MINIMAL MODEL PROGRAM FOR PROJECTIVE MORPHISMS BETWEEN COMPLEX ANALYTIC SPACES

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Abstract. We discuss the minimal model program for projective morphisms of complex analytic spaces. Roughly speaking, we show that the results obtained by Birkar–Cascini–Hacon–M[c]Kernan hold true for projective morphisms between complex analytic spaces. We also treat some related topics.

Contents

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1. Introduction

This paper is the first step of the minimal model program for projective morphisms of complex analytic spaces.

In [BCHM] and [HacM], Birkar, Cascini, Hacon, and McKernan established many important results on the minimal model program for quasi-projective kawamata log terminal pairs defined over the complex number field (see [BCHM, Theorems A, B, C, D, E, and F]). Thus we can run the minimal model program with scaling.

Theorem 1.1 (Minimal model program with scaling, see [BCHM, Corollary 1.4.2])**.** *Let $\pi\colon X \to Y$ be a projective morphism of normal quasi-projective varieties. Let (X, Δ) be a $\mathbb{Q}$-factorial kawamata log terminal pair, where $K_X + \Delta$ is $\mathbb{R}$-Cartier and Δ is π-big. Let C be an effective $\mathbb{R}$-divisor on X. If $K_X + \Delta + C$ is kawamata log terminal and π-nef, then we can run the $(K_X + \Delta)$-minimal model program over Y with scaling of C. The output of this minimal model program is a log terminal model (resp. a Mori fiber space) over Y when $K_X + \Delta$ is π-pseudo-effective (resp. not π-pseudo-effective).*

Hence we can easily check:

Theorem 1.2 ([BCHM, Theorem 1.2])**.** *Let (X, Δ) be a kawamata log terminal pair, where $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X \to Y$ be a projective morphism of quasi-projective varieties. If either Δ is π-big and $K_X + \Delta$ is π-pseudo-effective or $K_X + \Delta$ is π-big, then*

(1) *$K_X + \Delta$ has a log terminal model over Y,*
(2) *if $K_X + \Delta$ is π-big then $K_X + \Delta$ has a log canonical model over Y, and*
(3) *if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then*

$$R(X/Y, K_X + \Delta) := \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}\left(\lfloor m(K_X + \Delta) \rfloor\right)$$

is finitely generated as an $\mathcal{O}_Y$-algebra.

The main purpose of this paper is to generalize the results obtained in [BCHM] and [HacM] for projective morphisms of complex analytic spaces under some suitable assumptions. One of the main difficulties to translate [BCHM] and [HacM] into the complex analytic setting is to find a reasonable formulation. For the general understanding of the theory of minimal models in the complex analytic setting, see [KM, Example 2.17]. To the best knowledge of the author, the minimal model program for projective morphisms between complex analytic spaces is not discussed in standard literature.

Remark 1.3 (Minimal model program for compact Kähler threefolds)**.** In a series of papers (see [HP1], [HP2], and [CHP]), the theory of minimal models was generalized for compact Kähler threefolds (see also [HP3]). It is different from our direction and is another complex analytic generalization of the minimal model program. The minimal model theory for log surfaces in Fujiki's class $\mathcal{C}$ was described in [Fu12]. Based on the idea that the essence of the theory of minimal models is projectivity, we think that our formulation in this paper is more natural than the minimal model program for compact Kähler varieties.

Let us see an easy example.

Example 1.4. Let $\{P_k\}_{k \in \mathbb{N}}$ be a set of mutually distinct discrete points of $Y := \mathbb{C}^2$ and let $\pi\colon X \to Y$ be the blow-up whose center is $\{P_k\}_{k \in \mathbb{N}}$. We put $E_k := \pi^{-1}(P_k)$ for every k. Since the line bundle $\mathcal{O}_X\left(-\sum_{k \in \mathbb{N}} E_k\right)$ is π-ample, π is a projective bimeromorphic morphism of smooth complex surfaces. In this case, there are infinitely many mutually disjoint π-exceptional curves on X. Hence, there exists no naive generalization of the minimal model program working for this projective bimeromorphic morphism $\pi\colon X \to Y$.

By Example 1.4, it seems to be reasonable and indispensable to fix a compact subset W of Y with some good properties and only treat $\pi\colon X \to Y$ over some open neighborhood of W. We need some finiteness condition in order to formulate the minimal model program for projective morphisms of complex analytic spaces. In this paper, we will mainly consider a projective morphism $\pi\colon X \to Y$ of complex analytic spaces and a compact subset W of Y with the following properties:

(P1) X is a normal complex variety,
(P2) Y is a Stein space,
(P3) W is a Stein compact subset of Y, and
(P4) $W \cap Z$ has only finitely many connected components for any analytic subset Z which is defined over an open neighborhood of W.

Since we are trying to discuss the minimal model program, (P1) is indispensable. So we almost always assume that X is a normal complex variety. We note that the Steinness of Y, that is, (P2), is a substitute of the quasi-projectivity of Y in [BCHM] and that the quasi-projectivity of Y is indispensable in [BCHM]. Let $\mathcal{F}$ be a coherent sheaf on X such that $\pi_*\mathcal{F} \neq 0$. Then we see that $\Gamma(X, \mathcal{F}) = \Gamma(Y, \pi_*\mathcal{F}) \neq 0$ holds by Cartan's Theorem A. Here we need the Steinness of Y. We also note that the analytic space naturally associated to an affine scheme is Stein. Let us explain (P3) and (P4). A compact subset on an analytic space is said to be *Stein compact* if it admits a fundamental system of Stein open neighborhoods. Note that a holomorphically convex hull $\widehat{K}$ of a compact subset K on a Stein space is a Stein compact subset. Therefore, we can find many Stein compact subsets on a given Stein space. If W satisfies (P3) and we are only interested in objects defined over some open neighborhood of W, then we can freely replace Y with a small Stein open neighborhood of W. The condition in (P4) is not so easy to understand and looks somewhat artificial. However, it is a very natural condition. It is known that a Stein compact subset W satisfies (P4) if and only if $\Gamma(W, \mathcal{O}_Y)$ is noetherian by Siu's theorem (see [Si, Theorem 1]). If W is a Stein compact semianalytic subset, then it satisfies (P3) and (P4). Thus we see that there are many Stein compact subsets satisfying (P4) on a given Stein space Y. We consider the free abelian group $Z_1(X/Y; W)$ generated by the projective integral curves C on X such that $\pi(C)$ is a point of W. We take $C_1, C_2 \in Z_1(X/Y; W) \otimes_{\mathbb{Z}} \mathbb{R}$. If $C_1 \cdot \mathcal{L} = C_2 \cdot \mathcal{L}$ holds for every $\mathcal{L} \in \operatorname{Pic}(\pi^{-1}(U))$ and every open neighborhood U of W, then we write $C_1 \equiv_W C_2$. We set $N_1(X/Y; W) := Z_1(X/Y; W) \otimes_{\mathbb{Z}} \mathbb{R}/ \equiv_W$. Then, by (P4), we can check that $N_1(X/Y; W)$ is a finite-dimensional $\mathbb{R}$-vector space (see [Na3, Chapter II. 5.19. Lemma]). When $N_1(X/Y; W)$ is finite-dimensional, we can define the *Kleiman–Mori cone* $\overline{\mathrm{NE}}(X/Y; W)$ in $N_1(X/Y; W)$ for $\pi\colon X \to Y$ and W, that is, $\overline{\mathrm{NE}}(X/Y; W)$ is the closure of the convex cone in $N_1(X/Y; W)$ generated by projective integral curves C on X such that $\pi(C)$ is a point in W. Without any difficulties, we can establish Kleiman's ampleness criterion in the complex analytic setting. We further assume that there exists an $\mathbb{R}$-divisor Δ on X such that (X, Δ) is kawamata log terminal. Then we can formulate the cone theorem as usual

$$\overline{\mathrm{NE}}(X/Y; W) = \overline{\mathrm{NE}}(X/Y; W)_{K_X+\Delta \geq 0} + \sum_j R_j$$

and prove the contraction theorem for each $(K_X + \Delta)$-negative extremal ray R_j over some open neighborhood of W. This is essentially due to Nakayama (see [Na2]). By the above observations, we recognize that (P1), (P2), (P3), and (P4) are reasonable. From now on, we usually simply say that $\pi\colon X\to Y$ and W satisfies (P) if it satisfies (P1), (P2), (P3), and (P4).

Remark 1.5. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Then $D\geq 0$ on $\overline{\mathrm{NE}}(X/Y;W)$ means that $D\cdot C\geq 0$ for every projective integral curve C on X such that $\pi(C)$ is a point in W. Unfortunately, however, D is not necessarily nef over some open neighborhood of W even when $D\geq 0$ on $\overline{\mathrm{NE}}(X/Y;W)$. This fact often causes troublesome problems when we discuss the minimal model program for projective morphisms between complex analytic spaces.

In this paper, we will prove:

Theorem 1.6 (Main theorem, see Theorem 1.13 below). *Let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P). *Then Theorems A, B, C, D, E, and F in* [BCHM] *hold true with some suitable modifications.*

For the precise statement, see Theorems A, B, C, D, E, and F in Subsection 1.1 and Theorem 1.13 below. To the best knowledge of the author, our results are new even in dimension three. For projective morphisms of complex analytic spaces, the minimal model program with scaling (see Theorem 1.1) becomes as follows.

Theorem 1.7 (Minimal model program with scaling for projective morphisms of complex analytic spaces, see [BCHM, Corollary 1.4.2]). *Let $\pi\colon X\to Y$ be a projective surjective morphism of complex analytic spaces and let W be a compact subset of Y. Assume that Y is Stein and that W is a Stein compact subset of Y such that $\Gamma(W,\mathcal{O}_Y)$ is noetherian. Let (X,Δ) be a kawamata log terminal pair such that Δ is π-big and that X is $\mathbb{Q}$-factorial over W. If C is an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta+C$ is kawamata log terminal and it is nef over W. Then we can run the $(K_X+\Delta)$-minimal model program with scaling of C over Y around W. More precisely, we have a finite sequence of flips and divisorial contractions over Y starting from (X,Δ):*

$$(X,\Delta)=:(X_0,\Delta_0)\overset{\varphi_0}{\dashrightarrow}(X_1,\Delta_1)\overset{\varphi_1}{\dashrightarrow}\cdots\overset{\varphi_{m-1}}{\dashrightarrow}(X_m,\Delta_m),$$

where $\Delta_{i+1}:=(\varphi_i)_*\Delta_i$ for every $i\geq 0$, such that (X_m,Δ_m) is a log terminal model (*resp. has a Mori fiber space structure*) *over some open neighborhood of W when $K_X+\Delta$ is π-pseudo-effective* (*resp. not π-pseudo-effective*). *We note that each step φ_i exists only after shrinking Y around W suitably. Hence we have to replace Y with a small Stein open neighborhood of W repeatedly in the above process.*

By Theorem 1.7, we can prove:

Theorem 1.8 ([BCHM, Theorem 1.2]). *Let (X,Δ) be a kawamata log terminal pair, where $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P). *If either Δ is π-big and $K_X+\Delta$ is π-pseudo-effective or $K_X+\Delta$ is π-big, then*

(1) *$K_X+\Delta$ has a log terminal model over some open neighborhood of W,*
(2) *if $K_X+\Delta$ is π-big then $K_X+\Delta$ has a log canonical model over some open neighborhood of W, and*
(3) *if $K_X+\Delta$ is $\mathbb{Q}$-Cartier, then*

$$R(X/Y,K_X+\Delta)=\bigoplus_{m\in\mathbb{N}}\pi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra.

Of course, Theorem 1.8 is an analytic version of Theorem 1.2. We note that this paper is not self-contained. If the proof is essentially the same as the original one for quasi-projective varieties, then we will only explain how to modify arguments in [BCHM] and [HacM] in order to make them work for projective morphisms between analytic spaces satisfying (P). In this paper, we always assume that complex analytic spaces are *Hausdorff* and *second-countable*.

1.9 (Motivation)**.** Let us explain the motivation of this paper. We sometimes have to consider the following setting when we study degenerations of algebraic varieties. Let $\pi\colon X \to \Delta$ be a projective morphism from a complex manifold X onto a disc $\Delta = \{z \in \mathbb{C} \,|\, |z| < 1\}$ with connected fibers. Suppose that π is smooth over $\Delta \setminus \{0\}$ and $K_{X_z} \sim_{\mathbb{Q}} 0$ for every $z \in \Delta \setminus \{0\}$, where $X_z = \pi^{-1}(z)$, and that π^*0 is a reduced simple normal crossing divisor on X. Roughly speaking, $\pi\colon X \setminus \pi^{-1}(0) \to \Delta \setminus \{0\}$ is a smooth projective family of Calabi–Yau manifolds and $\pi\colon X \to \Delta$ is a semistable degeneration. Since Δ is not a quasi-projective algebraic variety, we can not directly use [BCHM] for the study of $\pi\colon X \to \Delta$. By using the results established in this paper, after slightly shrinking Δ around 0 repeatedly, we can construct a finite sequence of flips and divisorial contractions starting from X:

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_m$$

over Δ such that X_m is a minimal model of X over Δ. More precisely, X_m has only terminal singularities and K_{X_m} is $\mathbb{Q}$-linearly trivial. The above result is a typical application of our result in this paper, which is not covered by [BCHM]. It is a complex analytic generalization of the semistable minimal model program established in [Fu4].

1.10 (Background, see [Fu5, 3.5, 3.6])**.** In the traditional framework of the minimal model program, the most important and natural object is a quasi-projective kawamata log terminal pair (see [KMM], [KM], [Matk], [BCHM], [HacM], and so on). From the Hodge theoretic viewpoint, we think that there exists the following correspondence.

Kawamata log terminal pairs	$\Longleftrightarrow$	Pure Hodge structures

We have already used mixed Hodge structures on cohomology with compact support systematically for the study of minimal models (see [Fu5], [Fu9], and so on). Then we succeeded in greatly expanding the framework of the minimal model program. Roughly speaking, we established the following correspondence.

Quasi-log schemes	$\Longleftrightarrow$	Mixed Hodge structures

For the details of this direction, see also [Fu13], [FFL], [Fu14], and so on.

On the other hand, from the complex analytic viewpoint, we know the following correspondence.

Kawamata log terminal pairs	$\Longleftrightarrow$	L^2 condition

Hence, it is natural to think that we can generalize the minimal model program for quasi-projective kawamata log terminal pairs established in [BCHM] and [HacM] to the one in the complex analytic setting. We note that the projectivity plays a crucial role in the theory of minimal models. Therefore, it is reasonable to discuss the minimal model program for projective morphisms between complex analytic spaces. This naive idea is now realized in this paper.

Unfortunately, we have not established complex analytic methods to treat varieties whose singularities are worse than kawamata log terminal singularities yet. Thus, it is a challenging problem to consider some analytic generalization of the theory of quasi-log schemes (see [Fu9, Chapter 6] and [Fu15]).

1.11 (How to use (P))**.** Before we see the main results in Subsection 1.1, let us explain how to use (P) for the reader's convenience. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies (P). We usually consider an $\mathbb{R}$-divisor Δ on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. It sometimes happens that some properties hold true only over an open neighborhood U of W. Since W is a Stein compact subset of Y, we can always take a relatively compact Stein open neighborhood Y' of W in Y satisfying

$$W \subset Y' \Subset U \subset Y.$$

Of course, $\pi'\colon X'\to Y'$ and W satisfies (P), where $X' := \pi^{-1}(Y')$ and $\pi' := \pi|_{X'}$. We frequently replace $\pi\colon X \to Y$ with $\pi'\colon X'\to Y'$ without mentioning it explicitly. We note that the support of Δ is only locally finite by definition. In general, the support of Δ may have infinitely many irreducible components. By construction, X' is a relatively compact open subset of X. Hence the support of $\Delta|_{X'}$ is finite. On the other hand, let V be a relatively compact open neighborhood of W in Y. Since Y is Stein, we can take an Oka–Weil domain V' satisfying

$$W \subset V \subset \overline{V} \subset V' \subset \overline{V'} \subset Y.$$

By construction, V' can be seen as a complex analytic subspace of a polydisc. Hence we can take a semianalytic Stein compact subset W' such that

$$W \subset V \subset \overline{V} \subset W' \subset V' \subset \overline{V'} \subset Y.$$

Note that W' satisfies (P4) since it is semianalytic. Therefore, $\pi\colon X\to Y$ and W' satisfies (P) and W' contains a given relatively compact open neighborhood V of W. This argument is useful and indispensable when we try to check that some properties hold true over an open neighborhood of W. For example, when we prove that $K_X+\Delta$ is nef over some open neighborhood of W, we sometimes have to consider $\pi\colon X\to Y$ and W'.

1.1. **Main results.**

Here, we state the main results of this paper. The following theorems look very similar to those in [BCHM] although they treat complex analytic spaces.

Theorem A (Existence of pl-flips, [BCHM, Theorem A])**.** *Let $\varphi\colon X\to Z$ be an analytic pl-flipping contraction for a purely log terminal pair (X,Δ). Then the flip $\varphi^+\colon X^+\to Z$ of φ always exists.*

Theorem B (Special finiteness, [BCHM, Theorem B])**.** *Let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P). *Suppose that X is $\mathbb{Q}$-factorial over W. Let V be a finite-dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$*, which is defined over the rationals, let S be the sum of finitely many prime divisors and let A be a general π-ample $\mathbb{Q}$-divisor on X such that the number of the irreducible components of* $\operatorname{Supp} A$ *is finite. Let (X,Δ_0) be a divisorial log terminal pair such that $S\leq \Delta_0$. We fix a finite set $\mathfrak{C}$ of prime divisors on X. Then, after shrinking Y around W suitably, there are finitely many bimeromorphic maps $\phi_i\colon X \dashrightarrow Z_i$ over Y for $1\leq i\leq k$ with the following property. If U is an open neighborhood of W and $\phi\colon \pi^{-1}(U)\dashrightarrow Z$ is any weak log canonical model over W of*

$(K_X+\Delta)|_{\pi^{-1}(U)}$ such that Z is $\mathbb{Q}$-factorial over W, where $\Delta \in \mathcal{L}_{S+A}(V;\pi^{-1}(W))$, which only contracts elements of $\mathfrak{C}$ and which does not contract every component of S, then there exists an index $1 \le i \le k$ such that, after shrinking Y and U around W suitably, the induced bimeromorphic map $\xi\colon Z_i \dashrightarrow Z$ is an isomorphism in a neighborhood of the strict transform of S.

Theorem C (Existence of log terminal models, [BCHM, Theorem C]). *Let $\pi\colon X \to Y$ be a projective surjective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P). *Suppose that (X,Δ) is kawamata log terminal and that Δ is big over Y. If there exists an $\mathbb{R}$-divisor D on X such that $K_X+\Delta \sim_{\mathbb{R}} D \ge 0$, then (X,Δ) has a log terminal model over some open neighborhood of W.*

Theorem D (Nonvanishing theorem, [BCHM, Theorem D]). *Let (X,Δ) be a kawamata log terminal pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that Y is Stein. Assume that Δ is big over Y and that $K_X+\Delta$ is pseudo-effective over Y. Let U be any relatively compact Stein open subset of Y. Then there exists a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on $\pi^{-1}(U)$ such that $(K_X+\Delta)|_{\pi^{-1}(U)} \sim_{\mathbb{R}} D \ge 0$.*

Theorem E (Finiteness of models, [BCHM, Theorem E]). *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P). *We fix a general π-ample $\mathbb{Q}$-divisor $A \ge 0$ on X such that the number of the irreducible components of* $\operatorname{Supp} A$ *is finite. Let V be a finite-dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$ *which is defined over the rationals. Suppose that there is a kawamata log terminal pair (X,Δ_0). Then, after shrinking Y around W suitably, there are finitely many bimeromorphic maps $\psi_j\colon X \dashrightarrow Z_j$ over Y for $1 \le j \le l$ with the following property. If U is an open neighborhood of W and $\psi\colon \pi^{-1}(U) \dashrightarrow Z$ is a weak log canonical model of $(K_X+\Delta)|_{\pi^{-1}(U)}$ over W for some $\Delta \in \mathcal{L}_A(V;\pi^{-1}(W))$, then there exists an index $1 \le j \le l$ and an isomorphism $\xi\colon Z_j \to Z$ such that $\psi = \xi \circ \psi_j$ after shrinking Y and U around W suitably.*

Theorem F (Finite generation, [BCHM, Theorem F]). *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P). *Let $(X,\Delta = A+B)$ be a kawamata log terminal pair, where $A \ge 0$ is a π-ample $\mathbb{Q}$-divisor and $B \ge 0$. We assume that the number of the irreducible components of* $\operatorname{Supp}\Delta$ *is finite. If $K_X+\Delta$ is pseudo-effective over Y, then*

(1) *After shrinking Y around W suitably, the pair (X,Δ) has a log terminal model $\mu\colon X \dashrightarrow Z$ over Y. In particular if $K_X+\Delta$ is $\mathbb{Q}$-Cartier, then the log canonical ring*
$$R(X/Y, K_X+\Delta) = \bigoplus_{m\in\mathbb{N}} \pi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)$$
is a locally finitely generated graded $\mathcal{O}_Y$-algebra.

(2) *Let $V \subset \mathrm{WDiv}_{\mathbb{R}}(X)$ be the vector space spanned by the components of Δ. Then, after shrinking Y around W suitably, there is a constant $\delta > 0$ such that if G is a prime divisor contained in the stable base locus of $K_X+\Delta$ over Y and $\Xi \in \mathcal{L}_A(V;\pi^{-1}(W))$ such that $\|\Xi-\Delta\| < \delta$, then G is contained in the stable base locus of $K_X+\Xi$ over Y.*

(3) *Let $V' \subset V$ be the smallest affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$ *containing Δ, which is defined over the rationals. Then, after shrinking Y around W suitably, there exists a constant $\eta > 0$ and a positive integer $r > 0$ such that if $\Xi \in V'$ is any divisor*

and k is any positive integer such that $\|\Xi - \Delta\| < \eta$ and $k(K_X + \Xi)/r$ is Cartier, then every component of $\mathrm{Fix}(k(K_X + \Xi))$ *is a component of the stable base locus of $K_X + \Delta$ over Y.*

1.2. **How to prove the main results.** The formulation of Theorem C is not appropriate for our inductive treatment of the main results in this paper. Therefore, we prepare a somewhat artificial statement, which is a slight generalization of Theorem C. We will use it instead of Theorem C in the inductive proof of the main results.

Theorem G (Existence of good log terminal models)**.** *Let $\pi\colon X \to Y$ be a projective surjective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Assume that $\pi\colon X \to Y^\flat \to Y$ such that $Y^\flat$ is projective over Y. Suppose that (X, Δ) is kawamata log terminal and that Δ is big over Y. If there exists an $\mathbb{R}$-divisor D on X such that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$, then, after shrinking Y around W suitably, there exists a bimeromorphic contraction $\phi\colon X \dashrightarrow Z$ over $Y^\flat$ such that ϕ is $(K_X + \Delta)$-negative, Z is $\mathbb{Q}$-factorial over W, $K_Z + \Gamma$ is semiample over $Y^\flat$, where $\Gamma = \phi_*\Delta$. This means that (Z, Γ) is a good log terminal model of (X, Δ) over $Y^\flat$.*

As an obvious remark, we have:

Remark 1.12. If we put $Y^\flat = Y$ in Theorem G, then we obtain Theorem C as a special case of Theorem G. Therefore, it is sufficient to prove Theorem G.

Note that Theorem A_n refers to Theorem A in the case when the dimension of X is n. In [BCHM] and [HacM], Theorem A, Theorem B, Theorem C, Theorem D, Theorem E, and Theorem F were proved by induction on n as follows.

- Theorem F_{n-1} implies Theorem A_n.
- Theorem E_{n-1} implies Theorem B_n.
- Theorem A_n and Theorem B_n imply Theorem C_n.
- Theorem D_{n-1}, Theorem B_n and Theorem C_n imply Theorem D_n.
- Theorem C_n and Theorem D_n imply Theorem E_n.
- Theorem C_n and Theorem D_n imply Theorem F_n.

Our strategy in this paper is essentially the same as that of [BCHM]. However, it is slightly simpler. We first note that we can easily check:

- Theorem D_n holds true for arbitrary n.
- Theorem G_n implies Theorem C_n for arbitrary n.

Hence it is sufficient to prove Theorem A, Theorem B, Theorem E, Theorem F, and Theorem G by induction on n as follows.

- Theorem F_{n-1} implies Theorem A_n.
- Theorem E_{n-1} implies Theorem B_n.
- Theorem A_n and Theorem B_n imply Theorem G_n.
- Theorem G_n implies Theorem E_n.
- Theorem G_n implies Theorem F_n.

Although there are some new difficulties, the proof of each step is similar to the original one in [BCHM] and [HacM]. Precisely speaking, we make great efforts to find a suitable formulation in order to make the original proof work with only some minor modifications.

The correct statement of Theorem 1.6 should be:

Theorem 1.13 (Main theorem)**.** *Theorems A, B, C, D, E, F, and G hold true in any dimension.*

1.3. **Some other results.** We have already known that many results follow from [BCHM]. We can prove that some of them hold true even in the complex analytic setting if we take some suitable modifications.

Once we have Theorem G, it is not difficult to prove the existence of kawamata log terminal flips in the complex analytic setting.

Theorem 1.14 (Existence of kawamata log terminal flips)**.** *Let (X, Δ) be a kawamata log terminal pair. Let $\varphi\colon X \to Z$ be a small projective surjective morphism of normal complex varieties. Then the flip $\varphi^+\colon X^+ \to Z$ of φ always exists. This means that there exists the following commutative diagram:*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi^+} & \\ & Z & \end{array}$$

where

(1) *$\varphi^+\colon X^+ \to Z$ is a small projective morphism of normal complex varieties, and*
(2) *$K_{X^+} + \Delta^+$ is φ^+-ample, where $\Delta^+ := \phi_*\Delta$.*

Note that (X^+, Δ^+) is automatically a kawamata log terminal pair.

Remark 1.15. Theorem 1.14 generalizes Mori's flip theorem (see [Mo3, (0.4.1) Flip Theorem]). Roughly speaking, Mori coarsely classified three-dimensional flipping contractions analytically and checked the existence of three-dimensional terminal flips.

The next one is a result on partial resolutions of singularities for complex varieties.

Theorem 1.16 (Existence of canonicalizations)**.** *Let X be a complex variety. Then there exists a projective bimeromorphic morphism $f\colon Z \to X$, which is the identity map over a nonempty Zariski open subset where X has only canonical singularities, from a normal complex variety Z with only canonical singularities such that K_Z is f-ample.*

If $K_X + \Delta$ is not pseudo-effective over Y, then we can run the minimal model program with scaling to get a Mori fiber space.

Theorem 1.17 ([BCHM, Corollary 1.3.3])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Let (X, Δ) be a divisorial log terminal pair such that X is $\mathbb{Q}$-factorial over W. Suppose that $K_X + \Delta$ is not pseudo-effective over Y. Then we can run a $(K_X + \Delta)$-minimal model program and finally obtain a Mori fiber space over some open neighborhood of W.*

We can prove the finite generation theorem for kawamata log terminal pairs in full generality in the complex analytic setting.

Theorem 1.18 ([BCHM, Corollary 1.1.2])**.** *Let (X, Δ) be a kawamata log terminal pair, where $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Then*

$$R(X/Y, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} \pi_*\mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra.

The original algebraic version of Theorem 1.19 below was first obtained in [Fu4] as an application of [BCHM].

Theorem 1.19 ([Fu4, Theorem 1.3])**.** *Let (X, Δ) be a divisorial log terminal pair and let $\pi\colon X \to Y$ be a projective morphism onto a disc $Y = \{z \in \mathbb{C} \,|\, |z| < 1\}$ with connected fibers. Assume that $(K_X + \Delta)|_F \sim_{\mathbb{R}} 0$ holds for an analytically sufficiently general fiber F of π. We further assume that W is a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian and that X is $\mathbb{Q}$-factorial over W. Then we can run the $(K_X + \Delta)$-minimal model program over Y in a neighborhood of W with ample scaling. More precisely, we have a finite sequence of flips and divisorial contractions over Y starting from (X, Δ):*

$$(X, \Delta) =: (X_0, \Delta_0) \overset{\varphi_0}{\dashrightarrow} (X_1, \Delta_1) \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{m-1}}{\dashrightarrow} (X_m, \Delta_m),$$

where $\Delta_{i+1} := (\varphi_i)_*\Delta_i$ for every $i \geq 0$, such that (X_m, Δ_m) is a log terminal model over some open neighborhood of W. Moreover, $K_{X_m} + \Delta_m \sim_{\mathbb{R}} (\pi_m)^*D$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on Y after shrinking Y around W suitably, where $\pi_m\colon X_m \to Y$. We note that each step φ_i exists only after shrinking Y around W suitably.

Remark 1.20. In Theorem 1.19, $W = \{z \in \mathbb{C} \,|\, |z| \leq r\}$ for $0 \leq r < 1$ is a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian.

The following theorem is an analytic version of *dlt blow-ups*. In the recent developments of the minimal model theory for higher-dimensional algebraic varieties, dlt blow-ups are very useful and important.

Theorem 1.21 (Dlt blow-ups, I)**.** *Let X be a normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let U be any relatively compact Stein open subset of X and let V be any relatively compact open subset of U. Then we can take a Stein compact subset W of U such that $\Gamma(W, \mathcal{O}_X)$ is noetherian, $V \subset W$, and, after shrinking X around W suitably, we can construct a projective bimeromorphic morphism $f\colon Z \to X$ from a normal complex variety Z with the following properties:*

1. *Z is $\mathbb{Q}$-factorial over W,*
2. *$a(E, X, \Delta) \leq -1$ for every f-exceptional divisor E on Z, and*
3. *$\left(Z, \Delta_Z^{<1} + \operatorname{Supp} \Delta_Z^{\geq 1}\right)$ is divisorial log terminal, where $K_Z + \Delta_Z = f^*(K_X + \Delta)$.*

Note that if (X, Δ) is log canonical then $\Delta_Z = \Delta_Z^{<1} + \operatorname{Supp} \Delta_Z^{\geq 1}$ holds.

We can use Theorem 1.21 for the study of log canonical singularities, which are not necessarily algebraic. The following result is a generalization of Theorem 1.18 for Kähler manifolds. When Y is a point, Theorem 1.22 is [Fu8, Theorem 1.8].

Theorem 1.22 (Finite generation for Kähler manifolds, see [Fu8])**.** *Let $\pi\colon X \to Y$ be a proper morphism from a Kähler manifold X to a complex analytic space Y. Let Δ be an effective $\mathbb{Q}$-divisor on X such that $\lfloor \Delta \rfloor = 0$ and that $\operatorname{Supp} \Delta$ is a simple normal crossing divisor on X. Then*

$$R(X/Y, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra.

The assumption that $\lfloor \Delta \rfloor = 0$ holds in Theorem 1.22 is very important. The following conjecture is widely open even when $\pi\colon X \to Y$ is a projective morphism between quasi-projective varieties (see [FG2]).

Conjecture 1.23. *Let $\pi\colon X \to Y$ be a proper morphism from a Kähler manifold X to a complex analytic space Y. Let Δ be a boundary $\mathbb{Q}$-divisor on X such that* $\operatorname{Supp}\Delta$ *is a simple normal crossing divisor on X. Then*

$$R(X/Y, K_X+\Delta)=\bigoplus_{m\in\mathbb{N}}\pi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra.

Let X be a normal complex variety and let $L \subset K$ be compact subsets of X. Assume that X is $\mathbb{Q}$-factorial at K. Unfortunately, however, X is not necessarily $\mathbb{Q}$-factorial at L. Therefore, the following theorem seems to be much more useful than we expected and is indispensable.

Theorem 1.24 (Small $\mathbb{Q}$-factorializations)**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P)*. Suppose that there exists Δ such that (X,Δ) is kawamata log terminal. Then, after shrinking Y around W suitably, there exists a small projective bimeromorphic contraction morphism $f\colon Z \to X$ from a normal complex variety Z such that Z is projective over Y and is $\mathbb{Q}$-factorial over W.*

As an obvious corollary of Theorem 1.24, we have:

Corollary 1.25. *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Let $\phi\colon X \dashrightarrow Z$ be a log terminal model over W. Let W' be a Stein compact subset of Y such that $W' \subset W$ and that $\Gamma(W', \mathcal{O}_Y)$ is noetherian. Then there exists a log terminal model $\phi'\colon X \dashrightarrow Z'$ over W' after shrinking Y around W' suitably.*

We further assume that there is an open neighborhood U of W' such that $U \Subset W$. Then $\phi'\colon X \dashrightarrow Z'$ is a log terminal model over some open neighborhood of W'.

We can also prove:

Theorem 1.26. *Let (X,Δ) be a kawamata log terminal pair and let D be an integral Weil divisor on X. Then $\bigoplus_{m\in\mathbb{N}}\mathcal{O}_X(mD)$ is a locally finitely generated graded $\mathcal{O}_X$-algebra.*

Theorem 1.26 is a complete generalization of [Kaw2, Theorem 6.1] (see also [Fu8, Theorem 7.2]). The next result will be indispensable for further studies of the minimal model program for projective morphisms of complex analytic spaces.

Theorem 1.27 (Dlt blow-ups, II)**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P)*. Let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Then, after shrinking Y around W suitably, we can construct a projective bimeromorphic morphism $f\colon Z\to X$ from a normal complex variety Z with the following properties:*

(1) *Z is projective over Y and is $\mathbb{Q}$-factorial over W,*
(2) *$a(E,X,\Delta)\le -1$ for every f-exceptional divisor E on Z, and*
(3) *$\left(Z, \Delta_Z^{<1}+\operatorname{Supp}\Delta_Z^{\geq 1}\right)$ is divisorial log terminal, where $K_Z+\Delta_Z=f^*(K_X+\Delta)$.*

We note that $\Delta_Z=\Delta_Z^{<1}+\operatorname{Supp}\Delta_Z^{\geq 1}$ holds when (X,Δ) is log canonical.

We can generalize [Kaw5, Theorem 1] as follows.

Theorem 1.28 (see [Bir2, Corollary 3.3])**.** *Let $\pi_1\colon X_1\to Y$ and $\pi_2\colon X_2 \to Y$ be projective morphisms such that Y is Stein and let W be a Stein compact subset of Y such that*

$\Gamma(W, \mathcal{O}_Y)$ is noetherian. Let (X_1, Δ_1) and (X_2, Δ_2) be two kawamata log terminal pairs such that $K_{X_1}+\Delta_1$ and $K_{X_2}+\Delta_2$ are nef over W, X_1 and X_2 are $\mathbb{Q}$-factorial over W, X_1 and X_2 are isomorphic in codimension one, and Δ_2 is the strict transform of Δ_1. Then, after shrinking Y around W suitably, X_1 and X_2 are connected by a sequence of flops with respect to $K_{X_1}+\Delta_1$.

Remark 1.29. Precisely speaking, the proof of Theorem 1.28 shows that there exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D_1 on X_1 such that (X_1, Δ_1+D_1) is kawamata log terminal, X_1 and X_2 are connected by a finite sequence of flips with respect to $K_{X_1}+\Delta_1+D_1$, and $K_{X_1}+\Delta_1$ is numerically trivial over W in each flip.

On the abundance conjecture, we have:

Theorem 1.30 (Abundance theorem, see Theorem 23.2)**.** *Assume that the abundance conjecture holds for projective kawamata log terminal pairs in dimension n.*

Let $\pi\colon X \to Y$ be a projective surjective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P)*. Assume that $K_X+\Delta$ is nef over Y and* $\dim X - \dim Y = n$*. Then $K_X+\Delta$ is π-semiample over some open neighborhood of W.*

Remark 1.31. The abundance conjecture for projective kawamata log terminal pairs was completely solved affirmatively in dimension ≤ 3. Therefore, in Theorem 1.30, $K_X+\Delta$ is π-semiample over some open neighborhood of W when $\dim X - \dim Y \leq 3$. This result seems to be new even when $\dim X = 3$.

We make some remarks on the proof of our results in this paper.

Remark 1.32. We will use the fact that every extremal ray is spanned by a rational curve of low degree (see [BCHM, Theorem 3.8.1] and Theorem 9.2) for the proof of Theorems A, B, C, E, F, and G. Note that in [BCHM] it was only used for the proof of the finiteness of negative extremal rays (see [BCHM, Corollary 3.8.2]). However, it is well known as a part of the cone theorem (see also Theorems 7.2 and 7.3). Therefore, [BCHM] is independent of Mori's bend and break technique, which relies on methods in positive characteristic.

Remark 1.33. We can easily reduce Theorem D to the case where Y is a point. In this case, X is projective and then Theorem D becomes a special case of [BCHM, Theorem D].

In [BP, Section 3], Păun proved a slightly weaker version of the nonvanishing theorem for projective varieties (see [BP, Theorem 1.5]). The proof is complex analytic and is independent of the theory of minimal models. By combining it with [CKP, Theorem 0.1 and Corollary 3.3], we can recover the nonvanishing theorem for projective varieties in full generality (see [BCHM, Theorem D]). By adopting this approach, Theorem D in this paper becomes completely independent of the framework of the minimal model program.

Anyway, we can prove Theorem D without any difficulties by using some known results.

Remark 1.34. In [CaL], Cascini and Lazić directly proved the finite generation of adjoint rings by using Hironaka's resolution and the Kawamata–Viehweg vanishing theorem. Their proof does not use the minimal model program. Then, in [CoL], Corti and Lazić recovered many results on the minimal model program from [CaL, Theorem A]. The author does not know whether this approach works or not in our complex analytic setting.

Remark 1.35. In dimension three, some results were formulated and established in the complex analytic setting (see, for example, [Kaw2], [Mo3], and [KM, Chapter 6]). It is not surprising because the theory of minimal models for 3-folds originally owes to the study of various singularities (see, for example, [Mo1], [Kaw2], and [Mo3]). Although the classification of surface singularities is indispensable for adjunction, we do not need any classifications of higher-dimensional singularities in [BCHM]. Hence, we think that the minimal model program in dimension ≥ 4 has been formulated and studied only for algebraic varieties.

We look at the organization of this paper. Section 2 is a long preliminary section, where we will collect and explain some basic definitions and results on complex analytic spaces. We think that the reader can understand that (P4) is reasonable. To the best knowledge of the author, some of the results in this section seem to be new. In Section 3, we will explain singularities of pairs in the complex analytic setting. The definitions of singularities become slightly complicated in the complex analytic setting. In Section 4, we will define Kleiman–Mori cones and establish Kleiman's ampleness criterion in the complex analytic setting. Here, the property (P4) plays a crucial role. Section 5 is a very short section, where we explain only two vanishing theorems. From Section 6 to Section 8, we will establish several basepoint-free theorems, the cone and contraction theorem, and so on, in the complex analytic setting. This part is essentially due to Nakayama (see [Na2]). We note that we have to treat $\mathbb{R}$-divisors. Therefore, some parts are harder than the classical setting discussed in [Na2]. In Section 9, we will prove that every negative extremal ray is spanned by a rational curve of low degree. Note that we need some result obtained by Mori's bend and break technique, which relies on methods in positive characteristic. The result in this section will play an important role in the subsequent sections. In Sections 11 and 12, we will prepare various basic definitions to establish the main results of this paper. We closely follow the treatment of [BCHM]. However, we have to reformulate some of them in order to make them suitable for our complex analytic setting. Hence we strongly recommend the reader to read these sections carefully. In Section 13, we will explain the minimal model program with scaling in the complex analytic setting in detail. It is very useful for various geometric applications. From Section 14 to Section 19, we will prove Theorems A, B, C, D, E, F, and G. Although there are many technical differences between the original proof for quasi-projective varieties (see [BCHM] and [HacM]) and the one given here in the complex analytic setting, the strategy of the whole proof is the same. In some parts, we will only explain how to modify the original proof in order to make it work in our complex analytic setting. In Section 20, we will prove almost all the theorems given in Section 1. We think that the reader who is familiar with the minimal model program for quasi-projective varieties can understand this section without any difficulties. In the last three sections, we will treat some advanced topics. In Section 21, we will briefly discuss a canonical bundle formula in the complex analytic setting and prove Theorems 1.18 and 1.22. This section needs some deep results on the theory of variations of Hodge structure. Hence the topic in Section 21 is slightly different from the other sections. In Section 22, we will discuss the minimal model program with scaling again. Then we will prove Theorem 1.28 as an easy application. In Section 23, we will explain how to reduce the abundance conjecture for projective morphisms between complex analytic spaces to the original abundance conjecture for projective varieties (see Theorem 1.30). Note that we will only treat kawamata log terminal pairs in Section 23. The abundance conjecture for log canonical pairs looks much harder than the one for kawamata log terminal pairs.

