R}$-linear combination of π-semi-ample line bundles over some open neighborhood U of W. This is what we wanted. □

We have a supplementary result, which is a generalization of [Fu15, Theorem 1.4].

Theorem 9.4. *Let $[X,\omega]$ be a quasi-log complex analytic space with $X_{-\infty}=\emptyset$, that is, $[X,\omega]$ is a quasi-log canonical pair. Let $\pi\colon X\to S$ be a projective morphism of complex analytic spaces and let W be a compact subset of S such that the dimension of $N^1(X/S;W)$ is finite. Suppose that $\pi\colon X\to S$ is decomposed as*

$$\pi\colon X \xrightarrow{f} S^\flat \xrightarrow{g} S$$

such that $S^\flat$ is projective over S. Let $\mathcal{A}_{S^\flat}$ be a g-ample line bundle on $S^\flat$. Let R be an $(\omega+(\dim X+1)f^*\mathcal{A}_{S^\flat})$-negative extremal ray of $\overline{\mathrm{NE}}(X/S;W)$. Then R is an ω-negative extremal ray of $\overline{\mathrm{NE}}\left(X/S^\flat;g^{-1}(W)\right)$, that is, $R\cdot f^*\mathcal{A}_{S^\flat}=0$.

Sketch of Proof of Theorem 9.4. We put $\mathcal{L}:=f^*\mathcal{A}_{S^\flat}$. We may assume that $\dim X\geq 1$. Since $\mathcal{L}$ is π-nef over W, R is an ω-negative extremal ray of $\overline{\mathrm{NE}}(X/S;W)$. After shrinking S around W suitably, we obtain a contraction morphism $\varphi_R\colon X\to Z$ over S associated to R by the cone and contraction theorem (see Theorem 9.2). It is sufficient to prove that $\mathcal{L}\cdot R=0$ holds. We will get a contradiction by supposing that $\mathcal{L}\cdot R>0$ holds. If there exists a positive-dimensional irreducible component X' of X such that $\varphi_R(X')$ is a point. Then, by applying the argument in Step 1 in the proof of [Fu15, Theorem 1.4] to $[X',\omega|_{X'}]$, we get a contradiction. Hence, we may assume that $\dim\varphi_R(X')\geq 1$ holds for every positive-dimensional irreducible component X' of X. By adjunction (see Theorem 4.4), $[X',\omega|_{X'}]$ is a quasi-log canonical pair. By considering $[X',\omega|_{X'}]$, we may assume that X is irreducible. We take a point $P\in Z$ with $\varphi_R^{-1}(P)\geq 1$. By Lemma 5.6, after shrinking Z around P suitably, we can take an effective $\mathbb{R}$-Cartier divisor G' on Z such that $[X,\omega+\varphi_R^*G']$ naturally becomes a quasi-log complex analytic space, there exists a positive-dimensional qlc center C of $[X,\omega+\varphi_R^*G']$ with $\varphi(C)=P$, $\dim\mathrm{Nqlc}(X,\omega+\varphi_R^*G')\leq 0$, and $\mathrm{Nqlc}(X,\omega+\varphi_R^*G')=\emptyset$ outside $\varphi_R^{-1}(P)$. Then, by adjunction (see Theorem 4.4), we can construct a projective irreducible quasi-log scheme $[X'',\omega'':=\omega|_{X''}]$ such that $\varphi_R(X'')=P$, $-\omega''$ is ample, $\dim X''\geq 1$, and $\dim X''_{-\infty}\leq 0$. By construction, $\mathcal{L}|_{X''}$ is ample and $\omega''+r\mathcal{L}|_{X''}$ is numerically trivial for some $r>\dim X+1$. This is a contradiction by [Fu15, Lemma 12.3]. Anyway, we obtain that $\mathcal{L}\cdot R=0$. This is what we wanted. □

As an obvious corollary of Theorem 9.4, we have:

Corollary 9.5. *Let $[X,\omega]$ be a quasi-log complex analytic space with $X_{-\infty}=\emptyset$, that is, $[X,\omega]$ is a quasi-log canonical pair. Let $\pi\colon X\to S$ be a projective morphism of complex analytic spaces and let $\mathcal{A}$ be any π-ample line bundle on X. Then $\omega+(\dim X+1)\mathcal{A}$ is always nef over S.*

Corollary 9.5 is a generalization of [Fu15, Corollary 1.5] (see also [Fu10, Corollaries 1.2 and 1.8]).

