ON FINITENESS OF RELATIVE LOG PLURICANONICAL REPRESENTATIONS

OSAMU FUJINO

Abstract. We prove the finiteness of relative log pluricanonical representations in the complex analytic setting. As an application, we discuss the abundance conjecture for semi-log canonical pairs within this framework. Furthermore, we establish the existence of log canonical flips for complex analytic spaces. Roughly speaking, we reduce the abundance conjecture for semi-log canonical pairs to the case of log canonical pairs in the complex analytic setting. Moreover, we show that the abundance conjecture for projective morphisms of complex analytic spaces can be reduced to the classical abundance conjecture for projective varieties.

Contents

1. Introduction

The present paper aims to address a missing component of the minimal model program for projective morphisms between complex analytic spaces (see [Fuj12], [Fuj13], [Fuj14], [Fuj15], [Fuj17], [FF], [DHP], [EH2], [LM], [EH3], [H5], and others). Broadly speaking, this work can be viewed as a complex analytic generalization of [FG] (see also [Fuj1]). One of the main objectives of the present paper is to establish the following result related to the abundance conjecture.

Theorem 1.1 (Abundance theorem for semi-log canonical pairs in the complex analytic setting, cf. [FG, Theorem 1.5])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces, let W be a compact subset of Y, and let (X, Δ) be a semi-log canonical pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\nu\colon X^\nu \to X$ be the normalization. Assume that $K_{X^\nu} + \Theta := \nu^*(K_X + \Delta)$ is $\pi \circ \nu$-semiample over some open neighborhood of W. Then there exists an open neighborhood U of W and a divisible positive integer m such that $\mathcal{O}_X(m(K_X + \Delta))$ is π-generated over U.*

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In order to prove Theorem 1.1, we need:

Theorem 1.2 (Finiteness of relative log pluricanonical representations, I, cf. [FG, Theorem 1.1])**.** *Let $\pi\colon X \to Y$ be a projective morphism from a* (*not necessarily connected*) *normal complex analytic space X onto a complex variety Y such that (X, Δ) is log canonical and that every irreducible component of X is dominant onto Y. Let m be a positive integer such that $m(K_X+\Delta)$ is Cartier and $\pi_*\mathcal{O}_X(m(K_X+\Delta)) \neq 0$. Assume that $K_X+\Delta$ is π-semiample. Then the image of*

$$\rho_m\colon \operatorname{Bim}(X/Y, \Delta) \to \operatorname{Aut}_{\mathcal{O}_Y}\left(\pi_*\mathcal{O}_X(m(K_X+\Delta))\right)$$

is a finite group, where $\operatorname{Bim}(X/Y, \Delta)$ is the group of all B-bimeromorphic maps of (X, Δ) over Y.

As an easy consequence of Theorem 1.2, we have a useful corollary. We will use it in the proof of Theorem 1.1.

Corollary 1.3 (Finiteness of relative log pluricanonical representations, II, cf. [FG, Theorem 1.1])**.** *Let (X, Δ) be an equidimensional* (*not necessarily connected*) *log canonical pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let m be a positive integer such that $m(K_X+\Delta)$ is Cartier and $\pi_*\mathcal{O}_X(m(K_X+\Delta)) \neq 0$. Assume that $K_X+\Delta$ is π-semiample. Let W be a compact subset of Y and let U be a semianalytic Stein open subset of Y with $U \subset W$. Let $\operatorname{Bim}(X/Y, \Delta; W)$ be the group of all B-bimeromorphic maps g defined over some open neighborhood U_g of W. In this setting, we can consider*

$$\rho^{WU}_m\colon \operatorname{Bim}(X/Y, \Delta; W) \to \operatorname{Aut}_{\mathcal{O}_U}\left(\pi_*\mathcal{O}_{\pi^{-1}(U)}(m(K_X+\Delta))\right).$$

Then $\rho^{WU}_m(\operatorname{Bim}(X/Y, \Delta; W))$ is a finite group.

As an application of Theorem 1.1, we have:

Theorem 1.4 (Freeness for nef and log abundant log canonical bundles, cf. [FG, Theorem 1.6])**.** *Let (X, Δ) be a semi-log canonical pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Assume that $K_X+\Delta$ is $\mathbb{Q}$-Cartier and is π-nef and π-log abundant with respect to (X, Δ) over Y. Let W be a compact subset of Y. Then there exists a positive integer m such that $\mathcal{O}_X(m(K_X+\Delta))$ is π-generated over some open neighborhood of W.*

Theorem 1.4 is well known when $\pi\colon X \to Y$ is algebraic (see [FG, Theorem 1.6]). As mentioned above, we prove Theorem 1.4 as a consequence of Theorem 1.1. In our proof of Theorem 1.4 presented in the present paper, we make use of a kind of canonical bundle formula (see [Fuj3] and [Fuj7]). Therefore, the result is not entirely obvious. When $K_X+\Delta$ is only assumed to be $\mathbb{R}$-Cartier, we have the following theorem.

Theorem 1.5. *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Let U be an open subset of Y and let L be a compact subset of Y such that $L \subset U \subset W$. Let (X, Δ) be a log canonical pair such that $K_X+\Delta$ is π-nef and π-log abundant with respect to (X, Δ) over Y. Then $K_X+\Delta$ is π-semiample over some open neighborhood of L.*

We note that Stein compact subsets play an important role in [Fuj12].

Remark 1.6 (Stein compact subsets)**.** A compact subset on an analytic space is said to be *Stein compact* if it admits a fundamental system of Stein open neighborhoods. Let W be a Stein compact subset on a complex analytic space Y. Then, by Siu's theorem,

$\Gamma(W, \mathcal{O}_Y)$ is noetherian if and only if $W \cap Z$ has only finitely many connected components for any analytic subset Z which is defined over an open neighborhood of W. Hence, if W is a Stein compact semianalytic subset of a complex analytic space Y, then $\Gamma(W, \mathcal{O}_Y)$ is always noetherian.

By combining Theorem 1.5 with [EH2, Theorem 1.2], we can prove the existence of log canonical flips in the complex analytic setting. We learned it from Kenta Hashizume.

Theorem 1.7 (Existence of log canonical flips)**.** *Let $\varphi\colon X \to Z$ be a small projective bimeromorphic morphism of normal complex varieties such that (X, Δ) is log canonical and that $-(K_X + \Delta)$ is φ-ample. Then we have a commutative diagram*

$$\begin{tikzcd}(X, \Delta) \arrow[rr, dashed, "\phi"] \arrow[dr, "\varphi"'] & & (X^+, \Delta^+) \arrow[dl, "\varphi^+"] \\ & Z & \end{tikzcd}$$

satisfying the following properties:

(i) *$\varphi^+\colon X^+ \to Z$ is a small projective bimeromorphic morphism of normal complex varieties,*
(ii) *(X^+, Δ^+) is log canonical, where Δ^+ is the strict transform of Δ on X^+, and*
(iii) *$K_{X^+} + \Delta^+$ is φ^+-ample.*

We usually simply say that $\phi\colon (X, \Delta) \dashrightarrow (X^+, \Delta^+)$ is a log canonical flip.

By combining Theorem 1.5 with [EH2, Theorem 1.3], we establish the existence of good dlt blow-ups in the complex analytic setting. This result immediately yields the inversion of adjunction for log canonicity (see [Fuj16, Theorem 1.1] and Theorem 6.3).

Theorem 1.8 (Good dlt blow-ups)**.** *Let X be a normal complex variety X and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Note that Δ is not necessarily a boundary $\mathbb{R}$-divisor. Let W be a Stein compact subset of X such that $\Gamma(W, \mathcal{O}_X)$ is noetherian. Then, after shrinking X around W suitably, we can construct a projective bimeromorphic morphism $f\colon Z \to X$ from a normal complex variety Z with the following properties:*

(i) *Z is $\mathbb{Q}$-factorial over W,*
(ii) *$a(E, X, \Delta) \leq -1$ for every f-exceptional divisor E on Z, and*
(iii) *$(Z, \Delta_Z^{<1} + \operatorname{Supp} \Delta_Z^{\geq 1})$ is divisorial log terminal, where $K_Z + \Delta_Z = f^*(K_X + \Delta)$.*

Moreover, we put

$$\Delta_Z^{\dagger} := \Delta_Z^{<1} + \operatorname{Supp} \Delta_Z^{\geq 1} = \Delta_Z^{\leq 1} + \operatorname{Supp} \Delta_Z^{>1}.$$

Then we have

$$K_Z + \Delta_Z^{\dagger} = f^*(K_X + \Delta) - G,$$

where

$$G = \Delta_Z^{\geq 1} - \operatorname{Supp} \Delta_Z^{\geq 1} = \Delta_Z^{>1} - \operatorname{Supp} \Delta_Z^{>1}.$$

In this setting, we can make $f\colon Z \to X$ satisfy:

(iv) *$-G$ is f-nef over W.*

We further assume that

$$L \subset U' \subset W' \subset U \subset W,$$

where W' is a Stein compact subset of X such that $\Gamma(W', \mathcal{O}_X)$ is noetherian, U and U' are open subsets of X, and L is a compact subset of X. Then, by Theorem 1.5, we have:

(v) *$-G$ is f-semiample over some open neighborhood of L.*

In Section 6, we also present another refinement of dlt blow-ups (see Theorem 6.1), derived as a consequence of [EH2], which is closely related to Theorem 1.8. Both Theorems 1.8 and 6.1 are useful for the study of complex analytic singularities. As applications of these theorems, we discuss the ACC for log canonical thresholds (see Theorem 6.2) and the inversion of adjunction for log canonicity (see Theorem 6.3) in the complex analytic setting.

For the reader's convenience, we recall the abundance conjecture for projective log canonical pairs. It is well known that the abundance conjecture is among the most important and profound conjectures in the theory of minimal models.

Conjecture 1.9 (Abundance conjecture for projective log canonical pairs)**.** *Let (X, Δ) be a projective log canonical pair such that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semiample.*

It is well known that Conjecture 1.9 has already been solved in $\dim X \leq 3$. When $\dim X \geq 4$, it is still widely open. By Theorem 1.4, we have:

Theorem 1.10 (cf. [Fuj12, Theorem 1.30])**.** *Assume that Conjecture 1.9 holds in dimension n. Let $\pi\colon X \to Y$ be a projective surjective morphism of normal complex varieties with $\dim X \leq n$ and let (X, Δ) be a log canonical pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Assume that $K_X + \Delta$ is π-nef. Let W be a compact subset of Y. Then there exists a positive integer m such that $\mathcal{O}_X(m(K_X + \Delta))$ is π-generated over some open neighborhood of W.*

When $K_X + \Delta$ is only $\mathbb{R}$-Cartier, we have:

Corollary 1.11. *Assume that Conjecture 1.9 holds in dimension n. Let $\pi\colon X \to Y$ be a projective surjective morphism of normal complex varieties with $\dim X \leq n$ and let (X, Δ) be a log canonical pair. Assume that $K_X + \Delta$ is π-nef. Let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Let U be an open subset of Y and let L be a compact subset of Y with $L \subset U \subset W$. Then $K_X + \Delta$ is π-semiample over some open neighborhood of L.*

Theorem 1.10 and Corollary 1.11 show that the abundance conjecture for projective morphisms of complex analytic spaces can be reduced to the original abundance conjecture for projective varieties. Therefore, in order to address the abundance conjecture for projective morphisms between complex analytic spaces, it is sufficient to resolve Conjecture 1.9. In the case where (X, Δ) is a kawamata log terminal pair, Theorem 1.10 has already been established in [Fuj12, Theorem 1.30].

Note that in the present paper, we employ the minimal model program for projective morphisms between complex analytic spaces, as established in [Fuj12], [EH1], and [EH2]. Additionally, we make use of the vanishing theorems proved in [Fuj13] (see also [Fuj17] and [FF]). However, we do not employ Kollár's gluing theory as presented in [K], since it is currently unclear whether it applies to complex analytic spaces.