As is well known, the recent developments of the theory of minimal models heavily owe to many ideas and results obtained by Shokurov. They are scattered in his papers (see, for example, [Sh1], [Sh2], and [Sh3]). In this paper, we will freely use them without referring to Shokurov's original papers.

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The set of integers (resp. rational numbers, real numbers, complex numbers) is denoted by $\mathbb{Z}$ (resp. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$). The set of nonnegative integers (resp. positive integers, positive rational numbers, positive real numbers, nonnegative real numbers) is denoted by $\mathbb{N}$ (resp. $\mathbb{Z}_{>0}$, $\mathbb{Q}_{>0}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$).

2. Preliminaries

In this section, we will collect some basic definitions and explain various standard results. We note that every complex analytic space in this paper is assumed to be *Hausdorff* and *second-countable*. The books [BS], [Fi], and [GuR] are standard references of complex analytic geometry for algebraic geometers. A relatively new book by Noguchi (see [No]) is a very accessible textbook on several complex variables and complex analytic spaces. Demailly's book (see [D]) is also helpful and contains a proof of Grauert's theorem on direct images of coherent sheaves. We will freely use Serre's GAGA principle (see, for example, [Tay, Chapter 13] and [SGA1, Exposé XII]) throughout this paper.

Let us start with the definition of *holomorphically convex hulls*.

Definition 2.1 (Holomorphically convex hulls)**.** Let X be an analytic space and let K be a compact subset of X. The *holomorphically convex hull* $\widehat{K}$ of K in X is the set

$$\widehat{K} = \left\{ x \in X \;\middle|\; |f(x)| \leq \sup_{z \in K} |f(z)| \text{ for every } f \in \Gamma(X, \mathcal{O}_X) \right\}.$$

We note that $K \subset \widehat{K}$ always holds by definition. A compact subset K of X is said to be *holomorphically convex* in X if $\widehat{K} = K$ holds.

Let us recall the definition of *Stein spaces* for the reader's convenience. We note that the analytic space naturally associated to an affine scheme is Stein

Definition 2.2 (Stein spaces)**.** A complex analytic space X is said to be *Stein* if

(i) the global sections of $\mathcal{O}_X$ separate points in X,
(ii) for each $x \in X$, the maximal ideal of $\mathcal{O}_{X,x}$ is generated by a set of global sections of $\mathcal{O}_X$, and
(iii) X is holomorphically convex, that is, $\widehat{K}$ is compact for every compact subset K of X, where $\widehat{K}$ is the holomorphically convex hull of K in X.

The notion of *Stein compact subsets* plays a crucial role in this paper.

Definition 2.3 (Stein compact subsets)**.** A compact subset K of a complex analytic space is called *Stein compact* if it admits a fundamental system of Stein open neighborhoods.

The notion of *Oka–Weil domains* is very useful when we construct desired Stein open neighborhoods.

Definition 2.4 (Oka–Weil domains, see [GuR, Chapter VII, Section A, 2′. Definition])**.** Let X be a complex analytic space. An *Oka–Weil domain* on X is a relatively compact open subset W such that there exists a holomorphic map φ defined in a neighborhood of $\overline{W}$, and with values in $\mathbb{C}^n$, such that $\varphi|_W$ is a biholomorphic mapping onto a closed complex analytic subspace of a polydisc in $\mathbb{C}^n$. We note that W itself is Stein.

By the following lemma, we know that we can find many Stein compact subsets on a given Stein space.

Lemma 2.5. *Let K be a compact subset of a Stein space X. Let $\widehat{K}$ be a holomorphically convex hull of K in X Then $\widehat{K}$ is a Stein compact subset of X.*

Proof. Since X is holomorphically convex, $\widehat{K}$ is a compact subset of X. Let U be any open subset of X with $\widehat{K} \subset U$. Then we can take an Oka–Weil domain W of X such that $\widehat{K} \subset W \subset \overline{W} \subset U$ (see [GuR, Chapter VII, Section A, 3. Proposition]). This means that $\widehat{K}$ admits a fundamental system of Stein open neighborhoods since W is a Stein space. Hence $\widehat{K}$ is a Stein compact subset of X. □

Throughout this paper, we freely use Cartan's Theorems A and B without mentioning it explicitly. We include them here for the reader's convenience.

Theorem 2.6 (Cartan's Theorems)**.** *Let X be a Stein space and let $\mathcal{F}$ be a coherent sheaf on X. Then*

(1) (Cartan's Theorem A). $\Gamma(X, \mathcal{F})$ *generates $\mathcal{F}_x$ at every point $x \in X$, and*
(2) (Cartan's Theorem B). $H^i(X, \mathcal{F}) = 0$ *holds for every $i > 0$.*

The following cohomological characterization of Stein spaces is useful and may help algebraic geometers understand the definition of Stein spaces.

Theorem 2.7. *Let X be a complex analytic space. Then X is Stein if and only if $H^1(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on X.*

Proof. If X is Stein, then $H^1(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on X by Cartan's Theorem B. On the other hand, if $H^1(X, \mathcal{I}) = 0$ for every coherent ideal sheaf $\mathcal{I}$ on X, then it is an easy exercise to check that (i) and (ii) in Definition 2.2 hold true. Let K be a compact subset of X. Suppose that the holomorphically convex hull $\widehat{K}$ of K is not compact. Then we can take a discrete sequence $\{x_k\}_{k\in\mathbb{N}} \subset \widehat{K}$. Note that $V := \{x_k \,|\, 0 \leq k < \infty\}$ is a closed analytic subspace of X. Hence the defining ideal sheaf $\mathcal{I}_V$ of V is a coherent sheaf on X. Thus $H^1(X, \mathcal{I}_V) = 0$ holds by assumption. This implies that the natural map $H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X/\mathcal{I}_V)$ is surjective. Therefore, we can take $f \in H^0(X, \mathcal{O}_X)$ such that $f(x_n) = n$. On the other hand,

$$n = |f(x_n)| \leq \sup_{x\in K} |f(x)| < \infty$$

for every n since $x_n \in \widehat{K}$. This is a contradiction. Thus, $\widehat{K}$ is always compact, that is, (iii) holds true. We finish the proof. □

As an easy consequence of Theorem 2.7, we have:

Theorem 2.8. *Let $f\colon Z \to X$ be a finite morphism between complex analytic spaces. If X is Stein, then so is Z.*

Proof. Let $\mathcal{F}$ be any coherent sheaf on Z. Since $f\colon Z \to X$ is finite, $f_*\mathcal{F}$ is coherent and $H^1(Z, \mathcal{F}) = H^1(X, f_*\mathcal{F}) = 0$ holds by the Steinness of X. Hence, by Theorem 2.7, Z is Stein. □

We have already explained that every Stein space X has many good properties. Unfortunately, however, $\Gamma(X, \mathcal{O}_X)$ is not noetherian if X does not consist of only finitely many points.

Example 2.9. Let X be a Stein space and let $\{P_k\}_{k\in\mathbb{N}}$ be a set of mutually distinct discrete points of X. Then $Z_n := \{P_n, P_{n+1}, \ldots\}$ can be seen as a closed analytic subspace of X for every $n \in \mathbb{N}$. Let $\mathcal{I}_n$ be the defining ideal sheaf of Z_n on X. It is well known that $\mathcal{I}_n$ is a coherent sheaf on X. We put $\mathfrak{a}_n := \Gamma(X, \mathcal{I}_n)$ for every n. Then

$$\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots \subsetneq \mathfrak{a}_n \subsetneq \cdots$$

is a strictly increasing sequence of ideals of $\Gamma(X, \mathcal{O}_X)$. This means that $\Gamma(X, \mathcal{O}_X)$ is not noetherian.

Siu's theorem clarifies the meaning of the condition in (P4).

Theorem 2.10 ([Si, Theorem 1])**.** *Let K be a Stein compact subset of a complex analytic space X. Then $K \cap Z$ has only finitely many connected components for any analytic subset Z which is defined over an open neighborhood of K if and only if*

$$\mathcal{O}_X(K) = \Gamma(K, \mathcal{O}_Y) = \varinjlim_{K \subset U} \Gamma(U, \mathcal{O}_X),$$

where U runs through all the open neighborhoods of K, is noetherian.

Proof. For the details, see, for example, [BS, Chapter V, §3]. □

One point is a Stein compact subset satisfying (P4).

Example 2.11. Let X be a complex analytic space and let P be any point of X. Then P is a Stein compact subset of X and $\mathcal{O}_{X,P} = \Gamma(P, \mathcal{O}_X)$ is noetherian.

The Cantor set is a Stein compact subset which does not satisfy (P4).

Example 2.12. We note that $\mathbb{C}$ is Stein and that any open subset of $\mathbb{C}$ is also Stein since it is holomorphically convex. We put $X = \{z \in \mathbb{C} \,|\, |z| < 2\}$ and consider the Cantor set $\mathcal{C}$. It is easy to see that $\mathcal{C}$ ($\subset [0,1] \subset X$) is a Stein compact subset of X and that X is Stein. We can easily check that for any given $x_1, x_2 \in \mathcal{C}$ there exists $x_3 \notin \mathcal{C}$ with $x_3 \in [x_1, x_2]$. Hence $\mathcal{C}$ does not satisfy (P4). Thus, $\Gamma(\mathcal{C}, \mathcal{O}_X)$ is not noetherian by Theorem 2.10 . More explicitly, we put

$$\mathfrak{a}_n := \left\{ f \in \Gamma(\mathcal{C}, \mathcal{O}_X) \;\middle|\; f(z) = 0 \text{ for any } z \in \mathcal{C} \cap \left[0, \frac{1}{3^n}\right] \right\}$$

for every $n \in \mathbb{N}$. Then we can check that $\mathfrak{a}_n \subsetneq \mathfrak{a}_{n+1}$ holds for every $n \in \mathbb{N}$. Therefore, we get a strictly increasing sequence of ideals of $\Gamma(\mathcal{C}, \mathcal{O}_X)$:

$$\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \subsetneq \cdots \subsetneq \mathfrak{a}_n \subsetneq \cdots .$$

This implies that $\Gamma(\mathcal{C}, \mathcal{O}_X)$ is not noetherian.

We supplement Theorem 2.8.

Theorem 2.13. *Let $f\colon Z \to X$ be a finite morphism of complex analytic spaces such that X is Stein. Let K be a Stein compact subset of X such that $\Gamma(K, \mathcal{O}_X)$ is noetherian. Then $f^{-1}(K)$ is a Stein compact subset of Z such that $\Gamma(f^{-1}(K), \mathcal{O}_Z)$ is noetherian.*

Proof. Since f is finite, $f^{-1}(K)$ is a compact subset of Z. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a fundamental system of Stein open neighborhoods of K. By Theorem 2.8, Z is Stein and $f^{-1}(U_\lambda)$ is also Stein for every $\lambda \in \Lambda$. Hence $\{f^{-1}(U_\lambda)\}_{\lambda\in\Lambda}$ is a fundamental system of Stein open neighborhoods of $f^{-1}(K)$. On the other hand, since f is finite, $f_*\mathcal{O}_Z$ is a coherent sheaf on X. By the Stein compactness of K, there exist a Stein open neighborhood U of K and a surjection

$$\mathcal{O}_U^{\oplus N} \to f_*\mathcal{O}_Z|_U \to 0$$

for some positive integer N. This implies the surjection

$$\Gamma(K, \mathcal{O}_X)^{\oplus N} \to \Gamma(K, f_*\mathcal{O}_Z) \to 0.$$

Hnece $\Gamma(K, f_*\mathcal{O}_Z)$ is a finitely generated $\Gamma(K, \mathcal{O}_X)$-module. We note that $\Gamma(K, f_*\mathcal{O}_Z) = \Gamma(f^{-1}(K), \mathcal{O}_Z)$ and that $\Gamma(K, \mathcal{O}_X)$ is noetherian. Thus, $\Gamma(f^{-1}(K), \mathcal{O}_Z)$ is noetherian. This means that $f^{-1}(K)$ is a Stein compact subset of Z such that $\Gamma(f^{-1}(K), \mathcal{O}_Z)$ is noetherian. □

Remark 2.14. Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). As an easy consequence of Theorems 2.8 and 2.13, we usually may assume that π is surjective and that Y is a Stein variety by replacing Y with $\pi(X)$. For some purposes, we sometimes replace Y with its normalization and further assume that Y is a normal Stein variety. By taking the Stein factorization of $\pi\colon X \to Y$, we sometimes further assume that π has connected fibers and that Y is a normal Stein variety, that is, $\pi\colon X \to Y$ is a contraction of normal complex varieties.

We note:

Definition 2.15. A proper morphism $\pi\colon X \to Y$ of normal complex varieties is called a *contraction* if $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$ holds.

When we enlarge a given Stein compact subset satisfying (P4) slightly, we need the following lemma.

Lemma 2.16. *Let X be a Stein space and let K be a holomorphically convex compact subset of X. If U is any open neighborhood of K, then there exists an Oka–Weil domain V, defined by global holomorphic functions on X, such that $K \subset V \subset \overline{V} \subset U$. Note that V can be seen as a closed complex analytic subspace of a polydisc $\Delta(0,r) = \{(z_1, \dots, z_n) \,|\, |z_i| < r$ for $1 \le i \le n\}$ for some $r > 0$ and $n \in \mathbb{Z}_{>0}$. We put $L := V \cap \overline{\Delta}(0, r-\varepsilon)$ with $0 < \varepsilon < r$. Then L is compact, semianalytic, and holomorphically convex in V. In particular, L is a Stein compact subset such that $\Gamma(L, \mathcal{O}_X)$ is noetherian. Furthermore, if U' is a relatively compact open neighborhood of K in X, then we can choose U, V, and L such that*

$$K \subset U' \subset L \subset V \subset \overline{V} \subset U \subset X$$

holds.

Proof. For the existence of a desired Oka–Weil domain V, see, for example, [GuR, Chapter VII, Section A, 3. Proposition]. By definition, L is obviously compact and semianalytic. Since L is defined by $|z_1| \le r-\varepsilon, \dots, |z_n| \le r-\varepsilon$ such that $z_i \in \Gamma(V, \mathcal{O}_V)$ for every i, it is easy to see that L is holomorphically convex in V. By Lemma 2.5, L is Stein compact. Since L is compact and semianalytic, it is well known that L satisfies (P4) (see, for example, [BM1, Corollary 2.7 (2)]). Thus, $\Gamma(L, \mathcal{O}_X)$ is noetherian by Theorem 2.10.

Or, by applying [Fr, Théorème (I,9)] to L, we obtain that $\Gamma(L, \mathcal{O}_X)$ is noetherian. By the above construction, the last statement is obvious. □

We frequently use the following property of coherent sheaves on complex analytic spaces.

Lemma 2.17. *Let $\mathcal{F}$ be a coherent sheaf on a complex analytic space X and let*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$$

be an increasing chain of coherent subsheaves. Then this chain is stationary over any relatively compact subset of X.

Proof. See, for example, [Fi, 0.40. Proposition and Corollary]. □

Note that the arguments in [Kau, §2 Basic theorems on coherent $\mathcal{O}$-modules] work for Stein compact subsets K satisfying (P4) with obvious modifications.

Definition 2.18 ($\mathcal{O}_X$-exhaustions, see [Kau, 2.9 Definition])**.** Let $\mathcal{M}$ be an $\mathcal{O}_X$-module on a complex analytic space X. An $\mathcal{O}_X$*-exhaustion* of $\mathcal{M}$ is an increasing sequence

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_k \subset \cdots \subset \mathcal{M}$$

of coherent sub-$\mathcal{O}_X$-modules such that $\mathcal{M} = \bigcup_k \mathcal{M}_k$.

Lemma 2.19 (see [Kau, 2.10 Proposition])**.** *Let K be a Stein compact subset of a complex analytic space X such that $\Gamma(K, \mathcal{O}_X)$ is noetherian. Let $\mathcal{M}$ and $\mathcal{M}'$ be $\mathcal{O}_X$-modules on X which admit $\mathcal{O}_X$-exhaustions. If $\phi \colon \mathcal{M} \to \mathcal{M}'$ is a surjective $\mathcal{O}_X$-homomorphism, then the induced map $\Gamma(K, \mathcal{M}) \to \Gamma(K, \mathcal{M}')$ is surjective.*

Proof. For the details, see the proof of [Kau, 2.10 Proposition]. Although K is a polydisc in [Kau], the proof of [Kau, 2.10 Proposition] works in our setting. □

Since we are working on complex analytic spaces, we note:

Remark 2.20. Let $\mathcal{F}_m$ be a coherent sheaf on a complex analytic space X for every $m \in \mathbb{N}$. Then the natural map

$$\bigoplus_{m\in\mathbb{N}} \Gamma(X, \mathcal{F}_m) \to \Gamma\left(X, \bigoplus_{m\in\mathbb{N}} \mathcal{F}_m\right)$$

is not necessarily an isomorphism. Fortunately,

$$\bigoplus_{m\in\mathbb{N}} \Gamma(K, \mathcal{F}_m) \simeq \Gamma\left(K, \bigoplus_{m\in\mathbb{N}} \mathcal{F}_m\right)$$

holds for any compact subset K of X.

Example 2.21. Let X be a noncompact complex analytic space and let $\{x_m\}_{m\in\mathbb{N}}$ be a discrete sequence of X. Let $\mathbb{C}(x_m) := (i_{x_m})_*\mathbb{C}$ be a skyscraper sheaf, where $i_{x_m} : x_m \hookrightarrow X$ is the inclusion map. Then $\bigoplus_{m\in\mathbb{N}} \mathbb{C}(x_m)$ is a coherent sheaf on X. In this case,

$$\Gamma\left(X, \bigoplus_{m\in\mathbb{N}} \mathbb{C}(x_m)\right) = \prod_{m\in\mathbb{N}} \mathbb{C} = \prod_{m\in\mathbb{N}} \Gamma(X, \mathbb{C}(x_m)).$$

Hence, the natural map

$$\bigoplus_{m\in\mathbb{N}} \Gamma(X, \mathbb{C}(x_m)) \to \Gamma\left(X, \bigoplus_{m\in\mathbb{N}} \mathbb{C}(x_m)\right)$$

is not an isomorphism.

In this paper, we will have to treat graded $\mathcal{O}_X$-algebras on a complex analytic spaces. So we prepare some definitions and basic properties.

Definition 2.22 (see [Na3, Chapter II. §1. b. Spec and Proj])**.** Let X be a complex analytic space and let $\mathbb{C}[x] = \mathbb{C}[x_1, \cdots, x_l]$ be the polynomial ring of l-variables $x = (x_1, \cdots, x_l)$. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is called *of finite presentation* if there exists a surjective $\mathcal{O}_X$-algebra homomorphism

$$\mathcal{O}_X[x] = \mathcal{O}_X[x_1, \cdots, x_l] = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[x] \twoheadrightarrow \mathcal{A}$$

for some l whose kernel is generated by a finite number of polynomials belonging to $H^0(X, \mathcal{O}_X)[x]$. If $\mathcal{A}|_{U_\lambda}$ is of finite presentation for an open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, then $\mathcal{A}$ is called *locally of finite presentation*.

The notion of locally finitely generated graded $\mathcal{O}_X$-algebras is indispensable.

Definition 2.23 (Locally finitely generated graded $\mathcal{O}_X$-algebras)**.** Let X be a complex analytic space. An $\mathcal{O}_X$-algebra $\mathcal{A} = \bigoplus_{m \in \mathbb{N}} \mathcal{A}_m$ is called a *finitely generated graded $\mathcal{O}_X$-algebra* if there exists a surjective $\mathcal{O}_X$-algebra homomorphism

$$\mathcal{O}_X[x] = \mathcal{O}_X[x_1, \cdots, x_l] = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[x] \twoheadrightarrow \mathcal{A}$$

for some l such that x_i is mapped to a homogeneous element of $H^0(X, \mathcal{A})$ for every i. If $\mathcal{A}|_{U_\lambda}$ is a finitely generated graded $\mathcal{O}_{U_\lambda}$-algebra for some open covering $X = \bigcup_{\lambda \in \Lambda} U_\lambda$, then $\mathcal{A}$ is called a *locally finitely generated graded $\mathcal{O}_X$-algebra*.

We note the following basic property of locally finitely generated graded $\mathcal{O}_X$-algebras.

Lemma 2.24 (see [Na3, Chapter II. 1.6. Corollary])**.** *Let X be a complex analytic space and let $\mathcal{A} = \bigoplus_{m \in \mathbb{N}} \mathcal{A}_m$ be a locally finitely generated graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_m$ are all coherent $\mathcal{O}_X$-modules. Then $\mathcal{A}$ is locally of finite presentation.*

Before we prove Lemma 2.24, we note:

Remark 2.25. Any locally finitely generated graded $\mathcal{O}_X$-algebra $\mathcal{A} = \bigoplus_{m \in \mathbb{N}} \mathcal{A}_m$ in this paper satisfies the condition that $\mathcal{A}_m$ is a coherent $\mathcal{O}_X$-module for every $m \in \mathbb{N}$. We do not treat the case where $\mathcal{A}_m$ is not a coherent $\mathcal{O}_X$-module.

Let us prove Lemma 2.24.

Proof of Lemma 2.24. We take an arbitrary point $P \in X$ and replace X with a small Stein open neighborhood of P. Then we have an exact sequence

$$\phi\colon \mathcal{O}_X[x] \to \mathcal{A} \to 0$$

for $x = (x_1, \cdots, x_l)$ such that x_i is mapped to a homogeneous element of $H^0(X, \mathcal{A})$ for every i. We take a relatively compact Stein open neighborhood U of P and a Stein compact subset K of Y such that $U \subset K$ and $\Gamma(K, \mathcal{O}_Y)$ is noetherian. Then

$$\phi(K)\colon \mathcal{O}_X(K)[x] \to \mathcal{A}(K) \to 0$$

is exact by Lemma 2.19. Since $\mathcal{O}_X(K)$ is noetherian, the kernel of $\phi(K)$ is generated by weighted homogeneous polynomials $f_1, \ldots, f_N$ in $\mathcal{O}_X(K)[x]$. Hence we obtain a Stein open neighborhood U' of K and a homomorphism

$$\psi := (f_1, \cdots, f_N)\colon \bigoplus_{i=1}^{N} \mathcal{O}_{U'}[x](-\deg f_i) \to \mathcal{O}_{U'}[x]$$

of graded $\mathcal{O}_{U'}[x]$-modules such that the image of $\psi(K)$ is $(\operatorname{Ker}\phi)(K)$. By construction, there exists the following natural surjection

$$\alpha_m\colon (\operatorname{Coker}\psi)_m \twoheadrightarrow \mathcal{A}_m|_{U'}$$

for every $m \in \mathbb{N}$, where $(\operatorname{Coker}\psi)_m$ is the degree m part of $\operatorname{Coker}\psi$. By construction again, we can check that $(\operatorname{Coker}\psi)_m(K) \simeq \mathcal{A}_m(K)$ holds for every $m \in \mathbb{N}$. This implies that the kernel of α_m is zero on U. Hence $\mathcal{A}|_U$ is of finite presentation. Therefore, $\mathcal{A}$ is locally of finite presentation. $\square$

The lemma below is important and will be used repeatedly without mentioning it explicitly. The proof is much harder than that of the corresponding statement for algebraic varieties (see [ADHL, Corollary 1.1.2.6]).

Lemma 2.26. *Let X be a complex analytic space and let $\mathcal{A} = \bigoplus_{m\in\mathbb{N}} \mathcal{A}_m$ be a graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_m$ is a coherent $\mathcal{O}_X$-module for every m and $\mathcal{A}(U)$ is an integral domain for every nonempty connected open subset U of X. We put $\mathcal{A}^{(d)} := \bigoplus_{m\in\mathbb{N}} \mathcal{A}_{dm}$. Then $\mathcal{A}$ is a locally finitely generated graded $\mathcal{O}_X$-algebra if and only if so is $\mathcal{A}^{(d)}$.*

Proof. Since $\mathcal{O}_X(U)$ is not necessarily noetherian, the proof is not so obvious.

Step 1. We assume that $\mathcal{A}^{(d)}$ is a locally finitely generated graded $\mathcal{O}_X$-algebra. We take an arbitrary point $P \in X$. By shrinking X around P, there exists a surjective $\mathcal{O}_X$-algebra homomorphism

$$\mathcal{O}_X[x] = \mathcal{O}_X[x_1, \cdots, x_l] \twoheadrightarrow \mathcal{A}^{(d)} \tag{2.1}$$

for some l such that x_i is mapped to a homogeneous element of $H^0(X, \mathcal{A}^{(d)})$ for every i. We take an open neighborhood U of $P \in X$ and a Stein compact subset K of X such that $P \in U \subset K$ and that K satisfies (P4). Without loss of generality, we may assume that K is connected. Since $\mathcal{O}_X[x_1, \cdots, x_l]$ and $\mathcal{A}^{(d)}$ admit $\mathcal{O}_X$-exhaustions, we obtain the surjection

$$\mathcal{O}_X(K)[x_1, \cdots, x_l] \to \mathcal{A}^{(d)}(K) \to 0 \tag{2.2}$$

induced by (2.1). This means that $\mathcal{A}^{(d)}(K)$ is a finitely generated graded $\mathcal{O}_X(K)$-algebra. Note that $\mathcal{O}_X(K)$ is noetherian and that $\mathcal{A}(K)$ is an integral domain. Therefore, we see that $\mathcal{A}(K)$ is a finitely generated graded $\mathcal{O}(K)$-algebra (see, for example, [CaL, Lemma 2.25 (ii)] and [ADHL, Proposition 1.1.2.5]). Since K is a Stein compact subset, for any nonnegative integer m, $\mathcal{A}_m(K)$ generates $\mathcal{A}_{m,x}$ for every $x \in K$. Hence we can find a surjective $\mathcal{O}_U$-algebra homomorphism

$$\mathcal{O}_U[y] = \mathcal{O}_U[y_1, \cdots, y_k] \twoheadrightarrow \mathcal{A}|_U$$

for some k such that y_j is mapped to a homogeneous element of $H^0(U, \mathcal{A})$ for every j. This means that $\mathcal{A}$ is a locally finitely generated graded $\mathcal{O}_X$-algebra.

Step 2. As in Step 1, we take an arbitrary point $P \in X$ and shrink X around P. Then there exists a surjective $\mathcal{O}_X$-algebra homomorphism

$$\mathcal{O}_X[x] = \mathcal{O}_X[x_1, \cdots, x_l] \twoheadrightarrow \mathcal{A} \tag{2.3}$$

for some l such that x_i is mapped to a homogeneous element of $H^0(X, \mathcal{A})$ for every i. In this case, it is easy to see that there exists a surjective $\mathcal{O}_X$-algebra homomorphism

$$\mathcal{O}_X[y] = \mathcal{O}_X[y_1, \cdots, y_k] \twoheadrightarrow \mathcal{A}^{(d)}$$

for some k such that y_j is mapped to a homogeneous element of $H^0(X, \mathcal{A}^{(d)})$ for every j (see, for example, [CaL, Lemma 2.25 (i)] and [ADHL, Proposition 1.1.2.4]). Hence $\mathcal{A}^{(d)}$ is a locally finitely generated graded $\mathcal{O}_X$-algebra.

We finish the proof. □

In order to construct flips and log canonical models in the category of complex analytic spaces, we need the notion of Projan. For the details of Projan, see [Na3, §1.b. Spec and Proj].

Remark 2.27 (Projan)**.** Let X be a complex analytic space and let $\mathcal{A} = \bigoplus_{m \in \mathbb{N}} \mathcal{A}_m$ be a locally finitely generated graded $\mathcal{O}_X$-algebra such that $\mathcal{A}_m$ is a coherent $\mathcal{O}_X$-modules for every m. Then we can define an analytic space $\operatorname{Projan}_X \mathcal{A}$ which is proper over X. More generally, we can define $\operatorname{Projan}_X \mathcal{A}$ under a weaker assumption that $\mathcal{A}$ is locally of finite presentation. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We note that $\operatorname{Projan}_X \operatorname{Sym} \mathcal{F}$ is usually denoted by $\mathbb{P}_X(\mathcal{F})$, where $\operatorname{Sym} \mathcal{F} = \bigoplus_{m \in \mathbb{N}} \operatorname{Sym}^m \mathcal{F}$.

Now we can define projective morphisms of complex analytic spaces.

Definition 2.28. Let $\pi \colon X \to Y$ be a proper morphism of complex analytic spaces and let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is said to be *π-very ample* or *relatively very ample over Y* if $\mathcal{L}$ is *π-free*, that is, $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ is surjective, and the induced morphism $X \to \mathbb{P}_Y(f_*\mathcal{L})$ over Y is a closed embedding. A line bundle $\mathcal{L}$ on X is called *π-ample* or *ample over Y* if for any point $y \in Y$ there are an open neighborhood U of y and a positive integer m such that $\mathcal{L}^{\otimes m}|_{\pi^{-1}(U)}$ is relatively very ample over U. Let D be a Cartier divisor on X. Then we say that D is *π-very ample*, *π-free*, and *π-ample* if the line bundle $\mathcal{O}_X(D)$ is so, respectively. We note that π is said to be *projective* when there exists a π-ample line bundle on X.

For the basic properties of π-ample line bundles, see [BS, Chapter IV] and [Na3, Chapter II. §1. c. Ample line bundles].

Definition 2.29 (Semiampleness)**.** Let $\pi \colon X \to Y$ be a proper morphism of complex analytic spaces and let $\mathcal{L}$ be a line bundle on X. If there exist an open covering $Y = \bigcup_{\lambda \in \Lambda} U_\lambda$ and positive integers m_λ such that $\mathcal{L}^{\otimes m_\lambda}|_{\pi^{-1}(U_\lambda)}$ is $\pi|_{\pi^{-1}(U_\lambda)}$-free for every $\lambda \in \Lambda$, then $\mathcal{L}$ is called *π-semiample* or *relatively semiample over Y*. Let D be a Cartier divisor on X. If $\mathcal{O}_X(D)$ is π-semiample, then D is called *π-semiample* or *relatively semiample over Y*.

Here, we recall the precise definition of *bimeromorphic maps* for the sake of completeness.

Definition 2.30 (Meromorphic maps)**.** A *meromorphic map* $f \colon X \dashrightarrow Y$ of complex analytic varieties is defined by the graph $\Gamma_f \subset X \times Y$ such that Γ_f is a subvariety of $X \times Y$ and that the first projection is an isomorphism over a Zariski open dense subset of X. Note that a *Zariski open subset* is the complement of an analytic subset. If further the second projection $\Gamma_f \to Y$ is proper and is an isomorphism over a Zariski open dense subset of Y, then $f \colon X \dashrightarrow Y$ is called a *bimeromorphic map.* We say that a bimeromorphic map $f \colon X \dashrightarrow Y$ of normal complex varieties is a *bimeromorphic contraction* if f^{-1} does not contract any divisors. If in addition f^{-1} is also a bimeromorphic contraction, then we say that f is a *small* bimeromorphic map. Let $f \colon X \to Y$ be a bimeromorphic morphism of complex normal varieties, equivalently, $f \colon X \dashrightarrow Y$ is a bimeromorphic map of normal complex varieties and the first projection $\Gamma_f \to X$ is an isomorphism. Then, we put $\operatorname{Exc}(f) := \{x \in X \mid f \text{ is not an isomorphism at } x\}$ and call it the *exceptional locus* of f.

Remark 2.31. Let X be a complex analytic space and let U be a Zariski open subset of X. Let V be a Zariski open subset of U. Unfortunately, V is not necessarily a Zariski open subset of X. This is because the analytic subset $\Gamma := U \setminus V$ of U can not always be extended to an analytic subset of X.

In this paper, we discuss the minimal model program. Therefore, we need $\mathbb{Q}$-*divisors* and $\mathbb{R}$-*divisors*.

Definition 2.32 (Divisors, $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors)**.** Let X be a normal complex variety. A *prime divisor* on X is an irreducible and reduced closed subvariety of codimension one. An $\mathbb{R}$-*divisor* D on X is a formal sum

$$D = \sum_i a_i D_i,$$

where D_i is a prime divisor on X with $D_i \neq D_j$ for $i \neq j$, $a_i \in \mathbb{R}$ for every i, and the *support*

$$\operatorname{Supp} D := \bigcup_{a_i \neq 0} D_i$$

is a closed analytic subset of X. In other words, the formal sum $\sum_i a_i D_i$ is locally finite. If $a_i \in \mathbb{Z}$ (resp. $a_i \in \mathbb{Q}$) for every i, then D is called a *divisor* (resp. $\mathbb{Q}$-*divisor*) on X. Note that a divisor is sometimes called an *integral Weil divisor* in order to emphasize the condition that $a_i \in \mathbb{Z}$ for every i. If $0 \leq a_i \leq 1$ (resp. $a_i \leq 1$) holds for every i, then an $\mathbb{R}$-divisor D is called a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor.

Let $D = \sum_i a_i D_i$ be an $\mathbb{R}$-divisor on X such that D_i is a prime divisor for every i with $D_i \neq D_j$ for $i \neq j$. The *round-down* $\lfloor D \rfloor$ of D is defined to be the divisor

$$\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i.$$

The *round-up* and the *fractional part* of D are defined to be

$$\lceil D \rceil := -\lfloor -D \rfloor, \quad \text{and} \quad \{D\} := D - \lfloor D \rfloor,$$

respectively. We put

$$D^{=1} = \sum_{a_i = 1} D_i, \quad D^{<1} := \sum_{a_i < 1} a_i D_i, \quad \text{and} \quad D^{\geq 1} := \sum_{a_i \geq 1} a_i D_i.$$

We sometimes use

$$D_+ := \sum_{a_i > 0} a_i D_i, \quad \text{and} \quad D_- := -\sum_{a_i < 0} a_i D_i \geq 0.$$

By definition, $D = D_+ - D_-$ holds.

Let D be an $\mathbb{R}$-divisor on X and let x be a point of X. If D is written as a finite $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of Cartier divisors on some open neighborhood of x, then D is said to be $\mathbb{R}$-*Cartier at* x (resp. $\mathbb{Q}$-*Cartier at* x). If D is $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) at x for every $x \in X$, then D is said to be $\mathbb{R}$-*Cartier* (resp. $\mathbb{Q}$-*Cartier*). Note that a $\mathbb{Q}$-Cartier $\mathbb{R}$-divisor D is automatically a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor by definition. If D is a finite $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of Cartier divisors on X, then we sometimes say that D is a *globally* $\mathbb{R}$-*Cartier* $\mathbb{R}$-*divisor* (resp. *globally* $\mathbb{Q}$-*Cartier* $\mathbb{Q}$-*divisor*).

Example 2.33 below shows a big difference between divisors on algebraic varieties and those on complex analytic spaces.

Example 2.33 (Weierstrass)**.** Let D be a divisor on $\mathbb{C}$. We note that $\operatorname{Supp} D$ may be any discrete subset of $\mathbb{C}$. By the classical Weierstrass theorem, we can construct a meromorphic function f on $\mathbb{C}$ such that $\operatorname{div}(f) = D$.

Definition 2.34. Let X be a normal variety. A real vector space spanned by the prime divisors on X is denoted by $\mathrm{WDiv}_{\mathbb{R}}(X)$. It has a canonical basis given by the prime divisors. Let D be an element of $\mathrm{WDiv}_{\mathbb{R}}(X)$. Then the sup norm of D with respect to this basis is denoted by $\|D\|$. Note that an $\mathbb{R}$-divisor D is an element of $\mathrm{WDiv}_{\mathbb{R}}(X)$ if and only if $\operatorname{Supp} D$ has only finitely many irreducible components.

We need the notion of semiample $\mathbb{Q}$-divisors.

Definition 2.35 (Relatively semiample $\mathbb{Q}$-divisors)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that X is a normal variety. A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X is called a *π-semiample $\mathbb{Q}$-divisor* on X if it is a finite $\mathbb{Q}_{>0}$-linear combination of π-semiample Cartier divisors on X.

The following lemma is very important.

Lemma 2.36. *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that X is a normal complex variety and let D be a π-semiample $\mathbb{Q}$-divisor on X. Then $\bigoplus_{m\in\mathbb{N}} \pi_*\mathcal{O}_X(\lfloor mD\rfloor)$ is a locally finitely generated graded $\mathcal{O}_Y$-algebra. In particular, if $\mathcal{L}$ is a π-ample line bundle on X, then $\bigoplus_{m\in\mathbb{N}} \pi_*\mathcal{L}^{\otimes m}$ is a locally finitely generated graded $\mathcal{O}_Y$-algebra.*

Proof. Throughout this proof, we fix a point $y \in Y$ and repeatedly shrink Y around y. In Step 1, we will reduce the problem to the case where $\mathcal{O}_X(D)$ is π-ample. In Step 2, we will prove the desired finite generation.

Step 1. By shrinking Y around y, we may assume that there exists a positive integer d such that $\mathcal{O}_X(dD)$ is π-free. By Lemma 2.26, we may further assume that $\mathcal{O}_X(D)$ is π-free by replacing D with dD. We consider a contraction morphism over Y associated to $\mathcal{O}_X(D)$ and take the Stein factorization. Then there exist a contraction morphism $\varphi\colon X \to Z$ over Y with $\varphi_*\mathcal{O}_X \simeq \mathcal{O}_Z$ and some π_Z-ample line bundle $\mathcal{L}$ on Z, where $\pi_Z\colon Z \to Y$ is the structure morphism, such that $\mathcal{O}_X(D) \simeq \varphi^*\mathcal{L}$. Since $\pi_*\mathcal{O}_X(mD) \simeq (\pi_Z)_*\mathcal{L}^{\otimes m}$ holds for every m, we may further assume that $\mathcal{O}_X(D)$ is π-ample by replacing X and $\mathcal{O}_X(D)$ with Z and $\mathcal{L}$, respectively.