Proof of Corollary 9.5. Let $P\in S$ be any point. We put $W:=\{P\}$. Then we can check that the dimension of $N^1(X/S;W)$ is finite (see Remark 2.7). Assume that there exists an $(\omega+(\dim X+1)\mathcal{A})$-negative extremal ray R of $\overline{\mathrm{NE}}(X/S;W)$. We apply Theorem 9.4 by putting $S^\flat:=S$. Then we obtain $R\cdot\mathcal{A}=0$. This is a contradiction since $\mathcal{A}$ is ample. Therefore, there are no $(\omega+(\dim X+1)\mathcal{A})$-negative extremal rays. This implies that $\omega+(\dim X+1)\mathcal{A}$ is π-nef over W. Since P is any point of S, we obtain that $\omega+(\dim X+1)\mathcal{A}$ is nef over S. We finish the proof. □

9.1. **On ω-negative extremal rational curves.** The following results easily follow from [Fu9]. They generalize [Ka]. We explicitly state them here for the reader's convenience.

We think that [Fu9] shows that the framework of quasi-log structures is useful. We note that the results in this subsection depend on Mori's bend and break method.

Theorem 9.6 (see [Fu15, Theorem 13.1])**.** *Let $\varphi\colon X \to Z$ be a projective morphism of complex analytic spaces such that $[X, \omega]$ is a quasi-log complex analytic space. Assume that $-\omega$ is φ-ample. Let P be an arbitrary point of Z. Let E be any positive-dimensional irreducible component of $\varphi^{-1}(P)$ such that $E \not\subset \mathrm{Nqlc}(X, \omega)$. Then E is covered by possibly singular rational curves ℓ with*

$$0 < -\omega \cdot \ell \leq 2 \dim E.$$

In particular, E is uniruled.

Sketch of Proof of Theorem 9.6. If E is an irreducible component of X, then $[E, \omega|_E]$ is a projective quasi-log scheme by adjunction (see Theorem 4.4) and Lemma 5.3 since $\varphi(E) = P$. Hence, the statement follows from [Fu9, Theorem 1.12]. From now on, we assume that E is not an irreducible component of X. We take an irreducible component X' of X such that $E \subset X'$. By adjunction and Lemma 5.3 again, $[X', \omega|_{X'}]$ is a quasi-log complex analytic space. By replacing X with X', we may assume that X is irreducible. By Lemma 5.5, after shrinking Z around P suitably, we can take an effective $\mathbb{R}$-Cartier divisor G on Z such that $[X, \omega + \varphi^*G]$ naturally becomes a quasi-log complex analytic space and that E is a qlc center of $[X, \omega + \varphi^*G]$. By adjunction (see Theorem 4.4) and Lemma 5.3 (see also the proof of [Fu15, Theorem 13.1]), $[E, \omega|_E]$ is a projective quasi-log scheme such that $-\omega|_E$ is ample since $\varphi(E) = P$. Thus, by [Fu9, Theorem 1.12], we have the desired properties. □

Hence, we have:

Theorem 9.7 (Lengths of ω-negative extremal rational curves, see [Fu15, Theorem 13.2])**.** *Let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces such that $[X, \omega]$ is a quasi-log complex analytic space and let W be a compact subset of S such that the dimension of $N^1(X/S; W)$ is finite. If R is an ω-negative extremal ray of $\overline{\mathrm{NE}}(X/S; W)$ which is relatively ample at $\mathrm{Nqlc}(X, \omega)$, that is, $R \cap \overline{\mathrm{NE}}(X/S; W)_{\mathrm{Nqlc}(X,\omega)} = \{0\}$, then there exists a possibly singular rational curve ℓ spanning R with*

$$0 < -\omega \cdot \ell \leq 2 \dim X.$$

Sketch of Proof of Theorem 9.7. By the cone and contraction theorem (see Theorem 9.2), we have a contraction morphism $\varphi_R\colon X \to Z$ over S associated to R after shrinking S around W suitably. By construction, $-\omega$ is φ_R-ample and $\varphi_R\colon \mathrm{Nqlc}(X, \omega) \to \varphi_R(\mathrm{Nqlc}(X, \omega))$ is finite. Thus, by Theorem 9.6, we can find a rational curve ℓ in a fiber of φ_R with $0 < -\omega \cdot \ell \leq 2 \dim X$. This ℓ is a desired rational curve spanning R. □

Theorem 9.7 will play a crucial role in Subsection 10.2. We close this subsection with a complex analytic generalization of [Fu9, Theorem 1.14].