Remark 1.12 (see Lemma 4.11)**.** We can easily check that Theorem 1.1 recovers [HX, Theorem 2], which is the original algebraic version of this problem. Hence the present paper gives an alternative proof of [HX, Theorem 2] without using Kollár's gluing theory in [K].

Remark 1.13. Based on [Fuj12], [EH1], [EH2], and the present paper, we believe that various results of the minimal model program for log canonical pairs can be formulated and proved in the complex analytic setting. We do not discuss these results here. For details and further developments, see [EH1], [EH2], [EH3], [H4], [H5], and the references therein. Since [EH1] and [EH2] do not rely on the results of the present paper, we may freely use their results here. In contrast, [EH3] and [H5] do depend on the results of this paper, and therefore will not be cited or used in the arguments below.

Remark 1.14. In [Fuj6], we demonstrated that the minimal model theory for algebraic surfaces can be developed under significantly weaker assumptions than those required in higher dimensions. A comparable result holds for projective morphisms between complex analytic spaces. Moriyama (see [M]) offers a detailed treatment of the minimal model theory for surfaces in the complex analytic setting.

We now outline the organization of the present paper. In Section 2, we review basic definitions and results that are essential for the development of the present paper. Section 3 is devoted to the finiteness of relative log pluricanonical representations. Our proof of Theorem 1.2 relies on the finiteness of log pluricanonical representations for projective log canonical pairs, as established in [FG]. In Section 4, we address the abundance conjecture for semi-log canonical pairs in the complex analytic setting. More specifically, we prove Theorem 1.1, which is one of the main results of the present paper. In Section 5, we prove Theorem 1.4 as an application of Theorem 1.1, and then deduce Theorem 1.10 as a straightforward consequence of Theorem 1.4. We also establish Theorems 1.5, 1.7, and Corollary 1.11. In Section 6, we discuss dlt blow-ups as applications of the minimal model program for log canonical pairs, as established in [EH2]. We prove two generalizations of dlt blow-ups (see Theorems 1.8 and 6.1). As a direct application of Theorem 6.1, we address the ACC for log canonical thresholds in the setting of complex analytic spaces (see Theorem 6.2). Moreover, we show that Theorem 1.8 allows us to quickly recover the inversion of adjunction for log canonicity (see Theorem 6.3). In the final section, Section 7, we provide some supplementary comments on [Fuj1] and [FG] for the reader's convenience. We also correct some minor issues in these papers.

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In the present paper, every complex analytic space is assumed to be *Hausdorff* and *second-countable*. A reduced and irreducible complex analytic space is called a *complex variety*. We will freely use the basic results on complex analytic geometry in [BS] and [Fis]. For the minimal model program for projective morphisms between complex analytic spaces, see [Fuj12] (see also [EH1] and [EH2]). For the basic definitions and results in the theory of minimal models for higher-dimensional algebraic varieties, see [Fuj4] and [Fuj11] (see also [KM] and [K]). In the present paper, we sometimes use semianalytic sets. For the basic properties of semianalytic sets, see [BieM1].

2. Preliminaries

In this section, we collect some basic definitions and results necessary for the present paper. We begin with the following fundamental definitions.

Definition 2.1 ([Fuj12, Definition 2.32] and [Fuj14, 2.1.6])**.** Let X be a normal complex variety and let $D=\sum_i a_iD_i$ be an $\mathbb{R}$-divisor on X such that D_i is a prime divisor on X for every i with $D_i \neq D_j$ for $i \neq j$. We put

$$\lfloor D\rfloor := \sum_i \lfloor a_i\rfloor D_i, \quad \lceil D\rceil := -\lfloor -D\rfloor, \quad \text{and} \quad \{D\} := D-\lfloor D\rfloor.$$

We also put

$$D^{=1} := \sum_{a_i=1} D_i, \quad D^{<1} := \sum_{a_i<1} a_iD_i, \quad \text{and} \quad D^{>1} := \sum_{a_i>1} a_iD_i.$$

Similarly, we can define $D^{\leq 1}$ and $D^{\geq 1}$. We note that D is called a *boundary* $\mathbb{Q}$*-divisor* (resp. a *subboundary* $\mathbb{Q}$*-divisor*) when $a_i \in \mathbb{Q}$ and $0 \leq a_i \leq 1$ (resp. $a_i \leq 1$) for every i.

Let us recall the definitions of log canonical pairs and log canonical strata. For a detailed discussion of the singularities of pairs, see [Fuj4], [Fuj11], [Fuj12, Section 3], [Fuj14, Section 2.1], [K], and others. Although there are some subtle issues regarding the complex analytic singularities of pairs, we do not repeat the details here.

Definition 2.2 (Log canonical pairs and log canonical strata, see [Fuj12, Definition 3.1] and [Fuj14, 2.1.1])**.** Let X be a normal complex analytic space and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. If $a(E,X,\Delta) \geq -1$ (resp. > -1) holds for any proper bimeromorphic morphism $f\colon Y \to X$ from a normal complex analytic space Y and every f-exceptional divisor E, then (X,Δ) is called a *log canonical* (resp. *purely log terminal*) *pair*. If (X,Δ) is purely log terminal and $\lfloor \Delta\rfloor = 0$, then we say that (X,Δ) is a *kawamata log terminal pair*.

Let (X,Δ) be a log canonical pair. The image of E with $a(E,X,\Delta)=-1$ for some $f\colon Y\to X$ is called a *log canonical center* of (X,Δ). A closed subset S of X is called a *log canonical stratum* of (X,Δ) if S is an irreducible component of X or a log canonical center of (X,Δ).

Definition 2.3 (Non-lc loci)**.** Let X be a normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Then the *non-lc locus* of (X,Δ), denoted by $\mathrm{Nlc}(X,\Delta)$, is the smallest closed subset Z of X such that the complement $(X\setminus Z, \Delta|_{X\setminus Z})$ is log canonical.

Let us recall the definition of divisorial log terminal pairs in the complex analytic setting (see [Fuj12, Definition 3.7]). Note that [KM, Definition 2.37, Proposition 2.40, Theorem 2.44] is helpful.

Definition 2.4 (Divisorial log terminal pairs)**.** Let X be a normal complex analytic space and let Δ be a boundary $\mathbb{R}$-divisor on X such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. If there exists a proper bimeromorphic morphism $f\colon Y\to X$ from a smooth complex variety Y such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f)\cup \operatorname{Supp} f_*^{-1}\Delta$ are simple normal crossing divisors on Y and that the discrepancy coefficient $a(E,X,\Delta) > -1$ holds for every f-exceptional divisor E, then (X,Δ) is called a *divisorial log terminal pair*. We note that $\mathrm{Exc}(f)$ denotes the *exceptional locus* of f.

We note that Definitions 2.2 and 2.4 work for a finite disjoint union of normal complex varieties. In Definitions 2.2 and 2.4, X is not necessarily connected. It is well known that a divisorial log terminal pair is a log canonical pair.

Remark 2.5. If we shrink X to a relatively compact open subset of X in Definition 2.4, then we can assume that f is a composite of a finite sequence of blow-ups. In particular, f is projective. For the details, see [Fuj12, Lemma 3.9] and [BieM2].

Let us define semi-log canonical pairs and semi-divisorial log terminal pairs in the complex analytic setting.

Definition 2.6 (Semi-log canonical pairs and semi-divisorial log terminal pairs)**.** Let X be an equidimensional reduced complex analytic space that is normal crossing in codimension one and satisfies Serre's S_2 condition. Let Δ be an effective $\mathbb{R}$-divisor on X such that the singular locus of X does not contain any irreducible components of $\operatorname{Supp}\Delta$. In this situation, the pair (X, Δ) is called a *semi-log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and
(2) (X^ν, Θ) is log canonical, where $\nu\colon X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta := \nu^*(K_X + \Delta)$.

Let (X, Δ) be a semi-log canonical pair in the above sense. If each irreducible component of X is normal and (X^ν, Θ) is divisorial log terminal, then we say that (X, Δ) is a *semi-divisorial log terminal pair* (an *sdlt pair*, for short). Let S be a closed subset of X. We say that S is a *semi-log canonical stratum* of (X, Δ) if and only if S is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ). When (X, Δ) is log canonical, then a semi-log canonical stratum S is called a *log canonical stratum* of (X, Δ) (see Definition 2.2).

For various results on algebraic (resp. complex analytic) semi-log canonical pairs, see [Fuj10] (resp. [Fuj15]).

Remark 2.7. Note that the definition of semi-divisorial log terminal pairs in Definition 2.6 is different from [K, Definition 5.19]. Our definition is a direct analytic generalization of the one in [Fuj1] (see [Fuj1, Definition 1.1]).

The following lemma is well known when X is an algebraic variety. We state it here explicitly for the sake of completeness.

Lemma 2.8. *Let (X, Δ) be a divisorial log terminal pair. We put $S := \lfloor \Delta \rfloor$ and $K_S + \Delta_S := (K_X + \Delta)|_S$ by adjunction. Then (S, Δ_S) is semi-divisorial log terminal in the sense of Definition 2.6. More precisely, let $S = S_1 + \cdots + S_l$ be the irreducible decomposition. We put $T := S_1 + \cdots + S_l$ for some l with $1 \leq l \leq k$. Then T is Cohen–Macaulay and is simple normal crossing in codimension one. In particular, every irreducible component of S is normal. We put $K_{S_i} + \Delta_{S_i} := (K_X + \Delta)|_{S_i}$ by adjunction for every i. Then (S_i, Δ_{S_i}) is divisorial log terminal. Thus we see that (T, Δ_T), where $K_T + \Delta_T := (K_X + \Delta)|_T$ by adjunction, is semi-divisorial log terminal.*

Proof. By [RRV], we can apply the proof of [Fuj11, Theorem 3.13.6] to our setting with some suitable modifications (see also Remark 2.5). Then we obtain that T is Cohen–Macaulay. It is obvious that T is simple normal crossing in codimension one. Hence we can easily check all the other statements. $\square$

We will repeatedly use Lemma 2.9 in subsequent sections.

Lemma 2.9. *Let (X, Δ) be a log canonical pair such that $(X, \Delta - \lfloor \Delta \rfloor)$ is kawamata log terminal. We put $S := \lfloor \Delta \rfloor$ and $K_S + \Delta_S := (K_X + \Delta)|_S$ by adjunction. Then S is Cohen–Macaulay and is semi-log canonical.*

Proof. Since $(X, \Delta - S)$ is kawamata log terminal, X has only rational singularities. Therefore, X is Cohen–Macaulay. Since S is $\mathbb{Q}$-Cartier, $\mathcal{O}_X(-S)$ is Cohen–Macaulay. This implies that $\mathcal{O}_S$ is Cohen–Macaulay. For the details, see [KM, Corollary 5.25], [K, Corollaries 2.62, 2.63, and 2.88], and others. By adjunction, we see that (S, Δ_S) is semi-log canonical. $\square$

We need nef and log abundant divisors in Theorem 1.4.

Definition 2.10 (Nef and abundant line bundles)**.** Let $\pi \colon X \to Y$ be a projective surjective morphism from a normal complex variety X onto a complex variety Y. Let $\mathcal{L}$ be a π-nef line bundle on X. If $\kappa(F, \mathcal{L}|_F) = \nu(F, \mathcal{L}|_F)$ holds for analytically sufficiently general fibers F, then $\mathcal{L}$ is said to be *π-nef and π-abundant over Y*. Similarly, we can define π-nef and π-abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

Remark 2.11. In Definition 2.10, if $\mathcal{L}$ is π-semiample, then it is easy to see that $\mathcal{L}$ is π-nef and π-abundant over Y.

We will freely use the following elementary lemma.