Step 2. From now on, we assume that $\mathcal{O}_X(D)$ is π-ample. By Lemma 2.26, we may further assume that $\mathcal{O}_X(D)$ is π-very ample. Therefore, after shrinking Y around y suitably, there exists the following commutative diagram

$$\begin{array}{ccccc} X & \overset{\iota}{\hookrightarrow} & Y \times \mathbb{P}^N & \xrightarrow[p_2]{} & \mathbb{P}^N \\ & \overset{\pi}{\searrow} & \downarrow{\scriptstyle p_1} & & \\ & & Y & & \end{array}$$

such that $\mathcal{O}_X(D) \simeq \iota^* p_2^*\mathcal{O}_{\mathbb{P}^N}(1)$. We put $\mathcal{N} := p_2^*\mathcal{O}_{\mathbb{P}^N}(1)$. Then there exists a positive integer m_0 such that $(p_1)_*\mathcal{N}^{\otimes m} \to \pi_*\mathcal{O}_X(mD)$ is surjective for every $m \geq m_0$. We note that

$$(p_1)_*\mathcal{N}^{\otimes m} \simeq (p_1)_*p_2^*\mathcal{O}_{\mathbb{P}^N}(1) \simeq \mathcal{O}_Y[X_0, \cdots, X_N]_m,$$

where $\mathcal{O}_Y[X_0, \cdots, X_N]_m$ is the degree m part of $\mathcal{O}_Y[X_0, \cdots, X_N]$. Since $\pi_*\mathcal{O}_X(mD)$ is a coherent $\mathcal{O}_Y$-module for every $0 \leq m < m_0$, after replacing Y with a small Stein open neighborhood of y if necessary, we see that there is a surjective $\mathcal{O}_Y$-algebra homomorphism

$$\mathcal{O}_Y[X_0, \cdots, X_N, X_{N+1}, \cdots, X_{N+M}] \twoheadrightarrow \bigoplus_{m\in\mathbb{N}} \pi_*\mathcal{O}_X(mD)$$

for some M such that each X_i is mapped to an element of $H^0(Y, \pi_*\mathcal{O}_X(m_iD))$ for some $m_i \in \mathbb{N}$. This means that $\bigoplus_{m\in\mathbb{N}} \pi_*\mathcal{O}_X(mD)$ is a locally finitely generated graded $\mathcal{O}_Y$-algebra.

By Step 1 and Step 2, $\bigoplus_{m\in\mathbb{N}} \pi_*\mathcal{O}_X(\lfloor mD\rfloor)$ and $\bigoplus_{m\in\mathbb{N}} \pi_*\mathcal{L}^{\otimes m}$ are both locally finitely generated graded $\mathcal{O}_Y$-algebras. □

For almost all applications, we may assume that any $\mathbb{R}$-divisor has finitely many components by the following lemma.

Lemma 2.37. *Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (resp. $\mathbb{Q}$-divisor) on a normal complex variety X. Let U be any relatively compact open subset of X. Then D is a finite $\mathbb{R}$-linear ($\mathbb{Q}$-linear) combination of Cartier divisors in a neighborhood of $\overline{U}$, that is, D is a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor (globally $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor) in a neighborhood of $\overline{U}$.*

Proof. Without loss of generality, we may assume that D is a finite $\mathbb{R}$-linear combination of prime divisors by shrinking X. We write $D = \sum_{i=1}^{k} a_iD_i$, where $a_i \in \mathbb{R}$ and D_i is a prime divisor for every i with $D_i \neq D_j$ for $i \neq j$. We consider the following $\mathbb{R}$-vector space

$$V = \{x_1D_1 + \cdots + x_kD_k \,|\, x_i \in \mathbb{R} \text{ for every } i\} \simeq \mathbb{R}^k.$$

We take an arbitrary point $x \in X$. Then there exists an open neighborhood U_x of x such that $D|_{U_x}$ is a finite $\mathbb{R}$-linear combination of Cartier divisors. Hence we can find an affine subspace V^x of V defined over the rationals such that $D \in V^x$ and that any member of V^x is $\mathbb{R}$-Cartier in a neighborhood of x. Since U is relatively compact, there exists an affine subspace Σ of V defined over the rationals such that $D \in \Sigma$ and that every element of Σ is $\mathbb{R}$-Cartier in a neighborhood of $\overline{U}$. Hence, by the standard argument, we can write D as a finite $\mathbb{R}$-linear combination of Cartier divisors in a neighborhood of $\overline{U}$. When D is a $\mathbb{Q}$-divisor, it can be written as a finite $\mathbb{Q}$-linear combination of Cartier divisors in a neighborhood of $\overline{U}$. Thus, we get the desired statement. □

The definition of $\mathbb{Q}$-*factoriality* is very subtle.

Definition 2.38 ($\mathbb{Q}$-factoriality, see [Na2, Definition 4.13])**.** Let X be a normal complex variety and let K be a compact subset of X. Then X is said to be $\mathbb{Q}$-*factorial at K* if every prime divisor defined on an open neighborhood U of K is $\mathbb{Q}$-Cartier at any point $x \in K$.

Let $\pi\colon X \to Y$ be a projective morphism and let W be a compact subset of Y. If X is $\mathbb{Q}$-factorial at $\pi^{-1}(W)$, then we usually say that X is $\mathbb{Q}$-*factorial over W*.

Remark 2.39. Let $\pi\colon X \to Y$ be a projective morphism and let W be a compact subset of Y. We take a compact subset W' of Y with $W' \subset W$. It is very important to note that X is not necessarily $\mathbb{Q}$-factorial over W' even if X is $\mathbb{Q}$-factorial over W. This is because there may exist a divisor defined over an open neighborhood of $\pi^{-1}(W')$ which can not be extended to a divisor defined over an open neighborhood of $\pi^{-1}(W)$.

We adopt the following definition of *linear, $\mathbb{Q}$-linear, and $\mathbb{R}$-linear equivalences* in this paper. Although it may be somewhat artificial, it is sufficient for our minimal model program for projective morphisms of complex analytic spaces.

Definition 2.40 (Linear, $\mathbb{Q}$-linear, and $\mathbb{R}$-linear equivalences)**.** Two $\mathbb{R}$-divisors D_1 and D_2 are said to be *linearly equivalent* if $D_1 - D_2$ is a principal Cartier divisor. The linear equivalence is denoted by $D_1 \sim D_2$. Two $\mathbb{R}$-divisors D_1 and D_2 are said to be $\mathbb{R}$-*linearly equivalent* (resp. $\mathbb{Q}$-*linearly equivalent*) if $D_1 - D_2$ is a *finite* $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of principal Cartier divisors. When D_1 is $\mathbb{R}$-linearly (resp. $\mathbb{Q}$-linearly) equivalent to D_2, we write $D_1 \sim_{\mathbb{R}} D_2$ (resp. $D_1 \sim_{\mathbb{Q}} D_2$).

Example 2.41. Let X be a noncompact complex manifold with $\dim X = 1$. Then it is known that X is always Stein. We assume that $H^2(X, \mathbb{Z}) = 0$ holds. Let D be an $\mathbb{R}$-divisor on X such that $\operatorname{Supp} D$ is finite. Then $D \sim_{\mathbb{R}} 0$, that is, D is $\mathbb{R}$-linearly trivial. On the other hand, if $\operatorname{Supp} D$ is not finite, then D is not necessarily $\mathbb{R}$-linearly trivial in the sense of Definition 2.40. As in Example 2.33, if D is an integral Weil divisor on X, then $D \sim 0$ always holds.

We will use the following lemma in the proof of Theorem F (3).

Lemma 2.42 (see [Kaw2, Lemma 1.12] and [Na3, Chapter II. 2.12. Lemma])**.** *Let X be a normal complex variety with only rational singularities and let K be a compact subset of X. Let D_i be an integral Weil divisor on X such that D_i is $\mathbb{Q}$-Cartier at K for $1 \leq i \leq k$. Then there exists a positive integer m such that mD_i is Cartier on some open neighborhood of K for every $1 \leq i \leq k$.*

Proof. We take an arbitrary point $x \in K$. By [Na3, Chapter II. 2.12. Lemma] (see [Kaw2, Lemma 1.12]), there exists a positive integer m_x such that $m_x D_i$ is Cartier at x for $1 \leq i \leq k$. This means that there exists an open neighborhood U_x of x such that $m_x D_i$ is Cartier on U_x for every $1 \leq i \leq k$. Since K is compact, we can take a positive integer m and an open neighborhood U of K such that mD_i is Cartier for every $1 \leq i \leq k$. This is what we wanted. □

In this paper, we usually consider the case where the base space Y is Stein and the morphism $\pi \colon X \to Y$ is projective. In this setting, we have many good properties.

Remark 2.43. Let $\pi \colon X \to Y$ be a projective morphism from a normal complex variety X to a Stein space Y. We take a π-ample line bundle $\mathcal{A}$ on X. Let ω_X be the canonical sheaf of X (see Definition 3.1 below). Since there exists a sufficiently large positive integer m such that

$$H^0(X, \omega_X \otimes \mathcal{A}^{\otimes m}) \simeq H^0(Y, \pi_*(\omega_X \otimes \mathcal{A}^{\otimes m})) \neq 0$$

and

$$H^0(X, \mathcal{A}^{\otimes m}) \simeq H^0(Y, \pi_* \mathcal{A}^{\otimes m}) \neq 0,$$

we can always take a Weil divisor K_X on X satisfying $\omega_X \simeq \mathcal{O}_X(K_X)$. As usual, we call it the *canonical divisor* of X. More generally, let $\mathcal{L}$ be a line bundle (resp. reflexive sheaf of rank one) on X. By the same argument as above, we can take a Cartier (resp. Weil) divisor D on X such that $\mathcal{L} \simeq \mathcal{O}_X(D)$.

2.44 (Ample, semiample, big, pseudo-effective, and nef $\mathbb{R}$-divisors)**.** In our framework of the minimal model program for projective morphisms of complex analytic spaces, we have to use $\mathbb{R}$-divisors. Hence we need the following definitions: Definitions 2.45, 2.46, 2.47, and 2.48. We state them explicitly here for the sake of completeness.

Definition 2.45 (Ample and semiample $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors, see Definition 2.35)**.** Let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces. A finite $\mathbb{R}_{>0}$-linear (resp. $\mathbb{Q}_{>0}$-linear) combination of π-ample Cartier divisors is called a π-*ample* $\mathbb{R}$-*divisor* (resp. π-*ample* $\mathbb{Q}$-*divisor*). A finite $\mathbb{R}_{>0}$-linear (resp. $\mathbb{Q}_{>0}$-linear) combination of

π-semiample Cartier divisors is called a *π-semiample $\mathbb{R}$-divisor* (resp. *π-semiample $\mathbb{Q}$-divisor*).

In this paper, we adopt the following definition of big $\mathbb{R}$-divisors.

Definition 2.46 (Bigness)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that X is a normal complex variety. When Y is Stein, an $\mathbb{R}$-divisor D is said to be *big over* Y or *π-big* if $D \sim_{\mathbb{R}} A + B$, where A is a π-ample $\mathbb{R}$-divisor and B is an effective $\mathbb{R}$-divisor. In general, if $D|_{\pi^{-1}(U)}$ is big over U for any Stein open subset of Y, then D is said to be *big over* Y or *π-big*. We note that D is not necessarily $\mathbb{R}$-Cartier.

Definition 2.47 (Pseudo-effective $\mathbb{R}$-divisors)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that X is a normal complex variety. An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X is said to be *pseudo-effective over* Y or *π-pseudo-effective* if $D + A$ is big over Y for every π-ample $\mathbb{R}$-divisor A on X.

Definition 2.48 (Nefness)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that X is a normal complex variety and let W be a compact subset of Y. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. If $D \cdot C \geq 0$ for every projective integral curve C on X such that $\pi(C)$ is a point, then D is said to be *π-nef* or *nef over* Y. If $D \cdot C \geq 0$ for every projective integral curve C on X such that $\pi(C)$ is a point of W, then D is said to be *nef over* W or *π-nef over* W.

Let us recall the definition of *analytically meagre subsets*. As we saw in Remark 2.31, the notion of Zariski open subsets does not work well in the category of complex analytic spaces. So we frequently have to use analytically meagre subsets.

Definition 2.49 (Analytically meagre subsets)**.** A subset $\mathcal{S}$ of a complex analytic space X is said to be *analytically meagre* if

$$\mathcal{S} \subset \bigcup_{n \in \mathbb{N}} Y_n,$$

where each Y_n is a locally closed analytic subset of X of codimension ≥ 1.

Definition 2.50 (Analytically sufficiently general points and fibers)**.** Let X be a complex analytic space. We say that a property P holds for an analytically sufficiently general point $x \in X$ when P holds for every point x contained in $X \setminus \mathcal{S}$ for some analytically meagre subset $\mathcal{S}$ of X.

Let $f\colon X \to Y$ be a morphism of analytic spaces. Similarly, we say that a property P holds for an *analytically sufficiently general fiber* of $f\colon X \to Y$ when P holds for $f^{-1}(y)$ for every $y \in Y \setminus \mathcal{S}$, where $\mathcal{S}$ is some analytically meagre subset of Y.

We sometimes use the notion of general π-ample $\mathbb{Q}$-divisors.

Definition 2.51. Let $\pi\colon X \to Y$ be a projective morphism from a normal variety X to a Stein space Y. Let A be a π-ample $\mathbb{Q}$-divisor on X. We say that A is a *general* π-ample $\mathbb{Q}$-divisor on X if there exist

(i) a large and divisible positive integer k such that kA is π-very ample,
(ii) a finite-dimensional linear subspace V of $H^0(X, \mathcal{O}_X(kA))$ which generates $\mathcal{O}_X(kA)$, and
(iii) some analytically meagre subset $\mathcal{S}$ of $\Lambda := (V \setminus \{0\})/\mathbb{C}^\times$,

such that $A = \frac{1}{k}A'$ for some $A' \in \Lambda \setminus \mathcal{S}$. Note that $\Lambda \simeq \mathbb{P}^N$ for some $N \in \mathbb{N}$.

Remark 2.52. Let $\pi\colon X \to Y$ be a projective morphism from a normal complex variety X to a Stein space Y. Let $\sigma\colon Z \to X$ be a projective bimeromorphic morphism from a smooth variety Z and let Σ be a simple normal crossing divisor on Z. Let $\{p_i\}_{i\in\mathbb{N}}$ be a set of points of X. Let A be a general π-ample $\mathbb{Q}$-divisor in the sense of Definition 2.51. Then, by Bertini's theorem (see [Man, (II.5) Theorem] and [Fu11, Theorem 3.2]), we may assume that $\sigma^{-1}_*A = \sigma^*A$ holds, $\Sigma + A'$ is a simple normal crossing divisor on Z, where $A = \frac{1}{k}A'$ as in Definition 2.51, and $p_i \notin \operatorname{Supp} A$ for every i, and so on.

The final result in this section is a very useful lemma. We will repeatedly use this lemma in the subsequent sections.

Lemma 2.53 ([BCHM, Lemma 3.2.1] and [HasH, Lemma 2.10])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex varieties such that X is normal and that Y is Stein. Let D be a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, that is, a finite $\mathbb{R}$-linear combination of Cartier divisors on X. Let F be an analytically sufficiently general fiber of $\pi\colon X \to Y$. Assume that $D' := D|_F \sim_{\mathbb{R}} B' \geq 0$ holds for some $\mathbb{R}$-divisor B' on F. Then there exists a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on X such that $D \sim_{\mathbb{R}} B \geq 0$.*

We closely follow the proof of [HasH, Lemma 2.10]. In the proof of Lemma 2.53, we will freely use the semicontinuity theorem and the base change theorem described in [BS, Chapter III], whose proof is much harder than the proof of the corresponding statements for algebraic varieties (see [Har, Chapter III, Section 12]).

Definition 2.54 (Iitaka–Kodaira dimensions)**.** Let X be a normal projective variety and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then $\kappa(X, D)$ denotes the *Iitaka–Kodaira dimension* of D.

Proof of Lemma 2.53. We can take a nonempty Zariski open subset U of Y such that $\pi\colon X \to Y$ is flat over U.

Step 1. We fix a representation $D = \sum_{i=1}^n r_i D_i$ of D as a finite $\mathbb{R}$-linear combination of Cartier divisors. Since F is an analytically sufficiently general fiber of $\pi\colon X \to Y$, we may assume that $D_i|_F$ are well defined as integral Weil divisors for all i. For any closed point $y \in U$, the fiber of π over y is denoted by X_y. For any $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{Q}^n$, we set $D_{\boldsymbol{p}} = \sum_{i=1}^n p_i D_i$. We fix a positive integer $k_{\boldsymbol{p}}$ such that $k_{\boldsymbol{p}} D_{\boldsymbol{p}}$ is Cartier. For every $\boldsymbol{p} \in \mathbb{Q}^n$ and every $m \in \mathbb{Z}_{>0}$, we put

$$S_{\boldsymbol{p},m} = \left\{ z \in U \;\middle|\; \dim H^0(X_y, \mathcal{O}_{X_y}(mk_{\boldsymbol{p}} D_{\boldsymbol{p}}|_{X_y})) = 0 \right\}.$$

Then $S_{\boldsymbol{p},m} = \emptyset$ holds or $U \setminus S_{\boldsymbol{p},m} = \emptyset$ is analytically meagre by the upper semicontinuity theorem. We set

$$J := \{(\boldsymbol{p}, m) \,|\, \boldsymbol{p} \in \mathbb{Q}^n, m \in \mathbb{Z}_{>0}, S_{\boldsymbol{p},m} \neq \emptyset\},$$

and put

$$W = \bigcap_{(\boldsymbol{p},m)\in J} S_{\boldsymbol{p},m}.$$

Then $U \setminus W$ is analytically meagre. Hence $Y \setminus W$ is also analytically meagre. We may assume that $F = X_{y_0}$ for some $y_0 \in W$ since F is an analytically sufficiently general fiber of $\pi\colon X \to Y$. Then, for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D_{\boldsymbol{p}'}$ associated to $\boldsymbol{p}' = (p'_1, \ldots, p'_n) \in \mathbb{Q}^n$, an inequality $\kappa(F, D_{\boldsymbol{p}'}|_F) \geq 0$ holds if and only if $D_{\boldsymbol{p}'} \sim_{\mathbb{Q}} E_{\boldsymbol{p}'}$ for some $E_{\boldsymbol{p}'} \geq 0$. Indeed, $\kappa(F, D_{\boldsymbol{p}'}|_F) \geq 0$ if and only if $y_0 \notin S_{\boldsymbol{p}',m}$ for some m. By the above definitions of W and J, the condition $y_0 \notin S_{\boldsymbol{p}',m}$ is equivalent to $S_{\boldsymbol{p}',m} = \emptyset$ since $y_0 \in W$. Since $k_{\boldsymbol{p}'} D_{\boldsymbol{p}'}$ is a Cartier divisor and Y is Stein, by the construction of $S_{\boldsymbol{p}',m}$, it is easy to check that $S_{\boldsymbol{p}',m} = \emptyset$ for some m if and only if $D_{\boldsymbol{p}'} \sim_{\mathbb{Q}} E_{\boldsymbol{p}'}$ for some $E_{\boldsymbol{p}'} \geq 0$.

Step 2. From our assumption that $D' \sim_{\mathbb{R}} B' \geq 0$, there are positive real numbers $a_1, \dots, a_s$, effective integral Weil divisors $E_1, \dots, E_s$ on F, real numbers $b_1, \dots, b_t$, and meromorphic functions $\phi_1, \dots, \phi_t$ on F such that

$$D|_F = \sum_{i=1}^{n} r_i D_i|_F = \sum_{j=1}^{s} a_j E_j + \sum_{k=1}^{t} b_k \cdot \mathrm{div}(\phi_k)$$

holds as $\mathbb{R}$-divisors on F. We consider the following set

$$\left\{ \boldsymbol{v}' = \big((r'_i)_i, (a'_j)_j, (b'_k)_k\big) \in \mathbb{R}^n \times (\mathbb{R}_{\geq 0})^s \times \mathbb{R}^t \,\middle|\, \sum_{i=1}^{n} r'_i D_i|_F = \sum_{j=1}^{s} a'_j E_j + \sum_{k=1}^{t} b'_k \cdot \mathrm{div}(\phi_k) \right\},$$

which contains $\boldsymbol{v} := \big((r_i)_i, (a_j)_j, (b_k)_k\big)$. Since all $D_i|_F$ are well defined as integral Weil divisors, we can find positive real numbers $\alpha_1, \dots, \alpha_{l_0}$ and rational points $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{l_0}$ in the above set such that $\sum_{l=1}^{l_0} \alpha_l = 1$ and $\sum_{l=1}^{l_0} \alpha_l \boldsymbol{v}_l = \boldsymbol{v}$. This shows that there are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $D^{(1)}, \dots, D^{(l_0)}$ on X such that $\sum_{l=1}^{l_0} \alpha_l D^{(l)} = D$ and $\kappa(F, D^{(l)}|_F) \geq 0$ for every $1 \leq l \leq l_0$. We note that each $D^{(l)}$ is a finite $\mathbb{Q}$-linear combination of Cartier divisors by construction. By the argument in Step 1, for every $1 \leq l \leq l_0$, there exists a $\mathbb{Q}$-divisor $E^{(l)} \geq 0$ on X with $D^{(l)} \sim_{\mathbb{Q}} E^{(l)}$. We put $B = \sum_{l=1}^{l_0} \alpha_l E^{(l)}$. Then we have $D \sim_{\mathbb{R}} B \geq 0$.

We finish the proof. □

We close this section with an important remark on [BCHM, Lemma 3.2.1].

Remark 2.55. In [BCHM], [BCHM, Lemma 3.2.1] plays a crucial role. We think that the quasi-projectivity is indispensable in the framework of [BCHM] since we have to assume that U is quasi-projective in [BCHM, Lemma 3.2.1]. Let $\pi\colon X \to U$ be a projective morphism of normal complete algebraic varieties with connected fibers such that X is a smooth projective variety and $\mathrm{Pic}(U) = \{0\}$. Let D be a Cartier divisor on X such that $-D$ is effective, $D \neq 0$, and $\pi(D) \subsetneq U$. Then there exists no effective $\mathbb{R}$-divisor B on X satisfying $D \sim_{\mathbb{R},U} B \geq 0$. This means that [BCHM, Lemma 3.2.1] does not always hold true without assuming the quasi-projectivity of U.

3. Singularities of pairs

In this section, we will define singularities of pairs in the complex analytic setting. The definition is essentially the same as the one for algebraic varieties.

Definition 3.1 (Singularities of pairs)**.** Let X be a normal complex variety. The *canonical sheaf* ω_X of X is the unique reflexive sheaf whose restriction to X_{sm} is isomorphic to the sheaf $\Omega^n_{X_{\mathrm{sm}}}$, where X_{sm} is the smooth locus of X and $n = \dim X$. Let Δ be an $\mathbb{R}$-divisor on X. We say that $K_X + \Delta$ is $\mathbb{R}$-Cartier at $x \in X$ if there exist an open neighborhood U_x of x and a Weil divisor K_{U_x} on U_x with $\mathcal{O}_{U_x}(K_{U_x}) \simeq \omega_X|_{U_x}$ such that $K_{U_x} + \Delta|_{U_x}$ is $\mathbb{R}$-Cartier at x. We simply say that $K_X + \Delta$ is $\mathbb{R}$-Cartier when $K_X + \Delta$ is $\mathbb{R}$-Cartier at any point $x \in X$. Unfortunately, we can not define K_X globally with $\mathcal{O}_X(K_X) \simeq \omega_X$. It only exists locally on X. However, we use the symbol K_X as a formal divisor class with an isomorphism $\mathcal{O}_X(K_X) \simeq \omega_X$ and call it the *canonical divisor* of X if there is no danger of confusion.

Let $f\colon Y \to X$ be a proper bimeromorphic morphism between normal complex varieties. Suppose that $K_X + \Delta$ is $\mathbb{R}$-Cartier in the above sense. We take a small Stein open subset

U of X where $K_U + \Delta|_U$ is a well-defined $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on U. In this situation, we can define $K_{f^{-1}(U)}$ and K_U such that $f_*K_{f^{-1}(U)} = K_U$. Then we can write

$$K_{f^{-1}(U)} = f^*(K_U + \Delta|_U) + E_U$$

as usual. Note that E_U is a well-defined $\mathbb{R}$-divisor on $f^{-1}(U)$ such that $f_*E_U = \Delta|_U$. Then we have the following formula

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

as in the algebraic case. We note that $\sum_E a(E, X, \Delta)E$ is a globally well-defined $\mathbb{R}$-divisor on Y such that $(\sum_E a(E, X, \Delta)E)\,|_{f^{-1}(U)} = E_U$ although K_X and K_Y are well defined only locally.

If Δ is a boundary $\mathbb{R}$-divisor and $a(E, X, \Delta) \geq -1$ holds for any $f\colon Y \to X$ and every f-exceptional divisor E, then (X, Δ) is called a *log canonical* pair. If (X, Δ) is log canonical and $a(E, X, \Delta) > -1$ for any $f\colon Y \to X$ and every f-exceptional divisor E, then (X, Δ) is called a *purely log terminal* pair. If (X, Δ) is purely log terminal and $\lfloor \Delta \rfloor = 0$, then (X, Δ) is called a *kawamata log terminal* pair. When $\Delta = 0$ and $a(E, X, 0) \geq 0$ (resp. > 0) for any $f\colon Y \to X$ and every f-exceptional divisor E, we simply say that X has only *canonical singularities* (resp. *terminal singularities*).

Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. The image of E with $a(E, X, \Delta) \leq -1$ for some $f\colon Y \to X$ is called a *non-kawamata log terminal center* of (X, Δ). The image of E with $a(E, X, \Delta) = -1$ for some $f\colon Y \to X$ such that (X, Δ) is log canonical around general points of $f(E)$ is called a *log canonical center* of (X, Δ). When (X, Δ) is log canonical, a closed subset of X is a log canonical center of (X, Δ) if and only if it is a non-kawamata log terminal center of (X, Δ) by definition. In the above setting, (X, Δ) is kawamata log terminal if and only if there are no non-kawamata log terminal centers of (X, Δ).

Remark 3.2. If we only assume that Δ is a subboundary $\mathbb{R}$-divisor on X in the above definition of log canonical pairs and kawamata log terminal pairs, then (X, Δ) is said to be a *sub log canonical pair* and *sub kawamata log terminal pair*, respectively. We will use sub log canonical pairs and sub kawamata log terminal pairs in Section 21.

Remark 3.3. Let X be a normal algebraic variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let X^{an} be the complex analytic space naturally associated to X and let Δ^{an} be the $\mathbb{R}$-divisor on X associated to Δ. Then $(X^{\mathrm{an}}, \Delta^{\mathrm{an}})$ is terminal, canonical, kawamata log terminal, purely log terminal, and log canonical in the sense of Definition 3.1 if and only if (X, Δ) is terminal, canonical, kawamata log terminal, purely log terminal, and log canonical in the usual sense, respectively. For the details, see, for example, [Matk, Proposition 4-4-4].

In this paper, we need the following local definition of *log canonical singularities* and *kawamata log terminal singularities*.

Definition 3.4. Let X be a normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X. We say that (X, Δ) is *log canonical* (resp. *kawamata log terminal*) *at* $x \in X$ if there exits an open neighborhood U_x of x such that $(U_x, \Delta|_{U_x})$ is a log canonical pair (resp. kawamata log terminal pair). We note that (X, Δ) is log canonical (resp. kawamata log terminal) in the sense of Definition 3.1 if and only if (X, Δ) is log canonical (resp. kawamata log terminal) at any point x of X.

Let K be a compact subset of X. Then we say that (X,Δ) is *log canonical* (resp. *kawamata log terminal*) *at* K if (X,Δ) is log canonical (resp. kawamata log terminal) at any point x of K. We note that (X,Δ) is log canonical (resp. kawamata log terminal) at K if and only if there exits an open neighborhood U of K such that $(U,\Delta|_U)$ is log canonical (resp. kawamata log terminal).

The following lemma is very fundamental.

Lemma 3.5 ([BCHM, Lemma 3.7.2])**.** *Let X be a normal complex variety and let V be a finite-dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$*, which is defined over the rationals. Let K be a compact subset of X. Then*

$$\mathcal{L}(V;K) := \{\Delta \in V \mid K_X+\Delta \textit{ is log canonical at } K\}$$

is a rational polytope. Moreover, there exists an open neighborhood U of K such that $(U,\Delta|_U)$ is log canonical for every $\Delta\in\mathcal{L}(V;K)$.

Proof. We note that the set of divisors Δ such that $K_X+\Delta$ is $\mathbb{R}$-Cartier at K forms an affine subspace V' of V. Since V is defined over the rationals, we can easily see that V' is also defined over the rationals (see the proof of Lemma 2.37). Hence, by replacing V with V', we may assume that $K_X+\Delta$ is $\mathbb{R}$-Cartier at K for every $\Delta\in V$. Since V is finite-dimensional, there is an open neighborhood U' of K such that $K_X+\Delta$ is $\mathbb{R}$-Cartier on U' for every $\Delta\in V$. By replacing X with U', we may assume that $K_X+\Delta$ is $\mathbb{R}$-Cartier for every $\Delta\in V$. Let Θ be the union of the support of any element of V. By shrinking X around K, we can take a projective birational morphism $f\colon Y\to X$ from a smooth complex variety Y such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f)\cup\operatorname{Supp} f^{-1}_*\Theta$ are simple normal crossing divisors on Y. Thus, we can easily check that $\mathcal{L}(V;K)$ is a rational polytope. Let $\Delta_1,\dots,\Delta_k$ be the vertices of $\mathcal{L}(V;K)$. Then (X,Δ_i) is log canonical on some open neighborhood U_i of K for every i. We put $U:=\bigcap_{i=1}^k U_i$. Then (X,Δ) is log canonical on U for every $\Delta\in\mathcal{L}(V;K)$. Hence U is a desired open neighborhood of K. □

We note the following elementary property. We explicitly state it for the sake of completeness.

Lemma 3.6. *Let (X,Δ) be a log canonical pair and let C be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $(X,\Delta+C)$ is log canonical. Let ε be any positive real number such that $0<\varepsilon\le 1$. Then V is a log canonical center of (X,Δ) if and only if V is a log canonical center of $(X,\Delta+(1-\varepsilon)C)$.*

Proof. This is obvious by definition. □

The definition of *divisorial log terminal pairs* is very subtle. We adopt the following definition, which is suitable for our purposes.

Definition 3.7 (Divisorial log terminal pairs)**.** Let X be a normal complex variety and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. If there exists a proper bimeromorphic morphism $f\colon Y\to X$ from a smooth complex variety Y such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f)\cup\operatorname{Supp} f^{-1}_*\Delta$ are simple normal crossing divisors on Y and that the discrepancy coefficient $a(E,X,\Delta)>-1$ holds for every f-exceptional divisor E, then (X,Δ) is called a *divisorial log terminal pair*.

Remark 3.8. By definition, we can easily check that a divisorial log terminal pair is a log canonical pair. Let (X,Δ) be a kawamata log terminal pair and let U be any relatively compact open subset of X. Then we can easily check that $(U,\Delta|_U)$ is a divisorial log terminal pair.

The morphism f in Definition 3.7 can be taken as a composite of blow-ups over any relatively compact open subset.

Lemma 3.9. *Let (X, Δ) be a divisorial log terminal pair. Then there exists a morphism $\sigma\colon Z \to X$ such that, for any relatively compact open subset X' of X,*

$$g := \sigma|_{\sigma^{-1}(X')}\colon Z' := \sigma^{-1}(X') \to X'$$

is a composite of a finite sequence of blow-ups, $\mathrm{Exc}(g)$ *and* $\mathrm{Exc}(g) \cup \mathrm{Supp}\, g_*^{-1}\Delta$ *are simple normal crossing divisors on Z', $a(E, X', \Delta|_{X'}) > -1$ holds for every g-exceptional divisor E. In particular, we can take an effective divisor F on Z' such that* $\mathrm{Exc}(g) = F$ *and that $-F$ is g-very ample.*

Proof. It is sufficient to apply the resolution of singularities explained in [BM2, Sections 12 and 13]. For the details, see [BM2, Theorems 13.3 and 12.4]. See also [Kol2, 3.44 (Analytic spaces)], [W], and [Kol3, Theorem 10.45 and Proposition 10.49]. □

Our definition of divisorial log terminal pairs is compatible with the usual definition of divisorial log terminal pairs for algebraic varieties.

Lemma 3.10. *Let X be a normal algebraic variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let X^{an} be the complex analytic space naturally associated to X and let Δ^{an} be the $\mathbb{R}$-divisor on X associated to Δ. Then (X, Δ) is divisorial log terminal in the usual sense if and only if $(X^{\mathrm{an}}, \Delta^{\mathrm{an}})$ is divisorial log terminal in the sense of Definition 3.7.*

Sketch of Proof of Lemma 3.10. If (X, Δ) is divisorial log terminal in the usual sense, then it is obvious that $(X^{\mathrm{an}}, \Delta^{\mathrm{an}})$ is divisorial log terminal in the sense of Definition 3.7. From now on, we assume that $(X^{\mathrm{an}}, \Delta^{\mathrm{an}})$ is divisorial log terminal in the sense of Definition 3.7. Let $f\colon Y \to X^{\mathrm{an}}$ be a projective morphism from a smooth complex variety as in Definition 3.7. We put $Z' := f(\mathrm{Exc}(f))$, which is a closed analytic subset of X^{an}. Then $X^{\mathrm{an}} \setminus Z'$ is smooth and the support of $\Delta^{\mathrm{an}}|_{X^{\mathrm{an}} \setminus Z'}$ is a simple normal crossing divisor on $X^{\mathrm{an}} \setminus Z'$. We can check that if $g\colon V \to X$ is a projective bimeromorphic morphism from a smooth complex variety V and E is a prime divisor on V such that $g(E) \subset Z'$ then $a(E, X, \Delta) > -1$ holds by the proof of [KM, Proposition 2.40]. Let Z be the smallest closed algebraic subset of X such that $X \setminus Z$ is smooth and the support of $\Delta|_{X \setminus Z}$ is a simple normal crossing divisor on $X \setminus Z$. Then $Z \subset Z'$ holds by definition. Hence, as in the proof of [Matk, Proposition 4-4-4], we see that (X, Δ) is divisorial log terminal in the usual sense. □

Of course, Definition 3.7 is not analytically local.

Example 3.11. Let X be a smooth algebraic surface and let C be an irreducible curve on X with only one singular point P. Assume that P is a node. It is obvious that $(X \setminus P, C|_{X \setminus P})$ is divisorial log terminal, but (X, C) is not divisorial log terminal. However, there exists a small open neighborhood U of P such that $(U, C|_U)$ is divisorial log terminal in the sense of Definition 3.7. Note that $C|_U$ is a simple normal crossing divisor on U if U is a small open neighborhood of P in X.

The final theorem in this section is more or less well known to the experts. We will use it in the proof of Theorem F (3).

Theorem 3.12 (see [KMM, Theorem 1-3-6] and [Na3, Chapter VII. §1])**.** *If (X, Δ) is divisorial log terminal, then X has only rational singularities.*

Proof. The arguments in [Fu9, 3.14 Elkik–Fujita vanishing theorem] work with some minor modifications if we use Grothendieck duality for proper morphisms of complex analytic spaces (see [RRV]). We note that we have necessary vanishing theorems in the complex analytic setting (see Section 5 below). □

4. Cones

In this section, we will define various cones and explain Kleiman's ampleness criterion for projective morphisms between complex analytic spaces.

Throughout this section, let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y. Let $Z_1(X/Y;W)$ be the free abelian group generated by the projective integral curves C on X such that $\pi(C)$ is a point of W. Let U be any open neighborhood of W. Then we can consider the following intersection pairing

$$\cdot : \mathrm{Pic}\bigl(\pi^{-1}(U)\bigr) \times Z_1(X/Y;W) \to \mathbb{Z}$$

given by $\mathcal{L}\cdot C \in \mathbb{Z}$ for $\mathcal{L} \in \mathrm{Pic}(\pi^{-1}(U))$ and $C \in Z_1(X/Y;W)$. We say that $\mathcal{L}$ is *π-numerically trivial over* W when $\mathcal{L}\cdot C = 0$ for every $C \in Z_1(X/Y;W)$. We take $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}(\pi^{-1}(U))$. If $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ is π-numerically trivial over W, then we write $\mathcal{L}_1 \equiv_W \mathcal{L}_2$ and say that $\mathcal{L}_1$ is numerically equivalent to $\mathcal{L}_2$ over W. We put

$$\widetilde{A}(U,W) := \mathrm{Pic}\bigl(\pi^{-1}(U)\bigr) /{\equiv_W}$$

and define

$$A^1(X/Y;W) := \varinjlim_{W\subset U} \widetilde{A}(U,W),$$

where U runs through all the open neighborhoods of W. The following lemma due to Nakayama is a key result of the minimal model program for projective morphisms between complex analytic spaces.

Lemma 4.1 (Nakayama's finiteness, see [Na3, Chapter II. 5.19. Lemma])**.** *Assume that $W\cap Z$ has only finitely many connected components for every analytic subset Z defined over an open neighborhood of W. Then $A^1(X/Y;W)$ is a finitely generated abelian group.*

Proof. For the details, see the proof of [Na3, Chapter II. 5.19. Lemma] and Theorem 4.7 in Subsection 4.1 below. □

Remark 4.2. Note that [Na3, Chapter II. 5.19. Lemma], that is, Lemma 4.1 above, is a correction of [Na2, Proposition 4.3 and Lemma 4.4]. In Lemma 4.1, W is not assumed to be Stein compact. Here, we only assume that W is a compact subset of Y satisfying (P4). We will discuss Lemma 4.1 in detail in Subsection 4.1 below.

Under the assumption of Lemma 4.1, we can define the *relative Picard number* $\rho(X/Y;W)$ to be the rank of $A^1(X/Y;W)$. We put

$$N^1(X/Y;W) := A^1(X/Y;W) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $A_1(X/Y;W)$ be the image of

$$Z_1(X/Y;W) \to \mathrm{Hom}_{\mathbb{Z}}\bigl(A^1(X/Y;W), \mathbb{Z}\bigr)$$

given by the above intersection pairing. Then we set

$$N_1(X/Y;W) := A_1(X/Y;W) \otimes_{\mathbb{Z}} \mathbb{R}.$$

As usual, we can define the *Kleiman–Mori cone*

$$\overline{\mathrm{NE}}(X/Y;W)$$

of $\pi\colon X \to Y$ over W, that is, $\overline{\mathrm{NE}}(X/Y;W)$ is the closure of the convex cone in $N_1(X/Y;W)$ spanned by the projective integral curves C such that $\pi(C)$ is a point of W. We also define $\mathrm{Amp}(X/Y;W)$ to be the cone in $N^1(X/Y;W)$ generated by line bundles L such that $L|_{\pi^{-1}(U)}$ is π-ample for some open neighborhood U of W. An element $\zeta \in N^1(X/Y;W)$ is called *π-nef over W* or *nef over W* if $\zeta \geq 0$ on $\overline{\mathrm{NE}}(X/Y;W)$. Even when ζ is nef over W, it is not clear whether ζ is nef over some open neighborhood of W or not.

Remark 4.3 (see [Le, Theorem 1.2])**.** There exist a projective surjective morphism of algebraic varieties $\pi\colon X \to Y$ and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X such that $\{y \in Y \mid D|_{X_y} \text{ is nef}\}$ is not Zariski open. This means that the nefness is not an open condition. For a criterion of openness of a family of nef line bundles, see [Mw].

On the other hand, for $\zeta \in N^1(X/Y;W)$, $\zeta|_{X_w}$ is ample for every $w \in W$ if and only if ζ is ample over some open neighborhood of W. This is because the ampleness is an open condition (see, for example, [KM, Proposition 1.41] and [Na2, Proposition 1.4]). Note that Kleiman's ampleness criterion holds true in our complex analytic setting.