Theorem 9.8 (Rationally chain connectedness, see [Fu9, Theorem 1.14])**.** *Let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$ and let $[X, \omega]$ be a quasi-log complex analytic space. Assume that $-\omega$ is π-ample. Then $\pi^{-1}(P)$ is rationally chain connected modulo $\pi^{-1}(P) \cap X_{-\infty}$ for every point $P \in S$. In particular, if further $\pi^{-1}(P) \cap X_{-\infty} = \emptyset$ holds, that is, $[X, \omega]$ is quasi-log canonical in a neighborhood of $\pi^{-1}(P)$, then $\pi^{-1}(P)$ is rationally chain connected.*

Proof. There are no difficulties to adapt the arguments in [Fu9, Section 13] to our complex analytic setting here. For the details, see [Fu9, Section 13]. □

10. On analytic semi-log canonical pairs

In this section, we will explain how to use the framework of quasi-log complex analytic spaces for the study of semi-log canonical pairs. In [Fu4], the author proved that any quasi-projective semi-log canonical pair naturally has a quasi-log structure. The following theorem is a complex analytic generalization.

Theorem 10.1. *Let (X, Δ) be a semi-log canonical pair and let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces. Then, after replacing S with any relatively compact open subset of S, $[X, K_X + \Delta]$ naturally becomes a quasi-log complex analytic space such that $\mathrm{Nqlc}(X, K_X + \Delta) = \emptyset$ and that $C^\dagger$ is a qlc center of $[X, K_X + \Delta]$ if and only if $C^\dagger$ is a semi-log canonical center of (X, Δ).*

More precisely, we can construct a projective surjective morphism $f\colon (Z, \Delta_Z) \to X$ from an analytic globally embedded simple normal crossing pair (Z, Δ_Z) such that the natural map

$$\mathcal{O}_X \to f_*\mathcal{O}_Z(\lceil -(\Delta_Z^{<1})\rceil)$$

is an isomorphism and that $C^\dagger$ is the f-image of some stratum of (Z, Δ_Z) if and only if $C^\dagger$ is a semi-log canonical center of (X, Δ) or an irreducible component of X. Moreover, if every irreducible component of X has no self-intersection in codimension one, then we can make $f\colon Z \to X$ bimeromorphic.

Proof of Theorem 10.1. We take an arbitrary relatively compact open subset U of S. We will construct $f\colon (Z, \Delta_Z) \to X$ over U. In this proof, X^{ncp} denotes the largest open subset of X consisting of smooth points, double normal crossing points, and pinch points. Similarly, X^{snc} denotes the largest open subset of X consisting of smooth points and simple normal crossing points. Moreover, X^{snc2} is the largest open subset of X which has only smooth points and simple normal crossing points of multiplicity ≤ 2. We note that $\mathrm{Sing}\, X$ denotes the singular locus of X. From Step 1 to Step 8, we will explain how to construct a projective surjective morphism $f\colon (Z, \Delta_Z) \to X$ from an analytic globally embedded simple normal crossing pair (Z, Δ_Z) over U.

Step 1. By [BM2, Remark 1.6 and Theorem 1.18], after replacing S with any relatively compact open subset containing $\overline{U}$, we can take a morphism $f_1\colon X_1 \to X$, which is a finite composite of admissible blow-ups, such that

(i) $X_1 = X_1^{\mathrm{ncp}}$,
(ii) f_1 is an isomorphism over X^{ncp}, and
(iii) $\mathrm{Sing}\, X_1$ maps bimeromorphically onto the closure of $\mathrm{Sing}\, X^{\mathrm{ncp}}$.

We note that $X^{\mathrm{ncp}} = X^{\mathrm{snc2}}$ and $X_1 = X_1^{\mathrm{ncp}} = X_1^{\mathrm{snc2}}$ hold in the above construction when every irreducible component of X has no self-intersection in codimension one.