Lemma 2.12. *Let $\pi \colon X \to Y$ be a projective surjective morphism from a normal complex variety X onto a complex variety Y and let $\mathcal{L}$ be a π-nef and π-abundant line bundle on X. Let $p \colon Z \to X$ be a projective surjective morphism from a normal complex variety Z. Then $p^*\mathcal{L}$ is $(\pi \circ p)$-nef and $(\pi \circ p)$-abundant over Y.*

Definition 2.13 (Nef and log abundant line bundles)**.** Let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces and let (X, Δ) be a semi-log canonical pair. Let $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is *π-nef and π-log abundant with respect to (X, Δ) over Y* if and only if $\mathcal{L}|_{S^\nu}$ is nef and abundant over $\pi(S)$ for every semi-log canonical stratum S of (X, Δ), where $\mathcal{L}|_{S^\nu}$ denotes the pull-back of $\mathcal{L}$ to the normalization of S. Similarly, we can define π-nef and π-log abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors with respect to (X, Δ).

For $\mathbb{R}$-Cartier $\mathbb{R}$-divisors, we need the following definitions. In the present paper, we will use $\mathbb{R}$-Cartier $\mathbb{R}$-divisors only in Theorems 1.5, 1.7, 1.8, Corollary 1.11, and Section 6.

Definition 2.14 (Relatively abundant $\mathbb{R}$-Cartier $\mathbb{R}$-divisors)**.** Let $\pi \colon X \to Y$ be a projective morphism from a normal complex variety X onto a complex variety Y. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. If $\kappa_\sigma(F, D|_F) = \kappa_\iota(F, D|_F)$ holds for analytically sufficiently general fibers F, then D is said to be π-abundant over Y.

For the details of κ_σ and κ_ι, see [N, Chapter V, §2] and [Fuj11, Section 2.5], respectively. For the details of abundant divisors, see also [EH2, Subsection 2.6. Abundant divisor].

Definition 2.15 (Nef and log abundant $\mathbb{R}$-Cartier $\mathbb{R}$-divisors)**.** Let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces and let (X, Δ) be a log canonical pair. Let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We say that D is *π-nef and π-log abundant with respect to (X, Δ) over Y* if and only if $D|_{S^\nu}$ is nef and abundant over $\pi(S)$ for every log canonical stratum of (X, Δ), where $D|_{S^\nu}$ denotes the pull-back of D to the normalization of S.

Remark 2.16. A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D is π-nef and π-log abundant with respect to (X, Δ) over Y in the sense of Definition 2.15 if and only if it is π-nef and π-log abundant with respect to (X, Δ) over Y in the sense of Definition 2.13.

Let us introduce the notion of B-bimeromorphic maps, which is obviously a generalization of the notion of B-birational maps.

Definition 2.17 (B-bimeromorphic maps)**.** Let $\pi\colon X \to Y$ and $\pi'\colon X' \to Y$ be projective morphisms of complex analytic spaces and let (X, Δ) and (X', Δ') be log canonical pairs. We say that a bimeromorphic map $f\colon X \dashrightarrow X'$ over Y is *B-bimeromorphic over Y* if there exists a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ X \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & Y & \end{tikzcd}$$

such that Z is a normal complex analytic space, α and α' are proper bimeromorphic morphisms, and

$$\alpha^*(K_X + \Delta) = \alpha'^*(K_{X'} + \Delta')$$

holds. Let m be a positive integer such that $m(K_X + \Delta)$ and $m(K_{X'} + \Delta')$ are Cartier. Then we have

$$\begin{aligned} f^*\colon \pi'_*\mathcal{O}_{X'}(m(K_{X'} + \Delta')) &\xrightarrow{\alpha'^*} \pi'_*\alpha'_*\mathcal{O}_Z(\alpha'^*(m(K_{X'} + \Delta'))) \\ &\simeq \pi_*\alpha_*\mathcal{O}_Z(\alpha^*(m(K_X + \Delta))) \xrightarrow{(\alpha^*)^{-1}} \pi_*\mathcal{O}_X(m(K_X + \Delta)). \end{aligned}$$

We put

$$\mathrm{Bim}(X/Y, \Delta) := \{f \mid f\colon (X, \Delta) \dashrightarrow (X, \Delta) \text{ is } B\text{-bimeromorphic over } Y\}.$$

Then it is obvious that $\mathrm{Bim}(X/Y, \Delta)$ has a natural group structure.

Let W be a compact subset of Y. Then we put

$$\mathrm{Bim}(X/Y, \Delta; W) := \left\{ g \;\middle|\; \begin{array}{l} g \in \mathrm{Bim}\left(\pi^{-1}(U_g)/U_g, \Delta|_{\pi^{-1}(U_g)}\right) \text{ such that} \\ U_g \text{ is an open neighborhood of } W \end{array} \right\}.$$

Note that $\mathrm{Bim}(X/Y, \Delta; W)$ also has a natural group structure.

We make small remarks on Definition 2.17.

Remark 2.18. If Y is a point in Definition 2.17, then (X, Δ) is a projective log canonical pair and $\mathrm{Bim}(X/Y, \Delta)$ is nothing but $\mathrm{Bir}(X, \Delta)$ in [Fuj1] and [FG].

Remark 2.19. In Definition 2.17, X and X' are not necessarily irreducible. In the proof of Theorem 1.2, we have to treat $\mathrm{Bir}(X, \Delta)$ in the case where X is a disjoint union of normal projective varieties.

Remark 2.20. Let $(X, \Delta) =: \bigsqcup_i (X_i, \Delta_i)$ and $(X', \Delta') =: \bigsqcup_i (X'_i, \Delta'_i)$ be the irreducible decompositions. Let $f\colon X \dashrightarrow X'$ be a B-bimeromrophic map over Y as in Definition 2.17. Then, there exists a permutation σ such that

$$f_i := f|_{X_i}\colon X_i \dashrightarrow X'_{\sigma(i)}$$

is a B-bimeromorphic map over Y between irreducible log canonical pairs (X_i, Δ_i) and $(X'_{\sigma(i)}, \Delta'_{\sigma(i)})$. We note that $\pi(X_i) = \pi'(X'_{\sigma(i)})$ holds for every i.

Remark 2.21 (see [FG, Remark 2.15])**.** Let (X, Δ) and (X', Δ') be log canonical pairs. Let $f\colon (X, \Delta) \dashrightarrow (X', \Delta')$ be a B-bimeromorphic map over Y as in Definition 2.17. We assume that $(X, \Delta - \lfloor \Delta \rfloor)$ and $(X', \Delta' - \lfloor \Delta' \rfloor)$ are kawamata log terminal. We put $S := \lfloor \Delta \rfloor$ and $S' := \lfloor \Delta' \rfloor$. By replacing Y with a relatively compact open subset, we may assume that Z in Definition 2.17 is smooth and

$$\alpha^*(K_X + \Delta) =: K_Z + \Delta_Z := \alpha'^*(K_{X'} + \Delta')$$

such that $\operatorname{Supp} \Delta_Z$ is a simple normal crossing divisor on Z. We may further assume that α and α' are projective in Definition 2.17. We put $K_S + \Delta_S := (K_X + \Delta)|_S$ and $K_{S'} + \Delta_{S'} := (K_{X'} + \Delta')|_{S'}$. By applying α_* and α'_* to

$$0 \to \mathcal{O}_Z(\lceil -(\Delta_Z^{<1}) \rceil - \Delta_Z^{=1}) \to \mathcal{O}_Z(\lceil -(\Delta_Z^{<1}) \rceil) \to \mathcal{O}_{\Delta_Z^{=1}}(\lceil -(\Delta_Z^{<1}) \rceil) \to 0,$$

we have $\alpha_* \mathcal{O}_{\Delta_Z^{=1}} \simeq \mathcal{O}_S$ and $\alpha'_* \mathcal{O}_{\Delta_Z^{=1}} \simeq \mathcal{O}_{S'}$. Here we used

$$R^1\alpha_* \mathcal{O}_Z(\lceil -(\Delta_Z^{<1}) \rceil - \Delta_Z^{=1}) = R^1\alpha'_* \mathcal{O}_Z(\lceil -(\Delta_Z^{<1}) \rceil - \Delta_Z^{=1}) = 0,$$

which is nothing but the relative Kawamata–Viehweg vanishing theorem. Thus f induces an isomorphism

$$(\alpha^*)^{-1} \circ (\alpha')^* \colon \pi'_* \mathcal{O}_{S'}(m(K_{S'} + \Delta_{S'})) \xrightarrow{\sim} \pi_* \mathcal{O}_S(m(K_S + \Delta_S)).$$

We note that f does not necessarily induce a bimeromorphic map $S \dashrightarrow S'$ in the above setting.

Let us introduce the notion of B-pluricanonical representations in the relative complex analytic setting.

Definition 2.22 (B-pluricanonical representations)**.** Let X be a normal complex analytic space such that (X, Δ) is log canonical and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. Then we have a group homomorphism

$$\rho_m \colon \operatorname{Bim}(X/Y, \Delta) \to \operatorname{Aut}_{\mathcal{O}_Y}\left(\pi_* \mathcal{O}_X(m(K_X + \Delta))\right)$$

given by $\rho_m(g) = g^*$ for $g \in \operatorname{Bim}(X/Y, \Delta)$. It is called the *$B$-pluricanonical representation* or *log pluricanonical representation* for (X, Δ) over Y. When Y is a point, we have

$$\rho_m \colon \operatorname{Bir}(X, \Delta) \to \operatorname{Aut}_{\mathbb{C}}\left(H^0(X, \mathcal{O}_X(m(K_X + \Delta)))\right).$$

Theorem 1.2 is a generalization of the following theorem, which is one of the main results of [FG]. We note that we need it in the proof of Theorem 1.2. In [HX], Hacon and Xu independently proved a slightly weaker theorem (see [HX, Theorem 1]), which seems to be insufficient for the purpose of the present paper.

Theorem 2.23 ([FG, Theorem 1.1])**.** *Let (X, Δ) be a projective log canonical pair. Suppose that $m(K_X + \Delta)$ is Cartier and that $K_X + \Delta$ is semiample. Then $\rho_m(\operatorname{Bir}(X, \Delta))$ is a finite group.*

In the proof of Theorem 1.2, Burnside's theorem plays a crucial role. Hence we state it explicitly for the sake of completeness. For the proof, see, for example, [CR, (36.1) Theorem].

Theorem 2.24 (Burnside)**.** *Let G be a subgroup of $\mathrm{GL}(n, \mathbb{C})$. If the order of any element g of G is uniformly bounded, then G is a finite group.*

In order to prove Theorem 1.1, we need the notion of *admissible* and *preadmissible* sections, which are first introduced in [Fuj1].

Definition 2.25 (Admissible and preadmissible sections, see [Fuj1, Definition 4.1])**.** Let (X, Δ) be a semi-divisorial log terminal pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let W be a compact subset of Y. Let $X = \bigcup_i X_i$ be the irreducible decomposition. As usual,

$$\nu\colon X^\nu = \bigsqcup_i X_i \to \bigcup_i X_i = X$$

is the normalization with

$$\nu^*(K_X + \Delta) = K_{X^\nu} + \Theta =: \bigsqcup_i (K_{X_i} + \Theta_i).$$

Let S be the disjoint union of $\lfloor \Theta_i \rfloor$'s. We put

$$K_S + \Delta_S := (K_{X^\nu} + \Theta)|_S.$$

Then, by adjunction, (S, Δ_S) is semi-divisorial log terminal. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. Let U be a semianalytic Stein open subset of Y with $U \subset W$. In particular, the number of the connected components of U is finite (see, for example, [BieM1, Corollary 2.7]). We put $X_U := \pi^{-1}(U)$ and $S_U := S \cap (\pi \circ \nu)^{-1}(U)$. Then we define *preadmissible* and *admissible* sections inductively as follows.

(1) $s \in H^0(X_U, \mathcal{O}_X(m(K_X + \Delta))) \simeq H^0(U, \pi_*\mathcal{O}_X(m(K_X + \Delta)))$ is *preadmissible* if the restriction $\nu^* s|_{S_U} \in H^0(S_U, \mathcal{O}_S(m(K_S + \Delta_S)))$ is admissible.

(2) $s \in H^0(X_U, \mathcal{O}_X(m(K_X + \Delta)))$ is *admissible* if s is preadmissible and $g^*(s|_{X_j}) = s|_{X_i}$ holds for every B-bimeromorphic map $g\colon (X_i, \Theta_i) \dashrightarrow (X_j, \Theta_j)$ defined over some open neighborhood U_g of W for every i, j.