Theorem 4.4 (Kleiman's criterion, see [Na2, Proposition 4.7])**.** *Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces and let W be a compact subset of Y such that W satisfies* (P4). *Then we have*

$$\mathrm{Amp}(X/Y;W) = \left\{\zeta \in N^1(X/Y;W) \mid \zeta > 0 \text{ on } \overline{\mathrm{NE}}(X/Y;W) \setminus \{0\}\right\}.$$

Sketch of Proof of Theorem 4.4. We note that the ample cone is an open convex cone in $N^1(X/Y;W)$. Hence we can easily check that it is contained in the right hand side. Therefore, it is sufficient to prove the opposite inclusion. We take a π-ample Cartier divisor A on X. Let ζ be an element of $N^1(X/Y;W)$ such that $\zeta > 0$ on $\overline{\mathrm{NE}}(X/Y;W) \setminus \{0\}$. Then $\zeta - \varepsilon A > 0$ on $\overline{\mathrm{NE}}(X/Y;W) \setminus \{0\}$ for some small positive rational number ε. This implies that $(\zeta - \varepsilon A)|_{\pi^{-1}(w)}$ is nef for every $w \in W$. Since $A|_{\pi^{-1}(w)}$ is ample, $\zeta|_{\pi^{-1}(w)} = (\zeta - \varepsilon A)|_{\pi^{-1}(w)} + A|_{\pi^{-1}(w)}$ is ample for every $w \in W$. We note that we can use Kleiman's ampleness criterion on $\pi^{-1}(w)$ since $\pi^{-1}(w)$ is projective. Hence, by the standard argument (see, for example, [FMi, Section 6]), we can write ζ as a finite $\mathbb{R}_{>0}$-linear combination of π-ample Cartier divisors over some open neighborhood of W. This is what we wanted. ☐

We can define *movable cones* in our complex analytic setting (see [Kaw2, Section 2]).

Definition 4.5 (see [Fu4, Definition 2.1])**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that X is a normal complex variety. A Cartier divisor D on $\pi^{-1}(U)$, where U is some open neighborhood of W, is called *π-movable over W* if $\pi_*\mathcal{O}_{\pi^{-1}(U)}(D) \neq 0$ and if the cokernel of the natural homomorphism $\pi^*\pi_*\mathcal{O}_{\pi^{-1}(U)}(D) \to \mathcal{O}_{\pi^{-1}(U)}(D)$ has a support of codimension ≥ 2. We define $\overline{\mathrm{Mov}}(X/Y;W)$ as the closure of the convex cone in $N^1(X/Y;W)$ generated by the classes of π-movable Cartier divisors over W. Note that $\overline{\mathrm{Mov}}(X/Y;W)$ is usually called the *movable cone* of $\pi\colon X \to Y$ and W.

We can easily see that a kind of negativity lemma holds.

Lemma 4.6 (Negativity lemma, see [Fu14, Lemma 3.8])**.** *Let $\pi\colon X \to Y$ be a projective bimeromorphic contraction morphism of normal complex varieties and let W be a compact subset of Y. Let E be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $E \in \overline{\mathrm{Mov}}(X/Y;W)$. Let U be any open subset of Y with $U \subset W$. If $-\pi_*E|_U$ is effective, then $-E|_{\pi^{-1}(U)}$ is effective.*

Proof. For the details, see the proof of [Fu14, Lemma 3.8]. □

We will use Lemma 4.6 in order to terminate minimal model programs with scaling.

4.1. **Nakayama's finiteness.** In this subsection, we give a detailed proof of Nakayama's finiteness (see Lemma 4.1), which is the starting point of the minimal model program for projective morphisms between complex analytic spaces, for the sake of completeness. We will closely follow Nakayama's original proof in [Na3]. The reader who is not interested in the proof can skip this subsection.

Let us recall the statement of Nakayama's finiteness for the reader's convenience.

Theorem 4.7 (Nakayama's finiteness, see Lemma 4.1 and [Na3, Chapter II. 5.19. Lemma])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y. Assume that $W \cap Z$ has only finitely many connected components for every analytic subset Z defined over an open neighborhood of W. Then $A^1(X/Y;W)$ is a finitely generated abelian group.*

In this subsection, an $\mathbb{R}$-*line bundle* on a complex analytic space X means an element of $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$. For simplicity of notation, we write the group law of $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ additively.

Definition 4.8. Let $\pi\colon X \to Y$ be a projective surjective morphism of complex analytic spaces. An $\mathbb{R}$-line bundle $\mathcal{L}$ on X is called π-*ample* if it is a finite $\mathbb{R}_{>0}$-linear combination of π-ample line bundles on X. Let Z be any subset of Y. An $\mathbb{R}$-line bundle $\mathcal{L}$ on X is called π-*nef over* Z and π-*numerically trivial over* Z if $\mathcal{L}|_{X_y}$ is nef and numerically trivial for every $y \in Z$, respectively, where $X_y := \pi^{-1}(y)$.

Let us see numerically trivial $\mathbb{R}$-line bundles on smooth projective varieties.

4.9 (Characterization of numerically trivial $\mathbb{R}$-line bundles on smooth projective varieties)**.** Let X be a smooth projective variety. We consider the following long exact sequence:

$$\cdots \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_X) \longrightarrow \cdots$$

given by $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$. The image of

$$c_1\colon \mathrm{Pic}(X) \simeq H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$$

is denoted by $\mathrm{NS}(X)$. It is usually called the *Neron–Severi group* of X. We note that a line bundle $\mathcal{L}$ on X is numerically trivial if and only if $c_1(\mathcal{L})$ is a torsion element in $\mathrm{NS}(X)$ (see, for example, [La1, Corollary 1.4.38]). Hence c_1 induces the following isomorphism:

$$\mathrm{Pic}(X)/\!\equiv \xrightarrow{\sim} \mathrm{NS}(X)/(\text{torsion}),$$

where $\equiv$ denotes the *numerical equivalence*. As usual, we put

$$N^1(X) = \{\mathrm{Pic}(X)/\!\equiv\} \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let us consider an $\mathbb{R}$-line bundle $\mathcal{L}$ on X. We can define the first Chern class $c_1(\mathcal{L})$ in $H^2(X,\mathbb{R})$ since $\mathcal{L}$ is a finite $\mathbb{R}$-linear combination of line bundles. If $\mathcal{L}$ is numerically trivial, then it is easy to see that $\mathcal{L}$ is a finite $\mathbb{R}$-linear combination of numerically trivial line bundles on X. Therefore, $\mathcal{L}$ is numerically equivalent to zero, that is, $\mathcal{L} \equiv 0$, if and only if $c_1(\mathcal{L}) = 0$ in $H^2(X,\mathbb{R})$.

Let us start with the following basic properties of ample and nef $\mathbb{R}$-line bundles. We can check them without any difficulties.

Lemma 4.10 (see [Na3, Chapter II. 5.14. Lemma])**.** *Let $\pi\colon X \to Y$ be a projective surjective morphism of complex analytic spaces such that Y is irreducible and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X.*

(1) *Assume that $\mathcal{L}|_{X_{y_0}}$ is ample for some $y_0 \in Y$. Then there exists an open neighborhood U of y_0 such that $\mathcal{L}|_{\pi^{-1}(U)}$ is ample over U and that $Y \setminus U$ is an analytically meagre subset of Y.*
(2) *Assume that $\mathcal{L}|_{X_{y_0}}$ is nef for some $y_0 \in Y$. Then there exists an analytically meagre subset $\mathcal{S}$ such that $\mathcal{L}$ is π-nef over $Y \setminus \mathcal{S}$ with $y_0 \notin \mathcal{S}$.*
(3) *Assume that $\mathcal{L}|_{X_{y_0}}$ is numerically trivial for some $y_0 \in Y$. Then there exists an analytically meagre subset $\mathcal{S}$ such that $\mathcal{L}$ is π-numerically trivial over $Y \setminus \mathcal{S}$ with $y_0 \notin \mathcal{S}$.*

Proof. (1) We can write $\mathcal{L} = \sum_{i=1}^k a_i \mathcal{L}_i$ such that $a_i \in \mathbb{R}_{>0}$ and $\mathcal{L}_i|_{X_{y_0}}$ is an ample line bundle for every i. If U_i is a desired open neighborhood of y_0 for $\mathcal{L}_i$, then $U := \bigcap_{i=1}^k U_i$ has the desired property for $\mathcal{L}$. Hence it is sufficient to prove this statement under the assumption that $\mathcal{L}$ is a line bundle. In this case, it is well known that there exists an open neighborhood V of y_0 such that $\mathcal{L}|_{\pi^{-1}(V)}$ is π-ample over V (see, for example, [KM, Proposition 1.41] and [Na2, Proposition 1.4]). Therefore, if m is a sufficiently large positive integer, then $\pi^*\pi_*\mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m}$ is surjective on a neighborhood of X_{y_0}. We put $Y^\dagger := Y \setminus \pi\left(\mathrm{Supp}\left(\mathrm{Coker}\, \pi^*\pi_*\mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m}\right)\right)$. Then $Y^\dagger$ is a Zariski open subset of Y and $\pi^*\pi_*\mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m}$ is surjective over $Y^\dagger$. We consider the induced map $\varphi\colon X^\dagger := \pi^{-1}(Y^\dagger) \to \mathbb{P}_{Y^\dagger}\left((\pi_*\mathcal{L}^{\otimes m})|_{Y^\dagger}\right)$ over $Y^\dagger$. Since m is sufficiently large, we may assume that φ is a closed embedding over some open neighborhood of y_0. In particular, φ is flat in a neighborhood of X_{y_0}. Hence, there exists a Zariski open subset U of $Y^\dagger$ such that φ is flat and finite over U (see [BS, Chapter V. Theorem 4.5]). This implies that $\mathcal{L}|_{\pi^{-1}(U)}$ is ample over U. We can check that U is an open neighborhood of y_0 with the desired properties.

(2) We take an ample line bundle $\mathcal{A}$. Then $(m\mathcal{L} + \mathcal{A})\,|_{X_{y_0}}$ is ample for every positive integer m. By (1), we can take an open neighborhood U_m of y_0 such that $(m\mathcal{L} + \mathcal{A})\,|_{\pi^{-1}(U_m)}$ is ample over U_m and that $Y \setminus U_m$ is an analytically meagre subset for every m. We put $\mathcal{S} := \bigcup_{m \in \mathbb{Z}_{>0}} (Y \setminus U_m)$. Then $\mathcal{S}$ is an analytically meagre subset and $\mathcal{L}$ is π-nef over $Y \setminus \mathcal{S}$. By construction, $y_0 \notin \mathcal{S}$. This is what we wanted.

(3) By assumption, $\mathcal{L}|_{X_{y_0}}$ and $-\mathcal{L}|_{X_{y_0}}$ are both nef. By (2), there exists analytically meagre subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{L}$ is π-nef over $Y \setminus \mathcal{S}_1$ and $-\mathcal{L}$ is π-nef over $Y \setminus \mathcal{S}_2$. We put $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$. Then $\mathcal{S}$ is also an analytically meagre subset and $\mathcal{L}$ is π-numerically trivial over $Y \setminus \mathcal{S}$ with $y_0 \notin \mathcal{S}$. □

Remark 4.11. In Lemma 4.10 (2) and (3), we can write $Y \setminus \mathcal{S} = \bigcap_{i \in \mathbb{Z}_{>0}} U_i$ such that U_i is an open neighborhood of y_0 and $Y \setminus U_i$ is an analytically meagre subset for every $i \in \mathbb{Z}_{>0}$.

The following lemma is a key lemma for the proof of Nakayama's finiteness.

Lemma 4.12 (see [Na3, Chapter II. 5.14. Lemma])**.** *Let $\pi\colon X \to Y$ be a projective surjective morphism between complex manifolds such that Y is connected and let $\mathcal{L}$ be an $\mathbb{R}$-line bundle on X. Assume that $\mathcal{L}|_{X_{y_0}}$ is numerically trivial for some $y_0 \in Y$. Then $\mathcal{L}$ is π-numerically trivial over Y.*

Proof. By Lemma 4.10 (3), there exists an analytically meagre subset $\mathcal{S}$ such that $\mathcal{L}$ is π-numerically trivial over $Y \setminus \mathcal{S}$. Therefore, it is sufficient to prove the statement under the extra assumption that Y is a polydisc and y_0 is its origin. We can define the first

Chern class $c_1(\mathcal{L})$ in $H^2(X,\mathbb{R})$ since $\mathcal{L}$ is a finite $\mathbb{R}$-linear combination of line bundles. It is sufficient to prove that $c_1(\mathcal{L}) = 0$ in $H^2(X,\mathbb{R})$. Since π is smooth and proper, X is diffeomorphic to $Y \times F$, where $F := X_{y_0}$, and $\pi\colon X \to Y$ is diffeomorphic to the first projection $p_1\colon Y \times F \to Y$ (see, for example, [Kod, Theorem 2.5]). On the other hand,

$$H^2(Y \times F, \mathbb{R}) \simeq H^2(F, \mathbb{R}) \tag{4.1}$$

holds as a very special case of Künneth formula (see, for example, [BT, (5.9)]), which can be checked by Poincaré's lemma. By the isomorphism $H^2(X,\mathbb{R}) \simeq H^2(Y \times F,\mathbb{R})$ and (4.1), $c_1(\mathcal{L}) = p_2^*c_1(\mathcal{L}|_{X_{y_0}})$, where $p_2\colon Y \times F \to F$ is the second projection. Since $c_1(\mathcal{L}|_{X_{y_0}}) = 0$, we obtain $c_1(\mathcal{L}) = 0$. This implies that $\mathcal{L}$ is π-numerically trivial over Y. This is what we wanted. □

By Lemmas 4.10 and 4.12, we can prove:

Theorem 4.13 (see [Na3, Chapter II. 5.14. Lemma])**.** *Let $\pi\colon X \to Y$ be a projective surjective morphism between normal complex varieties and let W be a compact subset of W. For a point $y_0 \in W$, after shrinking Y around W suitably, there exists a Zariski open subset U of Y containing y_0 having the following property: If an $\mathbb{R}$-line bundle on X is π-numerically trivial over the point $y_0 \in Y$, then it is π-numerically trivial over U.*

Proof. Throughout this proof, we will repeatedly shrink Y around W suitably without mentioning it explicitly. By taking a resolution of singularities, we have a projective surjective morphism $X_0 \to X$ from a smooth complex variety X_0. Let π_0 be the composition of $X_0 \to X$ and $\pi\colon X \to Y$. Let $Y_1 \subset Y$ be an analytic subset such that $\dim Y_1 < \dim Y$, $Y \setminus Y_1$ is smooth, and π_0 is smooth over $Y \setminus Y_1$. Let $X_1 \to \pi_0^{-1}(Y_1)$ be a projective bimeromorphic morphism from a smooth complex analytic space X_1 obtained by taking resolutions of singularities of irreducible components of $\pi_0^{-1}(Y_1)$. Then we obtain a sequence of analytic subsets

$$Y =: Y_0 \supset Y_1 \supset \cdots \supset Y_l \supset Y_{l+1}$$

and projective surjective morphisms $\pi_i\colon X_i \to Y_i$, and projective surjective morphisms $X_i \to \pi_{i-1}^{-1}(Y_i)$ for $1 \le i \le l$ such that

(i) $\dim_y Y_i < \dim_y Y_{i+1}$ holds at any point $y \in Y_i$,
(ii) $Y_i \setminus Y_{i+1}$ is smooth,
(iii) π_i is smooth over $Y_i \setminus Y_{i+1}$,
(iv) π_i is nothing but the composition $X_i \to \pi_{i-1}^{-1}(Y_i) \to Y_i$,
(v) π_i is projective, and
(vi) $y_0 \in Y_l \setminus Y_{l+1}$.

Let S be any connected component of $Y_i \setminus Y_{i+1}$ for some $i \le l$ such that $y_0 \notin \overline{S}$, where $\overline{S}$ is the topological closure of S in Y. We note that $\overline{S}$ is an analytic subset of Y by Remmert's extension theorem (see, for example, [GrR, Chapter 9, §4, 2. Extension Theorem for Analytic Sets] and [No, Theorem 6.8.1]) since $\dim_y Y_{i+1} < \dim_y Y_i$ holds at any point $y \in Y_{i+1}$. Let $U \subset Y$ be the Zariski open subset whose complement is the union of all such $\overline{S}$ for all i and of Y_{l+1}. By Lemma 4.10 (3), there exists an analytically meagre subset $\mathcal{S}$ of Y such that $\mathcal{L}$ is π-numerically trivial over $Y \setminus \mathcal{S}$ with $y_0 \notin \mathcal{S}$. Let T be any connected component of $Y_i \setminus Y_{i+1}$ for some $i \le l$ with $y_0 \in \overline{T}$. Then $T \cap (Y \setminus \mathcal{S}) \neq \emptyset$ by Remark 4.11. Thus, by Lemma 4.12, $\mathcal{L}$ is π-numerically trivial over U. This is what we wanted. □

Let us prove Theorem 4.7. In the proof of Theorem 4.7, we will use the argument in the proof Theorem 4.13.

Proof of Theorem 4.7. As in the proof of Theorem 4.13, we construct a finite sequence of analytic subsets

$$Y =: Y_0 \supset Y_1 \supset \cdots \supset Y_k$$

and projective surjective morphisms $\pi_i \colon X_i \to Y_i$ satisfying (i)–(v) in the proof of Theorem 4.13. Let $W_{i,j}$ be the connected components of $W \cap Y_i$ for $1 \leq j \leq k_i$. We take a point $w_{i,j} \in W_{i,j} \setminus Y_{i+1}$ for any (i,j) with $W_{i,j} \not\subset Y_{i+1}$. Then it is sufficient to show that

$$A^1(X/Y;W) \to \bigoplus \mathrm{NS}\left(\pi_i^{-1}(w_{i,j})\right)/(\mathrm{torsion}) \tag{4.2}$$

is injective, where $\mathrm{NS}\left(\pi_i^{-1}(w_{i,j})\right)$ is the Neron–Severi group of $\pi_i^{-1}(w_{i,j})$. Let $\mathcal{L}$ be a line bundle on $\pi^{-1}(U)$, where U is an open neighborhood of W. We will prove that $\mathcal{L}$ is π-numerically trivial over W under the assumption that $\mathcal{L}$ is π-numerically trivial over all $w_{i,j}$. Since $\mathcal{L}$ is π-numerically trivial over $w_{i,j}$, $\mathcal{L}$ is π-numerically trivial over $U_{i,j} \setminus Y_{i+1}$ by Lemma 4.12, where $U_{i,j}$ is the connected component of $Y_i \cap U$ containing $w_{i,j}$. We note that $W \cap Y_i \subset \bigcup_j U_{i,j}$. Hence, $\mathcal{L}$ is π-numerically trivial over $W = \bigcup_i W \cap Y_i$. This is what we wanted, that is, (4.2) is injective. Therefore, we obtain that $A^1(X/Y;W)$ is a finitely generated abelian group since it is a subgroup of $\bigoplus \mathrm{NS}\left(\pi_i^{-1}(w_{i,j})\right)/(\mathrm{torsion})$. □

5. Vanishing theorems

In this section, we will treat some vanishing theorems. Fortunately, the necessary vanishing theorems have already been established. We explain only two vanishing theorems here for the reader's convenience. The first one is the Kawamata–Viehweg vanishing theorem for projective morphisms between complex varieties.

Theorem 5.1 (Kawamata–Viehweg vanishing theorem for projective morphisms of complex varieties)**.** *Let X be a smooth complex variety and let $\pi \colon X \to Y$ be a projective morphism of complex varieties. Assume that D is an $\mathbb{R}$-divisor on X such that D is π-nef and π-big and that $\mathrm{Supp}\{D\}$ is a simple normal crossing divisor on X. Then $R^i\pi_*\mathcal{O}_X(K_X + \lceil D \rceil) = 0$ for every $i > 0$.*

Sketch of Proof of Theorem 5.1. When D is a $\mathbb{Q}$-divisor, this statement is well known. It follows from [Na2, Theorem 3.4], [Fu6, Corollary 1.4], and so on. Let $y \in Y$ be any point. It is sufficient to prove that $R^i\pi_*\mathcal{O}_X(K_X + \lceil D \rceil) = 0$ holds for $i > 0$ on some open neighborhood U_y of y. Therefore, we will freely shrink Y around y without mentioning it explicitly. We take a projective bimermorphic morphism $f \colon Z \to X$ and can reduce the problem to the case where D is a $\mathbb{Q}$-divisor which is ample over Y. This reduction step is well known (see, for example, Step 2 in the proof of [Na2, Theorem 3.4], the proof of [KMM, Theorem 1-2-3], and so on). Hence we obtain the desired vanishing theorem. □

The second one is essentially the same as the first one. However, we think that this formulation is useful for some applications.

Theorem 5.2. *Let (X, Δ) be a divisorial log terminal pair and let $\pi \colon X \to Y$ be a projective morphism of complex varieties. Let D be a $\mathbb{Q}$-Cartier integral Weil divisor on X such that $D - (K_X + \Delta)$ is π-ample. Then $R^i\pi_*\mathcal{O}_X(D) = 0$ holds for every $i > 0$.*

Sketch of Proof of Theorem 5.2. By Lemma 3.9 and Theorem 5.1, the proof of [KMM, Theorem 1-2-5] works. For the details, see [KMM]. □

The reader can find various useful vanishing theorems for projective morphisms between complex varieties in [Fu6], [FMa], [Matm], [Fu11], and so on.

6. Basepoint-free theorems, I

In this section, we will collect some necessary basepoint-free theorems for Cartier divisors.

Let us start with Shokurov's nonvanishing theorem for smooth projective varieties, which can be proved by using Hironaka's resolution of singularities and the Kawamata–Viehweg vanishing theorem.

Theorem 6.1 (Shokurov's nonvanishing theorem)**.** *Let X be a smooth projective variety and let D be a nef Cartier divisor on X. Let A be an $\mathbb{R}$-divisor on X such that $pD+A-K_X$ is ample for some positive integer p, $\lceil A\rceil \geq 0$, and* $\mathrm{Supp}\{A\}$ *is a simple normal crossing divisor. Then there exists some positive integer m_0 such that $H^0(X, \mathcal{O}_X(mD+\lceil A\rceil)) \neq 0$ holds for every integer $m \geq m_0$.*

Sketch of Proof of Theorem 6.1. By perturbing the coefficients of A slightly, we may assume that A is a $\mathbb{Q}$-divisor. Then this statement is a special case of [KMM, Theorem 2-1-1] because $pD + A - K_X$ is automatically nef and big. For the details, see the proof of [KMM, Theorem 2-1-1]. □

The following formulation of the basepoint-free theorem is suitable for our purposes in this paper. It is the well-known Kawamata–Shokurov basepoint-free theorem when $\pi\colon X \to Y$ is algebraic.

Theorem 6.2 (Basepoint-free theorem)**.** *Let $\pi\colon X \to Y$ be a projective morphism from a normal complex variety to a complex analytic space Y and let Δ be an $\mathbb{R}$-divisor on X such that (X, Δ) is divisorial log terminal. Let D be a π-nef Cartier divisor on X such that $aD - (K_X + \Delta)$ is π-ample for some positive integer a. Then, for any relatively compact open subset U of Y, there exists a positive integer m_0, which depends on U, such that*

$$\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$$

is surjective over U for every integer $m \geq m_0$.

Sketch of Proof of Theorem 6.2. By Lemma 3.9 and Theorem 6.1, the proof of [KMM, Theorem 3-1-1] works over U. Note that we can not consider the generic fiber of $\pi^{-1}(U) \to U$ since it is a projective morphism of complex analytic spaces. Therefore, we apply Theorem 6.1 to an analytically sufficiently general fiber of $\pi^{-1}(U) \to U$ when we prove $\pi_*\mathcal{O}_X(mD) \neq 0$ for every large positive integer m. For the details, see the proof of [KMM, Theorem 3-1-1]. □

Remark 6.3. If (X, Δ) is kawamata log terminal in Theorem 6.2, then the same statement holds under a slightly weaker assumption that $aD - (K_X + \Delta)$ is π-nef and π-big. This is almost obvious by the proof of Theorem 6.2 (see also the proof of [KMM, Theorem 3-1-1]).

In this paper, the following variant of the basepoint-free theorem is indispensable. Theorem 6.4 is Kollár's effective basepoint-freeness for projective morphisms of complex analytic spaces.

Theorem 6.4 (Effective basepoint-free theorem, see [BCHM, Theorem 3.9.1])**.** *Fix a positive integer n. Then there exists a positive integer m with the following properties.*

Let $\pi\colon X \to Y$ be a projective morphism from a normal complex variety X to a complex analytic space Y and let D be a π-nef Cartier divisor on X such that $D-(K_X+\Delta)$ is π-nef and π-big, where (X, Δ) is kawamata log terminal with $\dim X = n$.

Then, mD is π-free, that is,

$$\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$$

is surjective.

Sketch of Proof of Theorem 6.4. It is sufficient to prove the statement over any small relatively compact Stein open subset U of Y. We will freely replace Y with a small open subset without mentioning it explicitly. By the standard argument (see, for example, the proof of [BCHM, Theorem 3.9.1]), we may further assume that $D-(K_X+\Delta)$ is π-ample and that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. The modified basepoint-freeness method explained in [Kol1, 2.1] works with some minor modifications. Note that we treat a projective morphism $\pi\colon X \to Y$. Therefore, when we prove that some sheaf is not zero, we restrict it to an analytically sufficiently general fiber. We can not use the generic fiber since we consider projective morphisms between complex analytic spaces. For the details, see [Kol1, Section 2]. □

Theorem 6.5 below is essentially due to Nakayama (see [Na2, Theorem 4.10]), which will play a crucial role in this paper.

Theorem 6.5 (see [Na2, Theorem 4.10])**.** *Let $\pi\colon X \to Y$ be a projective surjective morphism between complex analytic spaces and let W be a Stein compact subset of Y and let (X, Δ) be a kawamata log terminal pair. Let D be a Cartier divisor on X. Assume that D is nef over W, that is, $D\cdot C \geq 0$ for every projective curve C such that $\pi(C)$ is a point of W. We further assume that $aD-(K_X+\Delta)$ is π-ample for some positive real number a. Then there exist an open neighborhood U of W and a positive integer m_0 such that*

$$\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$$

is surjective over U for every integer $m \geq m_0$.

Before we prove Lemma 6.5, we prepare an easy lemma. We describe it for the sake of completeness.

Lemma 6.6. *Let $\pi\colon X \to Y$ be a projective surjective morphism between complex varieties and let $\mathcal{L}$ be a line bundle on X. Assume that $\mathcal{L}|_{\pi^{-1}(y_0)}$ is nef for some $y_0 \in Y$. Then there exists an analytically meagre subset $\mathcal{S}$ such that $\mathcal{L}|_{\pi^{-1}(y)}$ is nef for every $y \in Y \setminus \mathcal{S}$.*

Proof of Lemma 6.6. We take a π-ample line bundle $\mathcal{H}$ on X. Then $\mathcal{L}^{\otimes n}\otimes\mathcal{H}|_{\pi^{-1}(y_0)}$ is ample for every $n \in \mathbb{Z}_{>0}$. For each n, there exist a positive integer m_n and an open neighborhood U_n of y_0 such that

$$\pi^*\pi_*(\mathcal{L}^{\otimes nm_n}\otimes\mathcal{H}^{\otimes m_n}) \to \mathcal{L}^{\otimes nm_n}\otimes\mathcal{H}^{\otimes m_n}$$

is surjective on $\pi^{-1}(U_n)$ since $\mathcal{L}^{\otimes n}\otimes\mathcal{H}$ is ample over some open neighborhood of y_0 (see, for example, [Na2, Proposition 1.4]). We put

$$\mathcal{F}_n := \operatorname{Coker}\left(\pi^*\pi_*(\mathcal{L}^{\otimes nm_n}\otimes\mathcal{H}^{\otimes m_n}) \to \mathcal{L}^{\otimes nm_n}\otimes\mathcal{H}^{\otimes m_n}\right).$$

Then $\pi_*\mathcal{F}_n$ is a coherent sheaf on Y and $\mathcal{S}_n := \operatorname{Supp}\mathcal{F}_n$ is a closed analytic subset of Y with $y_0 \notin \mathcal{S}_n$. By construction, $\mathcal{L}^{\otimes nm_n}\otimes\mathcal{H}^{\otimes m_n}$ is π-free over $Y\setminus\mathcal{S}_n$. In particular,

$\mathcal{L}^{\otimes n}\otimes\mathcal{H}$ is π-nef over $Y\setminus\mathcal{S}_n$. We put $\mathcal{S}:=\bigcup_{n\in\mathbb{Z}_{>0}}\mathcal{S}_n$. If $(\mathcal{L}^{\otimes n}\otimes\mathcal{H})\,|_{\pi^{-1}(y)}$ is nef for every $n\in\mathbb{Z}_{>0}$, then $\mathcal{L}|_{\pi^{-1}(y)}$ is nef. Therefore, $\mathcal{S}$ is the desired analytically meagre subset of Y. □

Let us see the proof of Theorem 6.5.

Sketch of Proof of Theorem 6.5. Let B be a Cartier divisor on X such that B is nef over W and let A be any π-ample $\mathbb{R}$-divisor on X. Then $A+B$ is π-ample over some open neighborhood of W. By this easy observations, we see that the usual proof of the basepoint-free theorem (see Sketch of Proof of Theorem 6.2) works over some open neighborhood U of W. We note that we can use the nonvanishing theorem (see Theorem 6.1) on an analytically sufficiently general fiber by Lemma 6.6. Of course, we may have to replace U with a smaller open neighborhood of W finitely many times throughout the proof. □

We will treat some basepoint-free theorems for $\mathbb{R}$-Cartier divisors in Section 8.

7. Cone and contraction theorem

This section will be devoted to the cone and contraction theorem for projective morphisms of complex analytic spaces.

Let us start with the rationality theorem. We need the following formulation. The proof is essentially the same as that for algebraic varieties.

Theorem 7.1 (Rationality theorem, see [Na2, Theorem 4.11])**.** *Let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P)*. Let Δ be an effective $\mathbb{Q}$-divisor on X such that (X,Δ) is divisorial log terminal and that $a(K_X+\Delta)$ is Cartier in a neighborhood of $\pi^{-1}(W)$ for some positive integer a. Let H be a π-ample Cartier divisor on X. Assume that $K_X+\Delta$ is not π-nef over W. Then*

$$r:=\max\{t\in\mathbb{R}\,|\,H+t(K_X+\Delta)\ \textit{is π-nef over } W\}$$

is a positive rational number. Furthermore, expressing $r/a=u/v$ with $u,v\in\mathbb{Z}_{>0}$ and $(u,v)=1$, we have $v\le a(d+1)$, where $d=\max_{w\in W}\dim\pi^{-1}(w)$.

Sketch of Proof of Theorem 7.1. The proof of [KMM, Theorem 4-1-1] works with some minor modifications. As usual, we will freely replace Y with a relatively compact Stein open neighborhood of W throughout this proof. By an easy reduction argument, we may further assume that H is π-very ample. We put

$$M(x,y):=xH+ya(K_X+\Delta)$$

and

$$\Lambda(x,y):=\operatorname{Supp}\left(\operatorname{Coker}\pi^*\pi_*\mathcal{O}_X(M(x,y))\to\mathcal{O}_X(M(x,y))\right).$$

It is not difficult to see that $\Lambda(x,y)$ is the same subset of X for (x,y) sufficiently large and $0<ya-xr<1$. We call it Λ_0. Moreover, let $I\subset\mathbb{Z}^2$ be the set of (x,y) for which $0<ya-xr<1$ and $\Lambda(x,y)=\Lambda_0$. Then I contains all sufficiently large (x,y) with $0<ya-xr<1$ (for the details, see Claim 1 in the proof of [Fu5, Theorem 15.1]). If $r\notin\mathbb{Q}$ or $v>r(d+1)$, then we can find (x',y') sufficiently large and $0<y'a-x'r<1$ with $\Lambda(x',y')\subsetneq\Lambda_0$ (for the details, see the proof of [KMM, Theorem 4-1-1] or Step 7–Step 11 in [KM, Section 3.4]). This is a contradiction. Hence, we get the desired properties of r. We note that we can not consider generic fibers. Therefore, when we check that some sheaf is not a zero sheaf in the above argument, we restrict it to an analytically sufficiently general fiber. □

It is very well known that the cone and contraction theorem is a consequence of the rationality theorem (see Theorem 7.1) and the basepoint-free theorem (see Theorem 6.5).

Theorem 7.2 (Cone and contraction theorem, see [Na2, Theorem 4.12])**.** *Let (X, Δ) be a divisorial log terminal pair. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Then we have*

$$\overline{\mathrm{NE}}(X/Y; W) = \overline{\mathrm{NE}}(X/Y; W)_{K_X+\Delta\geq 0} + \sum_j R_j$$

with the following properties.

(1) *Let A be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many R_j's included in $(K_X + \Delta + A)_{<0}$. In particular, the R_j's are discrete in the half space $(K_X + \Delta)_{<0}$.*

(2) *Let R be a $(K_X + \Delta)$-negative extremal ray. Then, after shrinking Y around W suitably, there exists a contraction morphism $\varphi_R\colon X \to Z$ over Y satisfying:*
   (i) *Let C be a projective integral curve on X such that $\pi(C)$ is a point in W. Then $\varphi_R(C)$ is a point if and only if $[C] \in R$.*
   (ii) $\mathcal{O}_Z \xrightarrow{\sim} (\varphi_R)_*\mathcal{O}_X$.
   (iii) *Let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L} \cdot C = 0$ for every curve C with $[C] \in R$. Then there is a line bundle $\mathcal{M}$ on Z such that $\mathcal{L} \simeq \varphi_R^*\mathcal{M}$.*

Sketch of Proof of Theorem 7.2. When $K_X+\Delta$ is $\mathbb{Q}$-Cartier in a neighborhood of $\pi^{-1}(W)$, we have the desired properties as a consequence of the rationality theorem (see Theorem 7.1) and the basepoint-free theorem (see Theorem 6.5). This part is well known (for the details, see, for example, [KMM, Theorem 4-2-1]). From now on, we assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier but is not $\mathbb{Q}$-Cartier. As usual, we will freely shrink Y around W without mentioning it explicitly. By the standard argument, we can find effective $\mathbb{Q}$-divisors $\Delta_1, \dots, \Delta_k$ on X and positive real numbers $r_1, \dots, r_k$ with $\sum_{i=1}^k r_i = 1$ such that $K_X + \Delta_i$ is $\mathbb{Q}$-Cartier and (X, Δ_i) is divisorial log terminal for every i and that $\sum_{i=1}^k r_i\Delta_i = \Delta$ holds. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(W/Y; W)$. Then there exists some i_0 such that R is a $(K_X + \Delta_{i_0})$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y; W)$. We have already known that extremal rays are discrete in the half space $(K_X + \Delta_i)_{<0}$ for every i since $K_X + \Delta_i$ is $\mathbb{Q}$-Cartier. Hence it is not difficult to see that all the desired properties hold true even when $K_X + \Delta$ is only $\mathbb{R}$-Cartier. □

In this paper, we sometimes treat log canonical pairs. Thus, we need:

Theorem 7.3. *Let (X, Δ) be a log canonical pair. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. We assume that there exists Δ_0 on X such that (X, Δ_0) is kawamata log terminal. Let A be a π-ample $\mathbb{R}$-divisor on X. Then there are only finitely many $(K_X + \Delta + A)$-negative extremal rays of $\overline{\mathrm{NE}}(X/Y; W)$.*

Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y; W)$. Then the contraction morphism $\varphi_R\colon X \to Z$ associated to R as in Theorem 7.2 (2) exists.

Proof. We take a sufficiently small positive rational number ε and consider

$$K_X + \Delta + A = K_X + (1-\varepsilon)\Delta + \varepsilon\Delta_0 + (A - \varepsilon(\Delta_0 - \Delta)).$$

Since ε is sufficiently small, $A - \varepsilon(\Delta_0 - \Delta)$ is still π-ample. On the other hand, the pair $(X, (1-\varepsilon)\Delta + \varepsilon\Delta_0)$ is kawamata log terminal. Hence, by Theorem 7.2 (1), there are only finitely many $(K_X+\Delta+A)$-negative extremal rays. Let R be a $(K_X+\Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y;W)$. Then we can see it as a $(K_X+(1-\varepsilon)\Delta+\varepsilon\Delta_0)$-negative extremal ray for some small positive rational number ε. Therefore, by Theorem 7.2 (2), we have the desired contraction morphism $\varphi_R\colon X\to Z$. □

8. Basepoint-free theorems, II

In this section, we will treat basepoint-free theorems for $\mathbb{R}$-divisors. We note that the use of $\mathbb{R}$-divisors is indispensable in the theory of minimal models.

Theorem 8.1 (Basepoint-free theorem for $\mathbb{R}$-divisors, see [BCHM, Theorem 3.9.1])**.** *Let $\pi\colon X\to Y$ be a projective morphism of normal complex variety X to a complex analytic space Y and let W be a Stein compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P)*. Let D be a π-nef $\mathbb{R}$-divisor on X such that $aD-(K_X+\Delta)$ is π-nef and π-big for some positive real number a, where (X,Δ) is kawamata log terminal.*

Then there exists an open neighborhood U of W such that $D|_{\pi^{-1}(U)}$ is semiample over U.

Sketch of Proof of Theorem 8.1. By replacing D with aD, we may assume that $a=1$. We take a small Stein open neighborhood U' of W and a Stein compact subset W' of Y such that $\Gamma(W',\mathcal{O}_Y)$ is noetherian and $U'\subset W'$. Throughout this proof, we will freely shrink Y around W' without mentioning it explicitly. By the standard argument (see, for example, the proof of [BCHM, Theorem 3.9.1]), we may further assume that $D-(K_X+\Delta)$ is π-ample and $K_X+\Delta$ is $\mathbb{Q}$-Cartier. We take a small π-ample $\mathbb{Q}$-divisor A on X such that $D-(K_X+\Delta+A)$ is still π-ample. By the cone theorem (see Theorem 7.2 (1)), there are only finitely many $(K_X+\Delta+A)$-negative extremal rays of $\overline{\mathrm{NE}}(X/Y;W')$. Hence, we can write $D=\sum_{i=1}^k r_iD_i$, where r_i is a positive real number, D_i is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X which is nef over W', and $D_i-(K_X+\Delta)$ is π-ample for every i (for the details, see the proof of [BCHM, Theorem 3.9.1]). We replace Y with U'. Then, by the usual basepoint-free theorem (see Theorem 6.2), $D_i|_{\pi^{-1}(U)}$ is semiample over U for some U and every i. This implies that $D|_{\pi^{-1}(U)}$ is semiample over U. □

We used the cone theorem (see Theorem 7.2) in the above proof of Theorem 8.1. So it is much deeper than the usual basepoint-free theorem for Cartier divisors (see Theorem 6.2).

By combining Lemma 6.5 with the argument in the proof of Theorem 8.1, we have:

Theorem 8.2. *Let $\pi\colon X\to Y$ be a projective bimeromorphic contraction morphism of complex analytic spaces and let $y\in Y$ be a point. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that D is numerically trivial over y, that is, $D\cdot C=0$ for every projective curve C on X such that $\pi(C)=y$. Assume that $aD-(K_X+\Delta)$ is π-nef for some positive real number a, where (X,Δ) is a kawamata log terminal pair. Then π_*D is $\mathbb{R}$-Cartier at y.*

Proof. We put $W=\{y\}$. Then W is a Stein compact subset of Y such that $\Gamma(W,\mathcal{O}_Y)$ is noetherian. We will freely shrink Y around W without mentioning it explicitly. As in the proof of Theorem 8.1, we may assume that $a=1$ and $D-(K_X+\Delta)$ is π-ample. By using the cone theorem as in the proof of Theorem 8.1, we can write $D=\sum_{i=1}^k r_iD_i$, where r_i is a positive real number, D_i is numerically trivial over y, and $D_i-(K_X+\Delta)$ is π-ample for every i. By replacing D_i with m_iD_i for some positive integer m_i, we may further assume that D_i is a Cartier divisor on X for every i. Then, by Theorem 6.5, π_*D_i is $\mathbb{Q}$-Cartier for every i. Hence π_*D is $\mathbb{R}$-Cartier. This is what we wanted. □

The final theorem in this section is suitable for our framework of the minimal model program. We only assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier and is nef over W in Theorem 8.3. The conclusion says that it is semiample over some open neighborhood of W.