Step 2. By replacing S with any relatively compact open subset containing $\overline{U}$ again, X_1 is projective over S by construction. Hence we can embed X_1 into $S \times \mathbb{P}^N$ over S for some $\mathbb{P}^N$. We pick a finite set $\mathcal{P} \subset X_1$ such that each irreducible component of $\mathrm{Sing}\, X_1$ contains a point of $\mathcal{P}$. Moreover, we may assume that each irreducible component of $\mathrm{Sing}(\pi\circ f_1)^{-1}(U)$ contains a point of $\mathcal{P}$. After shrinking S around $\overline{U}$, we take a sufficiently large positive integer d such that $I_{X_1} \otimes \mathcal{O}(d)$ is globally generated, where I_{X_1} is the defining ideal sheaf of X_1 on $S \times \mathbb{P}^N$ and $\mathcal{O}(d) := p^*\mathcal{O}_{\mathbb{P}^N}(d)$ with the second projection $p\colon S \times \mathbb{P}^N \to \mathbb{P}^N$. We take a complete intersection of $(\dim S + N - \dim X - 1)$ general members of $|I_{X_1} \otimes \mathcal{O}(d)|$. Then we have $X_1 \subset Y$ such that Y is smooth at every point of $\mathcal{P}$. Note that here we used the fact that X_1 has only hypersurface singularities near $\mathcal{P}$.

Step 3. After replacing S with a relatively compact open subset containing $\overline{U}$ suitably, we take a resolution $g\colon Y_2 \to Y$, which is a finite composite of admissible blow-ups and is an isomorphism over the largest Zariski open subset of Y on which Y is smooth (see [BM1, Theorem 13.3]). Let X_2 be the strict transform of X_1 on Y_2. We note that $f_2 := g|_{X_2}\colon X_2 \to X_1$ is an isomorphism over general points of any irreducible component of $\operatorname{Sing} X_1$ because Y is smooth at every point of $\mathcal{P}$ by construction.

Step 4. By applying [BM2, Remark 1.6 and Theorem 1.18] to $X_2 \subset Y_2$, after replacing S with any relatively compact open subset containing $\overline{U}$, we have a finite composite of admissible blow-ups $g_3\colon Y_3 \to Y_2$ from a smooth variety Y_3 such that $X_3 = X_3^{\mathrm{ncp}}$ holds, where X_3 is the strict transform of X_2 on Y_3, and that $\operatorname{Sing} X_3$ maps bimeromorphically onto $\operatorname{Sing} X_1$ by $f_2 \circ f_3$, where $f_3 := g_3|_{X_3}\colon X_3 \to X_2$. We note that we can make $X_3 = X_3^{\mathrm{ncp}} = X_3^{\mathrm{snc2}}$ hold by [BM1, Theorems 13.3 and 12.4] when $X_1 = X_1^{\mathrm{ncp}} = X_1^{\mathrm{snc2}}$ hold.

Step 5. We put

$$K_{X_1} + \Delta_1 = f_1^*(K_X + \Delta)$$

and

$$K_{X_3} + \Delta_3 = (f_1 \circ f_2 \circ f_3)^*(K_X + \Delta).$$

Since X_1 and X_3 have only Gorenstein singularities, Δ_1 and Δ_3 are well-defined $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X_1 and X_3, respectively. By construction, the singular locus of X_1 (resp. X_3) does not contain any irreducible components of $\operatorname{Supp} \Delta_1$ (resp. $\operatorname{Supp} \Delta_3$).

Step 6. Let C be an irreducible component of $X_3 \setminus X_3^{\mathrm{snc}}$. Then C is smooth and $\dim C = \dim X_3 - 1$ since $X_3 = X_3^{\mathrm{ncp}}$ holds. Let $\alpha\colon W \to Y_3$ be the blow-up along C and let V be $\alpha^{-1}(X_3)$ with the reduced structure. Then we can check that $\beta_*\mathcal{O}_V \simeq \mathcal{O}_{X_3}$, where $\beta := \alpha|_V$. We put

$$K_V + \Delta_V = \beta^*(K_{X_3} + \Delta_3).$$

We can easily check that $K_V = \beta^* K_{X_3}$ and $\Delta_V = \beta^*\Delta_3$. When α is the blow-up along a pinch points locus C, see [Fu4, Lemma 4.4] for the local description of $\alpha\colon W \to Y_3$. When α is the blow-up along a double normal crossing points locus C, it is easy to understand $\alpha\colon W \to Y_3$. By repeating this process finitely many times and replacing S with any relatively compact open subset containing $\overline{U}$, we obtain a projective bimeromorphic morphism $g_4\colon Y_4 \to Y_3$ from a smooth variety Y_4 and a simple normal crossing divisor X_4 on Y_4 such that $f_{4*}\mathcal{O}_{X_4} \simeq \mathcal{O}_{X_3}$, where $f_4 := g_4|_{X_4}$. We put