Then we put

$$\begin{aligned} &\mathrm{PA}\,(X_U, \mathcal{O}_X(m(K_X + \Delta))) \\ &:= \{s \mid s \in H^0(X_U, \mathcal{O}_X(m(K_X + \Delta))) \text{ is preadmissible}\} \end{aligned}$$

and

$$\begin{aligned} &\mathrm{A}\,(X_U, \mathcal{O}_X(m(K_X + \Delta))) \\ &:= \{s \mid s \in H^0(X_U, \mathcal{O}_X(m(K_X + \Delta))) \text{ is admissible}\}. \end{aligned}$$

We note that if Z is any analytic subset defined over some open neighborhood of W then $U \cap Z$ is a semianalytic Stein open subset of Z contained in $W \cap Z$. Thus the number of the connected components of $U \cap Z$ is finite (see, for example, [BieM1, Corollary 2.7]).

Let U' be a semianalytic Stein open subset of Y such that $U' \subset U$. We put $X_{U'} := \pi^{-1}(U')$. Then there exist natural restriction maps

$$\mathrm{PA}\,(X_U, \mathcal{O}_X(m(K_X + \Delta))) \to \mathrm{PA}\,(X_{U'}, \mathcal{O}_X(m(K_X + \Delta)))$$

and

$$\mathrm{A}\,(X_U, \mathcal{O}_X(m(K_X + \Delta))) \to \mathrm{A}\,(X_{U'}, \mathcal{O}_X(m(K_X + \Delta)))\,.$$

Remark 2.26. In Definition 2.25, the natural map

$$H^0(X_U, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(U, \pi_*\mathcal{O}_X(m(K_X + \Delta)))$$

is an isomorphism of topological vector spaces since U is Stein (see, for example, [P, Lemma II.1]).

The following remark is almost obvious by definition. We state it explicitly for the sake of completeness.

Remark 2.27. In Definition 2.25, if

$$s \in \mathrm{A}\,(X_U, \mathcal{O}_X(m(K_X+\Delta))) \quad (\text{resp. } \mathrm{PA}\,(X_U, \mathcal{O}_X(m(K_X+\Delta))),$$

then

$$s^l \in \mathrm{A}\,(X_U, \mathcal{O}_X(lm(K_X+\Delta))) \quad (\text{resp. } \mathrm{PA}\,(X_U, \mathcal{O}_X(lm(K_X+\Delta)))$$

for every positive integer l. Moreover, if $\mathrm{A}\,(X_U, \mathcal{O}_X(m(K_X+\Delta)))$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over U, then $\mathrm{A}\,(X_U, \mathcal{O}_X(lm(K_X+\Delta)))$ generates $\mathcal{O}_X(lm(K_X+\Delta))$ over U for every positive integer l. Similarly, if $\mathrm{PA}\,(X_U, \mathcal{O}_X(m(K_X+\Delta)))$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over U, then $\mathrm{PA}\,(X_U, \mathcal{O}_X(lm(K_X+\Delta)))$ generates $\mathcal{O}_X(lm(K_X+\Delta))$ over U for every positive integer l.

The following remark is obvious by definition.

Remark 2.28. In Definition 2.25, if (X,Δ) is kawamata log terminal, then any section $s \in H^0(X_U, \mathcal{O}_X(m(K_X+\Delta)))$ is preadmissible by definition.

In our complex analytic setting, we can reformulate Claim (A_n) and Claim (B_n) in the proof of [Fuj1, Lemma 4.9] as follows. We note that (X,Δ_X) is *sub log canonical* when X is smooth and Δ_X is a subboundary $\mathbb{Q}$-divisor such that $\operatorname{Supp}\Delta_X$ is a simple normal crossing divisor. For sub log canonical pairs, we can define log canonical centers as in Definition 2.2.

Lemma 2.29. *Let $p\colon Z\to X$ be a projective bimeromorphic morphism of smooth complex varieties and let $\pi\colon X\to Y$ be a projective morphism of complex varieties. Let W be a compact subset of Y. Let Δ_Z (resp. Δ_X) be a subboundary $\mathbb{Q}$-divisor on Z (resp. X) such that* $\operatorname{Supp}\Delta_Z$ *(resp.* $\operatorname{Supp}\Delta_X$*) is a simple normal crossing divisor on Z (resp. X). We assume that $K_Z+\Delta_Z=p^*(K_X+\Delta_X)$. Let m be a positive integer such that $m(K_X+\Delta_X)$ is Cartier.Then the following statements hold over some open neighborhood of W.*

(i) *If T is a log canonical center of (X,Δ_X), then there exists a log canonical center S of (Z,Δ_Z) such that $p\colon S\to T$ is bimeromorphic. In particular,*
$$p_*\mathcal{O}_S(m(K_S+\Delta_S))\simeq \mathcal{O}_T(m(K_T+\Delta_T)),$$
where $K_S+\Delta_S:=(K_Z+\Delta_Z)|_S$ and $K_T+\Delta_T:=(K_X+\Delta_X)|_T$ by adjunction.

(ii) *If S is a log canonical center of (Z,Δ_Z) such that $p\colon S\to \pi(S)=:T$ is not bimeromorphic, then there exists a log canonical center S' of (Z,Δ_Z) with $S'\subset S$ such that $p\colon S'\to T$ is bimeromorphic and the restriction map*
$$p_*\mathcal{O}_S(m(K_S+\Delta_S))\to p_*\mathcal{O}_{S'}(m(K_{S'}+\Delta_{S'})),$$
induced by the inclusion $S'\hookrightarrow S$ and adjunction, is an isomorphism, where $K_{S'}+\Delta_{S'}:=(K_Z+\Delta_Z)|_{S'}$ by adjunction. We note that
$$p_*\mathcal{O}_{S'}(m(K_{S'}+\Delta_{S'}))\simeq \mathcal{O}_T(m(K_T+\Delta_T))$$
obviously holds.

Sketch of Proof of Lemma 2.29. With suitable modifications, the proofs of Claims (A_n) and (B_n) in the proof of [Fuj1, Lemma 4.9] also work in our setting (see also [Fuj2, Lemma 7.2]). Therefore, we only sketch the argument here.

We can verify (i) by induction on the dimension of X. To prove (ii), by blowing up Z along the center S, we may assume that S is a divisor on Z. Using (i), we can reduce the problem to the case where $p\colon Z\to X$ is a finite composition of blow-ups with centers corresponding to S. The statement then follows by a direct check. $\square$

We will freely use Lemma 2.29 in subsequent sections.

3. Finiteness of relative log pluricanonical representations

In this section, we prove Theorem 1.2 and Corollary 1.3. We note that the proof of Theorem 1.2 relies on Theorem 2.23. We begin with an elementary lemma.

Lemma 3.1. *Let Y be a complex manifold, which is connected. Let*

$$\rho\colon G \to \mathrm{GL}(r, \mathcal{O}_Y)$$

be a group homomorphism. We further consider

$$\rho_y := \mathrm{ev}_y \circ \rho\colon G \to \mathrm{GL}(r, \mathcal{O}_Y) \to \mathrm{GL}(r, \mathbb{C}),$$

where ev_y is the evaluation map at $y \in Y$. We assume that $\operatorname{Im} \rho_y = \rho_y(G)$ is a finite group for every $y \in Y$. Then $\mathrm{ev}_y\colon \rho(G) \to \rho_y(G)$ is an isomorphism for every $y \in Y$. In particular, $\operatorname{Im} \rho = \rho(G)$ is a finite group.

Proof. It is obvious that $\mathrm{ev}_y\colon \rho(G) \to \rho_y(G)$ is surjective for every $y \in Y$. We take an arbitrary point $y_0 \in Y$. It is sufficient to prove that $\mathrm{ev}_{y_0}\colon \rho(G) \to \rho_{y_0}(G)$ is injective. We take $g \in \rho(G)$ such that $\mathrm{ev}_{y_0}(g) = E_r$, where E_r is the $r \times r$ identity matrix. Note that $\mathrm{ev}_y(g)$ is semisimple and every eigenvalue of $\mathrm{ev}_y(g)$ is a root of unity for every $y \in Y$ since $\rho_y(G)$ is a finite group by assumption. We consider the characteristic polynomial $\chi(t) := \det(tE_r - g)$. The coefficients of $\chi(t)$ are holomorphic and take values in K, where K is the subfield of $\mathbb{C}$ generated by all roots of unity. Hence they are constant. Since $\mathrm{ev}_{y_0}(g) = E_r$, we see that every eigenvalue of $\mathrm{ev}_y(g)$ is 1 for every $y \in Y$. This implies that $\mathrm{ev}_y(g) = E_r$ holds for every $y \in Y$ because $\mathrm{ev}_y(g)$ is semisimple. Hence we have $g = E_r$, that is, $\mathrm{ev}_{y_0}\colon \rho(G) \to \rho_{y_0}(G)$ is injective. We finish the proof. □

Theorem 3.2 is one of the most important results in the present paper.

Theorem 3.2. *Let $\pi\colon X \to Y$ be a projective morphism from a normal complex analytic space X onto a polydisc Y such that (X, Δ) is divisorial log terminal and that $K_X + \Delta$ is π-semiample. Let $\varphi\colon Z \to X$ be a projective bimeromorphic morphism from a smooth complex analytic space Z with $K_Z + \Delta_Z := \varphi^*(K_X + \Delta)$ such that $\pi \circ \varphi\colon Z \to Y$ is smooth and projective and that $\operatorname{Supp} \Delta_Z$ is a simple normal crossing divisor on Z and is relatively normal crossing over Y. Let m be a positive integer such that $m(K_X + \Delta)$ is Cartier. We assume that $R^i\pi_*\mathcal{O}_X(m(K_X + \Delta))$ is locally free for every i and $\pi_*\mathcal{O}_X(m(K_X + \Delta)) \simeq \mathcal{O}_Y^{\oplus r}$ for some positive integer r. We consider*

$$\rho_m\colon \operatorname{Bim}(X/Y, \Delta) \to \mathrm{GL}(r, \mathcal{O}_Y) \simeq \operatorname{Aut}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X(m(K_X + \Delta)))$$

and

$$\rho_{m,y} := \mathrm{ev}_y \circ \rho_m\colon \operatorname{Bim}(X/Y, \Delta) \to \mathrm{GL}(r, \mathcal{O}_Y) \to \mathrm{GL}(r, \mathbb{C}),$$

where ev_y is the evaluation map at $y \in Y$. Then $\operatorname{Im} \rho_{m,y}$ is a finite group for every $y \in Y$. Moreover,

$$\mathrm{ev}_y\colon \operatorname{Im} \rho_m \to \operatorname{Im} \rho_{m,y}$$

is an isomorphism for every $y \in Y$. In particular, $\operatorname{Im} \rho_m$ is a finite group.

We note that, in the above setting, $X_y := \pi^{-1}(y)$ is a normal projective scheme, (X_y, Δ_y) is divisorial log terminal, where $K_{X_y} + \Delta_y := (K_X + \Delta)|_{X_y}$, and

$$\mathrm{ev}_y\colon \pi_*\mathcal{O}_X(m(K_X + \Delta)) \to H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \tag{3.1}$$

by the base change theorem.

Remark 3.3. In Theorem 3.2, X is not necessarily connected.

Let us prove Theorem 3.2.

Proof of Theorem 3.2. By Lemma 3.1, it is sufficient to prove the finiteness of $\operatorname{Im}\rho_{m,y}$ for every $y \in Y$. In Step 1, we will prove the description (3.1) of the evaluation map ev_y. Then, in Step 2, we will prove the finiteness of $\operatorname{Im}\rho_{m,y}$.