Theorem 8.3. *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Let (X, Δ) be a log canonical pair. Assume that there exists Δ_0 such that (X, Δ_0) is kawamata log terminal. We further assume that $\Delta = A + B$, where A is π-ample, $A \geq 0$, and $B \geq 0$. If $K_X + \Delta$ is nef over W, then there exists an open neighborhood U of W such that $K_X + \Delta$ is semiample over U.*

Proof. Throughout this proof, we will shrink Y around W suitably without mentioning it explicitly. By assumption, we can take Δ' such that (X, Δ') is kawamata log terminal and $K_X + \Delta' \sim_{\mathbb{R}} K_X + \Delta$. Hence, by replacing Δ with Δ', we may assume that (X, Δ) is kawamata log terminal. Then (X, B) is kawamata log terminal and $2(K_X + \Delta) - (K_X + B)$ is ample over Y. We take a general π-ample $\mathbb{Q}$-divisor H on X such that $(X, B + H)$ is kawamata log terminal and that $2(K_X + \Delta) - (K_X + B + H)$ is still ample over Y. As in the proof of [BCHM, Theorem 3.9.1], by using Theorem 7.3, we take positive real numbers $r_1, \ldots, r_k$ and $\mathbb{Q}$-divisors $\Delta_1, \ldots, \Delta_k$ such that $K_X + \Delta_i$ is nef over W and $2(K_X + \Delta_i) - (K_X + B + H)$ is ample over Y for every i and that $\sum_{i=1}^{k} \Delta_i = \Delta$. By Theorem 6.5, we obtain that $K_X + \Delta_i$ is π-semiample for every i. This means that $K_X + \Delta$ is semiample over Y. This is what we wanted. □

We close this section with conjectures related to Theorem 8.3.

Conjecture 8.4. *Let (X, Δ) be a log canonical pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. We put*

$$\mathfrak{N} := \{y \in Y \mid (K_X + \Delta)|_{\pi^{-1}(y)} \text{ is nef}\}.$$

Then $\mathfrak{N}$ is open.

If we can establish the minimal model program for projective morphisms between complex analytic spaces in full generality, then we see that Conjecture 8.4 holds true.

Remark 8.5. In Conjecture 8.4, it is sufficient to prove that $\mathfrak{N}$ contains an open neighborhood of y_0 under the assumption that $(K_X + \Delta)|_{\pi^{-1}(y_0)}$ is nef. We take an open neighborhood U of y_0 and a Stein compact subset W of Y such that $y_0 \in U \subset W$ and that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. We will freely shrink Y around W. By Theorem 1.27, we can reduce the problem to the case where X is $\mathbb{Q}$-factorial over W and (X, Δ) is divisorial log terminal. Then we run a $(K_X + \Delta)$-minimal model program over Y around W. Note that $K_X + \Delta$ is π-pseudo-effective since $(K_X + \Delta)|_{\pi^{-1}(y_0)}$ is nef. If the above minimal model program terminates after finitely many steps, then it is easy to see that $K_X + \Delta$ is nef over some open neighborhood of y_0. Hence Conjecture 8.4 would be absolutely correct.

Conjecture 8.4 is also related to the following abundance conjecture for projective morphisms of complex analytic spaces.

Conjecture 8.6 (Abundance conjecture)**.** *Let (X, Δ) be a log canonical pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Assume that $K_X + \Delta$ is π-nef, that is, $(K_X + \Delta) \cdot C \geq 0$ for every projective integral curve C on X such that $\pi(C)$ is a point. Then $K_X + \Delta$ is π-semiample.*

As is well known, the abundance conjecture (see Conjecture 8.6) is widely open even when Y is a point. We will treat the abundance conjecture in Section 23.

9. Lengths of extremal rational curves

In this paper, we will repeatedly use the fact that every extremal ray is spanned by a rational curve of low degree, which is essentially due to Kawamata (see [Kaw3, Theorem 1]). Note that Kawamata's result comes from the result obtained by Mori's bend and break technique, which relies on methods in positive characteristic.

Theorem 9.1 (see [Kaw3, Theorem 1] and [Fu14, Theorem 1.12])**.** *Let (X, Δ) be a kawamata log terminal pair and let $\varphi\colon X \to Z$ be a projective morphism of complex analytic spaces such that $-(K_X + \Delta)$ is φ-ample. Let P be an arbitrary point of Z. Let E be any positive-dimensional irreducible component of $\varphi^{-1}(P)$. Then E is covered by possibly singular rational curves ℓ with*

$$0 < -(K_X + \Delta) \cdot \ell \leq 2 \dim E.$$

In particular, E is uniruled.

Here, we quickly reduce Theorem 9.1 to [Fu14, Theorem 1.12]. So we use the framework of quasi-log schemes.

Proof of Theorem 9.1. If $\varphi(X) = P$, then $E = X$ holds. In this case, the statement follows from [Fu14, Theorem 1.12] since $[X, K_X + \Delta]$ is a quasi-log scheme. From now on, we may assume that $\varphi(X) \neq P$. We shrink Z around P. Then we can take an effective $\mathbb{R}$-Cartier divisor B on Z such that $(X, \Delta + \varphi^*B)$ is kawamata log terminal outside $\varphi^{-1}(P)$ and that E is a log canonical center of $(X, \Delta + \varphi^*B)$. We consider the non-kawamata log terminal locus $V := \mathrm{Nklt}(X, \Delta + \varphi^*B)$. Note that $\varphi(V) = P$. Let $f\colon Y \to X$ be a projective bimeromorphic morphism from a smooth variety Y such that $K_Y + \Delta_Y = f^*(K_X + \Delta + \varphi^*B)$ and that $\mathrm{Supp}\, \Delta_Y$ is a simple normal crossing divisor on Y. We put $U := \Delta_Y^{=1}$ and $(K_Y + \Delta_Y)|_U = K_U + \Delta_U$ by adjunction. Note that U is projective since $\varphi \circ f(U) = P$. We consider the following short exact sequence:

$$0 \to \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor) \to \mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + U) \to \mathcal{O}_U(\lceil -\Delta_U^{<1} \rceil - \lfloor \Delta_U^{>1} \rfloor) \to 0.$$

By the Kawamata–Viehweg vanishing theorem (see Theorem 5.1), $R^1 f_*\mathcal{O}_Y(-\lfloor \Delta_Y \rfloor) = 0$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J}(X, \Delta + \varphi^*B) & \longrightarrow & \mathcal{J}_{\mathrm{NLC}}(X, \Delta + \varphi^*B) & \longrightarrow & f_*\mathcal{O}_U(\lceil -\Delta_U^{<1} \rceil - \lfloor \Delta_U^{>1} \rfloor) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{J}(X, \Delta + \varphi^*B) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_V & \longrightarrow & 0,
\end{array}$$

where $\mathcal{J}(X, \Delta + \varphi^*B) = f_*\mathcal{O}_Y(-\lfloor \Delta_Y \rfloor)$ is the *multiplier ideal sheaf* of $(X, \Delta + \varphi^*B)$ and $\mathcal{J}_{\mathrm{NLC}}(X, \Delta + \varphi^*B) = f_*\mathcal{O}_Y(-\lfloor \Delta_Y \rfloor + U)$, which is called the *non-lc ideal sheaf* of $(X, \Delta + \varphi^*B)$, is an ideal sheaf that defines the non-log canonical locus of $(X, \Delta + \varphi^*B)$. Hence $\mathcal{J} := f_*\mathcal{O}_U(\lceil -\Delta_U^{<1} \rceil - \lfloor \Delta_U^{>1} \rfloor)$ is an ideal sheaf on V such that

$$\mathcal{O}_X / \mathcal{J}_{\mathrm{NLC}}(X, \Delta + \varphi^*B) = \mathcal{O}_V / \mathcal{J}.$$

Therefore, since U is projective,

$$\bigl(V, (K_X + \Delta + \varphi^*B)|_V, f\colon (U, \Delta_U) \to V\bigr)$$

gives a quasi-log scheme structure on $[V, (K_X + \Delta + \varphi^*B)|_V]$ (see [Fu10, Theorem 4.9]) such that E is a qlc stratum of $[V, (K_X + \Delta + \varphi^*B)|_V]$ by construction. Then, by [Fu14, Theorem 1.12], E is covered by rational curves ℓ with

$$0 < -(K_X + \Delta + \varphi^*B) \cdot \ell \leq 2 \dim E$$

and is uniruled. Since $\varphi^*B \cdot \ell = 0$, we obtain the desired statement. □

By combining Theorem 9.1 with a standard argument, we have the following theorem, which is well known when $f\colon X \to Y$ is a projective morphism between quasi-projective varieties.

Theorem 9.2 (see [BCHM, Theorem 3.8.1])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Suppose that (X, Δ) is a log canonical pair. Suppose that there is an $\mathbb{R}$-divisor Δ_0 such that (X, Δ_0) is kawamata log terminal. If R is a $(K_X + \Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y; W)$, then there exists a rational curve ℓ spanning R such that*

$$0 < -(K_X + \Delta) \cdot \ell \leq 2 \dim X.$$

Proof. By assumption, we can find $\mathbb{R}$-divisors Δ_i with $\lim_{i\to\infty} \Delta_i = \Delta$ such that (X, Δ_i) is kawamata log terminal. By replacing π by the contraction defined by the extremal ray R, we may further assume that $-(K_X + \Delta)$ is π-ample. By Theorem 9.1, for some $P \in Y$, we can find a rational curve ℓ_i in $\pi^{-1}(P)$ such that

$$0 < -(K_X + \Delta_i) \cdot \ell_i \leq 2 \dim X$$

for every $i \gg 0$. We note that $\pi^{-1}(P)$ is projective. We take a π-ample $\mathbb{Q}$-divisor A on X such that $-(K_X + \Delta + A)$ is also π-ample. In particular, $-(K_X + \Delta_i + A)$ is π-ample for every $i \gg 0$. Hence

$$0 < A \cdot \ell_i = (K_X + \Delta_i + A) \cdot \ell_i - (K_X + \Delta_i) \cdot \ell_i < 2 \dim X.$$

This means that the curves ℓ_i belong to a bounded family. Thus, possibly passing to a subsequence, we may assume that $\ell = \ell_i$ is constant. Therefore, we have

$$-(K_X + \Delta) \cdot \ell = \lim_{i\to\infty} -(K_X + \Delta_i) \cdot \ell \leq 2 \dim X.$$

This is what we wanted. □

The following easy observation is very useful.

Theorem 9.3. *Let (X, Δ) be a log canonical pair. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. We assume that there exists Δ_0 on X such that (X, Δ_0) is kawamata log terminal. Suppose that*

$$\pi\colon X \xrightarrow{g} Y^\flat \xrightarrow{h} Y$$

such that $Y^\flat$ is projective over Y. Let H be a general h-ample $\mathbb{Q}$-divisor on $Y^\flat$ with $H \cdot C > 2 \dim X$ for every projective curve C such that $h(C)$ is a point. Let R be a $(K_X + \Delta + g^*H)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y; W)$ and let $\varphi_R\colon X \to Z$ be the contraction morphism over Y associated to R. Then $\varphi_R\colon X \to Z$ is a contraction morphism over $Y^\flat$, that is, $Z \to Y$ factors through $Y^\flat$.

Proof. We note that we can see R as a $(K_X+\Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y;W)$ since g^*H is nef over Y. Therefore, by Theorem 9.2, R is spanned by a rational curve ℓ on X such that $0 < -(K_X+\Delta)\cdot\ell \leq 2\dim X$. If $g(\ell)$ is a curve, then $(K_X+\Delta+g^*H)\cdot\ell > 0$ since $\ell\cdot g^*H > 2\dim X$. Therefore, this means that $g(\ell)$ is a point. Hence, the contraction morphism $\varphi_R\colon X\to Z$ exists over $Y^\flat$. □

The next lemma is an easy consequence of Theorem 9.3. We will repeatedly use it in the subsequent sections.

Lemma 9.4. *Let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P). *Assume that (X,Δ) is divisorial log terminal and that X is $\mathbb{Q}$-factorial over W. Suppose that*

$$\pi\colon X \xrightarrow{g} Y^\flat \xrightarrow{h} Y$$

such that $Y^\flat$ is projective over Y. Let H be a general h-ample $\mathbb{Q}$-divisor on $Y^\flat$ with $H\cdot C > 2\dim X$ for every projective curve C such that $h(C)$ is a point. Let

$$(X_0,\Delta_0) \overset{\phi_0}{\dashrightarrow} (X_1,\Delta_1) \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} (X_i,\Delta_i) \overset{\phi_i}{\dashrightarrow} \cdots$$

be a $(K_X+\Delta+g^*H)$-minimal model program over Y starting from $(X_0,\Delta_0):=(X,\Delta)$. Then it is a $(K_X+\Delta)$-minimal model program over $Y^\flat$.

Proof. We apply Theorem 9.3 to each extremal contraction. Then we can see that it is a minimal model program over $Y^\flat$. □

By combining Theorem 9.2 with Theorem 8.1, we have:

Theorem 9.5. *Let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies* (P). *Suppose that*

$$\pi\colon X \xrightarrow{g} Y^\flat \xrightarrow{h} Y$$

such that $Y^\flat$ is projective over Y. Assume that (X,Δ) is kawamata log terminal such that Δ is π-big. We further assume that $K_X+\Delta$ is g-nef. Then there exists an open neighborhood U of W such that $(K_X+\Delta)|_{\pi^{-1}(U)}$ is semiample over $h^{-1}(U)$.

Proof. We take a relatively compact Stein open subset U' of Y and a Stein compact subset W' of Y such that $W\subset U'\subset W'$ and $\Gamma(W',\mathcal{O}_Y)$ is noetherian. From now on, we will freely shrink Y around W' suitably without mentioning it explicitly. Since Δ is π-big, there exists Δ' such that $\Delta'\sim_{\mathbb{R}}\Delta$, $\Delta'=A+B$, $A\geq 0$, A is π-maple, $B\geq 0$, and (X,Δ') is kawamata log terminal. Let H be a general h-ample $\mathbb{Q}$-divisor on $Y^\flat$ with $H\cdot C>2\dim X$ for every projective curve C such that $h(C)$ is a point. Then $K_X+\Delta+g^*H$ is nef over W'. Hence $2(K_X+\Delta+g^*H)-(K_X+B)$ is π-ample. We apply Theorem 8.1 to $K_X+\Delta+g^*H$ over U'. Then there exists an open neighborhood U of W such that $K_X+\Delta+g^*H$ is semiample over U. This implies that $K_X+\Delta$ is semiample over $h^{-1}(U)$. □

10. Real linear systems, stable base loci, and augmented base loci

In the theory of minimal models, we have to treat $\mathbb{R}$-divisors. Throughout this section, we always assume that $\pi\colon X\to Y$ is a projective morphism of normal varieties and let W be a Stein compact subset of Y. We further assume that Y is Stein for simplicity. An $\mathbb{R}$-divisor on X may have infinitely many irreducible components. Hence we frequently have to restrict it to a relatively compact open subset of X in order to make the number of the irreducible components finite.

Let us start with the definition of *real linear systems* and *stable base loci*.

Definition 10.1 (Real linear systems and stable base loci)**.** Let D be an $\mathbb{R}$-divisor on X. Then we put

$$|D/Y|_{\mathbb{R}} = \{C \geq 0 \,|\, C \sim_{\mathbb{R}} D\}$$

and call it the *real linear system* associated to D over Y. We sometimes simply write $|D|_{\mathbb{R}}$ to denote $|D/Y|_{\mathbb{R}}$ if there is no danger of confusion. The *stable base locus* of D over Y is the Zariski closed subset $\mathbf{B}(D/Y)$ given by the intersection of the support of the elements of $|D/Y|_{\mathbb{R}}$. If $|D/Y|_{\mathbb{R}} = \emptyset$, then we put $\mathbf{B}(D/Y) = X$. Similarly, we consider

$$|D/Y|_{\mathbb{Q}} = \{C \geq 0 \,|\, C \sim_{\mathbb{Q}} D\}.$$

and define the Zariski closed subset $\mathbf{B}(D/Y)_{\mathbb{Q}}$ as the intersection of the support of the elements of $|D/Y|_{\mathbb{Q}}$. If $|D/Y|_{\mathbb{Q}} = \emptyset$, then we put $\mathbf{B}(D/Y)_{\mathbb{Q}} = X$. We note that the inclusion $\mathbf{B}(D/Y) \subset \mathbf{B}(D/Y)_{\mathbb{Q}}$ holds since $|D/Y|_{\mathbb{Q}} \subset |D/Y|_{\mathbb{R}}$.

We make an important remark on the definition of $\mathbf{B}(D/Y)$ and $\mathbf{B}(D/Y)_{\mathbb{Q}}$.

Remark 10.2 (see [BCHM, Remark 3.5.2])**.** In Definition 10.1, $\mathbf{B}(D/Y)$ and $\mathbf{B}(D/Y)_{\mathbb{Q}}$ are only defined as closed analytic subsets.

We will repeatedly use the following basic property of $\mathbf{B}(D/Y)$ implicitly.

Lemma 10.3. *Let U be any Stein open subset of Y. If $D \sim_{\mathbb{R}} C \geq 0$, then $D|_{\pi^{-1}(U)} \sim_{\mathbb{R}} C|_{\pi^{-1}(U)} \geq 0$ obviously holds. Hence the inclusion*

$$\mathbf{B}(D|_{\pi^{-1}(U)}/U) \subset \mathbf{B}(D/Y)|_{\pi^{-1}(U)}$$

always holds true.

Proof. This is obvious. □

When we treat $\mathbb{Q}$-divisors, we need:

Lemma 10.4 (see [BCHM, Lemma 3.5.3])**.** *Let D be an integral Weil divisor. Then we have the following inclusions*

$$\mathbf{B}(D/Y)_{\mathbb{Q}} \supset \mathbf{B}(D/Y),$$

and

$$\mathbf{B}(D|_{\pi^{-1}(U)}/U)_{\mathbb{Q}} \subset \mathbf{B}(D/Y)|_{\pi^{-1}(U)}$$

for any relatively compact Stein open subset U of Y.

Proof. Since $|D/Y|_{\mathbb{Q}} \subset |D/Y|_{\mathbb{R}}$, the first inclusion $\mathbf{B}(D/Y)_{\mathbb{Q}} \supset \mathbf{B}(D/Y)$ is obvious. We take $x \in \pi^{-1}(U)$ such that $x \notin \mathbf{B}(D/Y)|_{\pi^{-1}(U)}$. Then, by the proof of [BCHM, Lemma 3.5.3], we can check that $x \notin \mathbf{B}(D|_{\pi^{-1}(U)}/U)_{\mathbb{Q}}$. Hence the desired second inclusion $\mathbf{B}(D|_{\pi^{-1}(U)}/U)_{\mathbb{Q}} \subset \mathbf{B}(D/Y)|_{\pi^{-1}(U)}$ holds. □

Although it may be dispensable, as in the algebraic case, we have:

Lemma 10.5. *Let A be any π-ample divisor on X and let U be any relatively compact open subset of Y. Then $\mathbf{B}((D - \varepsilon A)/Y)|_{\pi^{-1}(U)}$ is independent of ε if $0 < \varepsilon \ll 1$.*

Proof. It is sufficient to note that

$$\mathbf{B}((D - \varepsilon_1 A)/Y) \subset \mathbf{B}((D - \varepsilon_2 A)/Y)$$

holds for $0 < \varepsilon_1 < \varepsilon_2$ by definition. On a relatively compact open subset $\pi^{-1}(U)$, the loci $\mathbf{B}((D - \varepsilon A)/Y)$ stabilize for sufficiently small $\varepsilon > 0$ (see Lemma 2.17). □

Let us define *augmented base loci*.

Definition 10.6 (Augmented base loci)**.** The *augmented base locus* of D over Y is the Zariski closed subset

$$\mathbf{B}_+(D/Y) := \bigcap_{\varepsilon>0} \mathbf{B}((D-\varepsilon A)/Y),$$

where A is some π-ample divisor on X. It is not difficult to see that $\mathbf{B}_+(D/Y)$ is independent of the choice of A. We note that D is π-big if and only if $\mathbf{B}_+(D/Y) \subsetneq X$ holds.

We recall the definition of *fixed divisors*. We need it in Theorem F (3).

Definition 10.7. Let D be an integral Weil divisor on X. We put

$$|D| := \{C \geq 0 \,|\, C \sim D\}.$$

Then $\mathrm{Fix}(D)$ denotes the *fixed divisor* of D so that

$$|D| = |D - \mathrm{Fix}(D)| + \mathrm{Fix}(D),$$

where the base locus of $|D - \mathrm{Fix}(D)|$ contains no divisors. If $\mathrm{Fix}(D) = 0$, then D is said to be *mobile*.

Since we are mainly interested in the minimal model program over some open neighborhood of W, the following definition is useful.

Definition 10.8 (Stable base divisors)**.** A divisor E defined on $\pi^{-1}(U)$, where U is an open neighborhood of W, is called a *stable base divisor of D near W* if $E|_{\pi^{-1}(U')} \subset \mathbf{B}(D|_{\pi^{-1}(U')}/U')$ holds for any Stein open neighborhood U' of W with $U' \subset U$.

For our purposes, we have to reformulate [BCHM, Proposition 3.5.4] as follows. We will use Lemma 10.9 in the proof of Theorem G.

Lemma 10.9 (see [BCHM, Proposition 3.5.4])**.** *Let $\pi\colon X \to Y$ be a projective morphism of normal complex varieties and let W be a Stein compact subset of Y. Let $D \geq 0$ be an $\mathbb{R}$-divisor on X. Then, after replacing Y with a Stein open neighborhood of W suitably, we can find $\mathbb{R}$-divisors M and F on X such that*

(1) *$M \geq 0$ and $F \geq 0$,*
(2) *$D \sim_{\mathbb{R}} M + F$,*
(3) *every component of* $\mathrm{Supp}\, F$ *is a stable base divisor of D near W,*
(4) *if B is a component of* $\mathrm{Supp}\, M$*, then some multiple is mobile.*

Proof. The proof of [BCHM, Proposition 3.5.4, Lemma 3.5.5, and Lemma 3.5.6] works in our setting with some minor modifications. As we mentioned above, an $\mathbb{R}$-divisor on X may have infinitely many irreducible components. Therefore, we have to replace Y with a relatively compact open neighborhood of W in the proof of this lemma. For the details, see the proof of [BCHM, Proposition 3.5.4]. □

11. Some basic definitions and properties, I

In this section, we will explain some basic definitions which are indispensable for the main results and their proof.

Let us start with the definition of *D-nonpositivity* and *D-negativity*.

Definition 11.1 ([BCHM, Definition 3.6.1]). Let $\phi\colon X \dashrightarrow Z$ be a bimeromorphic contraction of normal complex varieties and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $D' := \phi_*D$ is also $\mathbb{R}$-Cartier. We say that ϕ is *D-nonpositive* (resp. *D-negative*) if for some common resolution $p\colon V \to X$ and $q\colon V \to Y$, we may write

$$p^*D = q^*D' + E$$

where E is effective and q-exceptional (resp. E is effective, q-exceptional, and the support of E contains the strict transform of the ϕ-exceptional divisors).

The so-called *negativity lemma* (see, for example, [BCHM, Lemma 3.6.2]) holds true in our complex analytic setting. This is because everything follows from the negative definiteness of intersection form of contractible curves on surfaces (see, for example, [Matk, Theorem 4-6-1]). Therefore, from now on, we will freely use the negativity lemma for projective morphisms of normal complex varieties. Note that the results obtained in [BCHM, Lemmas 3.6.2, 3.6.3, and 3.6.4] hold true in our complex analytic setting with some obvious modifications.

Let us define *semiample models* and *ample models* following [BCHM].

Definition 11.2 ([BCHM, Definition 3.6.5]). Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X.

Let $f\colon X \dashrightarrow Z$ be a bimeromorphic contraction over Y after shrinking Y around W suitably.

- We say that $f\colon X \dashrightarrow Z$ is a *semiample model of D over some open neighborhood of W* if, after shrinking Y around W suitably, Z is a normal variety and is projective over Y, f is D-nonpositive, and $H := f_*D$ is semiample over Y.

Let $g\colon X \dashrightarrow Z$ be a meromorphic map over Y after shrinking Y around W suitably.

- We say that $g\colon X \dashrightarrow Z$ is the *ample model of D over some open neighborhood of W* if, after shrinking Y around W suitably, Z is a normal variety and is projective over Y, and there exists an ample $\mathbb{R}$-divisor H over Y on Z such that if $p\colon V \to X$ and $q\colon V \to Z$ resolve the indeterminacy of g then q is a contraction morphism and we can write $p^*D \sim_{\mathbb{R}} q^*H + E$, where $E \geq 0$ and $B \geq E$ holds for every $B \in |p^*D/Y|_{\mathbb{R}}$.

The basic properties of semiample models and ample models are summarized as follows.

Lemma 11.3 (see [BCHM, Lemma 3.6.6]). *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P). *Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X.*

(1) *If $g_i\colon X \dashrightarrow X_i$, $i = 1, 2$, are two ample models of D over some open neighborhood of W, then there exists an isomorphism $\chi\colon X_1 \to X_2$ over some open neighborhood of W such that $g_2 = \chi \circ g_1$.*

(2) *If $f\colon X \dashrightarrow Z$ is a semiample model of D over some open neighborhood of W, then, after shrinking Y around W suitably, the ample model $g\colon X \dashrightarrow Z'$ of D over some open neighborhood of W exists and $g = h \circ f$, where $h\colon Z \to Z'$ is a contraction morphism and $f_*D \sim_{\mathbb{R}} h^*H$ holds such that H is an $\mathbb{R}$-divisor on Z' which is ample over some open neighborhood of W.*

(3) *If $f\colon X \dashrightarrow Z$ is a bimeromorphic map over some open neighborhood of W, then f is the ample model of D over some open neighborhood of W if and only if f is a*

semiample model of D over some open neighborhood of W and f_*D is ample over some open neighborhood of W.

Proof. For the details, see the proof of (1), (3), and (4) in [BCHM, Lemma 3.6.6]. □

The definition of *weak log canonical models* and *log terminal models* becomes subtle in our complex analytic setting.

Definition 11.4 ([BCHM, Definition 3.6.7])**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y. Suppose that $K_X + \Delta$ is log canonical and let $\phi\colon X \dashrightarrow Z$ be a bimeromorphic contraction of normal complex varieties over Y after shrinking Y around W suitably, where Z is projective over Y. We set $\Gamma = \phi_*\Delta$.

- (i) Z is a *weak log canonical model for $K_X + \Delta$ over W* if ϕ is $(K_X + \Delta)$-nonpositive over some open neighborhood of W and $K_Z + \Gamma$ is nef over W.
- (ii) Z is a *weak log canonical model for $K_X + \Delta$ over some open neighborhood of W* if, after shrinking Y around W suitably, ϕ is $(K_X + \Delta)$-nonpositive and $K_Z + \Gamma$ is nef over Y.
- (iii) Z is a *log terminal model for $K_X + \Delta$ over W* if ϕ is $(K_X + \Delta)$-negative over some open neighborhood of W, (Z, Γ) is divisorial log terminal, $K_Z + \Gamma$ is nef over W, and Z is $\mathbb{Q}$-factorial over W.
- (iv) Z is a *log terminal model for $K_X + \Delta$ over some open neighborhood of W* if, after shrinking Y around W suitably, ϕ is $(K_X + \Delta)$-negative, (Z, Γ) is divisorial log terminal, $K_Z + \Gamma$ is nef over Y, and Z is $\mathbb{Q}$-factorial over W.
- (v) Z is a *good log terminal model for $K_X + \Delta$ over some open neighborhood of W* if, after shrinking Y around W suitably, ϕ is $(K_X + \Delta)$-negative, (Z, Γ) is divisorial log terminal, $K_Z + \Gamma$ is semiample over Y, and Z is $\mathbb{Q}$-factorial over W.

We further assume that $\pi\colon X \to Y$ and W satisfies (P).

- (vi) Z is the *log canonical model for $K_X + \Delta$ over some open neighborhood of W* if ϕ is the ample model of $K_X + \Delta$ over some open neighborhood of W.

We give some remarks on Definitions 11.2 and 11.4.

Remark 11.5 (see [BCHM, Remark 3.6.8])**.** A log terminal model is sometimes simply called a *log minimal model* or a *minimal model*.

Remark 11.6. In Definitions 11.2 and 11.4 , we only require that $f\colon X \dashrightarrow Z$, $g\colon X \dashrightarrow Z$, and $\phi\colon X \dashrightarrow Z$ exist after replacing Y with a small open neighborhood of W suitably. If there is no danger of confusion, then we simply say that $\phi\colon X \dashrightarrow Z$ is a log terminal model (weak log canonical model, log canonical model, and so on) for $K_X + \Delta$ *over Y* when it is a log terminal model (weak log canonical model, log canonical model, and so on) for $K_X + \Delta$ over some open neighborhood of W.

In our complex analytic setting, the definition of *Mori fiber spaces* becomes as follows.

Definition 11.7 (Mori fiber spaces)**.** Let (X, Δ) be a divisorial log terminal pair. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). Let $f\colon X \to Z$ be a projective morphism of normal complex varieties over Y. Then $f\colon (X, \Delta) \to Z$ is a *Mori fiber space over Y* if

- (i) X is $\mathbb{Q}$-factorial over W,

(ii) f is a contraction morphism associated to a $(K_X+\Delta)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y;W)$, and
(iii) $\dim Z<\dim X$.

The following definition is essentially the same as [BCHM, Definition 1.1.4]. However, we need some modifications since we treat only curves mapped to points in W by π.

Definition 11.8 ([BCHM, Definition 1.1.4])**.** Let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces such that X is a normal variety and let W be a compact subset of Y. Let V be a finite-dimensional affine subspace of the real vector space $\mathrm{WDiv}_{\mathbb{R}}(X)$ spanned by the prime divisors on X. We fix an $\mathbb{R}$-divisor $A\geq 0$ on X such that $\mathrm{Supp}\,A$ has only finitely many irreducible components and define

$$\begin{aligned}
V_A&=\{\Delta\,|\,\Delta=A+B, B\in V\},\\
\mathcal{L}_A(V;\pi^{-1}(W))&=\{\Delta=A+B\in V_A\,|\,K_X+\Delta \text{ is log canonical at } \pi^{-1}(W) \text{ and } B\geq 0\},\\
\mathcal{E}_{A,\pi}(V;W)&=\left\{\Delta\in\mathcal{L}_A(V;\pi^{-1}(W))\;\middle|\; \begin{array}{l}K_X+\Delta \text{ is pseudo-effective over}\\ \text{some open neighborhood of } W\end{array}\right\},\\
\mathcal{N}^{\sharp}_{A,\pi}(V;W)&=\{\Delta\in\mathcal{L}_A(V;\pi^{-1}(W))\,|\,K_X+\Delta \text{ is nef over } W\}, \text{ and}\\
\mathcal{N}_{A,\pi}(V;W)&=\left\{\Delta\in\mathcal{L}_A(V;\pi^{-1}(W))\;\middle|\; \begin{array}{l}K_X+\Delta \text{ is nef over some open}\\ \text{neighborhood of } W\end{array}\right\}.
\end{aligned}$$

Given a bimeromorphic contraction $\phi\colon X\dashrightarrow Z$ after shrinking Y around W suitably, define

$$\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)=\left\{\Delta\in\mathcal{E}_{A,\pi}(V;W)\;\middle|\;\begin{array}{l}\phi \text{ is a weak log canonical model for } (X,\Delta)\\ \text{over } W\end{array}\right\},$$

and

$$\mathcal{W}_{\phi,A,\pi}(V;W)=\left\{\Delta\in\mathcal{E}_{A,\pi}(V;W)\;\middle|\;\begin{array}{l}\phi \text{ is a weak log canonical model for } (X,\Delta)\\ \text{over some open neighborhood of } W\end{array}\right\}.$$

Given a meromorphic map $\psi\colon X\dashrightarrow Z$ after shrinking Y around W suitably, define

$$\mathcal{A}_{\psi,A,\pi}(V;W)=\left\{\Delta\in\mathcal{E}_{A,\pi}(V;W)\;\middle|\;\begin{array}{l}\psi \text{ is the ample model for } (X,\Delta)\\ \text{over some open neighborhood of } W\end{array}\right\}.$$

We make some elementary remarks.

Remark 11.9. By the same argument as in the proof of Lemma 3.5, we can check that $\mathcal{L}_A(V;\pi^{-1}(W))$ in Definition 11.8 is a polytope. We further assume that A is a $\mathbb{Q}$-divisor and that V is defined over the rationals. Then $\mathcal{L}_A(V;\pi^{-1}(W))$ is a rational polytope.

Remark 11.10. By definition, it is easy to see that $\mathcal{N}^{\sharp}_{A,\pi}(V;W)$ and $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$ are closed subsets of $\mathcal{L}_A(V;\pi^{-1}(W))$.

We note the following elementary fact.

Remark 11.11. In Definition 11.8, let S be an effective $\mathbb{R}$-divisor on X such that $\mathrm{Supp}\,S$ has only finitely many irreducible components. If $\mathrm{Supp}\,A$ and $\mathrm{Supp}\,B$ have no common irreducible components for every $B\in V$, then

$$\mathcal{L}_{S+A}(V;\pi^{-1}(W))=\mathcal{L}_S(V_A;\pi^{-1}(W))$$

holds. Of course, if $\mathrm{Supp}\,S$ and $\mathrm{Supp}\,B$ have no common irreducible components for every $B\in V$, then

$$\mathcal{L}_{S+A}(V;\pi^{-1}(W))=\mathcal{L}_A(V_S;\pi^{-1}(W))$$

holds.

From now on, we assume that S is reduced. We put

$$V' := \{B \in V \mid \operatorname{Supp} B \text{ and } \operatorname{Supp} S \text{ have no common irreducible components}\}.$$

Then V' is an affine subspace of V such that

$$\mathcal{L}_{S+A}(V;\pi^{-1}(W)) = \mathcal{L}_{S+A}(V';\pi^{-1}(W)) = \mathcal{L}_{A}(V'_S;\pi^{-1}(W))$$

There are no difficulties to adapt [BCHM, Lemmas 3.6.9, 3.6.10, and 3.6.11] to our complex analytic setting. Roughly speaking, they are easy consequences of the negativity lemma. Hence we omit the details here. On the other hand, [BCHM, Lemma 3.6.12] is subtle and needs some reformulation for our purposes. We will discuss it in Section 12.

11.12 (see [BCHM, Lemmas 3.7.3, 3.7.4, and 3.7.5])**.** Note that [BCHM, Lemmas 3.7.3, 3.7.4, and 3.7.5] are very important. We need them to reduce various problems for log canonical pairs to simpler ones for kawamata log terminal pairs. We state them here explicitly in our complex analytic setting for the sake of completeness. In the following three lemmas 11.13, 11.14, and 11.15, we assume that $\pi\colon X \to Y$ is a projective morphism of complex analytic spaces such that X is a normal variety and Y is Stein and that W is a Stein compact subset of Y.

Lemma 11.13 (see [BCHM, Lemma 3.7.3])**.** *Let V be a finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ and let $A \geq 0$ be a π-big $\mathbb{R}$-divisor on X. Let $\mathcal{C} \subset \mathcal{L}_A(V;\pi^{-1}(W))$ be a polytope.*

If $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of (X,Δ) for every $\Delta \in \mathcal{C}$, then, after shrinking Y around W suitably, we can find a general π-ample $\mathbb{Q}$-divisor A' on X, a finite-dimensional affine subspace V' of $\mathrm{WDiv}_{\mathbb{R}}(X)$, and a translation

$$L\colon \mathrm{WDiv}_{\mathbb{R}}(X) \to \mathrm{WDiv}_{\mathbb{R}}(X),$$

by an $\mathbb{R}$-divisor T with $T \sim_{\mathbb{R}} 0$ such that $L(\mathcal{C}) \subset \mathcal{L}_{A'}(V';\pi^{-1}(W))$ and $(X,\Delta - A)$ and $(X, L(\Delta))$ have the same non-kawamata log terminal centers. Furthermore, if A is a $\mathbb{Q}$-divisor, then we may assume that $T \sim_{\mathbb{Q}} 0$ holds.

Lemma 11.14 (see [BCHM, Lemma 3.7.4])**.** *Let V be a finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals, and let A be a general π-ample $\mathbb{Q}$-divisor on X. Let S be a finite sum of prime divisors on X such that each irreducible component of S intersects with $\pi^{-1}(W)$. Suppose that there exists a divisorial log terminal pair (X,Δ_0) with $S = \lfloor \Delta_0 \rfloor$ and let $G \geq 0$ be any divisor whose support does not contain any non-kawamata log terminal centers of (X,Δ_0).*

Then, after shrinking Y around W suitably, we can find a general π-ample $\mathbb{Q}$-divisor A' on X, and affine subspace V' of $\mathrm{WDiv}_{\mathbb{R}}(X)$, which is defined over the rationals, and a rational affine linear isomorphism

$$L\colon V_{S+A} \to V'_{S+A'}$$

such that

- *L preserves $\mathbb{Q}$-linear equivalence,*
- *$L\left(\mathcal{L}_{S+A}(V;\pi^{-1}(W))\right)$ is contained in the interior of $\mathcal{L}_{S+A'}(V';\pi^{-1}(W))$,*
- *for any $\Delta \in L\left(\mathcal{L}_{S+A}(V;\pi^{-1}(W))\right)$, $K_X+\Delta$ is divisorial log terminal and $\lfloor \Delta \rfloor = S$, and*
- *for any $\Delta \in L\left(\mathcal{L}_{S+A}(V;\pi^{-1}(W))\right)$, the support of Δ contains the support of G.*

Lemma 11.15 (see [BCHM, Lemma 3.7.5]). *Let $(X, \Delta = A + B)$ be a log canonical pair, where $A \geq 0$ and $B \geq 0$.*

If A is π-big and $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of (X, Δ) and there exists a kawamata log terminal pair (X, Δ_0), then we can find a kawamata log terminal pair $(X, \Delta' = A' + B')$, where $A' \geq 0$ is a general π-ample $\mathbb{Q}$-divisor on X, $B' \geq 0$, and $K_X + \Delta' \sim_{\mathbb{R}} K_X + \Delta$. If in addition A is a $\mathbb{Q}$-divisor, then $K_X + \Delta' \sim_{\mathbb{Q}} K_X + \Delta$.

Here we omit the proof of Lemmas 11.13, 11.14, and 11.15. This is because there are no difficulties to translate the proof of [BCHM, Lemmas 3.7.3, 3.7.4, and 3.7.5] into our complex analytic setting.

In this paper, we are mainly interested in kawamata log terminal pairs (X, Δ) such that Δ is big over Y. For such pairs, we have some good properties.