$$K_{X_4} + \Delta_4 := f_4^*(K_{X_3} + \Delta_3).$$

If $X_3 = X_3^{\mathrm{snc2}}$ holds, then $X_3 \setminus X_3^{\mathrm{snc}}$ is empty. In this case, we have $Y_4 = Y_3$ and $X_4 = X_3$.

Step 7. We consider the following closed subset $\Sigma := \operatorname{Supp}(f_1 \circ f_2 \circ f_3 \circ f_4)^*\Delta$ of X_4. By [BM1, Theorems 13.3 and 12.4], after replacing S with any relatively compact open subset containing $\overline{U}$, we can construct a projective bimeromorphic morphism $g_5\colon Y_5 \to Y_4$ with the following properties.

(i) Let X_5 be the strict transform of X_4 on Y_5. Then $f_5 := g_5|_{X_5}\colon X_5 \to X_4$ is an isomorphism outside Σ with $f_{5*}\mathcal{O}_{X_5} \simeq \mathcal{O}_{X_4}$.

(ii) (X_5, Σ') is an analytic globally embedded simple normal crossing pair such that Σ' is reduced and contains $\operatorname{Supp} f_{5*}^{-1}\Delta_4$ and $\operatorname{Exc}(f_5)$, where $\operatorname{Exc}(f_5)$ is the exceptional locus of f_5.

Step 8. We replace S with U and put $M := Y_5$, $Z := X_5$, and $f := f_1\circ f_2\circ f_3\circ f_4\circ f_5\colon Z = X_5 \to X$. We define Δ_Z by

$$K_Z + \Delta_Z := f^*(K_X + \Delta).$$

By construction, f is a projective surjective morphism. Moreover, we see that f is a projective bimeromorphic morphism when every irreducible component of X has no self-intersection in codimension one.

Step 9. In this step, we will prove that $f_*\mathcal{O}_Z(\lceil -(\Delta_Z^{<1})\rceil) \simeq \mathcal{O}_X$ holds.

We first note that X satisfies Serre's S_2 condition and $\operatorname{codim}_X(X \setminus X^{\mathrm{ncp}}) \geq 2$ holds by assumption. Thus, we have $f_{1*}\mathcal{O}_{X_1} \simeq \mathcal{O}_X$. Since Δ is effective, $\lceil -(\Delta_1^{<1})\rceil$ is effective and f_1-exceptional, we see that $f_{1*}\mathcal{O}_{X_1}(\lceil -(\Delta_1^{<1})\rceil) \simeq \mathcal{O}_X$ holds. By construction, we can easily check that the following inclusions

$$\mathcal{O}_{X_1} \subset (f_2\circ f_3)_*\mathcal{O}_{X_3}(\lceil -(\Delta_3^{<1})\rceil) \subset \mathcal{O}_{X_1}(\lceil -(\Delta_1^{<1})\rceil)$$

hold. Therefore, we obtain

$$(f_1\circ f_2\circ f_3)_*\mathcal{O}_{X_3}(\lceil -(\Delta_3^{<1})\rceil) \simeq \mathcal{O}_X.$$

Let $\alpha\colon W \to Y_3$ be the blow-up in Step 6. When $\alpha\colon W\to Y_3$ is the blow-up along a pinch points locus, see [Fu4, Lemma 4.4] for the local description of α. When $\alpha\colon W \to Y_3$ is the blow-up along a double normal crossing points locus, it is easy to describe $\alpha\colon W \to Y_3$. Then we have

$$0 \leq \lceil -(\Delta_V^{<1})\rceil \leq \beta^*\left(\lceil -(\Delta_3^{<1})\rceil\right)$$

by $\Delta_V = \beta^*\Delta_3$. This implies

$$\mathcal{O}_{X_3} \subset \beta_*\mathcal{O}_V(\lceil -(\Delta_V^{<1})\rceil) \subset \mathcal{O}_{X_3}(\lceil -(\Delta_3^{<1})\rceil)$$

by $\beta_*\mathcal{O}_V \simeq \mathcal{O}_{X_3}$. By using it finitely many times, we get

$$\mathcal{O}_{X_3} \subset f_{4*}\mathcal{O}_{X_4}(\lceil -(\Delta_4^{<1})\rceil) \subset \mathcal{O}_{X_3}(\lceil -(\Delta_3^{<1})\rceil).$$