Step 1. We put $d := \dim Y$. We note that Y is a polydisc by assumption. We take general hyperplanes $H_1, \cdots, H_d$ on Y passing through y. Then $\left(X, \Delta + \sum_{i=1}^d \pi^* H_i\right)$ is a divisorial log terminal pair. We note that $(\pi \circ \varphi)^* \left(\sum_{i=1}^d H_i\right)$ and $\operatorname{Supp}\left(\Delta_Z + (\pi \circ \varphi)^* \left(\sum_{i=1}^d H_i\right)\right)$ are simple normal crossing divisors on Z. By construction, X_y is a log canonical center of $\left(X, \Delta + \sum_{i=1}^d \pi^* H_i\right)$. This implies that X_y is normal and (X_y, Δ_y) is divisorial log terminal. Since $R^i\pi_*\mathcal{O}_X(m(K_X + \Delta))$ is locally free for every i by assumption, we have

$$\pi_*\mathcal{O}_X(m(K_X + \Delta)) \otimes \mathbb{C}(y) \simeq H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y)))$$

by the base change theorem. Hence we have the desired description (3.1) of the evaluation map ev_y.

Step 2. We take an arbitrary element g of $\operatorname{Bim}(X/Y, \Delta)$. By Theorem 2.24, it is sufficient to prove that the order of $\rho_{m,y}(g) = \mathrm{ev}_y \circ \rho_m(g)$ is uniformly bounded. We make H_1 general in Step 1 and put $Y' := H_1$, $X' := \pi^* H_1$, and $K_{X'} + \Delta' := (K_X + X' + \Delta)|_{X'}$. Then the above g induces $g' \in \operatorname{Bim}(X'/Y', \Delta')$ such that $\mathrm{ev}_y \circ \rho_m(g) = \mathrm{ev}_y \circ \rho'_m(g')$ holds, where

$$\rho'_m \colon \operatorname{Bim}(X'/Y', \Delta') \to \operatorname{Aut}_{\mathcal{O}_{Y'}}\left(\pi_*\mathcal{O}_{X'}(m(K_{X'} + \Delta'))\right).$$

By repeating this process finitely many times, we may assume that Y is a disc. Hence X_y is a divisor on X.

We first assume that X_y is connected. Let l be the number of the log canonical strata of (X_y, Δ_y). We consider

$$\rho_m \colon \operatorname{Bir}(V, \Delta_V) \to \operatorname{Aut}_{\mathbb{C}}\left(H^0(V, \mathcal{O}_V(m(K_V + \Delta_V)))\right),$$

where (V, Δ_V) is a log canonical stratum of (X_y, Δ_y). Since $K_V + \Delta_V$ is semiample, $\rho_m\left(\operatorname{Bir}(V, \Delta_V)\right)$ is a finite group by Theorem 2.23. Then we put

$$k := \operatorname{lcm}\left\{\#\rho_m\left(\operatorname{Bir}(V, \Delta_V)\right) \mid (V, \Delta_V) \text{ is a log canonical stratum of } (X_y, \Delta_y)\right\}$$

Claim. $\rho_{m,y}(g)^{l!k} = E_r$ holds.

Proof of Claim. We consider log canonical strata (T, Δ_T) of (X_y, Δ_y) satisfying that the natural restriction map

$$H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \to H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \tag{3.2}$$

is an isomorphism. We put $t := \min \dim T$.

Let (T, Δ_T) be a t-dimensional log canonical stratum of (X_y, Δ_y) such that the natural restriction map (3.2) is an isomorphism. We consider the following commutative diagram as in Definition 2.17

$$\begin{tikzcd} & X^\dagger \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

where g is a B-bimeromorphic map of (X, Δ) over Y taken above. By shrinking Y around y, we may assume that $X^\dagger$ is smooth, α and β are projective, and

$$\alpha^*(K_X + \Delta) =: K_{X^\dagger} + \Delta_{X^\dagger} := \beta^*(K_X + \Delta)$$

such that $\operatorname{Supp} \Delta_{X^\dagger} \cup \operatorname{Supp}(\pi \circ \alpha)^* y$ is a simple normal crossing divisor on $X^\dagger$. We take $X^\dagger$ suitably. Then, by Lemma 2.29 (see also the proof of [Fuj1, Lemma 4.9] and [FG, Lemma 2.16]), we can find a log canonical stratum $(T', \Delta_{T'})$ of (X_y, Δ_y) and a commutative diagram

$$\begin{tikzcd} & T^\dagger \arrow[dl, "\alpha|_{T^\dagger}"'] \arrow[dr, "\beta|_{T^\dagger}"] & \\ (T, \Delta_T) & & (T', \Delta_{T'}) \end{tikzcd}$$

such that $\alpha|_{T^\dagger}$ and $\beta|_{T^\dagger}$ are proper birational and that

$$(\beta|_{T^\dagger}) \circ (\alpha|_{T^\dagger})^{-1} \colon (T, \Delta_T) \dashrightarrow (T', \Delta_{T'})$$

is a B-birational map of projective divisorial log terminal pairs. Note that there are only finitely many log canonical strata contained in X_y. Thus we can find t-dimensional log canonical strata (S_i, Δ_{S_i}) of (X_y, Δ_y) for $1 \leq i \leq p$ and a natural embedding

$$H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \hookrightarrow \bigoplus_i H^0(S_i, \mathcal{O}_{S_i}(m(K_{S_i} + \Delta_{S_i})))$$

such that g induces $\widetilde{g} \in \operatorname{Bir}(S, \Delta_S)$, where $(S, \Delta_S) := \bigsqcup_i (S_i, \Delta_{S_i})$, satisfying the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \arrow[r] \arrow[d, "\rho_{m,y}(g)"'] & \bigoplus_i H^0(S_i, \mathcal{O}_{S_i}(m(K_{S_i} + \Delta_{S_i}))) \arrow[d, "\rho_m(\widetilde{g})"] \\ 0 \arrow[r] & H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \arrow[r] & \bigoplus_i H^0(S_i, \mathcal{O}_{S_i}(m(K_{S_i} + \Delta_{S_i}))). \end{tikzcd}$$

We note the following description of $\rho_{m,y}(g)$. Let V be the union of the irreducible components of $(\Delta_{X^\dagger} + (\pi \circ \alpha)^* y)^{=1}$ mapped to y. We put

$$K_V + \Delta_V := (K_{X^\dagger} + \Delta_{X^\dagger} + (\pi \circ \alpha)^* y)|_V.$$

Then we can check that $\alpha_* \mathcal{O}_V \simeq \mathcal{O}_{X_y} \simeq \beta_* \mathcal{O}_V$ holds, which is an easy consequence of the strict support condition established in [Fuj13, Theorem 1.1 (i)] (see, for example, the proof of Lemma 4.2 below). Thus $\rho_{m,y}(g)$ can be written as

$$\begin{aligned} \rho_{m,y} \colon H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) &\xrightarrow{\beta^*} H^0(V, \mathcal{O}_V(m(K_V + \Delta_V))) \\ &\xrightarrow{(\alpha^*)^{-1}} H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))). \end{aligned}$$

Since $\rho_m(\widetilde{g})^{l!k} = \mathrm{id}$ on $\bigoplus_i H^0(S_i, \mathcal{O}_{S_i}(m(K_{S_i} + \Delta_{S_i})))$ by the definitions of l and k, we have $\rho_{m,y}(g)^{l!k} = E_r$. This is what we wanted. $\square$

We note that $l!k$ is independent of g. Therefore, Claim implies that $\operatorname{Im}\rho_{m,y}$, which is a subgroup of $\mathrm{GL}(r, \mathbb{C})$, is a finite group by Burnside's theorem (see Theorem 2.24). Thus we finish the proof under the assumption that X_y is connected.

From now, we assume that X_y is not connected. Let a denote the number of the connected components of X_y. Then $g^{a!}$ preserves each connected component of X_y. Thus, by the above argument, we can take a positive integer b such that $\rho_{m,y}(g)^b = E_r$ holds for every $g \in \mathrm{Bim}(X/Y, \Delta)$. Thus, by Burnside's theorem (see Theorem 2.24), we see that $\operatorname{Im}\rho_{m,y}$ is a finite group.

We finish the proof. $\square$

We can prove Theorem 1.2 as an easy application of Theorem 3.2.

Proof of Theorem 1.2. Let U be a nonempty open subset of Y. We consider the following commutative diagram

$$\begin{tikzcd} \rho_m\colon \mathrm{Bim}(X/Y, \Delta) \arrow[r] \arrow[d] & \mathrm{Aut}_{\mathcal{O}_Y}\left(\pi_*\mathcal{O}_X(m(K_X + \Delta))\right) \arrow[d] \\ \rho_m\colon \mathrm{Bim}(\pi^{-1}(U)/U, \Delta|_{\pi^{-1}(U)}) \arrow[r] & \mathrm{Aut}_{\mathcal{O}_U}\left(\pi_*\mathcal{O}_{\pi^{-1}(U)}(m(K_X + \Delta))\right). \end{tikzcd}$$

Note that the vertical arrows are natural restriction maps. It is obvious that the restriction map

$$\mathrm{Aut}_{\mathcal{O}_Y}\left(\pi_*\mathcal{O}_X(m(K_X + \Delta))\right) \to \mathrm{Aut}_{\mathcal{O}_U}\left(\pi_*\mathcal{O}_{\pi^{-1}(U)}(m(K_X + \Delta))\right)$$

is injective since Y is irreducible and every irreducible component of X is dominant onto Y. Hence, in order to prove Theorem 1.2, we can freely replace Y with a small nonempty open subset of Y. We take a Stein compact subset W of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Then, by [Fuj12, Theorems 1.21 and 1.27], we can take a dlt blow-up $\psi\colon (X', \Delta') \to (X, \Delta)$. By replacing $\pi\colon (X, \Delta) \to Y$ with $\pi' := \pi \circ \psi\colon (X', \Delta') \to Y$, we may further assume that (X, Δ) is divisorial log terminal. By taking a resolution of singularities of X (see, for example, [BieM2]) and shrinking Y suitably, we may assume that $\pi\colon (X, \Delta) \to Y$ satisfies all the conditions in Theorem 3.2. Then, by Theorem 3.2, $\rho_m\left(\mathrm{Bim}(X/Y, \Delta)\right)$ is a finite group. This is what we wanted. We finish the proof. $\square$

Let us prove Corollary 1.3, which is almost obvious by Theorem 1.2. We will use it in the proof of Theorem 1.1.

Proof of Corollary 1.3. We decompose $(X, \Delta) =: \bigsqcup_i (X_i, \Delta_i)$ such that $\pi_i := \pi|_{X_i}\colon X_i \to Y_i := \pi(X_i)$ is surjective and every irreducible component of X_i is dominant onto Y_i for every i. We may assume that $Y_i \neq Y_j$ for $i \neq j$. Since U is a semianalytic Stein open subset of Y with $U \subset W$, $Y_i \cap U$ is a finite disjoint union of semianalytic Stein open subsets of Y_i (see, for example, [BieM1, Corollary 2.7]). Let U' be a connected component of $Y_i \cap U$. Then, by Theorem 1.3, the image of

$$\rho_m\colon \mathrm{Bim}\left(\pi_i^{-1}(U')/U', \Delta_i|_{\pi_i^{-1}(U')}\right) \to \mathrm{Aut}_{\mathcal{O}_{U'}}\left(\pi_{i*}\mathcal{O}_{\pi_i^{-1}(U')}(m(K_{X_i} + \Delta_i))\right) \tag{3.3}$$

is a finite group. Note that there exists a natural restriction map

$$\mathrm{Bim}(X/Y, \Delta; W) \to \mathrm{Bim}\left(\pi_i^{-1}(U')/U', \Delta_i|_{\pi_i^{-1}(U')}\right). \tag{3.4}$$

By the natural restriction map (3.4),

$$\rho_m^{WU'}\colon \operatorname{Bim}(X/Y,\Delta;W)\to \operatorname{Aut}_{\mathcal{O}_{U'}}\left(\pi_{i*}\mathcal{O}_{\pi_i^{-1}(U')}(m(K_{X_i}+\Delta_i))\right)$$

factors through ρ_m in (3.3). Thus, we have

$$\rho_m^{WU'}(\operatorname{Bim}(X/Y,\Delta;W))\subset \rho_m\left(\operatorname{Bim}\left(\pi_i^{-1}(U')/U',\Delta_i|_{\pi_i^{-1}(U')}\right)\right).$$

Since $\rho_m^{WU}(\operatorname{Bim}(X/Y,\Delta;W))$ is contained in

$$\prod_{U'}\rho_m^{WU'}(\operatorname{Bim}(X/Y,\Delta;W)),$$

where U' runs over all connected components of $Y_i\cap U$ for all i. Hence we see that $\rho_m^{WU}\left(\operatorname{Bim}(X/Y,\Delta;W)\right)$ is a finite group. We finish the proof. □

4. Abundance for semi-log canonical pairs

In this section, we prove Theorem 1.1. Our strategy follows that of [Fuj1], except that we make use of the minimal model program for projective morphisms between complex analytic spaces established in [Fuj12], [EH1], and [EH2].