Lemma 11.16 ([BCHM, Lemma 3.9.3]). *Let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi \colon X \to Y$ and W satisfies* (P). *Suppose that (X, Δ) is a kawamata log terminal pair, where Δ is π-big. If $\phi \colon X \dashrightarrow Z$ is a weak log canonical model of $K_X + \Delta$ over W, then*

(1) *ϕ is a weak log canonical model of $K_X + \Delta$ over some open neighborhood of W,*
(2) *ϕ is a semiample model over some open neighborhood of W,*
(3) *after shrinking Y around W suitably, there exists a contraction morphism $h \colon Z \to Z'$ such that $K_Z + \Gamma \sim_{\mathbb{R}} h^*H$, for some $\mathbb{R}$-divisor H on Z', which is ample over Y, where $\Gamma = \phi_*\Delta$, and*
(4) *the ample model $\psi \colon X \dashrightarrow Z'$ of $K_X + \Delta$ over some open neighborhood of W exists.*

Proof. Throughout this proof, we will freely shrink Y around W without mentioning it explicitly. We put $\Gamma = \phi_*\Delta$. Then (Z, Γ) is kawamata log terminal by the negativity lemma. Since Δ is big, we can write $\Gamma \sim_{\mathbb{R}} A + B$ such that A is ample over Y, $A \geq 0$, $B \geq 0$, and $(Z, A + B)$ is kawamata log terminal. Then, by Theorem 8.3, we can check that $K_Z + \Gamma$ is semiample over Y. This means that ϕ is a weak log canonical model of $K_X + \Delta$ over Y and that $K_Z + \Gamma$ is semiample over Y. Hence we obtain (1) and (2). Since $K_Z + \Gamma$ is semiample over Y, we get a contraction morphism $h \colon Z \to Z'$ such that $\psi := h \circ \phi \colon X \dashrightarrow Z'$ is the ample model of (X, Δ) (see Lemma 11.3 (2)). Therefore, we have (3) and (4). □

The following theorem is very important.

Theorem 11.17 ([BCHM, Theorem 3.11.1]). *Let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi \colon X \to Y$ and W satisfies* (P). *Let V be a finite-dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$, *which is defined over the rationals. Fix a π-ample $\mathbb{Q}$-divisor A on X. Suppose that there exists a kawamata log terminal pair (X, Δ_0). Then $\mathcal{N}_{A,\pi}(V; W) = \mathcal{N}^\sharp_{A,\pi}(V; W)$ holds and the set of hyperplanes $R^\perp$ is finite in $\mathcal{L}_A(V; \pi^{-1}(W))$, as R ranges over the set of extremal rays of $\overline{\mathrm{NE}}(X/Y; W)$. In particular, $\mathcal{N}_{A,\pi}(V; W) = \mathcal{N}^\sharp_{A,\pi}(V; W)$ is a rational polytope.*

Sketch of Proof of Theorem 11.17. By Theorem 8.3, $K_X + \Delta$ is semiample over some open neighborhood of W for every $\Delta \in \mathcal{N}^\sharp_{A,\pi}(V; W)$. In particular, $K_X + \Delta$ is nef over some open neighborhood of W. This implies that $\mathcal{N}_{A,\pi}(V; W) = \mathcal{N}^\sharp_{A,\pi}(V; W)$ holds. On the other hand, the proof of [BCHM, Theorem 3.11.1] works by Theorem 7.3. Hence we see that $R^\perp$ is finite in $\mathcal{L}_A(V; \pi^{-1}(W))$ and $\mathcal{N}^\sharp_{A,\pi}(V; W)$ is a rational polytope. □

We prepare an easy lemma.

Lemma 11.18. *In Theorem 11.17, we consider $\Delta_1, \Delta_2 \in \mathcal{L}_A(V;\pi^{-1}(W))$. Let $f_i\colon X \to Z_i$ be a contraction morphism between normal varieties over Y such that $K_X+\Delta_i \sim_{\mathbb{R}} f_i^*D_i$ for some g_i-ample $\mathbb{R}$-divisor D_i on Z_i, where $g_i\colon Z_i \to Y$ is the structure morphism, for $i=1,2$. Then the following conditions are equivalent.*

(i) *Δ_1 and Δ_2 belong to the same interior of a unique face of $\mathcal{N}^{\sharp}_{A,\pi}(V;W)$.*
(ii) *Z_1 and Z_2 are isomorphic over some open neighborhood of W.*

Proof. We note that $\Delta_1, \Delta_2 \in \mathcal{N}^{\sharp}_{A,\pi}(V;W)$. If (ii) holds, then (i) obviously holds true. From now on, we will prove (ii) under the assumption that (i) holds. Let $\overline{Z}$ be the image of the map $(f_1, f_2)\colon X \to Z_1 \times_Y Z_2$ given by $x \mapsto (f_1(x), f_2(x))$ Let $p_i\colon \overline{Z} \to Z_i$ be the projection for $i=1,2$. We take any point $z_i \in g_i^{-1}(W)$. Then we can easily see that $p_i^{-1}(z_i)$ is a point by (i). By using the Stein factorization (see, for example, [BS, Chapter III, Corollary 2.13]), $p_i\colon \overline{Z} \to Z_i$ is an isomorphism over some open neighborhood of $g_i^{-1}(W)$. Hence Z_1 and Z_2 are isomorphic over some open neighborhood of W. This is what we wanted. □

As an easy consequence of Theorem 11.17, we have:

Corollary 11.19 ([BCHM, Corollary 3.11.2])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P). *Let V be a finite-dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$, *which is defined over the rationals. Fix a general π-ample $\mathbb{Q}$-divisor A on X. Suppose that there exists a kawamata log terminal pair (X, Δ_0). Let $\phi\colon X \dashrightarrow Z$ be any bimeromorphic contraction over Y. Then we obtain:*

(1) *$\mathcal{W}_{\phi,A,\pi}(V;W) = \mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$ holds and $\mathcal{W}_{\phi,A,\pi}(V;W)$ is a rational polytope.*

Moreover, we have:

(2) *There are finitely many contraction morphisms $f_i\colon Z \to Z_i$ over Y, $1 \le i \le k$, such that if $f\colon Z \to Z'$ is any contraction morphism over Y and there is an $\mathbb{R}$-divisor D on Z', which is ample over Y, such that $K_Z+\Gamma := \phi_*(K_X+\Delta) \sim_{\mathbb{R}} f^*D$ for some $\Delta \in \mathcal{W}_{\phi,A,\pi}(V;W)$, then there is an index $1 \le i \le k$ and an isomorphism $\eta\colon Z_i \to Z'$ such that $f = \eta \circ f_i$.*

Note that in (2) we require that f_i, f, D, and η exist only after shrinking Y around W suitably.

The proof of [BCHM, Corollary 3.11.2] works with some minor modifications.

Sketch of Proof of Corollary 11.19. Note that $\mathcal{L}_A(V;\pi^{-1}(W))$ is a rational polytope. Therefore, its span is an affine subspace of V_A, which is defined over the rationals. By replacing V, we may assume that $\mathcal{L}_A(V;\pi^{-1}(W))$ spans V_A. To prove that $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$ is a rational polytope, we may work locally about a divisor $\Delta \in \mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$. By Lemma 11.14, we may assume that (X,Δ) is kawamata log terminal. In this case, (Z,Γ) is automatically kawamata log terminal. We put $C := \phi_*A$. Then C is big over Y. Let $V^{\dagger} \subset \mathrm{WDiv}_{\mathbb{R}}(Z)$ be the image of V. By Lemmas 11.13 and 11.14, we can reduce the problem to the case where C is a ψ-ample $\mathbb{Q}$-divisor and Γ belongs to the interior of $\mathcal{L}_C(V^{\dagger};\psi^{-1}(W))$, where $\psi\colon Z \to Y$ is the structure morphism. By Theorem 11.17, $\mathcal{N}^{\sharp}_{C,\psi}(V^{\dagger},W) = \mathcal{N}_{C,\psi}(V^{\dagger},W)$ is a rational polytope. Hence we can easily check that $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W) = \mathcal{W}_{\phi,A,\pi}(V;W)$ holds and $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$ is a rational polytope. Thus we obtain (1). Let $f\colon Z \to Z'$ be a contraction morphism over some open neighborhood of W such that $\phi_*(K_X+\Delta) = K_Z+\Gamma \sim_{\mathbb{R}} f^*D$ for some ψ'-ample $\mathbb{R}$-divisor D on Z', where $\psi'\colon Z' \to Y$ is the structure morphism. Then Γ belongs to the interior of a unique face G of $\mathcal{N}^{\sharp}_{C,\psi}(V^{\dagger};W) = \mathcal{N}_{C,\psi}(V^{\dagger};W)$. Note that Δ belongs to the interior of a unique face F of $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W) = \mathcal{W}_{\phi,A,\pi}(V;W)$ and G is determined by F. Thus we can check that (2) holds true by Lemma 11.18. □

11.20 (see [BCHM, Lemma 3.10.11])**.** When we run a minimal model program, we have to check that several properties are preserved by flips and divisorial contractions.

Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). Assume that (X,Δ) is divisorial log terminal and that X is $\mathbb{Q}$-factorial over W. Let $\varphi\colon X \to Z$ be a bimeromorphic contraction morphism over Y associated to a $(K_X+\Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y;W)$. Let A be a π-big $\mathbb{R}$-divisor on X such that $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of (X,Δ).

Lemma 11.21 (Divisorial contractions)**.** *In the above setting, we further assume that φ is divisorial. Then, after shrinking Y around W suitably, we have the following properties.*

(1) *Z is $\mathbb{Q}$-factorial over W.*
(2) *(Z,Γ) is divisorial log terminal, where $\Gamma := \varphi_*\Delta$.*
(3) *$\mathrm{Exc}(\varphi)$ is a prime divisor on X.*
(4) *$\rho(Z/Y;W) = \rho(X/Y;W) - 1$.*
(5) *$\mathbf{B}_+(\varphi_*A/Y)$ does not contain any non-kawamata log terminal centers of (Z,Γ).*

Lemma 11.22 (Flips)**.** *In the above setting, we further assume that φ is a flipping contraction and that the flip $\varphi^+\colon X^+ \to Z$ of φ exists.*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ & \underset{\varphi}{\searrow} \quad \underset{\varphi^+}{\swarrow} & \\ & Z & \end{array}$$

Then, after shrinking Y and W suitably, we have the following properties.

(1) *X^+ is $\mathbb{Q}$-factorial over W.*
(2) *(X^+,Δ^+) is divisorial log terminal, where $\Delta^+ := \phi_*\Delta$.*
(3) *$\rho(X^+/Y;W) = \rho(Z/Y;W)+1 = \rho(X/Y;W)$.*
(4) *X^+ is projective over Y.*
(5) *$\mathbf{B}_+(\phi_*A/Y)$ does not contain any non-kawamata log terminal centers of (X^+,Δ^+).*

Proof of Lemmas 11.21 and 11.22. The proof for algebraic varieties works with only some obvious modifications even in the complex analytic setting. Here, we will only prove (5). There are no difficulties to prove the other properties. Let $f\colon X \dashrightarrow X'$ denote the divisorial contraction $\varphi\colon X \to Z$ in Lemma 11.21 or the flip $\phi\colon X \dashrightarrow X^+$ in Lemma 11.22. We will freely shrink Y around W suitably without mentioning it explicitly. We take a general π'-ample $\mathbb{Q}$-divisor C on X', where $\pi'\colon X' \to Y$ is the structure morphism. We may assume that $\mathbf{B}((A-\varepsilon f^{-1}_*C)/Y)$ does not contain any non-kawamata log terminal centers of (X,Δ) for some $0<\varepsilon<1$. Therefore, we have an effective $\mathbb{R}$-divisor D on X such that $D \sim_{\mathbb{R}} A-\varepsilon f^{-1}_*C$ and that $(X, \Delta' := \Delta+\varepsilon' D)$ is a divisorial log terminal pair for $0<\varepsilon' \ll 1$. Note that if $0<\varepsilon'\ll 1$ then R is still a $(K_X+\Delta')$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y;W)$. Therefore, $(X', f_*\Delta' = f_*\Delta + \varepsilon' f_*D)$ is still a divisorial log terminal pair. Hence the support of f_*D contains no non-kawamata log terminal centers of $(X', f_*\Delta)$.

Since $f_*A \sim_{\mathbb{R}} f_*D + \varepsilon C$, $\mathbf{B}_+(f_*A/Y)$ contains no non-kawamata log terminal centers of $(X', f_*\Delta')$. This is what we wanted. □

12. Some basic definitions and properties, II

In this section, we will treat [BCHM, Lemma 3.6.12] in the complex analytic setting. We change the formulation suitable for our complex analytic setting. The main result of this section is Lemma 12.3. For the proof of Lemma 12.3, we prepare two lemmas.

Let us start with *small projective $\mathbb{Q}$-factorializations* (see Theorem 1.24).

Lemma 12.1 (Small projective $\mathbb{Q}$-factorializations)**.** *Assume that Theorem G_n holds true.*

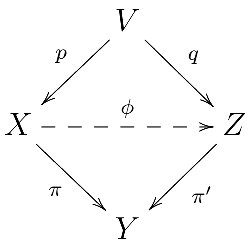
Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces with $\dim X = n$ *and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Assume that (X, Δ) is kawamata log terminal. Then, after shrinking Y around W suitably, there exists a small projective bimeromorphic contraction morphism $f\colon X' \to X$ such that X' is projective over Y and that X' is $\mathbb{Q}$-factorial over W.*

Proof. Throughout this proof, we will freely shrink Y around W suitably without mentioning it explicitly. By taking a resolution, we have a bimeromorphic contraction morphism $g\colon V \to X$ such that V is smooth, V is projective over Y, and $\mathrm{Exc}(g)$ and $\mathrm{Exc}(g) \cup \mathrm{Supp}\, g_*^{-1}\Delta$ are simple normal crossing divisors on V. Then we can take an $\mathbb{R}$-divisor Δ_V on V such that (V, Δ_V) is kawamata log terminal and that $K_V + \Delta_V = g^*(K_X + \Delta) + E$, where $E \geq 0$ and $\mathrm{Supp}\, E = \mathrm{Exc}(g)$. We take a general π-ample $\mathbb{Q}$-divisor H on X such that $K_X + \Delta + H \sim_{\mathbb{R}} D \geq 0$. We apply Theorem G_n to $K_V + \Delta_V + g^*H \sim_{\mathbb{R}} g^*D + E \geq 0$. Then we get a bimeromorphic contraction $\phi\colon V \dashrightarrow X'$ over X such that X' is $\mathbb{Q}$-factorial over W, X' is projective over Y, and $K_{X'} + \Gamma$ is nef over Y, where $\Gamma := \phi_*\Delta_V$. By the negativity lemma, we see that $f\colon X' \to X$ is small. This is what we wanted. □

By combining Lemma 12.1 with Theorem 8.2, we have:

Lemma 12.2. *Assume that Theorem G_n holds true.*

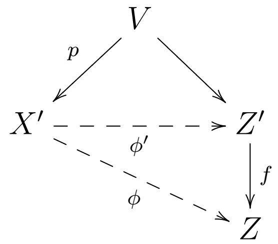
Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces with $\dim X = n$ *and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Let $\phi\colon X \dashrightarrow Z$ be a bimeromorphic contraction of normal complex varieties over Y such that (Z, Δ_Z) is kawamata log terminal for some Δ_Z. We consider the following commutative diagram:*



where p and q are projective bimeromorphic morphisms and V is a normal complex variety. Let H be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Z such that $H' := p_*q^*H$ is also $\mathbb{R}$-Cartier. Let B be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that B is numerically equivalent to H' over W. Then $\phi_*B := q_*p^*B$ is $\mathbb{R}$-Cartier over some open neighborhood of W and is numerically equivalent to H over W.

Proof. It is sufficient to prove that ϕ_*B is $\mathbb{R}$-Cartier over some open neighborhood of W, equivalently, ϕ_*B is $\mathbb{R}$-Cartier at any point z of $(\pi')^{-1}(W)$. We will freely shrink Y

around W without mentioning it explicitly. By applying Lemma 12.1 to $\pi'\colon Z \to Y$, we can construct a small projective bimeromorphic contraction morphism $f\colon Z' \to Z$ such that Z' is projective over Y and that Z' is $\mathbb{Q}$-factorial over W. By replacing V, we may assume that $q\colon V \to Z$ factors through Z'. Hence we have the following commutative diagram.



Since Z' is $\mathbb{Q}$-factorial over W, we may assume that ϕ'_*B is $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Z'. We put $K_{Z'} + \Delta_{Z'} = f^*(K_Z + \Delta_Z)$. Then $(Z', \Delta_{Z'})$ is kawamata log terminal since f is small. Note that ϕ'_*B is numerically equivalent to f^*H over W by construction. Therefore, ϕ'_*B is numerically trivial over z for any $z \in (\pi')^{-1}(W)$. Since $f\colon Z' \to Z$ is bimeromorphic, by replacing Z with a small Stein open neighborhood of some $z \in (\pi')^{-1}(W)$, we can take Θ on Z' such that (Z', Θ) is kawamata log terminal and that $-(K_{Z'} + \Theta)$ is ample over Z. Hence, by Theorem 8.2, ϕ_*B is $\mathbb{R}$-Cartier at z. Note that W is compact. Therefore, this means that ϕ_*B is $\mathbb{R}$-Cartier over some open neighborhood of W. This is what we wanted. □

The following lemma is essentially the same as [BCHM, Lemma 3.6.12]. We note that we do not assume that A is π-ample in Lemma 12.3 (see Remark 12.4 below).

Lemma 12.3 ([BCHM, Lemma 3.6.12])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P) *and that X is $\mathbb{Q}$-factorial over W and has only kawamata log terminal singularities. Let $\phi\colon X \dashrightarrow Z$ be a bimeromorphic contraction over Y and let A be an effective $\mathbb{R}$-divisor on X such that* $\operatorname{Supp} A$ *has only finitely many irreducible components. We assume one of the following conditions:*

(i) *Z is $\mathbb{Q}$-factorial over W, or*
(ii) *Theorem G_n holds, where $n = \dim X$.*

If V is any finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ *such that $\mathcal{L}_A(V; \pi^{-1}(W))$ spans* $\mathrm{WDiv}_{\mathbb{R}}(X)$ *modulo numerical equivalence over W and $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$ intersects the interior of $\mathcal{L}_A(V; \pi^{-1}(W))$, then*

$$\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W) = \overline{\mathcal{A}}_{\phi,A,\pi}(V;W)$$

holds, where $\overline{\mathcal{A}}_{\phi,A,\pi}(V;W)$ is the closure of $\mathcal{A}_{\phi,A,\pi}(V;W)$.

Let us prove Lemma 12.3.

Proof of Lemma 12.3. It is easy to see that

$$\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W) \supset \mathcal{A}_{\phi,A,\pi}(V;W)$$

holds. Since $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$ is closed, it follows that

$$\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W) \supset \overline{\mathcal{A}}_{\phi,A,\pi}(V;W).$$

In order to prove the opposite inclusion, it is sufficient to prove that a dense subset of $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$ is contained in $\mathcal{A}_{\phi,A,\pi}(V;W)$.

From now on, we will freely shrink Y around W suitably without mentioning it explicitly. We take Δ belonging to the interior of $\mathcal{W}^{\sharp}_{\phi,A,\pi}(V;W)$. We put $\Gamma := \phi_*\Delta$. Then (Z,Γ) is a weak log canonical model of (X,Δ) over W by definition. Since $\mathcal{L}_A(V;\pi^{-1}(W))$ spans $\mathrm{WDiv}_{\mathbb{R}}(X)$ modulo numerical equivalence over W, we can find $\Delta_0 \in \mathcal{L}_A(V;\pi^{-1}(W))$ such that $\Delta_0 - \Delta$ is numerically equivalent over W to $\mu\Delta$ for some $\mu > 0$. We consider

$$\Delta' := \Delta + \varepsilon\left((\Delta_0 - \Delta) - \mu\Delta\right) = (1-\varepsilon)\nu\Delta + \varepsilon\Delta_0,$$

where

$$\nu = \frac{1-\varepsilon-\varepsilon\mu}{1-\varepsilon} < 1.$$

Hence, Δ' is numerically equivalent to Δ over W and if $\varepsilon > 0$ is sufficiently small then Δ' is effective. Since $(X,\nu\Delta)$ is kawamata log terminal, it follows that (X,Δ') is also kawamata log terminal. We put $\Gamma' := \phi_*\Delta'$. If (i) holds, then $K_Z + \Gamma'$ is obviously $\mathbb{R}$-Cartier. If (ii) holds, then we can check that $K_Z + \Gamma'$ is $\mathbb{R}$-Cartier by Lemma 12.2.

Let H be a general π'-ample $\mathbb{Q}$-divisor on Z, where $\pi' \colon Z \to Y$ is the structure morphism. Let $p\colon U \to X$ and $q\colon U \to Z$ resolve the indeterminacy locus of ϕ. We put $H' := p_*q^*H$. It is obvious that ϕ is H'-nonpositive. We take $\Delta_1 \in \mathcal{L}_A(V;\pi^{-1}(W))$ such that $B := \Delta_1 - \Delta$ is numerically equivalent to $\eta H'$ over W for some $\eta > 0$. By replacing H with ηH, we may assume that $\eta = 1$. If (i) holds, then ϕ_*B is obviously $\mathbb{R}$-Cartier. If (ii) holds, then we can check that ϕ_*B is $\mathbb{R}$-Cartier by Lemma 12.2. Therefore, we obtain that ϕ is $(K_X+\Delta+\lambda B)$-nonpositive and $\phi_*(K_X+\Delta+\lambda B)$ is ample over Y for every $\lambda > 0$. On the other hand, we have

$$\Delta + \lambda B = \Delta + \lambda(\Delta_1 - \Delta) = (1-\lambda)\Delta + \lambda\Delta_1 \in \mathcal{L}_A(V;\pi^{-1}(W))$$

for every $\lambda \in [0,1]$. This implies that ϕ is the ample model of $K_X + \Delta + \lambda B$ over Y for every $\lambda \in (0,1]$. □

We close this section with a useful remark.

Remark 12.4. In Lemma 12.3, we further assume that $A = S + A'$, where S is reduced and $A' \geq 0$ is a general π-ample $\mathbb{Q}$-divisor on X, and that V is defined over the rationals. We put

$$V' := \{B \in V \mid \operatorname{Supp} B \text{ and } \operatorname{Supp} S \text{ have no common irreducible components}\}$$

as in Remark 11.11. Hence V' is also defined over the rationals. Then we have

$$\begin{aligned}\mathcal{W}^{\sharp}_{\phi,S+A',\pi}(V;W) &= \mathcal{W}^{\sharp}_{\phi,S+A',\pi}(V';W) = \mathcal{W}^{\sharp}_{\phi,A',\pi}(V'_S;W)\\ &= \mathcal{W}_{\phi,A',\pi}(V'_S;W) = \mathcal{W}_{\phi,S+A',\pi}(V';W) = \mathcal{W}_{\phi,S+A',\pi}(V;W)\end{aligned}$$

by Remark 11.11 and Corollary 11.19 (1). In particular,

$$\mathcal{W}^{\sharp}_{\phi,S+A',\pi}(V;W) = \mathcal{W}_{\phi,S+A',\pi}(V;W)$$

is a rational polytope since $\mathcal{W}^{\sharp}_{\phi,A',\pi}(V'_S;W) = \mathcal{W}_{\phi,A',\pi}(V'_S;W)$ is a rational polytope by Corollary 11.19 (1).

13. Minimal model program with scaling

In this section, we will explain the minimal model program with scaling. It is very important for various geometric applications.

13.1 (Minimal model program with scaling)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). Precisely speaking, X is a normal complex variety, Y is a Stein space, and W is a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Let (X, Δ) be a divisorial log terminal pair such that X is $\mathbb{Q}$-factorial over W and let C be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $(X, \Delta + C)$ is log canonical and that $K_X + \Delta + C$ is nef over W. We assume that one of the following conditions hold.

(i) $\Delta = S + A + B$, $S = \lfloor \Delta \rfloor$, $A \geq 0$ is π-big, $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of (X, Δ), and $B \geq 0$.
(ii) C is π-big and $\mathbf{B}_+(C/Y)$ does not contain any non-kawamata log terminal centers of (X, Δ).

We recall the following elementary fact for the reader's convenience.

Remark 13.2. Assume that (X, Δ) and $(X, \Delta + C)$ are both log canonical and that C is effective. Then V is a non-kawamata log terminal center of (X, Δ) if and only if V is a non-kawamata log terminal center of $(X, \Delta + \varepsilon C)$ for every $0 < \varepsilon < 1$.

Although we have already treated more general lemmas, we explicitly state an easy lemma for the sake of completeness.

Lemma 13.3. *Suppose that (i) holds true. Then, after shrinking Y around W suitably, we can find Δ' such that $K_X + \Delta \sim_{\mathbb{R}} K_X + \Delta'$, $\Delta' = A' + B'$, $A' \geq 0$ is a π-ample $\mathbb{Q}$-divisor, $B' \geq 0$, and (X, Δ') is kawamata log terminal. Suppose that (ii) holds true. Then, after shrinking Y around W suitably, for any $0 < \varepsilon < 1$, there exists Δ' such that $K_X + \Delta + \varepsilon C \sim_{\mathbb{R}} K_X + \Delta'$, $\Delta' = A' + B'$, $A' \geq 0$ is a π-ample $\mathbb{Q}$-divisor, $B' \geq 0$, and (X, Δ') is kawamata log terminal.*

Proof. Throughout this proof, we will freely shrink Y around W without mentioning it explicitly. We assume that (i) holds. By the assumption on $\mathbf{B}_+(A/Y)$, we can write $A \sim_{\mathbb{R}} A_1 + A_2$ such that A_1 is a π-ample $\mathbb{Q}$-divisor on X and A_2 does not contain any non-kawamata log terminal centers of (X, Δ). Then $K_X + S + A + B \sim_{\mathbb{R}} K_X + S + B + (1-\alpha)A + \alpha A_2 + \alpha A_1$ such that $(X, S + B + (1-\alpha)A + \alpha A_2)$ is divisorial log terminal for some positive rational number α with $0 < \alpha \ll 1$. By replacing A and B with αA_1 and $B + (1-\alpha)A + \alpha A_2$, respectively, we may assume that A itself is a π-ample $\mathbb{Q}$-divisor. Since $\beta S + \frac{1}{2}A$ is π-ample for some rational number β with $0 < \beta \ll 1$, we can take $A_3 \sim_{\mathbb{Q}} \beta S + \frac{1}{2}A$ such that $K_X + (1-\beta)S + \frac{1}{2}A + B + A_3$ is kawamata log terminal. If we put $A' = \frac{1}{2}A$ and $B' = (1-\beta)S + \frac{1}{2}A + B + A_3$, then $\Delta' = A' + B'$ satisfies the desired properties. From now on, we assume that (ii) holds. We note that V is a non-kawamata log terminal center of (X, Δ) if and only if V is a non-kawamata log terminal center of $(X, \Delta + \varepsilon C)$ for $0 < \varepsilon < 1$. Therefore, we can apply the above argument to $\Delta + \varepsilon C$. Thus we have a desired divisor Δ' on X. ☐

We put

$$\lambda := \inf\{\mu \in \mathbb{R}_{\geq 0} \,|\, K_X + \Delta + \mu C \text{ is nef over } W\}.$$

If $\lambda = 0$, equivalently, $K_X + \Delta$ is nef over W, then we stop. In this case, (X, Δ) itself is a log terminal model of (X, Δ) over W. If further (i) holds true, then (X, Δ) is a log terminal model of (X, Δ) over some open neighborhood of W by Theorem 8.3 (see also Lemma 11.16). Moreover, it is a good log terminal model of (X, Δ) over some open neighborhood of W.

Lemma 13.4. *If $\lambda > 0$ holds, then there exists a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y; W)$ such that $(K_X + \Delta + \lambda C) \cdot R = 0$.*

Proof. If (i) holds, then there are only finitely many $(K_X + \Delta)$-negative extremal rays by Theorems 7.2 and 7.3. Hence it is not difficult to find a desired extremal ray R. If (ii) holds, then we consider $K_X + \Delta + \varepsilon C$ for $0 < \varepsilon \ll 1$. By Lemma 13.3, after shrinking Y around W suitably, $K_X + \Delta + \varepsilon C \sim_{\mathbb{R}} K_X + \Delta'$ such that (X, Δ') is kawamata log terminal, $\Delta' = A' + B'$, $A' \geq 0$ is a π-ample $\mathbb{Q}$-divisor, and $B' \geq 0$. Hence, by Theorem 7.3, there are only finitely many $(K_X + \Delta + \varepsilon C)$-negative extremal rays. Thus, we can easily take a desired extremal ray R. □

From now on, we assume that $\lambda > 0$ holds. Let $\varphi \colon X \to Z$ be the extremal contraction over Y defined by R (see Theorems 7.2 and 7.3). We note that in general the contraction morphism $\varphi \colon X \to Z$ over Y exists only after shrinking Y around W suitably. If φ is not birational, then we have a Mori fiber space over Y (see Definition 11.7) and we stop.

Lemma 13.5. *Assume that Theorem G_n holds true.*
Let $\varphi \colon X \to Z$ be a flipping contraction associated to a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y; W)$ with $\dim X = n$. Then the flip $\varphi^+ \colon X^+ \to Z$ exists.

Proof. For the details, see Theorem 17.9 and its proof. □

If φ is birational, then either φ is divisorial and we replace X by Z or φ is small, that is, flipping, and we replace X by the flip X^+ (see Lemmas 11.21 and 11.22). In either case, $K_X + \Delta + \lambda C$ is nef over W and $K_X + \Delta$ is divisorial log terminal. Hence we may repeat the process under the assumption that Theorem G_n holds true for $n = \dim X$. In this way, we obtain a sequence of flips and divisorial contractions starting from $X_0 := X$:

$$X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} X_i \overset{\phi_i}{\dashrightarrow} X_{i+1} \overset{\phi_{i+1}}{\dashrightarrow} \cdots,$$

and a real numbers

$$1 \geq \lambda =: \lambda_0 \geq \lambda_1 \geq \cdots$$

such that $K_{X_i} + \Delta_i + \lambda_i C_i$ is nef over W, where $\Delta_i := (\phi_{i-1})_*\Delta_{i-1}$ and $C_i := (\phi_{i-1})_*C_{i-1}$ for every $i \geq 1$. We note that each step ϕ_i exists only after shrinking Y around W suitably. We can easily check that each step of this minimal model program preserves the conditions (i) and (ii) by the negativity lemma (see, for example, Lemmas 11.21, 11.22 and [BCHM, Lemma 3.10.11]). The above minimal model program is usually called the *minimal model program with scaling over Y around W*. We sometimes simply say that it is a *$(K_X + \Delta)$-minimal model program with scaling*. If (i) holds true and $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of $(X, \Delta + C)$, then this minimal model program always terminates after finitely many steps under the assumption that Theorem E holds true.

Theorem 13.6. *Assume that Theorem G_n and Theorem E_n hold true, where $n = \dim X$.*
Suppose that (i) holds. We further assume that $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of $(X, \Delta + C)$. Then the minimal model program with scaling explained above always terminates after finitely many steps.

Proof. By the proof of Lemma 13.3, we may assume that $\Delta = A + B$, $A \geq 0$ is a π-ample $\mathbb{Q}$-divisor, $B \geq 0$, (X, Δ) is kawamata log terminal, and $(X, \Delta + C)$ is still log canonical. By construction, after shrinking Y around W suitably, $(X_i, \Delta_i + \lambda_i C_i)$ is a weak log canonical model of $(X, \Delta + \lambda_i C)$ over W for every i. By Theorem E_n and the negativity lemma (see [BCHM, Lemma 3.10.12]), we know that there are no infinite sequences of flips and divisorial contractions. This is what we wanted. □

If C is π-ample, then we can run a minimal model program with scaling of C by (ii). We conjecture that the minimal model program with scaling always terminates after finitely many steps. Unfortunately, however, this conjecture is still widely open. The following easy lemma is useful for some geometric applications (see [Fu4]).

Lemma 13.7. *Assume that Theorem G_n and Theorem E_n hold true, where $n = \dim X$.*

Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). *Let (X, Δ) be a divisorial log terminal pair such that X is $\mathbb{Q}$-factorial over W and let $C \geq 0$ be a π-ample $\mathbb{R}$-divisor on X such that $(X, \Delta + C)$ is log canonical and that $K_X + \Delta + C$ is nef over W. We consider a $(K_X + \Delta)$-minimal model program with scaling of C over Y around W starting from $(X_0, \Delta_0) := (X, \Delta)$:*

$$X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} X_i \overset{\phi_i}{\dashrightarrow} X_{i+1} \overset{\phi_{i+1}}{\dashrightarrow} \cdots,$$

with

$$1 \geq \lambda =: \lambda_0 \geq \lambda_1 \geq \cdots$$

such that $K_{X_i} + \Delta_i + \lambda_i C_i$ is nef over W, $\Delta_i := (\phi_{i-1})_*\Delta_{i-1}$, and $C_i := (\phi_{i-1})_*C_{i-1}$ for every $i \geq 1$. We further assume that $K_X + \Delta$ is π-pseudo-effective. Then there exists i_0 such that $K_{X_{i_0}} + \Delta_{i_0} \in \overline{\mathrm{Mov}}(X_{i_0}/Y; W)$.

Proof. If the minimal model program terminates after finitely many steps, then $K_{i_0} + \Delta_{i_0}$ is nef over W for some i_0 since $K_X + \Delta$ is π-pseudo-effective. This means that $K_{X_{i_0}} + \Delta_{i_0} \in \overline{\mathrm{Mov}}(X_{i_0}/Y; W)$.

From now on, we assume that the minimal model program does not terminate. We put $\lambda_\infty := \lim_{i\to\infty} \lambda_i \geq 0$. If $\lambda_\infty > 0$, then the given minimal model program can be seen as a $(K_X + \Delta + \frac{\lambda_\infty}{2} C)$-minimal model program with scaling of C. Without loss of generality, we may assume that C does not contain any non-kawamata log terminal centers of $(X, \Delta + C)$ since C is π-ample. Hence, by Theorem 13.6, it must terminate. This is a contradiction. Therefore, we may assume that $\lambda_\infty = 0$. By replacing (X, Δ) with (X_{i_0}, Δ_{i_0}) for some i_0, we may further assume that every step of the $(K_X + \Delta)$-minimal model program is a flip. Let G_i be a $\mathbb{Q}$-divisor on X_i such that G_i is ample over Y. We assume that $G_{iX} \to 0$ in $N^1(X/Y; W)$ for $i \to \infty$, where G_{iX} is the strict transform of G_i on X. We note that $K_{X_i} + \Delta_i + \lambda_i C_i + G_i$ is ample over some open neighborhood of W for every i. Since $X \dashrightarrow X_i$ is an isomorphism in codimension one, the strict transform $K_X + \Delta + \lambda_i C + G_{iX}$ is in $\overline{\mathrm{Mov}}(X/Y; W)$ for every i. By taking $i \to \infty$, we obtain $K_X + \Delta \in \overline{\mathrm{Mov}}(X/Y; W)$. This is what we wanted. □

Anyway, if Theorem G_n and Theorem E_n hold true, then we can run the minimal model program with scaling explained in this section in dimension n, although we do not know whether it terminates or not.

14. Nonvanishing theorem; D_n

One of the most difficult results in [BCHM] is the nonvanishing theorem (see [BCHM, Theorem D]). Fortunately, we can generalize it for projective morphisms of complex varieties without any difficulties. For a completely different approach to the nonvanishing theorem (see [BCHM, Theorem D]), see [BP, Section 3] and [CKP, Theorem 0.1 and Corollary 3.3].

Theorem 14.1 (Nonvanishing theorem, [BCHM, Theorem D]). *Let (X,Δ) be a kawamata log terminal pair and let $\pi\colon X\to Y$ be a projective morphism of complex varieties such that Y is Stein. Assume that Δ is big over Y and that $K_X+\Delta$ is pseudo-effective over Y. Let U be any relatively compact Stein open subset of Y. Then there exists a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on $\pi^{-1}(U)$ such that $(K_X+\Delta)|_{\pi^{-1}(U)}\sim_{\mathbb{R}} D\geq 0$.*

Proof. By Lemma 2.37 , $(K_X+\Delta)|_{\pi^{-1}(U)}$ is a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $\pi^{-1}(U)$. We take an analytically sufficiently general fiber F of $\pi\colon X\to Y$. Then $(F,\Delta|_F)$ is kawamata log terminal, $(K_X+\Delta)|_F=K_F+\Delta|_F$, $\Delta|_F$ is big, and $K_F+\Delta|_F$ is pseudo-effective. Hence, by the nonvanishing theorem for projective varieties (see [BCHM, Theorem D]), there exists an effective $\mathbb{R}$-divisor D' on F such that $K_F+\Delta|_F\sim_{\mathbb{R}} D'\geq 0$. By Lemma 2.53, we can find a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on $\pi^{-1}(U)$ with $(K_X+\Delta)|_{\pi^{-1}(U)}\sim_{\mathbb{R}} D\geq 0$. This is what we wanted. □

Remark 14.2. In [BCHM, Section 6], Birkar, Cascini, Hacon, and McKernan proved [BCHM, Theorem D_n] by using [BCHM, Theorems B_n, C_n, and D_{n-1}].

We close this section with a very important conjecture.

Conjecture 14.3 (Nonvanishing conjecture). *Let X be a smooth projective variety such that K_X is pseudo-effective. Then there exists a positive integer m such that*

$$H^0(X,\mathcal{O}_X(mK_X))\neq 0.$$

For the details of Conjecture 14.3, see [Has]. Note that if Conjecture 14.3 holds true then the existence problem of minimal models for projective log canonical pairs is completely solved (see [Has]).

15. Existence of analytic pl-flips; $\mathrm{F}_{n-1}\Rightarrow \mathrm{A}_n$

In this section, we will see that [HacM] works with some minor modifications for projective morphisms between complex analytic spaces.

Let us start with the definition of *analytic pl-flipping contractions*.

Definition 15.1 (Analytic pl-flipping contractions). Let (X,Δ) be a purely log terminal pair and let $\varphi\colon X\to Z$ be a projective morphism of complex varieties. Then φ is called an *analytic pl-flipping contraction* if Δ is a $\mathbb{Q}$-divisor and

(i) φ is small,
(ii) $-(K_X+\Delta)$ is φ-ample,
(iii) $S=\lfloor\Delta\rfloor$ is irreducible and $-S$ is φ-ample, and
(iv) $a(K_X+\Delta)\sim bS$ holds for some positive integers a and b.

Remark 15.2. Here, we replaced the condition that the relative Picard number is one in the usual definition of pl-flipping contractions for algebraic varieties with (iv) in Definition 15.1. This is because the definition of relative Picard numbers is not so clear in the setting of Definition 15.1. Moreover, (iv) is sufficient for the proof of the existence of pl-flips.

We can define *analytic pl-flips*.

Definition 15.3 (Analytic pl-flip). Let $\varphi\colon(X,\Delta)\to Z$ be an analytic pl-flipping contraction as in Definition 15.1. The *flip* of $\varphi\colon(X,\Delta)\to Z$ is a small projective morphism $\varphi^+\colon X^+\to Z$ from a normal complex variety X^+ such that $K_{X^+}+\Delta^+$ is φ^+-ample, where Δ^+ is the strict transform of Δ. This flip is sometimes called the (*analytic*) *pl-flip* of $\varphi\colon (X,\Delta)\to Z$. It is not difficult to see that the existence of $\varphi^+\colon (X^+,\Delta^+)\to Z$ is equivalent to the condition that

$$\bigoplus_{m\in\mathbb{N}}\varphi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Z$-algebra. The flip φ^+ of φ is nothing but

$$X^+=\operatorname{Projan}_Z\bigoplus_{m\in\mathbb{N}}\varphi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)\to Z.$$

We prepare an easy but important lemma.