This implies

$$(f_1\circ f_2\circ f_3\circ f_4)_*\mathcal{O}_{X_4}(\lceil -(\Delta_4^{<1})\rceil) \simeq \mathcal{O}_X.$$

It is easy to see that

$$\mathcal{O}_{X_4} \subset f_{5*}\mathcal{O}_{X_5}(\lceil -(\Delta_5^{<1})\rceil) \subset \mathcal{O}_{X_4}(\lceil -(\Delta_4^{<1})\rceil).$$

Thus,

$$(f_1\circ f_2\circ f_3\circ f_4\circ f_5)_*\mathcal{O}_{X_5}(\lceil -(\Delta_5^{<1})\rceil) \simeq \mathcal{O}_X.$$

This means that

$$f_*\mathcal{O}_Z(\lceil -(\Delta_Z^{<1})\rceil) \simeq \mathcal{O}_X.$$

This is what we wanted.

Step 10. In this final step, we will see that $C^\dagger$ is a semi-log canonical center if and only if $C^\dagger$ is a qlc center of $[X, K_X + \Delta]$.

When f is bimeromorphic, the desired statement is almost obvious. Hence, we may assume that f is not bimeromorphic. In this case, we can directly check the above statement with the aid of [Fu4, Lemma 4.4].

We finish the proof. □

By Theorem 10.1, we can apply the results for quasi-log complex analytic spaces to semi-log canonical pairs. We see that the basepoint-free theorem and its variants (see Theorems 6.1, 7.1, 8.1, 8.2, and 8.3) hold true for semi-log canonical pairs. All the results established in Section 9 hold for semi-log canonical pairs. In particular, the cone and contraction theorem (see Theorem 9.2) holds for semi-log canonical pairs in the complex analytic setting.

10.1. **Vanishing theorems and torsion-freeness for semi-log canonical pairs.**

In this subsection, we will explicitly state the vanishing theorems and torsion-freeness for semi-log canonical pairs in the complex analytic setting for the reader's convenience.

Theorem 10.2 ([Fu4, Theorems 1.7 and 1.10])**.** *Let (X, Δ) be a semi-log canonical pair and let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces. Let D be a Cartier divisor on X, or a $\mathbb{Q}$-Cartier integral Weil divisor on X such that any irreducible component of* $\operatorname{Supp} D$ *is not contained in the singular locus of X. Assume that $D-(K_X+\Delta)$ is nef and log big over S with respect to (X, Δ). This means that $D-(K_X+\Delta)$ is nef over S and that $(D-(K_X+\Delta))|_C$ is big over $\pi(C)$ for every slc stratum C of (X, Δ). Then $R^i\pi_*\mathcal{O}_X(D)=0$ for every $i>0$.*

Sketch of Proof of Theorem 10.2. We take an arbitrary point $s \in S$. It is sufficient to prove $R^i\pi_*\mathcal{O}_X(D)=0$ for every $i>0$ on some open neighborhood of s. Hence, we can freely shrink S around s suitably. By [Ko2, 5.23], we can take a double cover $p\colon \widetilde{X} \to X$ and reduce the problem to the case where every irreducible component of X has no self-intersection in codimension one. Thus, we can construct a projective bimeromorphic morphism $f\colon (Z, \Delta_Z) \to X$ as in Theorem 10.1. By [BM1, Theorems 13.3 and 12.4], we may further assume that (Z, Σ) is an analytic globally embedded simple normal crossing pair with $\operatorname{Supp} f^*D \cup \operatorname{Supp} \Delta_Z \subset \Sigma$. Thus, by Theorem 3.5 (ii), we can apply the argument in the proof of [Fu4, Theorem 1.7 and Theorem 1.10]. □