The following lemma is well known and follows easily from the relative Kawamata–Viehweg vanishing theorem.

Lemma 4.1 (Connectedness lemma). *Let (X,Δ) be a log canonical pair and let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces with $\pi_*\mathcal{O}_X\simeq\mathcal{O}_Y$. Assume that $-(K_X+\Delta)$ is π-nef and π-big. Then $\operatorname{Nklt}(X,\Delta)\cap\pi^{-1}(y)$ is connected for every $y\in Y$, where $\operatorname{Nklt}(X,\Delta)$ denotes the non-kawamata log terminal locus of (X,Δ). In particular, if $(X,\Delta-\lfloor\Delta\rfloor)$ is kawamata log terminal, $\lfloor\Delta\rfloor\cap\pi^{-1}(y)$ is connected for every $y\in Y$.*

Proof. The usual proof in the algebraic setting can work with only some suitable modifications. This is because the Kawamata–Viehweg vanishing theorem holds for projective morphisms between complex analytic spaces. In this proof, we can freely shrink Y around y. We consider the following short exact sequence:

$$0\to\mathcal{J}(X,\Delta)\to\mathcal{O}_X\to\mathcal{O}_{\operatorname{Nklt}(X,\Delta)}\to 0,$$

where $\mathcal{J}(X,\Delta)$ denotes the multiplier ideal sheaf of (X,Δ). By the relative Kawamata–Viehweg–Nadel vanishing theorem, we have

$$0\to\pi_*\mathcal{J}(X,\Delta)\to\mathcal{O}_Y\to\pi_*\mathcal{O}_{\operatorname{Nklt}(X,\Delta)}\to 0.$$

This implies that $\operatorname{Nklt}(X,\Delta)\cap\pi^{-1}(y)$ is connected. □

The following lemma also asserts that the union of log canonical centers is connected under a suitable setting. Lemma 4.2 is substantially more difficult than Lemma 4.1. Its proof relies heavily on the strict support condition established in [Fuj13, Theorem 1.1 (i)] (see also [Fuj17] and [FF]).

Lemma 4.2. *Let (X,Δ) be a log canonical pair and let $\pi\colon X\to Y$ be a projective morphism of normal complex varieties with $\pi_*\mathcal{O}_X\simeq\mathcal{O}_Y$. Let W be a compact subset of Y. We assume that $K_X+\Delta\sim_{\mathbb{Q},\pi}0$ holds. We put $Y':=\bigcup_i\pi(C_i)\subsetneq Y$, where $\{C_i\}$ is a set of some log canonical centers of (X,Δ). Let X' be the union of the log canonical centers of (X,Δ) mapped to Y' by π. Then, after shrinking Y around W suitably, $\pi_*\mathcal{O}_{X'}\simeq\mathcal{O}_{Y'}$ holds. In particular, $\pi_*\mathcal{O}_{X'}\simeq\mathcal{O}_{Y'}$ holds on an open subset U contained in W.*

Proof. Throughout this proof, we will freely shrink Y around W without mentioning it explicitly. Let $p\colon Z\to X$ be a projective bimeromorphic morphism from a smooth complex variety Z with $K_Z+\Delta_Z:=p^*(K_X+\Delta)$ (see [BieM2]). We may assume that $(\pi\circ p)^{-1}(Y')$ and $p^{-1}(X')$ are simple normal crossing divisors on Z. We may further assume that the union of $(\pi\circ p)^{-1}(Y')$, $p^{-1}(X')$, and $\operatorname{Supp}\Delta_Z$ is contained in a simple normal crossing divisor on Z. Let V be the union of the irreducible components of $\Delta_Z^{=1}$ mapped to Y' by $\pi\circ p$. We put $A:=\lceil -(\Delta_Z^{<1})\rceil$, which is a p-exceptional effective divisor on Z. By assumption, we have

$$A-V-(K_Z+\Delta_Z^{=1}-V+\{\Delta_Z\})\sim_{\mathbb{Q},\pi\circ p}0.$$

We consider the following short exact sequence

$$0\to\mathcal{O}_Z(A-V)\to\mathcal{O}_Z(A)\to\mathcal{O}_V(A)\to 0.$$

We note that no log canonical centers of $(Z,\Delta_Z^{=1}-V+\{\Delta_Z\})$ map to Y' by construction. Then we have

$$0\to(\pi\circ p)_*\mathcal{O}_Z(A-V)\to\mathcal{O}_Y\to(\pi\circ p)_*\mathcal{O}_V(A)\to 0.$$

Here we used the strict support condition for $R^1(\pi\circ p)_*\mathcal{O}_Z(A-V)$ (see [Fuj13, Theorem 1.1 (i)]) in order to prove the connecting homomorphism

$$\delta\colon(\pi\circ p)_*\mathcal{O}_V(A)\to R^1(\pi\circ p)_*\mathcal{O}_Z(A-V)$$

is zero. This implies that $(\pi\circ p)_*\mathcal{O}_V(A)\simeq\mathcal{O}_{Y'}$ holds. Similarly, we have the short exact sequence

$$0\to p_*\mathcal{O}_Z(A-V)\to\mathcal{O}_X\to p_*\mathcal{O}_V(A)\to 0$$

since no log canonical centers of $(Z,\Delta_Z^{=1}-V+\{\Delta_Z\})$ map to X' by construction. This implies that $p_*\mathcal{O}_V(A)\simeq\mathcal{O}_{X'}$. Hence we have $\pi_*\mathcal{O}_{X'}\simeq\mathcal{O}_{Y'}$. We finish the proof of Lemma 4.2. □

As a straightforward corollary of Lemma 4.2, we obtain the following:

Corollary 4.3 ([Fuj1, Lemma 4.2])**.** *Let (X,Δ) be a divisorial log terminal pair and let $\pi\colon X\to Y$ be a projective morphism of normal complex varieties with $\pi_*\mathcal{O}_X\simeq\mathcal{O}_Y$. Let W be a compact subset of Y. We assume that $K_X+\Delta\sim_{\mathbb{Q},\pi}0$ holds. If $Y':=\pi(\lfloor\Delta\rfloor)\subsetneq Y$, then, after shrinking Y around W suitably, we have $\pi_*\mathcal{O}_{\lfloor\Delta\rfloor}\simeq\mathcal{O}_{Y'}$.*

Proof. Since (X,Δ) is divisorial log terminal, $\lfloor\Delta\rfloor$ is the union of all log canonical centers of (X,Δ). Therefore, by Lemma 4.2, we have $\pi_*\mathcal{O}_{\lfloor\Delta\rfloor}\simeq\mathcal{O}_{Y'}$. □

The following lemma, which serves as a toy model for Lemma 4.5 and Proposition 4.6 below, is sufficient for the purposes of [Fuj2], [Fuj5], [Fuj8], and [G2]. Therefore, we do not need to address any subtle issues when $K_X+\Delta$ is numerically trivial.

Lemma 4.4. *Let (X,Δ) be a projective $\mathbb{Q}$-factorial divisorial log terminal pair such that $K_X+\Delta\sim_{\mathbb{Q}}0$. Assume that $\lfloor\Delta\rfloor$ is not connected. Then $\lfloor\Delta\rfloor=S_1+S_2$ such that (S_i,Δ_{S_i}) is kawamata log terminal with $K_{S_i}+\Delta_{S_i}:=(K_X+\Delta)|_{S_i}$ for $i=1,2$ and that (S_1,Δ_1) is B-birationally equivalent to (S_2,Δ_{S_2}). In particular, (X,Δ) is purely log terminal.*

Proof. Note that $K_X+\Delta-\varepsilon\lfloor\Delta\rfloor$ is not pseudo-effective for a small positive rational number ε. By running a $(K_X+\Delta-\varepsilon\lfloor\Delta\rfloor)$-minimal model program with ample scaling (see [BCHM]), we finally get an extremal Fano contraction morphism, which is generically a $\mathbb{P}^1$-bundle with two disjoint sections. More precisely, we have

$$\begin{tikzcd} p\colon X \arrow[r, dashed] & X' \arrow[d, "\varphi"] \\ & V \end{tikzcd}$$

where $p\colon X \dashrightarrow X'$ is a finite sequence of flips and divisorial contractions and $\varphi\colon X' \to V$ is a $(K_{X'} + \Delta' - \varepsilon\lfloor\Delta'\rfloor)$-negative extremal Fano contraction with $\dim V = \dim X - 1$, where $\Delta' := p_*\Delta$. We know that the number of connected components of $\lfloor\Delta\rfloor$ is preserved by the above minimal model program, as we apply Lemma 4.1 at each step. Hence we obtain that $\lfloor\Delta'\rfloor = S'_1 + S'_2$, $\varphi\colon S'_i \to V$ is an isomorphism for $i = 1, 2$, and $S'_1 \cap S'_2 = \emptyset$. By using [AFKM, 12.3.4 Theorem], we can check that $\varphi\colon (S'_i, \Delta_{S'_i}) \to (V, P)$ is a B-birational isomorphism for some effective $\mathbb{Q}$-divisor P on V, where $K_{S'_i} + \Delta_{S'_i} := (K_{X'} + \Delta')|_{S'_i}$. Then, by Lemma 4.2, we see that there are no log canonical centers except $\lfloor\Delta'\rfloor$. This implies that (X', Δ') is purely log terminal. Hence, (X, Δ) is purely log terminal and (S_1, Δ_{S_1}) is B-birationally equivalent to (S_2, Δ_{S_2}). This is what we wanted. □

The following lemma is crucial.

Lemma 4.5. *Let (X', Δ') be a log canonical pair and let $\pi'\colon X' \to Y$ be a projective surjective morphism of normal complex varieties. Let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Assume that X' is $\mathbb{Q}$-factorial over W. Let $f'\colon X' \to Z$ be a projective surjective morphism of normal complex varieties over Y such that $K_{X'} + \Delta' \sim_{\mathbb{Q}, f'} 0$, and $\pi_Z\colon Z \to Y$ is projective, where π_Z is the structure morphism. Assume that $(X', \Delta' - \varepsilon\lfloor\Delta'\rfloor)$ is kawamata log terminal for some small positive rational number ε and there exists a $(K_{X'} + \Delta' - \varepsilon\lfloor\Delta'\rfloor)$-negative extremal Fano contraction $\varphi := \varphi_R\colon X' \to V$ over Z associated to an extremal ray R of $\overline{\mathrm{NE}}(X'/Z; \pi_Z^{-1}(W))$ with $\dim V = \dim X' - 1$. Note that V is $\mathbb{Q}$-factorial over W and has only kawamata log terminal singularities.*

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] \arrow[dr, "f'"] & V \arrow[d] \\ Y & Z \arrow[l, "\pi_Z"] \end{tikzcd}$$

Then the horizontal part $(\Delta')^h$ of $\lfloor\Delta'\rfloor$ with respect to φ satisfies one of the following conditions.

(I) $(\Delta')^h = D'_1$, *which is irreducible, and* $\deg[D'_1 : V] = 1$.
(II) $(\Delta')^h = D'_1 + D'_2$ *such that D'_i is irreducible and* $\deg[D'_i : V] = 1$ *for $i = 1, 2$.*
(III) $(\Delta')^h = D'_1$, *which is irreducible, and* $\deg[D'_1 : V] = 2$.