Lemma 15.4. *Let $\varphi\colon (X,\Delta)\to Z$ be an analytic pl-flipping contraction as in Definition 15.1. We put $T:=\varphi(S)$. Then T is normal and $\varphi\colon S\to T$ has connected fibers, that is, $\mathcal{O}_T\xrightarrow{\sim}\varphi_*\mathcal{O}_S$ holds. Hence, for any open subset U of Z such that $T|_U$ is connected, $T|_U$ and $S|_{\varphi^{-1}(U)}$ are normal irreducible varieties.*

Proof. In this proof, we do not need (iv) in Definition 15.1. We will only use (i), (ii), and (iii). We note that φ is bimeromorphic by (i). We consider the following short exact sequence

$$0\to\mathcal{O}_X(-S)\to\mathcal{O}_X\to\mathcal{O}_S\to 0.$$

Since $-S-(K_X+\Delta)$ is φ-ample and (X,Δ) is purely log terminal, we obtain $R^1\varphi_*\mathcal{O}_X(-S)=0$. This implies that

$$0\to\mathcal{O}_Y(-T)=\varphi_*\mathcal{O}_X(-S)\to\mathcal{O}_Y\to\varphi_*\mathcal{O}_S\to 0$$

is exact. Hence we get $\mathcal{O}_T\xrightarrow{\sim}\varphi_*\mathcal{O}_S$. Therefore, T is normal and $\varphi\colon S\to T$ has connected fibers. Note that every normal complex variety is locally irreducible. Thus, $T|_U$ is a normal irreducible complex variety. So, $S|_{\varphi^{-1}(U)}$ is also a normal irreducible variety. □

15.5 (Theorem $\mathrm{F}_{n-1}\Rightarrow$ Theorem A_n)**.** From now on, let us see how to modify some arguments in [HacM].

Step 1 (see [HacM, Section 3])**.** Let $\varphi\colon (X,\Delta)\to Z$ be an analytic pl-flipping contraction with $\dim X=n$. In order to prove the existence of the flip of φ, it is sufficient to check that

$$\bigoplus_{m\in\mathbb{N}}\varphi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Z$-algebra. Therefore, we take an arbitrary point $z\in Z$ and assume that Z is a Stein open neighborhood of z by shrinking Z (see Lemma 15.4). We can always take a Stein compact subset W of Z containing z such that $\Gamma(W,\mathcal{O}_Z)$ is noetherian.

The preliminary results in [HacM, Section 3] hold with some minor modifications with the aid of Lemma 2.26 . Hence, the existence problem of the flip φ^+ can be reduced to the condition that the restricted algebra is a finitely generated graded $\mathcal{O}_{\varphi(S)}$-algebra.

Step 2 (see [HacM, Section 4])**.** As in Step 1, we consider a projective morphism $\pi\colon X\to Z$ of normal complex varieties such that Z is Stein and that there exists a Stein compact subset W of Z such that $\Gamma(W,\mathcal{O}_Z)$ is noetherian. We take a relatively compact Stein open neighborhood U of W in Z. Then every argument in [HacM, Section 4] works in a neighborhood of $\pi\colon\pi^{-1}(\overline{U})\to\overline{U}$. This means that we can define multiplier ideal sheaves (see [HacM, Definition-Lemma 4.2]) and check the basic properties. Then we can establish [HacM, Theorem 4.1] for $\pi\colon \pi^{-1}(U) \to U$. All we need here are a relative Kawamata–Viehweg vanishing theorem and a suitable resolution theorem for complex analytic spaces.

Step 3 (see [HacM, Section 5]). Let us see [HacM, Section 5]. As in Step 2, we work over a neighborhood of $\pi\colon \pi^{-1}(\overline{U}) \to \overline{U}$. Then we can define *asymptotic multiplier ideal sheaves* (see [HacM, Definition-Lemma 5.2]) with the aid of Lemma 2.17. Thus, we can establish [HacM, Theorem 5.3], which is the main result of [HacM, Section 5], for $\pi\colon \pi^{-1}(U) \to U$. We note that the topics in [HacM, Sections 4 and 5] are independent of the theory of minimal models.

Step 4 (see [HacM, Section 6]). The main result of [HacM, Section 6], which is [HacM, Theorem 6.3], is a consequence of [HacM, Theorems 4.1 and 5.3]. Therefore, we can formulate and prove it for $\pi^{-1}(U) \to U$ without any difficulties, where U is a sufficiently small relatively compact Stein open neighborhood of a given Stein compact subset W of Z with $z \in W$. We note that we do need the assumption that $\Gamma(W, \mathcal{O}_Z)$ is noetherian in Steps 2, 3, and 4.

Step 5 (see [HacM, Section 7]). We can formulate [HacM, Theorem 7.1] in a neighborhood of $\pi^{-1}(W) \to W$. By taking a Stein open neighborhood U of W suitably, we can use Theorem F_{n-1} and the results in the previous sections for $\pi^{-1}(U) \to U$. In this step, we need the assumption that $\Gamma(W, \mathcal{O}_Z)$ is noetherian in order to apply Theorem F_n.

Step 6 (see [HacM, Section 8]). Note that [HacM, Section 8] is an easy consequence of [HacM, Section 7]. Therefore, we need no new ideas. Hence we obtain that

$$\bigoplus_{m\in\mathbb{N}} \varphi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Z$-algebra for every n-dimensional analytic pl-flipping contraction $\varphi\colon (X,\Delta) \to Z$ under the assumption that Theorem F_{n-1} holds.

Anyway, we have understood that Theorem F_{n-1} implies Theorem A_n, that is, the existence of analytic pl-flips in dimension n follows from Theorem F_{n-1}. This is a very important step of the whole proof of the main theorem (see Theorems 1.6 and 1.13).

16. Special finiteness; $\mathrm{E}_{n-1} \Rightarrow \mathrm{B}_n$

This section corresponds to [BCHM, Section 4]. We will check that the arguments in [BCHM, Section 4] can work with some minor obvious modifications. We do not need no new ideas here.

16.1 (Theorem $\mathrm{E}_{n-1} \Rightarrow$ Theorem B_n). Let us see [BCHM, Section 4] and make some comments.

Step 1. In [BCHM, Lemmas 4.1, 4.2, and 4.3], some elementary results are prepared. Although they are formulated for quasi-projective varieties, there are no difficulties to translate them into our complex analytic setting. Of course, we are interested in the situation where $\pi\colon X \to Y$ is a projective morphism of complex analytic spaces and W is a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P) and consider everything over some Stein open neighborhood of W.

We make an important remark for the reader's convenience.

Remark 16.2. When we formulate [BCHM, Lemma 4.1] for our complex analytic setting, there are no differences between the notion of weak log canonical models *over* W and that of weak log canonical models *over some open neighborhood of* W by Theorem 8.3.

Step 2. The main result in [BCHM, Section 4] is [BCHM, Lemma 4.4], where we prove Theorem B_n under the assumption that Theorem E_{n-1} holds. We note that we can use Lemma 11.14 instead of [BCHM, Lemma 3.7.4]. In the proof of [BCHM, Lemma 4.4], Y_i is $\mathbb{Q}$-factorial for every i by assumption. In our complex analytic setting, the corresponding condition becomes the one that Z_i is $\mathbb{Q}$-factorial over W for every i. Therefore, (i) in Lemma 12.3 is satisfied. Thus, we can use Lemma 12.3 instead of [BCHM, Lemma 3.6.12] and check that the arguments in the proof of [BCHM, Lemma 4.4] can be adapted for our complex analytic setting.

Hence we see that Theorem E_{n-1} implies Theorem B_n.

We close this section with a remark.

Remark 16.3. Theorem B is not in the first version of [BCHM] circulated in 2006, where the special termination, which is a more traditional approach originally due to Shokurov, is used. In [HacK], Hacon adopts the special termination instead of the special finiteness. For the details, see [HacK, 8.A Special termination] (see also [Fu3]).

17. Existence of log terminal models; A_n and $\mathrm{B}_n \Rightarrow \mathrm{G}_n$

This section corresponds to [BCHM, Section 5]. This part is not difficult once we know the existence of analytic pl-flips (see Theorem A) and the special finiteness (see Theorem B). Precisely speaking, we prove Theorem G_n, which is a slight generalization of Theorem C_n, under the assumption that Theorem A_n and Theorem B_n hold true.

17.1 (Theorem A_n and Theorem $\mathrm{B}_n \Rightarrow$ Theorem G_n)**.** Note that [BCHM, Lemmas 5.1, 5.2, 5.4, 5.5, and 5.6] hold true for our complex analytic setting with only minor suitable modifications. Since [BCHM, Section 5] is easily accessible for everyone who studies the minimal model program, there are no difficulties to translate it into our complex analytic setting.

The first lemma is an easy consequence of Theorem B_n.

Lemma 17.2 (see [BCHM, Lemma 5.1])**.** *Assume that Theorem B_n holds true.*
Let $\pi\colon X\to Y$ be a projective morphism between complex analytic spaces with

$$\pi\colon X \xrightarrow{g} Y^\flat \xrightarrow{h} Y$$

such that $Y^\flat$ is projective over Y and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies (P)*. Let H be a general h-ample $\mathbb{Q}$-divisor on $Y^\flat$ satisfying $H\cdot\ell > 2\dim X$ for every projective curve ℓ such that $h(\ell)$ is a point. Suppose that X is $\mathbb{Q}$-factorial over W with* $\dim X = n$,

$$K_X+\Delta+C = K_X+S+A+B+C$$

is nef over W and is divisorial log terminal with $A\geq 0$, $B\geq 0$, and $C\geq 0$, where S is a finite sum of prime divisors, and $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of $(X,\Delta+C)$. Then any sequence of flips and divisorial contractions for the $(K_X+\Delta+g^*H)$-minimal model program with scaling over Y around W which does not contract S, is eventually disjoint from S.

Proof. Although we made the formulation suitable for our complex analytic setting, the proof of [BCHM, Lemma 5.1] works. The desired statement is an almost direct consequence of Theorem B_n. For the details, see the proof of [BCHM, Lemma 5.1]. □

We note that the $(K_X+\Delta+g^*H)$-minimal model program over Y in Lemma 17.2 can be seen as a $(K_X+\Delta)$-minimal model program over $Y^\flat$ by Lemma 9.4.

Lemma 17.3 (see [BCHM, Lemma 5.2])**.** *Assume that Theorem A_n and Theorem B_n hold true.*

Let $\pi\colon X\to Y$ be a projective morphism between complex analytic spaces with

$$\pi\colon X \xrightarrow{g} Y^\flat \xrightarrow{h} Y$$

such that $Y^\flat$ is projective over Y and let W be a compact subset of Y such that $\pi\colon X\to Y$ and W satisfies (P)*. Let H be a general h-ample $\mathbb{Q}$-divisor on $Y^\flat$ satisfying $H\cdot\ell>2\dim X$ for every projective curve ℓ such that $h(\ell)$ is a point. Suppose that X is $\mathbb{Q}$-factorial over W with $\dim X=n$, $(X,\Delta+C=S+A+B+C)$ is a divisorial log terminal pair such that $\lfloor\Delta\rfloor=S$, $A\geq 0$ is big over Y, $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of $(X,\Delta+C)$ with $B\geq 0$ and $C\geq 0$. Suppose that there is an $\mathbb{R}$-divisor $D\geq 0$ whose support is contained in S and a real number $\alpha\geq 0$ such that*

$$K_X+\Delta+g^*H\sim_{\mathbb{R}} D+\alpha C.$$

If $K_X+\Delta+C$ is nef over W, then, after shrinking Y around W suitably, there is a log terminal model $\phi\colon X\dashrightarrow Z$ for $K_X+\Delta+g^*H$ over W, where ϕ is a bimeromorphic contraction over $Y^\flat$, such that $\mathbf{B}_+(\phi_*A/Y)$ does not contain any non-kawamata log terminal centers of $(Z,\Gamma:=\phi_*\Delta)$.

Proof. We can run the $(K_X+\Delta+g^*H)$-minimal model program over Y around W explained in Section 13. As usual, we put

$$\lambda:=\inf\{t\in\mathbb{R}_{\geq 0}\,|\,K_X+\Delta+g^*H+tC \text{ is nef over } W\}.$$

If $\lambda=0$, there is nothing to do. Otherwise, we can find a $(K_X+\Delta+g^*H)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y;W)$ such that $(K_X+\Delta+g^*H+\lambda C)\cdot R=0$. Let $\varphi_R\colon X\to W$ be the contraction morphism over Y associated to R. By Theorem 9.3, φ_R is a contraction morphism over $Y^\flat$. Since $\lambda>0$, $C\cdot R>0$. Hence we have $D\cdot R<0$. In particular, φ_R is always birational. When φ_R is divisorial, we can replace everything with its image. When φ_R is small, we can see it as an analytic pl-flipping contraction because $D\cdot R<0$ and $\operatorname{Supp}D\subset S=\lfloor\Delta\rfloor$. Therefore, by Theorem A_n, we know that the flip $\varphi_R^+\colon X^+\to Z$ exists. In this case, we replace X with X^+. Note that we have to replace Y with a relatively compact Stein open neighborhood of W in each step. Then the condition $\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of (X,Δ) is preserved by Lemmas 11.21 and 11.22. By construction, this minimal model program is not an isomorphism in a neighborhood of S. Hence it terminates by Lemma 17.2 and Theorem B_n. Thus, we finally get a log terminal model $\phi\colon X\dashrightarrow Z$. By Lemma 9.4, the above minimal model program can be seen as a $(K_X+\Delta)$-minimal model program over $Y^\flat$. Therefore, the bimeromorphic contraction $\phi\colon X\dashrightarrow Z$ is a bimeromorphic contraction over $Y^\flat$. □

We need the notion of *neutral models* in our complex analytic setting.

Definition 17.4 (see [BCHM, Definition 5.3])**.** Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces with

$$\pi\colon X \xrightarrow{g} Y^\flat \xrightarrow{h} Y$$

such that $Y^\flat$ is projective over Y and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). Let H be a general h-ample $\mathbb{Q}$-divisor on $Y^\flat$ satisfying $H\cdot\ell > 2\dim X$ for every projective curve ℓ such that $h(\ell)$ is a point. Let $(X, \Delta = A+B)$ be a divisorial log terminal pair with $A \geq 0$ and $B \geq 0$ such that X is $\mathbb{Q}$-factorial over W and let D be an effective $\mathbb{R}$-divisor on X. A *neutral model* over $Y^\flat$ for $(X, \Delta + g^*H)$ with respect to A and D is any bimeromorphic map $f\colon X \dashrightarrow Z$ over $Y^\flat$ such that

- f is a bimeromorphic contraction,
- the only divisors contracted by f are components of D,
- Z is $\mathbb{Q}$-factorial over W and is projective over Y,
- $\mathbf{B}_+(f_*A/Y)$ does not contain any non-kawamata log terminal centers of $(Z, \Gamma := f_*\Delta)$, and
- $K_Z + \Gamma + g_Z^*H$ is divisorial log terminal and is nef over W, where $g_Z\colon Z \to Y^\flat$ is the structure morphism.

Lemma 17.5 (see [BCHM, Lemma 5.4])**.** *Assume that Theorem A_n and Theorem B_n hold true.*

Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces with

$$\pi\colon X \xrightarrow{g} Y^\flat \xrightarrow{h} Y$$

such that $Y^\flat$ is projective over Y and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P)*. Let H be a general h-ample $\mathbb{Q}$-divisor on $Y^\flat$ satisfying $H\cdot\ell > 2\dim X$ for every projective curve ℓ such that $h(\ell)$ is a point. Let $(X, \Delta = A+B)$ be a divisorial log terminal pair and let D be an $\mathbb{R}$-divisor, where $A \geq 0$ is big over Y, $B \geq 0$, $D \geq 0$, and D and A have no common components. If*

(1) *$K_X + \Delta + g^*H \sim D$,*
(2) *X is smooth and G is a simple normal crossing divisor on X such that* $\operatorname{Supp}(\Delta + D) = G$*, and*
(3) *$\mathbf{B}_+(A/Y)$ does not contain any non-kawamata log terminal centers of (X, G),*

then, after shrinking Y around W suitably, $(X, \Delta + g^*H)$ has a neutral model over $Y^\flat$ with respect to A and D.

Proof. Although our formulation is slightly different from [BCHM, Lemma 5.4], the proof of [BCHM, Lemma 5.4] works by using Lemma 17.3 instead of [BCHM, Lemma 5.2]. We note that we have to shrink Y around W in each step of the proof. For the details, see the proof of [BCHM, Lemma 5.4]. □

Lemma 17.6 (see [BCHM, Lemma 5.5])**.** *Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces with*

$$\pi\colon X \xrightarrow{g} Y^\flat \xrightarrow{h} Y$$

such that $Y^\flat$ is projective over Y and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P)*. Let H be a general h-ample $\mathbb{Q}$-divisor on $Y^\flat$ satisfying $H\cdot\ell > 2\dim X$ for every projective curve ℓ such that $h(\ell)$ is a point. Let $(X, \Delta = A+B)$ be a divisorial log terminal pair such that X is $\mathbb{Q}$-factorial over W and let D be an $\mathbb{R}$-divisor, where $A \geq 0$ is big over Y, $B \geq 0$, and $D \geq 0$. If every component of D is either semiample* *over Y or a stable base divisor of $K_X + \Delta + g^*H$ near W and $f\colon X \dashrightarrow Z$ is a neutral model over $Y^\flat$ for $(X, \Delta + g^*H)$ with respect to A and D, then f is a log terminal model over U for some open neighborhood U of W. Moreover, $K_Z + \Gamma + g_Z^*H$ is semiample over U, where $\Gamma := f_*\Delta$ and $g_Z\colon Z \to Y^\flat$ is the structure morphism.*

Proof. The proof of [BCHM, Lemma 5.5] works with only some minor modifications. In the proof of this lemma, we have to shrink Y around W repeatedly. We note that $K_Z + \Gamma$ is nef over $h^{-1}(W)$ if and only if $K_Z + \Gamma + g_Z^*H$ is nef over W. We also note that $K_Z + \Gamma + g_Z^*H$ is semiample over some open neighborhood of W when $K_Z + \Gamma + g_Z^*H$ is nef over W. For the details, see the proof of [BCHM, Lemma 5.5]. □

By using the above lemmas, there are no difficulties to prove Theorem G_n under the assumption that Theorem A_n and Theorem B_n hold true.

Lemma 17.7 (see [BCHM, Lemma 5.6])**.** *Theorem A_n and Theorem B_n imply Theorem G_n.*

Proof. The proof of [BCHM, Lemma 5.6] works in our setting by using Lemmas 17.5 and 17.6 instead of [BCHM, Lemmas 5.4 and 5.5]. We note that we can use Lemma 10.9 instead of [BCHM, Proposition 3.5.4]. As usual, we have to replace Y with a relatively compact Stein open neighborhood of W finitely many times in the proof of this lemma. For the details, see the proof of [BCHM, Lemma 5.6]. □

Finally, we explicitly state the following obvious result for the sake of completeness.

Lemma 17.8. *Theorem G_n implies Theorem C_n for every n.*

Proof. It is sufficient to put $Y^\flat = Y$ and apply Theorem G_n. □

Anyway, we see that Theorem G_n and Theorem C_n hold under the assumption that Theorem A_n and Theorem B_n are true.

We need the existence theorem of $\mathbb{Q}$-factorial divisorial log terminal flips, which is an easy consequence of Theorem G, in order to run minimal model programs.

Theorem 17.9 (Existence of $\mathbb{Q}$-factorial divisorial log terminal flips)**.** *Assume that Theorem G_n holds true.*

Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces with $\dim X = n$ *and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. We further assume that (X, Δ) is divisorial log terminal and that X is $\mathbb{Q}$-factorial over W. Let $\varphi\colon X \to Z$ be a small projective bimeromorphic contraction morphism associated to a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y; W)$, that is, $\varphi\colon (X, \Delta) \to Z$ is a* flipping contraction *associated to R. Then, after shrinking Y around W suitably, the flip $\varphi^+\colon X^+ \to Z$ always exists.*

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X^+ \\ & {}_{\varphi}\searrow \quad \swarrow_{\varphi^+} & \\ & Z & \end{array}$$

This means that

(1) *$\varphi^+\colon X^+ \to Z$ is a small projective bimeromorphic contraction morphism, and*
(2) *$K_{X^+} + \Delta^+$ is φ^+-ample, where $\Delta^+ := \phi_*\Delta$.*

Moreover, (X^+, Δ^+) is divisorial log terminal, X^+ is $\mathbb{Q}$-factorial over W, and the equality $\rho(X^+/Y; W) = \rho(X/Y; W)$ holds.

Proof. We will freely shrink Y around W. We note that $(X, (1-\varepsilon)\Delta)$ is kawamata log terminal and $-(K_X + (1-\varepsilon)\Delta)$ is φ-ample for some $0 < \varepsilon \ll 1$. We take a general π_Z-ample $\mathbb{Q}$-divisor A on Z, where $\pi_Z \colon Z \to Y$ is the structure morphism, such that $(X, (1-\varepsilon)\Delta + \varphi^* A)$ is kawamata log terminal and $K_X + (1-\varepsilon)\Delta + \varphi^* A \sim_{\mathbb{R}} D \geq 0$. By Theorem G_n, $(X, (1-\varepsilon)\Delta)$ has a good log terminal model over Z. Hence $\varphi \colon (X, (1-\varepsilon)\Delta) \to Z$ has a flip $\phi \colon X \dashrightarrow X^+$. We can easily see that $\varphi^+ \colon (X^+, \Delta^+) \to Z$ is the flip of $\varphi \colon (X, \Delta) \to Z$ and satisfies all the desired properties. □

We close this section with an almost obvious remark, which may be useful for some applications.

Remark 17.10. In Theorem 17.9, (X, Δ) is assumed to be a divisorial log terminal pair. There are no difficulties to see that the existence of flips (see Theorem 17.9) also holds true under a slightly weaker assumption that (X, Δ) is log canonical and that there exists Δ_0 such that (X, Δ_0) is kawamata log terminal.

18. Finiteness of models; $\mathrm{G}_n \Rightarrow \mathrm{E}_n$

This section corresponds to [BCHM, Section 7], where Theorem E_n is proved under the assumption that Theorem C_n and Theorem D_n hold true. In our complex analytic setting, in Section 14, we have already checked that the nonvanishing theorem (see Theorem D) holds true in any dimension by reducing it to the original nonvanishing theorem formulated for algebraic varieties (see [BCHM, Theorem D]). Therefore, we can freely use Theorem D here. In this section, we will use Theorem G_n, which is slightly stronger than Theorem C_n, for the proof of Theorem E_n.

18.1 (Theorem $\mathrm{G}_n \Rightarrow$ Theorem E_n)**.** Let us see [BCHM, Section 7] in detail.

Step 1. The proof of [BCHM, Lemma 7.1] is well known and can work for our complex analytic setting. Note that it is sufficient to assume that Theorem G_n holds true since we can freely use Theorem D in arbitrary dimension as mentioned above. The correct formulation of [BCHM, Lemma 7.1] for our complex analytic setting is as follows.

Lemma 18.2 (see [BCHM, Lemma 7.1])**.** *Assume that Theorem G_n holds.*

Let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces with $\dim X = n$ *and let W be a compact subset of Y such that $\pi \colon X \to Y$ and W satisfies* (P)*. Let V be a finite-dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$*, which is defined over the rationals. Fix a general π-ample $\mathbb{Q}$-divisor A on X. Let $\mathcal{C} \subset \mathcal{L}_A(V; \pi^{-1}(W))$ be a rational polytope such that if $\Delta \in \mathcal{C}$ then (X, Δ) is kawamata log terminal at $\pi^{-1}(W)$.*

Then, after shrinking Y around W suitably, there are finitely many bimeromorphic maps $\phi_i \colon X \dashrightarrow Z_i$ over Y, $1 \leq i \leq k$, with the property that if $\Delta \in \mathcal{C} \cap \mathcal{E}_{A,\pi}(V; W)$, then there exists an index $1 \leq i \leq k$ such that ϕ_i is a log terminal model of $K_X + \Delta$ over some open neighborhood of W.

Proof. We have already proved Theorem D in any dimension. We can use Lemma 11.14 instead of [BCHM, Lemma 3.7.4]. Therefore, by using Theorem G_n instead of [BCHM, Theorem C_n], we see that the proof of [BCHM, Lemma 7.1] works in our complex analytic setting. Precisely speaking, we formulate Theorem G_n in order to make the proof of [BCHM, Lemma 7.1] work in the complex analytic setting. For the details, see the proof of [BCHM, Lemma 7.1]. □

Step 2. The proof of [BCHM, Lemma 7.2] uses [BCHM, Lemma 3.6.12]. In our case, we can use Lemma 12.3 instead of [BCHM, Lemma 3.6.12]. We note that (ii) in Lemma 12.3, which is nothing but Theorem G_n, is satisfied by assumption. As in Step 1, it is sufficient to assume Theorem G_n since we can freely use Theorem D in any dimension.

Lemma 18.3 ([BCHM, Lemma 7.2]). *Assume that Theorem G_n holds true.*

Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces with $\dim X = n$ and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P)*. Suppose that there is a kawamata log terminal pair (X, Δ_0). We fix a general π-ample $\mathbb{Q}$-divisor A on X. Let V be a finite-dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ which is defined over the rationals. Let $\mathcal{C} \subset \mathcal{L}_A(V; \pi^{-1}(W))$ be a rational polytope.*

Then, after shrinking Y around W suitably, there are finitely many bimeromorphic contractions $\psi_j\colon X \dashrightarrow Z_j$ over Y, $1 \leq j \leq l$, such that if $\psi\colon X \dashrightarrow Z$ is a weak log canonical model of $K_X + \Delta$ over W for some $\Delta \in \mathcal{C}$ then there exist an index $1 \leq j \leq l$ and an isomorphism $\xi\colon Z_j \to Z$ over some open neighborhood of W such that $\psi = \xi \circ \psi_j$ holds.

Proof. If we use Lemma 18.2, Corollary 11.19, and Lemma 12.3 instead of [BCHM, Lemma 7.1], [BCHM, Corollary 3.11.2], and [BCHM, Lemma 3.6.12], then the proof of [BCHM, Lemma 7.2] works in our complex analytic setting. The idea is as follows. By using Lemma 11.14 and so on, we can reduce the problem to the case where we can use Lemma 12.3. Let $\psi\colon X \dashrightarrow Z$ be a weak log canonical model over W. Then we can take Δ' such that $\psi\colon X \dashrightarrow Z$ is an ample model of (X, Δ'). Then, by Lemma 18.2 and Corollary 11.19, we obtain all the desired properties. For the details, see the proof of [BCHM, Lemma 7.2]. □

Step 3. The final step is obvious.

Lemma 18.4 ([BCHM, Lemma 7.3]). *Theorem G_n implies Theorem E_n.*

Proof. We note that $\mathcal{L}_A(V; \pi^{-1}(W))$ is a rational polytope. Therefore, it is sufficient to put $\mathcal{C} = \mathcal{L}_A(V; \pi^{-1}(W))$ in Lemma 18.3. □

Hence we see that Theorem E_n holds under the assumption that Theorem G_n holds true. This is what we wanted.

By the above arguments, we think that the reader can understand the reason why we prepared Theorem G, Corollary 11.19, and Lemma 12.3.

19. Finite generation; $\mathrm{G}_n \Rightarrow \mathrm{F}_n$

This section corresponds to [BCHM, Section 8]. Note that [BCHM, Section 8] is a very short section, which consists of only one lemma (see [BCHM, Lemma 8.1]). The proof of Theorem F_n given below is slightly more complicated than the original algebraic version in [BCHM, Section 8]. This is because we formulated everything only over some open neighborhood of a given compact subset of the base space.

19.1 (Theorem $\mathrm{G}_n \Rightarrow$ Theorem F_n). Here, we will prove Theorem F_n under the assumption that Theorem G_n holds true.

First, we will prove (1). In the proof of (1), we will freely shrink Y around W suitably without mentioning it explicitly. If $K_X + \Delta$ is π-pseudo-effective, then $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$ by Theorem D. Hence, by Theorem G_n, (X, Δ) has a good log terminal model (Z, Γ) over Y. Hence $K_Z + \Gamma$ is semiample over Y. We take any point $y \in Y$. By applying the above result to $\pi\colon X \to Y$ with $W := \{y\}$. Then we see that there exits an open neighborhood U_y of y such that (X, Δ) has a good log terminal model over U_y. By this observation, we obtain that $R(X/Y, K_X+\Delta)$ is a locally finitely generated graded $\mathcal{O}_Y$-algebra. Thus, we get (1) in Theorem F_n.

From now on, we will prove (2). Let $\mu\colon X \dashrightarrow Z$ be a good log terminal model for $K_X+\Delta$ over Y after shrinking Y around W (see Theorem G_n). Since G is a prime divisor contained in the stable base locus of $K_X+\Delta$ over Y, G is μ-exceptional. We take a small positive real number δ such that if $\|\Xi-\Delta\|<\delta$ then $(Z, \mu_*\Xi)$ is kawamata log terminal and $a(G, X, \Xi) < a(G, Y, \mu_*\Xi)$. If $K_X+\Xi$ is not π-pseudo-effective, then $\mathbf{B}((K_X+\Xi)/Y) = X$. Therefore, $G \subset \mathbf{B}((K_X+\Xi)/Y)$ is obvious. Hence we may assume that $K_X+\Xi$ is π-pseudo-effective. We take any point y of Y. Then there exists an open neighborhood U_y of y such that $(Z, \mu_*\Xi)$ has a good log terminal model over U_y by Theorem G_n. This means that (X, Ξ) has a good log terminal model over U_y. Hence we can easily check that $G|_{\pi^{-1}(U_y)} \subset \mathbf{B}((K_X+\Xi)|_{\pi^{-1}(U_y)}/U_y)$. Therefore, $G|_{\pi^{-1}(U_y)} \subset \mathbf{B}((K_X+\Xi)/Y)$. Thus, we obtain that $G \subset \mathbf{B}((K_X+\Xi)/Y)$ holds since y is any point of Y. This is (2).

Finally, we will prove (3). We take a good log terminal model $\mu\colon X \dashrightarrow Z$ of $K_X+\Delta$ over Y after shrinking Y around W (see Theorem G_n). By Corollary 11.19 (1), $\mathcal{W}_{\mu,A,\pi}(V'; W) = \mathcal{W}^{\sharp}_{\mu,A,\pi}(V'; W)$ is a rational polytope and $\Delta \in \mathcal{W}_{\mu,A,\pi}(V'; W)$. Therefore, we may assume that $K_Z+\mu_*\Xi$ is nef over Y for every $\Xi \in \mathcal{W}_{\mu,A,\pi}(V'; W)$ after shrinking Y around W again. Hence, after shrinking Y around W suitably, there exists a positive constant η such that if $\Xi \in V'$ and $\|\Xi-\Delta\|<\eta$ then $(Z, \mu_*\Xi)$ is kawamata log terminal and $K_Z+\mu_*\Xi$ is semiample over Y. By Theorem 3.12, Z has only rational singularities. Since Z is $\mathbb{Q}$-factorial over W, there is a positive integer l such that if $m(K_Z+\mu_*\Xi)$ is an integral Weil divisor then $lm(K_Z+\mu_*\Xi)$ is Cartier over some open neighborhood of W (see Lemma 2.42). By replacing Y with a small open neighborhood of W, we may assume that $lm(K_Z+\mu_*\Xi)$ is Cartier on Z. Therefore, by Theorem 6.4, there exists $r>0$ such that $rm(K_Z+\mu_*\Xi)$ is free over Y when $m(K_Z+\mu_*\Xi)$ is an integral Weil divisor. It follows that if $k(K_X+\Xi)/r$ is Cartier then every component of $\mathrm{Fix}(k(K_X+\Xi))$ is contracted by μ and so every component is in $\mathbf{B}((K_X+\Delta)/Y)$. We finish the proof of (3) in Theorem F_n.

In Section 21, we will prove Theorems 1.18 and 1.22, which are much more general than the finite generation in Theorem F (1).

20. Proof of theorems

In this section, we will prove theorems. Note that we postpone the proof of Theorems 1.18 and 1.22 until Section 21 because it needs some deep results from the theory of variations of Hodge structure. Theorem 1.28 will be proved in Section 22 after we explain some supplementary results on the minimal model program with scaling. Theorem 1.30 (see Theorem 23.2) will be treated in Section 23. Note that the proof of Theorem 1.30 uses Theorem 1.18.

Let us start with the proof of Theorems 1.6 and 1.13, which is now almost obvious.

Proof of Theorems 1.6 and 1.13. As we explained in Subsection 1.2, we will prove Theorems A, B, C, D, E, F, and G by induction on $\dim X$. In Section 14, we established Theorem D in arbitrary dimension. We use induction on $n=\dim X$. When $n=0$, all the statements are trivially true. From now on, we assume that Theorems A, B, C, E, F, G hold true when $\dim X \le n-1$. In Section 15, we proved Theorem A in $\dim X = n$. In Section 16, we obtained Theorem B in $\dim X = n$. Hence we can prove Theorem G in $\dim X = n$ by Section 17. Note that Theorem C_n is a very special case of Theorem G_n (see Lemma 17.8). By Sections 18 and 19, we have Theorems E and F in $\dim X = n$, respectively. This means that we have established Theorems A, B, C, D, E, F, and G in arbitrary dimension. □

From now on, we can freely use Theorems A, B, C, D, E, F, and G in arbitrary dimension. Therefore, we can freely use the minimal model program with scaling explained in Section 13.

Proof of Theorem 1.7. In this theorem, we only treat kawamata log pairs. Therefore, we do not need any extra assumptions on non-kawamata log terminal centers. We can freely use the minimal model program with scaling explained in Section 13. Note that the pseudo-effectivity over Y is preserved by the minimal model program. Therefore, this theorem is a special case of the minimal model program with scaling under the condition (i) explained in Section 13. We also note that the termination of the minimal model program follows from Theorem E (see the proof of Theorem 13.6). □

Theorem 1.8 is a direct generalization of [BCHM, Theorem 1.2] in the complex analytic setting.

Proof of Theorem 1.8. Throughout this proof, we will freely shrink Y around W suitably without mentioning it explicitly. By Theorems D and C, (X, Δ) has a log terminal model over Y. This is (1). By Lemma 11.16 (2), ϕ is a semiample model over Y. By Lemma 11.16 (3) and (4), we know that (X, Δ) has a log canonical model over Y when $K_X + \Delta$ is π-big. This is (2). We take an arbitrary point $y \in Y$. It is sufficient to prove (3) over some open neighborhood of $y \in Y$. Hence we may take a Stein compact subset W of Y such that $y \in W$ and that $\Gamma(W, \mathcal{O}_Y)$ is noetherian, and shrink Y and enlarge W suitably without mentioning it explicitly (see Lemma 2.16). We take a positive integer a such that $a(K_X + \Delta)$ and $a(K_Z + \Gamma)$ are both Cartier. Since $a(K_Z + \Gamma)$ is semiample over Y,

$$\bigoplus_{m\in\mathbb{N}} \pi_*\mathcal{O}_X(ma(K_X + \Delta)) \simeq \bigoplus_{m\in\mathbb{N}} (\pi_Z)_*\mathcal{O}_Z(ma(K_Z + \Gamma)),$$

where $\pi_Z\colon Z \to Y$ is the structure morphism, is a locally finitely generated graded $\mathcal{O}_Y$-algebra (see Lemma 2.36). Therefore, by Lemma 2.26 , $R(X/Y, K_X + \Delta)$ is a locally finitely generated graded $\mathcal{O}_Y$-algebra. This is (3). □

The existence of kawamata log terminal flips is a direct consequence of Theorem 1.8 (2).

Proof of Theorem 1.14. We take a point $z \in Z$ and consider a small Stein open neighborhood U of $z \in Z$. Then $(X^+, \Delta^+)|_{(\varphi^+)^{-1}(U)}$ is nothing but the log canonical model of $(X, \Delta)|_{\varphi^{-1}(U)}$ over U. Therefore, after shrinking U around z suitably, it exists by Theorem 1.8 (2) and is unique. Hence the desired flip $\varphi^+\colon (X^+, \Delta^+) \to Z$ exists globally. □

The existence of canonicalizations for complex variety is new.

Proof of Theorem 1.16. We take a point $x \in X$. Over some open neighborhood U of $x \in X$, there exist a projective bimeromorphic morphism $\pi\colon V \to U$ and a log canonical model $\pi'\colon V' \to U$ of $\pi\colon V \to U$ by Theorem 1.8 (2). Note that π' is projective bimeromorphic, $K_{V'}$ is π'-ample, and V' has only canonical singularities. We can easily check that π' is an isomorphism over a nonempty Zariski open subset where U has only canonical singularities. We also note that $\pi'\colon V' \to U$ is usually called a *canonical model* of V over U and is unique. Thus, the desired model $f\colon Z \to X$ exists globally. □

When $K_X+\Delta$ is not pseudo-effective, we see that we can always run a minimal model program and finally get a Mori fiber space.

Proof of Theorem 1.17. As usual, we will repeatedly shrink Y around W without mentioning it explicitly. We take a π-ample $\mathbb{Q}$-divisor C on X such that $K_X+\Delta+C$ is nef over Y and that $(X, \Delta+(1+a)C)$ is a divisorial log terminal pair for some positive real number a. We run a $(K_X+\Delta)$-minimal model program with scaling of C. Since $K_X+\Delta$ is not pseudo-effective, $K_X+\Delta+\varepsilon C$ is still not pseudo-effective for some $0<\varepsilon \ll 1$. We can see the above minimal model program as a $(K_X+\Delta+\varepsilon C)$-minimal model program with scaling of C. By Theorem 13.6, this minimal model program always terminates and then we finally get a Mori fiber space structure over Y. This is what we wanted. $\square$

We note that Theorems 1.19 and 1.21 for quasi-projective varieties are not treated in [BCHM]. We also note that a key ingredient of the proof of Theorems 1.19 and 1.21 is Lemma 13.7.

Proof of Theorem 1.19. Throughout this proof, we will freely shrink Y around W suitably without mentioning it explicitly. By Lemma 2.53, there exists a globally $\mathbb{R}$-Cartier $\mathbb{R}$-divisor B on X such that $K_X+\Delta\sim_{\mathbb{R}} B\geq 0$. Since $(K_X+\Delta)|_F\sim_{\mathbb{R}} 0$, we see that $\pi(B)\subsetneq Y$ holds. Hence we can write $K_X+\Delta\sim_{\mathbb{R}}\pi^*D+B'$, where D is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on Y, B' is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $\pi(B')\subsetneq Y$ and that $\operatorname{Supp} B'$ does not contain any fibers of π. Without loss of generality, we may assume that $\pi(B')\subsetneq W$ by shrinking Y around W suitably. We take a general π-ample $\mathbb{Q}$-divisor $C\geq 0$ on X such that $(X, \Delta+C)$ is divisorial log terminal and that $K_X+\Delta+C$ is nef over Y. Then we run a $(K_X+\Delta)$-minimal model program with scaling of C over Y around W starting from $X_0:=X$:

$$X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots .$$

In this case, any divisorial contraction contracts an irreducible component of $\operatorname{Supp} B'$. By Lemma 13.7, we finally obtain (X_m, Δ_m) such that $K_{X_m}+\Delta_m\in \overline{\operatorname{Mov}}(X_m/Y; W)$. By Zariski's lemma, we can check that $K_{X_m}+\Delta_m\sim_{\mathbb{R}}(\pi_m)^*D$ holds. This is what we wanted. $\square$

In the proof of Theorem 1.21, Lemma 4.6 plays an important role.