Theorem 10.3 ([Fu4, Theorem 1.11 and Remark 5.2])**.** *Let (X, Δ) be a semi-log canonical pair and let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces. Let $\mathcal{L}$ be a line bundle on X. Let X' be a union of some slc strata of (X, Δ) with the reduced structure and let $\mathcal{I}_{X'}$ be the defining ideal sheaf of X' on X. Assume that $\mathcal{L}-(K_X+\Delta)$ is nef over S and $(\mathcal{L}-(K_X+\Delta))|_C$ is big over $\pi(C)$ for every slc stratum C of (X, Δ) which is not contained in X'. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{L})=0$ holds for every $i>0$. In particular, $R^i\pi_*\mathcal{L}=0$ holds for every $i>0$ when $X'=\emptyset$.*

Sketch of Proof of Theorem 10.3. We take an arbitrary point $s \in S$. It is sufficient to prove that $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{L})=0$ holds for every $i>0$ on some open neighborhood of s. By Theorem 10.1, after shrinking S around s suitably, we may assume that $[X, K_X+\Delta]$ is a quasi-log complex analytic space such that C is a qlc stratum of $[X, K_X+\Delta]$ if and only if C is an slc stratum of (X, Δ). Hence, we obtain $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{L})=0$ for every $i>0$ by Theorem 4.8. □

We close this subsection with the torsion-freeness, that is, the strict support condition, for semi-log canonical pairs.

Theorem 10.4 ([Fu4, Theorem 1.12])**.** *Let (X, Δ) be a semi-log canonical pair and let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces. Let D be a Cartier divisor on X, or a $\mathbb{Q}$-Cartier integral Weil divisor on X such that any irreducible component of*

Supp D is not contained in the singular locus of X. Assume that $D-(K_X+\Delta)$ is π-semi-ample. Then every associated subvariety of $R^i\pi_*\mathcal{O}_X(D)$ is the π-image of some slc stratum of (X,Δ) for $i=0$ and 1.

We make a very important remark on [Fu4, Theorem 1.12].

Remark 10.5 (Correction of [Fu4, Theorem 1.12])**.** In [Fu4, Theorem 1.12], we claim that every associated prime of $R^i\pi_*\mathcal{O}_X(D)$ is the generic point of the π-image of some slc stratum of (X,Δ) for *every i*. Unfortunately, however, the latter half of the proof of [Fu4, Theorem 1.12] is insufficient. In the proof of [Fu4, Theorem 1.12], the $\pi|_A$-image of some slc stratum of $(A,\Delta|_A)$ is not necessarily the π-image of some slc stratum of (X,Δ). Hence, the correct statement of [Fu4, Theorem 1.12] is that every associated prime of $R^i\pi_*\mathcal{O}_X(D)$ is the generic point of some slc stratum of (X,Δ) for $i=0$ and 1.

Sketch of Proof of Theorem 10.4. As in the proof of Theorem 10.2, the former half of the proof of [Fu4, Theorem 1.12] works with some minor modifications. Hence we omit the details here. □

10.2. **On Shokurov's polytopes for semi-log canonical pairs.** In this final subsection, we will briefly explain Shokurov's polytopes for semi-log canonical pairs in the complex analytic setting. Here, we need the results in Subsection 9.1. Therefore, the results in this subsection also depend on Mori's bend and break.

Let $\pi\colon X\to S$ be a projective morphism between complex analytic spaces such that X is equidimensional and let W be a compact subset of S. Let V be a finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals. We put

$$\mathcal{L}(V;\pi^{-1}(W)) := \{D\in V \mid (X,D) \text{ is semi-log canonical at } \pi^{-1}(W)\}.$$

Then, as we saw in 2.4, it is known that $\mathcal{L}(V;\pi^{-1}(W))$ is a rational polytope in V defined over the rationals.

Definition 10.6 (Extremal curves)**.** Let $\pi\colon X\to S$ be a projective morphism of complex analytic spaces and let W be a compact subset of S such that the dimension of $N^1(X/S;W)$ is finite. A curve Γ on X is called *extremal over W* if the following properties hold.

(i) Γ generates an extremal ray R of $\overline{\mathrm{NE}}(X/S;W)$.
(ii) There exists a π-ample line bundle $\mathcal{H}$ over some open neighborhood of W such that

$$\mathcal{H}\cdot\Gamma=\min_{\ell}\{\mathcal{H}\cdot\ell\},$$

where ℓ ranges over curves generating R.