We define $\Delta_{D'_i}$ by

$$K_{D'_i} + \Delta_{D'_i} = (K_{X'} + \Delta')|_{D'_i}$$

for $i = 1, 2$. Let $\nu_i\colon D'^{\nu}_i \to D'_i$ be the normalization for $i = 1, 2$. We put

$$K_{D'^{\nu}_i} + \Delta_{D'^{\nu}_i} := \nu_i^*(K_{D'_i} + \Delta_{D'_i})$$

for $i = 1, 2$. After shrinking Y around W suitably, we have the following statements.

Case (I)**.** $\lfloor\Delta'\rfloor \cap \varphi^{-1}(v)$ is connected for every $v \in V$.

Case (II)**.** The number of the connected components of $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is at most two for every $v \in V$ and

$$(\varphi \circ \nu_2)^{-1} \circ (\varphi \circ \nu_1) : (D_1'^{\nu}, \Delta_{D_1'^{\nu}}) \dashrightarrow (D_2'^{\nu}, \Delta_{D_2'^{\nu}})$$

is a B-bimeromorphic map over V.

Case (III)**.** The number of the connected components of $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is at most two for every $v \in V$ and there exists a B-bimeromorphic map

$$\iota \colon (D_1'^{\nu}, \Delta_{D_1'^{\nu}}) \dashrightarrow (D_1'^{\nu}, \Delta_{D_1'^{\nu}})$$

over V with $\iota \neq \mathrm{id}$ and $\iota^2 = \mathrm{id}$.

Moreover, in (II) *and* (III)*, if* $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ *is not connected for some* $v \in V$*, then* (X', Δ') *is purely log terminal in a neighborhood of* $\varphi^{-1}(v)$*.*

More details on Cases (II) and (III) will be discussed in the following proof.

Proof of Theorem 4.5. We have $R^i\varphi_*\mathcal{O}_{X'} = 0$ by the relative Kawamata–Viehweg vanishing theorem. Therefore, we see that general fibers of $\varphi \colon X' \to V$ are $\mathbb{P}^1$. Hence the mapping degree of $(\Delta')^h$, the horizontal part of $\lfloor \Delta' \rfloor$, is at most two. Therefore, we have (I), (II), and (III).

In Case (I), $(\Delta')^h = D_1'$ is irreducible and φ-ample. Since φ is an extremal Fano contraction, the vertical part of $\lfloor \Delta' \rfloor$ is the pull-back of some effective $\mathbb{Q}$-divisor on V. Hence $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is connected for every $v \in V$.

In Case (II), we consider the following commutative diagram

$$\begin{tikzcd} D_i' \arrow[d] & D_i'^{\nu} \arrow[l, "\nu_i"'] \arrow[dl, "\rho_i"] \\ D_i^{\dagger} \arrow[d, "\psi_i"'] & \\ V & \end{tikzcd}$$

where $D_i' \to D_i^{\dagger} \to V$ is the Stein factorization for $i = 1, 2$. Since the mapping degree $\deg[D_i' : V] = 1$, $\psi_i \colon D_i^{\dagger} \to V$ is an isomorphism for $i = 1, 2$. We put

$$K_{D_i^{\dagger}} + \Delta_{D_i^{\dagger}} := \rho_{i*}(K_{D_i^{\nu}} + \Delta_{D_i^{\nu}})$$

for $i = 1, 2$. Then we can check that

$$\psi_2^{-1} \circ \psi_1 \colon (D_1^{\dagger}, \Delta_{D_1^{\dagger}}) \to (D_2^{\dagger}, \Delta_{D_2^{\dagger}})$$

is a B-bimeromorphic isomorphism. More precisely, by taking general hyperplane cuts and applying [AFKM, 12.3.4 Theorem] to our setting, we see that there exists an effective $\mathbb{Q}$-divisor P on V such that $\psi_i \colon (D_i^{\dagger}, \Delta_{D_i^{\dagger}}) \to (V, P)$ is a B-bimeromorphic isomorphism for $i = 1, 2$. Hence

$$(\varphi \circ \nu_2)^{-1} \circ (\varphi \circ \nu_1) : (D_1'^{\nu}, \Delta_{D_1'^{\nu}}) \dashrightarrow (D_2'^{\nu}, \Delta_{D_2'^{\nu}})$$

is a B-bimeromorphic map over V.

In Case (III), we consider the following commutative diagram

$$\begin{tikzcd} D'_1 \arrow[d] & D'^{\nu}_1 \arrow[l, "\nu_1"'] \arrow[d, "\rho_1"] \\ D^{\dagger}_1 \arrow[d] & D^{\dagger\nu}_1 \arrow[l, "\nu^{\dagger}_1"'] \arrow[dl] \\ V & \end{tikzcd}$$

where $D'_1 \to D^{\dagger}_1 \to V$ is the Stein factorization and $\nu^{\dagger}_1 \colon D^{\dagger\nu}_1 \to D^{\dagger}_1$ is the normalization. We put

$$K_{D^{\dagger\nu}_1} + \Delta_{D^{\dagger\nu}_1} := \rho_{1*}(K_{D'^{\nu}_1} + \Delta_{D'^{\nu}_1}).$$

Then there exists an isomorphism $\iota^{\dagger} \colon D^{\dagger\nu}_1 \to D^{\dagger\nu}_1$ over V such that $\iota^{\dagger} \neq \mathrm{id}$ and $(\iota^{\dagger})^2 = \mathrm{id}$ (see, for example, [EH1, Lemma 2.24 and Corollary 2.26]). Over a nonempty open subset of V over which $D^{\dagger}_1$ is a union of two sections, the situation is the same as in Case (II). Where $D^{\dagger}_1 \to V$ is a ramified double cover of smooth varieties, $D^{\dagger\nu}_1 \to D^{\dagger}_1$ is an isomorphism and the ramification locus is $\iota^{\dagger}$-invariant. Hence we can check that $\iota^{\dagger}$ preserves $\Delta_{D^{\dagger\nu}_1}$. Therefore, we obtain a B-bimeromorphic involution map

$$\iota \colon (D'^{\nu}_1, \Delta_{D'^{\nu}_1}) \dashrightarrow (D'^{\nu}_1, \Delta_{D'^{\nu}_1})$$

over V.

We assume that $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is not connected in (II) and (III). Then D'_i is finite over some open neighborhood of v. Therefore, $D'_i \to D^{\dagger}_i$ is an isomorphism for $i = 1, 2$. In particular, D'_i is normal for $i = 1, 2$. By Lemma 4.2, we can prove that there are no log canonical centers except $(\Delta')^h$ over some open neighborhood of v. This means that (X', Δ') is purely log terminal in a neighborhood of $\varphi^{-1}(v)$. This is what we wanted.

We finish the proof of Lemma 4.5. $\square$

By Lemma 4.5, we have:

Proposition 4.6. *Let (X, Δ) be a divisorial log terminal pair and let $\pi \colon X \to Y$ be a projective surjective morphism of normal complex varieties. Let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Assume that X is $\mathbb{Q}$-factorial over W. Let $f \colon X \to Z$ be a projective surjective morphism of normal complex varieties over Y such that $f_*\mathcal{O}_X \simeq \mathcal{O}_Z$, $K_X + \Delta \sim_{\mathbb{Q},f} 0$, and $\pi_Z \colon Z \to Y$ is projective, where π_Z is the structure morphism. We further assume that $\lfloor \Delta \rfloor \cap f^{-1}(z)$ is not connected for some $z \in \pi_Z^{-1}(W)$. Then, after shrinking Y around W suitably, the number of the connected components of $\lfloor \Delta \rfloor \cap f^{-1}(z)$ is at most two for every $z \in Z$. There exists a meromorphic map $q \colon X \dashrightarrow V$ over Z whose general fiber is $\mathbb{P}^1$ such that V is $\mathbb{Q}$-factorial over W and has only kawamata log terminal singularities. The horizontal part Δ^h of $\lfloor \Delta \rfloor$ with respect to q satisfies one of the following conditions.*

(i) *$\Delta^h = D_1$, which is irreducible, the mapping degree* $\deg[D_1 : V] = 2$, *and there is a B-bimermorphic involution on (D_1, Δ_{D_1}) over Z.*

(ii) *$\Delta^h = D_1 + D_2$ such that D_i is irreducible for $i = 1, 2$ and*

$$(q|_{D_2})^{-1} \circ (q|_{D_1}) \colon (D_1, \Delta_{D_1}) \dashrightarrow (D_2, \Delta_{D_2})$$

is a B-bimeromorphic map over Z.

We note that $K_{D_i}+\Delta_{D_i} := (K_X+\Delta)|_{D_i}$ *and* (D_i, Δ_{D_i}) *is divisorial log terminal for* $i = 1, 2$. *More precisely, by a* $(K_X + \Delta - \varepsilon\lfloor\Delta\rfloor)$*-minimal model program with ample scaling over* Z *around* $\pi_Z^{-1}(W)$*, after shrinking* Y *around* W *suitably, we have* $p\colon (X, \Delta) \dashrightarrow (X', \Delta')$ *over* Z *and* (X', Δ') *satisfies* (II) *or* (III) *in Lemma 4.5.*

$$\begin{tikzcd}
X \arrow[rr, dashed, "p"] \arrow[drr, dashed, "q"'] \arrow[dd, "\pi"'] \arrow[ddr, "f"] & & X' \arrow[d, "\varphi"] \\
& & V \arrow[dl] \\
Y & Z \arrow[l, "\pi_Z"] &
\end{tikzcd}$$

The reader can find more details in the following proof.

Proof of Proposition 4.6. The idea of the proof is very simple. By running a suitable minimal model program, we reduce the problem to Lemma 4.5. We note that we need the minimal model program established in [EH2] in Step 1. The minimal model program treated in [Fuj12] is sufficient for Step 2.

Step 1. In this step, we assume that $\lfloor\Delta\rfloor$ is not dominant onto Z. Under this assumption, we will prove that $\lfloor\Delta\rfloor \cap f^{-1}(z)$ is connected for every $z \in \pi_Z^{-1}(W)$.

We take an arbitrary point $z \in \pi_Z^{-1}(W)$ and a Stein compact subset W_z of Z such that $\Gamma(W_z, \mathcal{O}_Z)$ is noetherian and $z \in W_z$. By [EH2, Theorem 1.2], we can run a $(K_X + \Delta - \varepsilon\lfloor\Delta\rfloor)$-minimal model program with ample scaling over Z around W_z. We finally get a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, dashed, "p"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& Z &
\end{tikzcd}$$

around W_z such that p is a finite composite of flips and divisorial contrations and that $K_{X'} + \Delta' - \varepsilon\lfloor\Delta'\rfloor$ is nef over W_z, where $\Delta' := p_*\Delta$. This implies that $\lfloor\Delta'\rfloor \cap f'^{-1}(z)$ is connected, where $f'\colon X' \to Z$ is the structure morphism. More precisely, we have $\lfloor\Delta'\rfloor \cap f'^{-1}(z) = \emptyset$ or $f'^{-1}(z) \subset \operatorname{Supp}\lfloor\Delta'\rfloor$. Since the number of the connected components of $\lfloor\Delta\rfloor \cap f^{-1}(z)$ is preserved by the above minimal model program by Lemma 4.1, $\lfloor\Delta\rfloor \cap f^{-1}(z)$ is connected.