Proof of Theorem 1.21. We will freely shrink X suitably without mentioning it explicitly. By taking a resolution of singularities, we have a projective bimeromorphic morphism $\pi\colon Y\to X$ from a complex variety Y such that $\pi^{-1}(U)$ is smooth and $\operatorname{Exc}(\pi)$ and $\operatorname{Exc}(\pi)\cup \operatorname{Supp}\pi^{-1}_*\Delta$ are simple normal crossing divisors on $\pi^{-1}(U)$. Let E be any π-exceptional divisor such that $\pi(E)\cap U\neq\emptyset$. Then, by enlarging V suitably, we may assume that $\pi(E)\cap V\neq\emptyset$. By Lemma 2.16, we can take a Stein compact subset W of U such that $\Gamma(W, \mathcal{O}_X)$ is noetherian and that $V\subset W$. We write $K_Y+\Delta_Y=\pi^*(K_X+\Delta)$. Let $\Delta_Y=\sum_i a_i\Delta_i$ be the irreducible decomposition. We put

$$\Theta=\sum_{0<a_i<1} a_i\Delta_i+\sum_{a_i\geq 1}\Delta_i.$$

Then we have $K_Y+\Theta=\pi^*(K_X+\Delta)+F$ such that $-\pi_*F$ is effective. Let C be a general π-ample $\mathbb{Q}$-divisor on Y such that $(Y, \Theta+C)$ is divisorial log terminal and $K_Y+\Theta+C$ is nef over W. We run a $(K_Y+\Theta)$-minimal model program with scaling of C over X around

W. We put $(Y_0, \Theta_0) := (Y, \Theta)$, $F_0 := F$, and $C_0 := C$. Then we obtain a sequence of flips and divisorial contractions starting from (Y_0, Θ_0):

$$(Y_0, \Theta_0) \overset{\phi_0}{\dashrightarrow} (Y_1, \Theta_1) \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} (Y_i, \Theta_i) \overset{\phi_i}{\dashrightarrow},$$

where $\Theta_{i+1} := (\phi_i)_*\Theta_i$, $C_{i+1} := (\phi_i)_*C_i$, and $F_{i+1} := (\phi_i)_*F_i$, for every i, and a sequence of real numbers

$$1 \geq \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_i \geq \cdots \geq 0$$

such that $K_{Y_i} + \Theta_i + \lambda_i C_i$ is nef over W. By Lemma 13.7, we can prove that $K_{Y_m} + \Theta_m$ is in $\overline{\mathrm{Mov}}(Y_m/X; W)$ for some m. By the negativity lemma (see Lemma 4.6), we see that $-F_m \geq 0$ over V. Hence, $-F_m$ is effective over some open neighborhood of W. We put $Z := Y_m$, $f\colon Z \to X$, and $K_Z + \Delta_Z = f^*(K_X + \Delta)$. Then, (Z, Δ_Z) has all the desired properties. □

We have already proved Theorem 1.24 in Section 12.

Proof of Theorem 1.24. We have already known that Theorem G_n holds true for every n. Therefore, Theorem 1.24 is nothing but Lemma 12.1. □

By Theorem 1.24, Corollary 1.25 is almost obvious.

Proof of Corollary 1.25. We note that Z has only kawamata log terminal singularities over some open neighborhood of W. We apply Theorem 1.24 to $Z \to Y$ and W'. Then, after shrinking Y around W' suitably, there exists a small projective bimeromorphic contraction morphism $Z' \to Z$ such that Z' is projective over Y and is $\mathbb{Q}$-factorial over W'. Hence the induced bimeromorphic contraction $\phi'\colon X \dashrightarrow Z'$ satisfies the desired properties. □

The argument in the proof of Theorem 1.26 is more or less well known.

Proof of Theorem 1.26. We take an arbitrary point $x \in X$. It is sufficient to prove that $\bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(mD)$ is a finitely generated graded $\mathcal{O}_X$-algebra on some open neighborhood of x. By shrinking X around x and replacing D with a linearly equivalent integral Weil divisor, we may assume that D is effective. We take a relatively compact Stein open neighborhood U of x and a Stein compact subset W of X such that $U \subset W$ and that $\Gamma(W, \mathcal{O}_X)$ is noetherian. By Theorem 1.24, after shrinking X around W, there exists a small projective bimeromorphic morphism $f\colon Z \to X$ from a normal complex variety Z such that Z is $\mathbb{Q}$-factorial over W. We put $K_Z + \Delta_Z = f^*(K_X + \Delta)$. Then (Z, Δ_Z) is kawamata log terminal. Let D_Z be the strict transform of D on Z. By shrinking X around W, we may assume that D_Z is $\mathbb{Q}$-Cartier. We take a small rational number ε such that $(Z, \Delta_Z + \varepsilon D_Z)$ is still kawamata log terminal. From now on, we will freely shrink X around W without mentioning it explicitly. We take a general f-ample $\mathbb{Q}$-divisor H on Z such that $K_Z + \Delta_Z + \varepsilon D_Z + H$ is nef over W and $(Z, \Delta_Z + \varepsilon D_Z + H)$ is kawamata log terminal. We run a $(K_Z + \Delta_Z + \varepsilon D_Z)$-minimal model program with scaling of H over X around W. Then we get a finite sequence of flips starting from $(Z_0, \Delta_{Z_0} + \varepsilon D_{Z_0}) := (Z, \Delta_Z + \varepsilon D_Z)$:

$$Z_0 \overset{\phi_0}{\dashrightarrow} Z_1 \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} Z_i \overset{\phi_i}{\dashrightarrow} \cdots \overset{\phi_{m-1}}{\dashrightarrow} Z_m,$$

such that $\Delta_{Z_i} := (\phi_{i-1})_*\Delta_{Z_{i-1}}$ and $D_{Z_i} := (\phi_{i-1})_*D_{Z_{i-1}}$ for every $i \geq 1$ and that $K_{Z_m} + \Delta_{Z_m} + \varepsilon D_{Z_m}$ is nef over W. Note that $K_{Z_m} + \Delta_{Z_m} = f_m^*(K_X + \Delta)$ holds by construction, where $f_m\colon Z_m \to X$ is the structure morphism. Since f_m is bimeromorphic, we can take an effective $\mathbb{Q}$-divisor B on Z_m such that $-B$ is f_m-ample and that $(Z_m, \Delta_{Z_m} + B)$ is kawamata log terminal. Hence, by Theorem 6.5, D_{Z_m} is semiample over X. By considering a contraction morphism $Z_m \to Z'$ over X associated to D_{Z_m}, we obtain a small projective bimeromorphic contraction morphism $f'\colon Z' \to X$ from a normal variety Z' and an integral Weil divisor D' on Z' such that D' is ample over X and that $f'_*D' = D$ holds. Since f' is small, we obtain $f'_*\mathcal{O}_{Z'}(mD') = \mathcal{O}_X(mD)$ for every $m \in \mathbb{N}$. Since D' is ample over X, $\bigoplus_{m\in\mathbb{N}} f'_*\mathcal{O}_{Z'}(mD')$ is a locally finitely generated graded $\mathcal{O}_X$-algebra by Lemma 2.36. This means that $\bigoplus_{m\in\mathbb{N}} \mathcal{O}_X(mD)$ is a finitely generated graded $\mathcal{O}_X$-algebra on some open neighborhood of x. This is what we wanted. □

Now there are no difficulties to prove Theorem 1.27.

Proof of Theorem 1.27. We take an open neighborhood U of W and a Stein compact subset W' of Y such that $U \subset W'$ and that $\Gamma(W', \mathcal{O}_Y)$ is noetherian. Throughout this proof, we will freely shrink Y suitably without mentioning it explicitly. Let A be a general π-ample $\mathbb{Q}$-divisor on X satisfying that $A\cdot C > 2\dim X$ for every projective curve C on X such that $\pi(C)$ is a point. We take a resolution $g\colon X' \to X$ such that $\operatorname{Supp} g^{-1}_*\Delta \cup \operatorname{Exc}(g)$ and $\operatorname{Exc}(g)$ are simple normal crossing divisors on X' and that $\pi'\colon X' \to Y$ is projective. We write $K_{X'} + \Delta_{X'} = g^*(K_X + \Delta)$. Let $\Delta_{X'} = \sum_i a_i\Delta'_i$ be the irreducible decomposition. We put

$$\Theta = \sum_{0<a_i<1} a_i\Delta'_i + \sum_{a_i\geq 1} \Delta'_i.$$

Then we can write $K_{X'} + \Theta = g^*(K_X + \Delta) + F$ such that $-g_*F \geq 0$. We take a general π'-ample $\mathbb{Q}$-divisor H on X' such that $K_{X'} + \Theta + g^*A + H$ is nef over Y. We run a $(K_{X'} + \Theta + g^*A)$-minimal model program over Y around W' with scaling of H. Then we obtain a sequence of flips and divisorial contractions starting from $(X', \Theta_0) := (X', \Theta)$:

$$(X'_0, \Theta_0) \overset{\phi_0}{\dashrightarrow} (X'_1, \Theta_1) \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} (X'_i, \Theta_i) \overset{\phi_i}{\dashrightarrow},$$

where $\Theta_{i+1} := (\phi_i)_*\Theta_i$, $H_{i+1} := (\phi_i)_*H_i$, and $F_{i+1} := (\phi_i)_*F_i$, for every i, and a sequence of real numbers

$$1 \geq \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_i \geq \cdots \geq 0$$

such that $K_{X'_i} + \Theta_i + g_i^*A + \lambda_i H_i$ is nef over W', where $g_i\colon X'_i \to X$. We note that by Lemma 9.4 the above minimal model program can be seen as a $(K_{X'} + \Theta)$-minimal model program over X. By Lemma 13.7 and its proof, we can check that $K_{X'_m} + \Theta_m$ is in $\overline{\operatorname{Mov}}(X'_m/X; \pi^{-1}(W'))$ for some m. By applying the negativity lemma (see Lemma 4.6) to $g_m\colon X'_m \to X$, we can check that $-F_m$ is effective on $(\pi\circ g_m)^{-1}(U)$. If Y_m is not $\mathbb{Q}$-factorial over W, then we replace Y_m with its small projective $\mathbb{Q}$-factorialization by Theorem 1.24. Hence we obtain a desired $f\colon Z \to X$. □

21. A canonical bundle formula in the complex analytic setting

In this section, we will quickly discuss a canonical bundle formula in the complex analytic setting and prove Theorems 1.18 and 1.22. We need some deep results from the theory of variations of Hodge structure.

Let us start with a generalization of Kollár's famous torsion-freeness.

Theorem 21.1 (Torsion-freeness, see [Tak])**.** *Let $\pi\colon X \to Y$ be a proper morphism from a Kähler manifold X onto a complex analytic space Y. Then $R^i\pi_*\omega_X$ is torsion-free for every i.*

If Y is projective in Theorem 21.1, then X is a compact Kähler manifold. In this case, there are no difficulties to prove Theorem 21.1. Unfortunately, however, we have to treat the case where Y is a general noncompact complex analytic space. Hence the proof of Theorem 21.1 is much harder than that of Kollár's original torsion-freeness. For the details, see [Tak] (see also [Fu6], and [Matm]). By combining Steenbrink's geometric description of canonical extensions of Hodge filtrations (see [St1] and [St2]) with Theorem 21.1, we have:

Theorem 21.2 (Hodge filtrations, see [Na3, Chapter V. 3.7. Theorem (4)])**.** *Let $\pi\colon X \to Y$ be a proper morphism from a Kähler manifold X onto a smooth variety Y. Assume that there exists a simple normal crossing divisor Σ_Y on Y such that π is smooth over $Y \setminus \Sigma_Y$. Then $R^i\pi_*\omega_{X/Y}$ is characterized as the upper canonical extension of the bottom Hodge filtration for every i.*

The proof of [Na1, Theorem 1] works in the above complex analytic setting once we know the torsion-freeness (see Theorem 21.1). For the details of Nakayama's argument, we recommend the interested reader to see [Fu2, Subsection 3.1] and [FF, §7] although they treat much more general settings than Nakayama's.

In order to discuss a canonical bundle formula, we recall the definition of *discriminant* $\mathbb{Q}$*-divisors*.

Definition 21.3. Let $f\colon X \to Y$ be a proper surjective morphism from a normal variety X onto a smooth variety Z with $f_*\mathcal{O}_X \simeq \mathcal{O}_Z$. Let Θ be a $\mathbb{Q}$-divisor on X such that $K_X + \Theta$ is $\mathbb{Q}$-Cartier and that (X, Θ) is sub kawamata log terminal over a nonempty Zariski open subset of Z. Let P be a prime divisor on Z. We put

$$b_P := \max\{t \in \mathbb{Q} \,|\, (X, \Theta + tf^*P) \text{ is sub log canonical over general points of } P\}.$$

Then we set $B_Z := \sum_P (1-b_P)P$, where P runs over prime divisors on Z, and call B_Z the *discriminant* $\mathbb{Q}$*-divisor* of $f\colon (X, \Theta) \to Z$. We can easily check that B_Z is a well-defined $\mathbb{Q}$-divisor on Z satisfying $\lfloor B_Z \rfloor \leq 0$. Let $\mu\colon Z' \to Z$ be a projective bimeromorphic morphism from a smooth variety Z'. We consider the following commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{\ \sigma\ } & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xleftarrow[\ \mu\ ]{} & Z', \end{array}$$

where X' is the normalization of the main component of $X \times_Z Z'$. We define Θ' by $K_{X'} + \Theta' = \sigma^*(K_X + \Theta)$. We call $f'\colon (X', \Theta') \to Z'$ the *induced fibration* of $f\colon (X, \Theta) \to Z$ by $\mu\colon Z' \to Z$. We can define the discriminant $\mathbb{Q}$-divisor $B_{Z'}$ of $f'\colon (X', \Theta') \to Z'$. By construction, we see that $\sigma_* B_{Z'} = B_Z$.

The following theorem is a generalization of Ambro's result in the complex analytic setting (see [A, Theorem 0.2]).

Theorem 21.4. *Let $f\colon X \to Z$ be a proper morphism from a Kähler manifold X onto a smooth complex variety Z with $f_*\mathcal{O}_X \simeq \mathcal{O}_Z$. Let $g\colon Z \to Y$ be a projective morphism to a Stein space Y. Let Θ be a $\mathbb{Q}$-divisor on X such that $K_X + \Theta \sim_{\mathbb{Q}} f^*D$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Z,* $\operatorname{Supp}\Theta$ *is a simple normal crossing divisor on X, $\Theta = \Theta^{<1}$ holds over general points of Z, and* $\operatorname{rank} f_*\mathcal{O}_X(\lceil -\Theta^{<1} \rceil) = 1$*. Let y be any point of Y. By replacing Y with a relatively compact Stein open neighborhood of y suitably, we have a commutative* *diagram:*

$$\begin{array}{ccc} X & \xleftarrow{\sigma} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Z & \xleftarrow[\mu]{} & Z' \end{array}$$

with the following properties.

(1) $\mu\colon Z' \to Z$ *is a projective bimeromorphic morphism from a smooth variety* Z'.
(2) X' *is a desingularization of* $X \times_Z Z'$ *such that* X' *is Kähler with* $K_{X'} + \Theta' = \sigma^*(K_X + \Delta)$.
(3) *Let* $B_{Z'}$ *be the discriminant* $\mathbb{Q}$*-divisor of* $f'\colon (X', \Theta') \to Z'$. *We write* $\sigma^* D = K_{Z'} + B_{Z'} + M_{Z'}$. *Then* $M_{Z'}$ *is nef over* Y. *Note that* $M_{Z'}$ *is usually called the* moduli $\mathbb{Q}$-divisor *of* $f'\colon (X', \Theta') \to Z'$.
(4) *Let* $\nu\colon Z'' \to Z'$ *be any projective bimeromorphic morphism from a smooth variety* Z''. *Then we can define* $f''\colon (X'', \Theta'') \to Z''$, $B_{Z''}$, *and* $M_{Z''}$ *as in (3) with* $\nu^*\mu^* D = K_{Z''} + B_{Z''} + M_{Z''}$. *In this situation, after shrinking* Y *with a slightly smaller relatively compact Stein open neighborhood of* y *again,* $\nu^* M_{Z'} = M_{Z''}$ *holds with* $\nu_* K_{Z''} = K_{Z'}$ *and* $\nu_* B_{Z''} = B_{Z'}$.

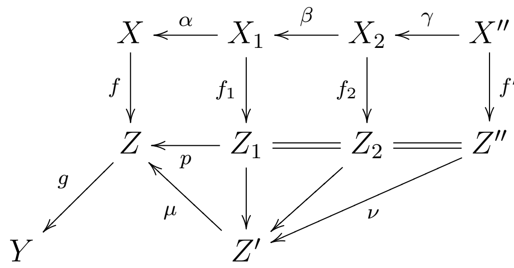
Proof. For the details, see [A, Section 5]. Although they treat much more general setting than Ambro's, [Fu13] and [FH] may help the reader understand the proof of this theorem. We note that Ambro's argument in [A] is different from Kawamata's in [Kaw4] and is closer to Mori's (see [Mo2, Section 5, Part II] and [Fu1, Section 4]). □

21.5 (A canonical bundle formula in the complex analytic setting, see [FMo])**.** Let $f\colon X \to Z$ be a proper morphism from a Kähler manifold X onto a smooth variety Z with $f_*\mathcal{O}_X \simeq \mathcal{O}_Z$ and let $g\colon Z \to Y$ be a projective morphism such that Y is Stein. Let Δ be an effective $\mathbb{Q}$-divisor on X such that $\operatorname{Supp}\Delta$ is a simple normal crossing divisor on X and that $\lfloor \Delta \rfloor = 0$. Suppose that $\kappa(F, (K_X + \Delta)|_F) = 0$ holds for an analytically sufficiently general fiber F of $f\colon X \to Z$. We fix an arbitrary point $y \in Y$. From now on, we sometimes replace Y with a smaller relatively compact Stein open neighborhood of y without mentioning it explicitly. Since $\kappa(F, (K_X + \Delta)|_F) = 0$, we obtain $g_*\left(f_*\mathcal{O}_X(m(K_X + \Delta)) \otimes \mathcal{A}\right) \neq 0$ for some divisible positive integer m and some g-ample line bundle $\mathcal{A}$ on Z. Hence we can write $K_X + \Delta \sim_{\mathbb{Q}} f^*D + B$ such that

(a) D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z,
(b) $f_*\mathcal{O}_X(\lfloor iB_+ \rfloor) \simeq \mathcal{O}_Z$ holds for every $i \geq 0$, and
(c) $\operatorname{codim}_Z f(\operatorname{Supp} B_-) \geq 2$.

We take a projective bimeromorphic morphism $\psi\colon X^\dagger \to X$ from a smooth variety $X^\dagger$ such that $\operatorname{Exc}(\psi) \cup \operatorname{Supp}\psi_*^{-1}\Delta \cup \operatorname{Supp}\psi_*^{-1}B$ is contained in a simple normal crossing divisor on $X^\dagger$. We put $K_{X^\dagger} + \Delta^\dagger = \psi^*(K_X + \Delta)$ and consider $K_{X^\dagger} + \Delta^\dagger_+ \sim_{\mathbb{Q}} \psi^* f^* D + \psi^* B + \Delta^\dagger_-$. By replacing $f\colon X \to Z$ with $f \circ \psi\colon X^\dagger \to Z$, we may further assume that the support of $\Theta := \Delta - B$ is a simple normal crossing divisor on Z. We apply Theorem 21.4 to $f\colon (X, \Theta) \to Z$. Then we have a projective bimeromorphic morphism $\mu\colon Z' \to Z$ satisfying the properties in Theorem 21.4. By Hironaka's flattening theorem (see [Hi]), we can take a projective bimeromorphic morphism $p\colon Z_1 \to Z$ from a smooth variety such that the main component of $X \times_Z Z_1$ is flat over Z_1. We may further assume that

$p\colon Z_1 \to Z$ factors through Z'. Then we consider the following big commutative diagram:



where X_1 is the main component of $X \times_Z Z_1$, X_2 is the normalization of X_1, and $\gamma\colon X'' \to X_2$ is a proper bimeromorphic morphism from a smooth variety X''. We put $h := \alpha \circ \beta \circ \gamma\colon X'' \to X$ and $K_{X''} + \Delta'' = h^*(K_X + \Delta)$. We may assume that there exist simple normal crossing divisors $\Sigma_{X''}$ and $\Sigma_{Z''}$ on X'' and Z'', respectively, such that $f''\colon X'' \to Z''$ is smooth over $Z'' \setminus \Sigma_{Z''}$, $\Sigma_{X''}$ is relatively simple normal crossing over $Z'' \setminus \Sigma_{Z''}$, $(f'')^{-1}\Sigma_{Z''} \subset \Sigma_{X''}$, and $\operatorname{Supp}\Delta'' \cup \operatorname{Supp} h^*B$ is contained in $\Sigma_{X''}$. Since $K_X + \Delta \sim_{\mathbb{Q}} f^*D + B$, we obtain $K_{X''} + \Delta'' \sim_{\mathbb{Q}} (f'')^*p^*D + h^*B$. We can write

$$\Delta''_- + (f'')^*p^*D + h^*B = (f'')^*D'' + B''$$

such that $\operatorname{codim}_{Z''} f''(\operatorname{Supp} B''_-) \geq 2$ and that $f''_*\mathcal{O}_{X''}(\lfloor iB''_+ \rfloor) \simeq \mathcal{O}_{Z''}$ for every $i \geq 0$, where $B'' = B''_+ - B''_-$ as usual. Hence, we can write

$$K_{X''} + \Delta''_+ \sim_{\mathbb{Q}} (f'')^*(K_{Z''} + B_{Z''} + M_{Z''}) + B''.$$

By construction, we can check that

(d) $M_{Z''} = \mu^* M_{Z'}$ is nef over Y, and
(e) $\operatorname{Supp} B_{Z''} \subset \Sigma_{Z''}$, $B_{Z''}$ is effective, and $\lfloor B_{Z''} \rfloor = 0$.

We put $\pi := g \circ f\colon X \to Y$ and $\pi''\colon X'' \to Y$. Let k be a divisible positive integer. Then

$$\begin{aligned}
\pi_*\mathcal{O}_X(k(K_X + \Delta)) &\simeq \pi''_*\mathcal{O}_{X''}(k(K_{X''} + \Delta'')) \\
&\simeq \pi''_*\mathcal{O}_{X''}(k(K_{X''} + \Delta''_+ + B''_-)) \\
&\simeq \pi''_*\mathcal{O}_{X''}((f'')^*(k(K_{Z''} + B_{Z''} + M_{Z''})) + kB''_+) \\
&\simeq g''_*\mathcal{O}_{Z''}(k(K_{Z''} + B_{Z''} + M_{Z''})),
\end{aligned}$$

where $g''\colon Z'' \to Y$. Here, we used the fact that $\Delta''_- + B''_-$ is effective and h-exceptional.

Remark 21.6. In [FMo], we used Kawamata's positivity theorem (see [Kaw4, Theorem 2]) to prove the nefness of the moduli part $M_{Z'}$. In Theorem 21.4, we adopted Ambro's formulation of klt-trivial fibrations (see Theorem 21.4 and [A]) instead of [Kaw4, Theorem 2].

Let us go to the proof of Theorems 1.18 and 1.22.

Proof of Theorems 1.18 and 1.22. Let y be any point of Y. Throughout this proof, we will feely replace Y with a relatively compact Stein open neighborhood of y. In Theorem 1.18, by taking a resolution of singularities, we may assume that X is smooth and $\operatorname{Supp}\Delta$ is a simple normal crossing divisor on X. Let $f\colon X \dashrightarrow Z$ be the Iitaka fibration with respect to $K_X + \Delta$ over Y. By replacing X and Z, we may assume that Z is a smooth variety and is projective over Y and that f is a morphism with $f_*\mathcal{O}_X \simeq \mathcal{O}_Z$. We use the canonical bundle formula discussed in 21.5. Then, by Lemma 2.26, it is sufficient to prove that

$$\bigoplus_{m \in \mathbb{N}} g''_*\mathcal{O}_{Z''}(mk(K_{Z''} + B_{Z''} + M_{Z''}))$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra. By construction, $K_{Z''} + B_{Z''} + M_{Z''}$ is big over Y. We can find $\Delta_{Z''}$ such that $(Z'', \Delta_{Z''})$ is kawamata log terminal and that

$$a(K_{Z''} + B_{Z''} + M_{Z''}) \sim b(K_{Z''} + \Delta_{Z''})$$

for some positive integers a and b. Hence, by Lemma 2.26 again, it is sufficient fo prove that

$$\bigoplus_{m\in\mathbb{N}} g''_*\mathcal{O}_{Z''}(\lfloor m(K_{Z''} + \Delta_{Z''})\rfloor)$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra. Since $K_{Z''}+\Delta_{Z''}$ is big over Y, it follows from Theorem 1.8 (3). Therefore, we get the desired result. □

22. Minimal model program with scaling revisited

In this section, we will discuss the minimal model program with scaling again for future usage. The original results for algebraic varieties are not covered by [BCHM]. Here, we will closely follow the presentation of [Bir1] and [Bir2].

Let us recall the definition of *extremal curves.*

Definition 22.1 (Extremal curves)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). A curve Γ on X is called *extremal over W* if the following properties hold.

(i) Γ generates an extremal ray R of $\overline{\mathrm{NE}}(X/Y; W)$.
(ii) There exists a π-ample Cartier divisor H on X such that

$$H \cdot \Gamma = \min\{H \cdot \ell\},$$

where ℓ ranges over curves generating R.

The following theorem is very useful when we run the minimal model program with scaling.

Theorem 22.2 (see [Fu9, Theorem 4.7.2])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Let V be a finite-dimensional affine subspace of* $\mathrm{WDiv}_{\mathbb{R}}(X)$*, which is defined over the rationals. Assume that there is an $\mathbb{R}$-divisor Δ_0 on X such that (X, Δ_0) is kawamata log terminal. We fix an $\mathbb{R}$-divisor $\Delta \in \mathcal{L}(V; \pi^{-1}(W))$. Then we can find positive real numbers α and δ, which depend on (X, Δ) and V, with the following properties.*

(1) *If Γ is any extremal curve over W and $(K_X + \Delta)\cdot\Gamma > 0$, then $(K_X + \Delta)\cdot\Gamma > \alpha$.*
(2) *If $D \in \mathcal{L}(V; \pi^{-1}(W))$, $\|D - \Delta\| < \delta$, and $(K_X + D)\cdot\Gamma \le 0$ for an extremal curve Γ over W, then $(K_X + \Delta)\cdot\Gamma \le 0$.*
(3) *Let $\{R_t\}_{t\in T}$ be any set of extremal rays of $\overline{\mathrm{NE}}(X/Y; W)$. Then*

$$\mathcal{N}_T := \{D \in \mathcal{L}(V; \pi^{-1}(W)) \,|\, (K_X + D)\cdot R_t \ge 0 \text{ for every } t \in T\}$$

is a rational polytope in V. In particular,

$$\mathcal{N}^\sharp_\pi(V; W) = \{\Delta \in \mathcal{L}(V; \pi^{-1}(W)) \,|\, K_X + \Delta \text{ is nef over } W\}$$

is a rational polytope.

Proof. This theorem is a formal consequence of Theorem 9.2 and Theorem 7.3. More precisely, (1) easily follows from Theorem 9.2. We can check that (2) holds true by using (1). By (2) and Theorem 7.3, we can prove (3). For the details, see, for example, the proof of [Fu9, Theorem 4.7.2]. □

By Theorem 22.2 (3) and Theorem 9.2, we can prove:

Theorem 22.3 (see [Fu9, Theorem 4.7.3])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies* (P)*. Let (X, Δ) be a log canonical pair and let H be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $(X, \Delta+H)$ is log canonical and that $K_X+\Delta+H$ is nef over W. Assume that there exists Δ_0 such that (X, Δ_0) is kawamata log terminal. Then, either $K_X + \Delta$ is nef over W or there is a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y; W)$ such that $(K_X + \Delta + \lambda H) \cdot R = 0$, where*

$$\lambda := \inf\{t \in \mathbb{R}_{\geq 0} \mid K_X + \Delta + tH \text{ is nef over } W\}.$$

Of course, $K_X + \Delta + \lambda H$ is nef over W.

Proof. The proof of [Fu9, Theorem 4.7.3] works without any modifications. □

By Theorems 22.2 and 22.3, the minimal model program with scaling explained in Section 13 becomes much more useful.

22.4 (Minimal model program with scaling)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a compact subset of Y such that $\pi\colon X \to Y$ and W satisfies (P). Let (X, Δ) be a log canonical pair such that X is $\mathbb{Q}$-factorial over W. Assume that there exists Δ_0 such that (X, Δ_0) is kawamata log terminal. Let H be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that $(X, \Delta + H)$ is log canonical and that $K_X + \Delta + H$ is nef over W. By Theorem 22.3, we can take a $(K_X + \Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y; W)$ such that $(K_X + \Delta + \lambda H) \cdot R = 0$ if $(K_X + \Delta)$ is not nef over W. We can consider the contraction morphism $\varphi_R\colon X \to Z$ associated to R over some open neighborhood of W by Theorem 7.3. By Remark 17.10, we know that the desired flip always exists. We note that we can always find Δ'_0 such that (X, Δ'_0) is kawamata log terminal and that R is a $(K_X + \Delta'_0)$-negative extremal ray of $\overline{\mathrm{NE}}(X/Y; W)$. Therefore, we can run a minimal model program similar to the one explained in Section 13. We call it the $(K_X + \Delta)$*-minimal model program with scaling of H over Y around W*. We sometimes simply say that it is the *minimal model program with scaling* if there is no danger of confusion.

It is well known that Theorem 1.28 is an easy consequence of the minimal model program with scaling. The main ingredient of the following proof of Theorem 1.28 is Theorem 22.2 (2).

Proof of Theorem 1.28. Throughout this proof, we will freely shrink Y around W suitably without mentioning it explicitly. Let H_2 be a general π_2-ample $\mathbb{Q}$-divisor on X_2 and let H_1 be its strict transform on X_1. Then there is a small positive real number δ such that $(X_1, \Delta_1 + \delta H_1)$ is kawamata log terminal. We take a general π_1-ample $\mathbb{Q}$-divisor H'_1 on X_1 such that $(X_2, \Delta_2 + \delta H_2 + \delta' H'_2)$ is kawamata log terminal for some positive real number δ', where H'_2 is the strict transform of H'_1. If δ is sufficiently small, $K_{X_1} + \Delta_1 + \delta H_1 + \delta' H'_1$ is nef over W. We can run the $(K_{X_1} + \Delta_1 + \delta H_1)$-minimal model program with scaling over Y around W (see 22.4). After finitely many flips, we finally end up with X_2. On the other hand, by Theorem 22.2 (2), we see that each step is a flop with respect to $K_{X_1} + \Delta_1$ if δ is sufficiently small. Therefore, we obtain the desired statement. □

23. On abundance conjecture

In this final section, we will treat the abundance conjecture for kawamata log terminal pairs in the complex analytic setting.

Let us recall the following famous conjecture, which is one of the most difficult conjectures in the theory of minimal models.

Conjecture 23.1 (Abundance conjecture for projective kawamata log terminal pairs)**.** *Let (X, Δ) be a projective kawamata log terminal pair such that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semiample.*

The main result of this section is as follows.

Theorem 23.2 (see Theorem 1.30)**.** *Assume that Conjecture 23.1 holds in dimension n.*
Let $\pi\colon X \to Y$ be a projective surjective morphism of normal complex varieties with $\dim X - \dim Y = n$ and let (X, Δ) be a kawamata log terminal pair. Assume that $K_X + \Delta$ is π-nef. Let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Then $K_X + \Delta$ is π-semiample over some open neighborhood of W.

Theorem 23.2 says that we can reduce the abundance conjecture for projective morphisms of complex analytic spaces to the original abundance conjecture for projective varieties. Before we prove Theorem 23.2, we prepare some lemmas. The following lemma is Wilson's theorem (see [La1, Theorem 2.3.9]) for projective morphisms of complex varieties.

Lemma 23.3. *Let $f\colon Z \to Y$ be a projective morphism from a smooth complex variety Z onto a normal Stein variety Y and let D be a Cartier divisor on X such that D is nef and big over Y. Let y be any point of Y. Then, by replacing Y with any relatively compact Stein open neighborhood of y, there exist a positive integer m_0 and an effective Cartier divisor B on Z such that $\mathcal{O}_Z(mD - B)$ is f-free.*

Proof. We can take an f-very ample Cartier divisor H on Z after replacing Y with any relatively compact Stein open neighborhood of y. Since D is big over Y, there exists a positive integer m_0 such that $m_0 D \sim A + B$, A is f-ample, $B \geq 0$, and $A - (K_X + (n+1)H)$ is f-ample with $n = \dim Z$. Then, $R^i f_* \mathcal{O}_Z(mD - B - iH) = 0$ holds for every $i > 0$ and every $m \geq m_0$ since $mD - B - iH - K_X \sim A - (K_X + (n+1)H) + (n+1-i)H$ is f-ample for $0 < i \leq n+1$ (see, for example, Theorem 5.1). Therefore, by Castelnuovo–Mumford regularity (see, for example, [La2, Example 1.8.24]), we obtain that $\mathcal{O}_Z(mD - B)$ is f-free for every $m \geq m_0$. □

As an easy consequence, we obtain:

Lemma 23.4. *Let $f\colon Z \to Y$ be a projective morphism from a smooth complex variety Z onto a normal Stein variety Y and let D be a Cartier divisor on X such that D is nef and big over Y. Assume that*

$$R(Z, D) := \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_Z(mD)$$

is a locally finitely generated graded $\mathcal{O}_Y$-algebra. Then D is f-semiample.

Lemma 23.4 is well known for normal projective varieties (see, for example, [La1, Theorem 2.3.15]).

Proof of Lemma 23.4. By taking the Stein factorization, we may assume that $f_* \mathcal{O}_Z \simeq \mathcal{O}_Y$. Suppose that, for every positive integer m, $f^* f_* \mathcal{O}_Z(mD) \to \mathcal{O}_Z(mD)$ is not surjective at $z \in Z$. We take an open neighborhood U of $f(z)$ and a Stein compact subset W of Y such that $f(z) \in U \subset W$ and that $\mathcal{O}_Y(W) = \Gamma(W, \mathcal{O}_Y)$ is noetherian. If we make U and W sufficiently small, then $\Gamma\left(W, \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_Z(mD)\right) \simeq \bigoplus_{m \in \mathbb{N}} f_* \mathcal{O}_Z(mD)(W)$ is a finitely generated $\mathcal{O}_Y(W)$-algebra. Therefore, there exists a positive integer l such that $\bigoplus_{m\in\mathbb{N}} f_*\mathcal{O}_Z(mlD)(W)$ is generated by $f_*\mathcal{O}_Z(lD)(W)$. Let V be a relatively compact Stein open neighborhood of W. Then, by Lemma 23.3, there exist $k > 0$ and $g \in \Gamma(f^{-1}(V), \mathcal{O}_Z(klD)) = \Gamma(V, f_*\mathcal{O}_Z(klD))$ such that $C = (g = 0)$ is an effective divisor on $f^{-1}(V)$ with $\operatorname{mult}_z C < k$. On the other hand, since $f_*\mathcal{O}_Z(klD)(W)$ is generated by $f_*\mathcal{O}_Z(lD)(W)$, $\operatorname{mult}_z C \geq k$ holds. It is a contradiction. This means that D is f-semiample. □

Let us prove Theorem 23.2.

Proof of Theorem 23.2. In Step 1, we will reduce the problem to the case where $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then, in Step 2, we will prove that it is semiample by using the finite generation of log canonical rings.

Step 1. We take a Stein open neighborhood U of W and a Stein compact subset W' such that $U \subset W'$ and that $\Gamma(W', \mathcal{O}_Y)$ is noetherian. By Theorem 22.2, after shrinking Y around W', we can find $\mathbb{Q}$-divisors $\Delta_1, \ldots, \Delta_l$ on X such that $K_X + \Delta = \sum_i r_i(K_X + \Delta_i)$, (X, Δ_i) is kawamata log terminal, $K_X + \Delta_i$ is nef over W', and $r_i \in \mathbb{R}_{>0}$ with $\sum_i r_i = 1$. Therefore, it is sufficient to prove that $K_X+\Delta_i$ is semiample over some open neighborhood of W. Hence, from now on, we may further assume that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Moreover, we may assume that there exists a positive integer k such that $k(K_X+\Delta)$ is Cartier. Without loss of generality, we may assume that $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$ by taking the Stein factorization.

Step 2. Let F be an analytically sufficiently general fiber of $\pi\colon X \to Y$. Then $(F, \Delta|_F)$ is kawamata log terminal with $K_F + \Delta|_F = (K_X + \Delta)|_F$. Hence, by assumption, $K_F + \Delta|_F$ is semiample. We put $L = k(K_X + \Delta)$. From now on, we will freely replace Y with a smaller Stein open neighborhood of W without mentioning it explicitly. We consider a meromorphic map $g\colon X \dashrightarrow Z_0$ over Y associated to $\pi^*\pi_*\mathcal{O}_X(mL) \to \mathcal{O}_X(mL)$ for some sufficiently large and divisible positive integer m such that $\dim Z_0 = \dim Y + \kappa(F, K_F + \Delta|_F)$. As in the proof of [Kaw1, Proposition 2.1], by using Hironaka's flattening theorem (see [Hi]), and so on, we can construct the following commutative diagram:

$$
\begin{array}{ccccccccc}
X & \xleftarrow{\mu_0} & X_1 & \xleftarrow{\mu_1} & X_2 & \xleftarrow{\mu_2} & X_3 & \xleftarrow{\mu_3} & X' \\
{\scriptstyle g}\,\vdots\downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle \phi} \\
Z_0 & \xleftarrow[\pi_0]{} & Z_1 & \xleftarrow[\pi_1]{} & Z & = & Z & = & Z
\end{array}
$$

which satisfies the following conditions.

(i) All the varieties in the diagram are projective over Y.
(ii) X_1, X', Z_1, and Z are smooth, and X_3 is normal.
(iii) μ_0, μ_1, μ_2, μ_3, and π_1 are projective bimeromorphic morphisms, g_1, g_2, g_3 and ϕ are surjective morphisms with connected fibers, and π_0 is a generically finite surjective morphism.
(iv) g_2 is flat, μ_2 is finite, and g_3 is equidimensional.

We put $\mu := \mu_0 \circ \mu_1 \circ \mu_2 \circ \mu_3 \colon X' \to X$. Then we finally get the following commutative diagram:

$$\begin{array}{ccccc} & & X' & & \\ & {\scriptstyle \mu}\swarrow & & \searrow{\scriptstyle \phi} & \\ X & & & & Z \\ & {\scriptstyle \pi}\searrow & & \swarrow{\scriptstyle f} & \\ & & Y & & \end{array}$$

such that

(a) X' and Z are projective over Y,
(b) X' and Z are smooth,
(c) μ is bimeromorphic and ϕ is a surjective morphism with connected fibers, and
(d) there exists a Cartier divisor D on Z such that D is nef and big over Y with $a\mu^*L \sim b\phi^*D$ for some positive integers a and b.

For the details, see the proof of [Kaw1, Proposition 2.1]. Since $R(X/Y, K_X+\Delta)$ is a locally finitely generated graded $\mathcal{O}_Y$-algebra by Theorem 1.18, $R(Z, D)$ is also a locally finitely generated graded $\mathcal{O}_Y$-algebra by Lemma 2.26. Hence, by Lemma 23.4, D is f-semiample. This means that L is π-semiample.

Anyway, $K_X + \Delta$ is a finite $\mathbb{R}_{>0}$-linear combination of semiample Cartier divisors over some open neighborhood of W. This is what we wanted. $\square$

The abundance conjecture for log canonical pairs in the complex analytic setting seems to be much more difficult than the one for kawamata log terminal pairs.

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