By Theorem 10.1 with Theorem 9.7, we have:

Lemma 10.7. *Let $\pi\colon X\to S$ be a projective morphism of complex analytic spaces and let (X,Δ) be a semi-log canonical pair. Let W be a compact subset of S such that the dimension of $N^1(X/S;W)$ is finite. Let R be a $(K_X+\Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/S;W)$. If Γ is an extremal curve over W generating R, then*

$$0<-(K_X+\Delta)\cdot\Gamma\leq 2\dim X$$

holds.

Sketch of Proof of Lemma 10.7. By Theorem 10.1, we may assume that $[X, K_X + \Delta]$ is a quasi-log canonical pair by shrinking S around W suitably. Then, by Theorem 9.7, we see that there exists a rational curve ℓ spanning R such that

$$0 < -(K_X + \Delta) \cdot \ell \leq 2 \dim X.$$

Therefore, we obtain

$$0 < -(K_X + \Delta) \cdot \Gamma = (-(K_X + \Delta) \cdot \ell) \cdot \frac{\mathcal{H} \cdot \Gamma}{\mathcal{H} \cdot \ell} \leq 2 \dim X.$$

This is what we wanted. □

We have already established the following theorems for log canonical pairs in [Fu15, Section 14], which may be useful for the minimal model program with scaling.

Theorem 10.8. *Let $\pi \colon X \to S$ be a projective morphism of complex analytic spaces such that X is equidimensional and let W be a compact subset of S such that the dimension of $N^1(X/S; W)$ is finite. Let V be a finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals. We fix an $\mathbb{R}$-divisor $\Delta \in \mathcal{L}(V; \pi^{-1}(W))$, that is, $\Delta \in V$ and (X, Δ) is semi-log canonical at $\pi^{-1}(W)$. Then we can find positive real numbers α and δ, which depend on (X, Δ) and V, with the following properties.*

(1) *If Γ is any extremal curve over W and $(K_X + \Delta) \cdot \Gamma > 0$, then $(K_X + \Delta) \cdot \Gamma > \alpha$.*
(2) *If $D \in \mathcal{L}(V; \pi^{-1}(W))$, $\|D - \Delta\| < \delta$, and $(K_X + D) \cdot \Gamma \leq 0$ for an extremal curve Γ over W, then $(K_X + \Delta) \cdot \Gamma \leq 0$.*
(3) *Let $\{R_t\}_{t \in T}$ be any set of extremal rays of $\overline{\mathrm{NE}}(X/S; W)$. Then*
$$\mathcal{N}_T := \{D \in \mathcal{L}(V; \pi^{-1}(W)) \mid (K_X + D) \cdot R_t \geq 0 \text{ for every } t \in T\}$$
is a rational polytope in V. In particular,
$$\mathcal{N}^{\sharp}_{\pi}(V; W) := \{\Delta \in \mathcal{L}(V; \pi^{-1}(W)) \mid K_X + \Delta \text{ is nef over } W\}$$
is a rational polytope.

Sketch of Proof of Theorem 10.8. By Theorem 10.1, we may assume that $[X, K_X + \Delta]$ is a quasi-log canonical pair. Hence, we can use the cone and contraction theorem (see Theorem 9.2). Thus, we can apply the proof of [Fu15, Theorem 14.3] with some minor modifications. We note that we need Lemma 10.7 for the proof of (2) and (3). □

We close this subsection with the following theorem, which is well known when $\pi \colon X \to S$ is algebraic and X is a normal variety.

Theorem 10.9. *Let $\pi \colon X \to S$ be a projective morphism of complex analytic spaces such that X is equidimensional and let W be a compact subset of S such that the dimension of $N^1(X/S; W)$ is finite. Let (X, Δ) be a semi-log canonical pair and let H be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $(X, \Delta + H)$ is semi-log canonical and that $K_X + \Delta + H$ is nef over W. Then, either $K_X + \Delta$ is nef over W or there is a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/S; W)$ such that $(K_X + \Delta + \lambda H) \cdot R = 0$, where*

$$\lambda := \inf\{t \geq 0 \mid K_X + \Delta + tH \text{ is nef over } W\}.$$

Of course, $K_X + \Delta + \lambda H$ is nef over W.

Sketch of Proof of Theorem 10.9. As for Theorem 10.8, we can use the proof of [Fu15, Theorem 14.4] with only some minor modifications. So we omit the details here. □

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