Step 2. In this step, we assume that $\lfloor\Delta\rfloor$ is dominant onto Z. Then $K_X + \Delta - \varepsilon\lfloor\Delta\rfloor$ is not pseudo-effective over Z. By [Fuj12, Theorem 1.1 and Lemma 9.4], we can run a $(K_X + \Delta - \varepsilon\lfloor\Delta\rfloor)$-minimal model program with ample scaling over Z around $W_Z := \pi_Z^{-1}(W)$. Then we obtain a finite sequence of divisorial contractions and flips

$$p\colon X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m =: X'$$

such that there exists a $(K_{X'}+\Delta'-\varepsilon\lfloor\Delta'\rfloor)$-negative extremal Fano contraction $\varphi\colon X' \to V$ over Z. If $\dim V \leq \dim X - 2$, then $\lfloor\Delta'\rfloor \cap \varphi^{-1}(v)$ is connected for every $v \in V$ since $\lfloor\Delta'\rfloor$ is φ-ample. This implies that $\lfloor\Delta'\rfloor \cap f'^{-1}(z)$ is connected for every $z \in \pi_Z^{-1}(W)$. Since the above minimal model program preserves the number of the connected components of $\lfloor\Delta\rfloor \cap f^{-1}(z)$ by Lemma 4.1, $\lfloor\Delta\rfloor \cap f^{-1}(z)$ is connected for every $z \in W$. Hence, from now, we may assume that $\dim V = \dim X - 1$. In this case, we have already described the situation in Lemma 4.5. Case (III) (resp. (II)) in Lemma 4.5 implies (i) (resp. (ii)).

We finish the proof of Proposition 4.6. □

Before presenting our gluing argument, we state an elementary but important lemma. Here, we need the finiteness of relative log pluricanonical representations (see Corollary 1.3).

Lemma 4.7 ([Fuj1, Lemma 4.9]). *Let (X, Δ) be an equidimensional (not necessarily connected) divisorial log terminal pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that $K_X + \Delta$ is π-semiample. Let W be a compact subset of Y and let U be a Stein open subset of Y with $U \subset W$. We further assume that U is semianalytic. We put $G := \rho_m^{WU}(\mathrm{Bim}(X/Y, \Delta; W))$. Then G is a finite group. We put $X_U := \pi^{-1}(U)$. If*

$$s \in \mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X + \Delta))\right),$$

then $g^*s|_{\lfloor \Delta \rfloor} = s|_{\lfloor \Delta \rfloor}$ and

$$g^*s \in \mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X + \Delta))\right)$$

for every $g \in G$. In particular,

$$\sum_{g \in G} g^*s \in A\left(X_U, \mathcal{O}_X(m(K_X + \Delta))\right),$$

$$\prod_{g \in G} g^*s \in A\left(X_U, \mathcal{O}_X(m|G|(K_X + \Delta))\right),$$

and

$$\prod_{g \in G} g^*s|_{\lfloor \Delta \rfloor} = \left(s|_{\lfloor \Delta \rfloor}\right)^{|G|}.$$

Of course,

$$\frac{1}{|G|} \sum_{g \in G} g^*s|_{\lfloor \Delta \rfloor} = s|_{\lfloor \Delta \rfloor}$$

holds.

Proof. By Corollary 1.3, G is a finite group. Then, by Lemma 2.29, it is not difficult to see that the proof of [Fuj1, Lemma 4.9] works in our complex analytic setting. Hence we omit the details here. □

The following lemma is required for our inductive gluing argument.

Lemma 4.8 ([Fuj1, Lemma 4.7]). *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that (X, Δ) is an equidimensional divisorial log terminal pair and that $K_X + \Delta$ is π-semiample. Let W be a compact subset of Y and let U be a semianalytic Stein open subset of Y with $U \subset W$. Assume that $\mathrm{PA}(X_U, \mathcal{O}_X(m(K_X + \Delta)))$ generates $\mathcal{O}_X(m(K_X + \Delta))$ over U. Then there exists a sufficiently divisible positive integer m' such that $\mathrm{A}(X_U, \mathcal{O}_X(m'm(K_X + \Delta)))$ generates $\mathcal{O}_X(m'm(K_X + \Delta))$ over U.*

Proof. We put $G := \rho_m^{WU}\left(\mathrm{Bim}(X/Y, \Delta; W)\right)$. Then G is a finite group by Corollary 1.3. We put $G := \{g_1, \cdots, g_N\}$ with $N := |G|$. Let σ_i be the ith elementary symmetric polynomial for $1 \le i \le N$. Then we have

$$\{s = 0\} \supset \bigcap_{j=1}^{N} \{g_j^*s = 0\} = \bigcap_{i=1}^{N} \{\sigma_i(g_1^*s, \cdots, g_N^*s) = 0\}.$$

By Lemma 4.7, we see that

$$\sigma_i(g_1^*s, \cdots, g_N^*s) \in A(X_U, \mathcal{O}_X(im(K_X+\Delta)))$$

for every i. Therefore, by considering

$$\sigma_i^{N!/i}(g_1^*s, \cdots, g_N^*s) \in \mathrm{A}(X_U, \mathcal{O}_X(N!m(K_X+\Delta)))$$

for $s \in \mathrm{PA}(X_U, \mathcal{O}_X(m(K_X+\Delta)))$, we can check that $\mathrm{A}(X_U, \mathcal{O}_X(N!m(K_X+\Delta)))$ generates $\mathcal{O}_X(N!m(K_X+\Delta))$ over U under the assumption that $\mathrm{PA}(X_U, \mathcal{O}_X(m(K_X+\Delta)))$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over U. Thus we obtain the desired statement of Lemma 4.8. We finish the proof. □

Proposition 4.9, which is essentially the same as [Fuj1, Proposition 4.5], is a key step of our gluing argument.

Proposition 4.9 ([Fuj1, Proposition 4.5])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that (X, Δ) is divisorial log terminal. Let U be a semianalytic Stein open subset of Y and let W be a Stein compact subset of Y with $U \subset W$ such that X is $\mathbb{Q}$-factorial over W and that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. We put $S := \lfloor \Delta \rfloor$, $X_U := \pi^{-1}(U)$, and $S_U := S|_{\pi^{-1}(U)}$. Assume that*

1. *$K_X + \Delta$ is π-semiample, and*
2. *$\mathrm{A}\left(S_U, \mathcal{O}_S(m_0(K_X+\Delta))\right)$ generates $\mathcal{O}_S(m_0(K_X+\Delta))$ over U for some positive integer m_0.*

If necessary, we replace U with a smaller semianalytic Stein open subset of Y. Then there exists a positive integer m_1 such that $m_1m_0 \in 2\mathbb{Z}$, the natural restriction map

$$\mathrm{PA}\left(X_U, \mathcal{O}_X(m_1m_0(K_X+\Delta))\right) \to \mathrm{A}\left(S_U, \mathcal{O}_S(m_1m_0(K_X+\Delta))\right)$$

is surjective, and $\mathrm{PA}\left(X_U, \mathcal{O}_X(m_1m_0(K_X+\Delta))\right)$ generates $\mathcal{O}_X(m_1m_0(K_X+\Delta))$ over U.

Since we are working with complex analytic spaces, there are additional technical difficulties. Nevertheless, the following proof is essentially the same as that of [Fuj1, Proposition 4.5]. We describe it here for the reader's convenience.

Proof of Proposition 4.9. It is sufficient to prove this proposition for each connected component of X. Hence we may assume that X is irreducible. Throughout this proof, we will freely shrink Y around W without mentioning it explicitly. We first take a relative Iitaka fibration $f\colon X \to Z$ over Y, that is, $f\colon X \to Z$ is a projective surjective morphism of normal complex analytic varieties such that $f_*\mathcal{O}_X \simeq \mathcal{O}_Z$ and that $\mathcal{O}_X(m(K_X+\Delta)) \simeq f^*\mathcal{L}$ holds for some positive integer m and a π_Z-ample line bundle $\mathcal{L}$ on Z, where $\pi_Z\colon Z \to Y$ is the structure morphism.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "\pi_Z"] \\ & Y & \end{tikzcd}$$

If $S = \lfloor \Delta \rfloor = 0$, then there is nothing to prove. Therefore, we may assume that $S = \lfloor \Delta \rfloor \neq 0$. Then we have the following four cases:

(1) Z is a point and S is connected,
(2) $\dim Z \geq 1$, $S \cap f^{-1}(z)$ is connected for every $z \in Z$, and $f(S) = Z$,
(3) $\dim Z \geq 1$, $S \cap f^{-1}(z)$ is connected for every $z \in Z$, and $f(S) \subsetneq Z$, and
(4) $S \cap f^{-1}(z)$ is not connected for some $z \in Z$.

Step 1. In this step, we will treat (1).

When Z is a point, X is projective and $K_X+\Delta\sim_{\mathbb{Q}}0$. We consider the following long exact sequence:

$$\begin{aligned}0 &\to H^0(X,\mathcal{O}_X(m_0(K_X+\Delta)-S))\to H^0(X,\mathcal{O}_X(m_0(K_X+\Delta)))\\ &\to H^0(S,\mathcal{O}_S(m_0(K_X+\Delta)))\to\cdots.\end{aligned}$$

Since $K_X+\Delta\sim_{\mathbb{Q}}0$ and $S\neq 0$, we obtain that $H^0(X,\mathcal{O}_X(m_0(K_X+\Delta)-S))=0$ and that the second and the third terms are one-dimensional. Hence we obtain the desired statement.

Step 2. In this step, we will treat (2).

By taking a divisible positive integer m such that $\mathrm{A}(S_U,\mathcal{O}_S(m(K_X+\Delta)))$ generates $\mathcal{O}_S(m(K_X+\Delta))$ over U and that $\mathcal{O}_X(m(K_X+\Delta))\simeq f^*\mathcal{L}$ holds for some π_Z-ample line bundle $\mathcal{L}$ on Z. If necessary, we replace U with a smaller relatively compact semianalytic Stein open subset of Y. By $\mathrm{A}\left(S_U,\mathcal{O}_S(m(K_X+\Delta))\right)$, we can construct a morphism $\Phi\colon S\to Z'$ over U. Since every curve in any fiber of $f|_S$ over U is mapped to a point by Φ, there exists a morphism $\Psi\colon Z\to Z'$ over U such that $\Psi\circ(f|_S)=\Phi$. Over U, there exists the following commutative diagram.

$$\begin{array}{ccc} X & \hookleftarrow & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Phi} \\ Z & \xrightarrow[\Psi]{} & Z' \end{array}$$

We note that

$$\Phi\colon S\xrightarrow{f|_S}Z\xrightarrow{\Psi}Z'$$

and that $f|_S$ is surjective with connected fibers. For any

$$s\in\mathrm{A}\left(S_U,\mathcal{O}_S(m(K_X+\Delta))\right),$$

we can take t such that $s=\Phi^*t$. We put $u:=f^*\Psi^*t$. Then

$$u\in\mathrm{PA}\left(X_U,\mathcal{O}_X(m(K_X+\Delta))\right)$$

such that $u|_S=s$. By construction, $\mathrm{PA}\left(X_U,\mathcal{O}_X(m(K_X+\Delta))\right)$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over U.

Step 3. In this step, we will treat (3).

This step is a relative version of [Fuj1, Lemma 4.3]. We take a divisible positive integer m such that $\mathcal{O}_X(m(K_X+\Delta))\simeq f^*\mathcal{L}$ for some π_Z-ample line bundle $\mathcal{L}$ on Z.

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_Z} & \\ & Y & \end{array}$$

We put $T:=f(S)\subsetneq Z$. Then $f_*\mathcal{O}_S\simeq\mathcal{O}_T$ by Corollary 4.3. Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_*\mathcal{O}_X(lm(K_X+\Delta)) & \longrightarrow & \pi_*\mathcal{O}_S(lm(K_S+\Delta_S)) \\ \simeq\uparrow & & \uparrow\simeq \\ \pi_{Z*}\mathcal{L}^{\otimes l} & \longrightarrow & \pi_{Z*}\left(\mathcal{L}^{\otimes l}|_T\right). \end{array} \tag{4.1}$$

Note that the vertical arrows are isomorphisms. If we replace U with a relatively compact semianalytic Stein open subset and make l sufficiently large, then $\mathcal{L}^{\otimes l}\otimes\mathcal{I}_T$ is π_Z-generated over U, where $\mathcal{I}_T$ is the defining ideal sheaf of T on Z, and $R^1\pi_{Z*}\left(\mathcal{L}^{\otimes l}\otimes\mathcal{I}_T\right)=0$ since $\mathcal{L}$ is π_Z-ample. Thus, by (4.1), we have the following short exact sequence:

$$\begin{aligned} 0 &\to \pi_*\mathcal{O}_X(lm(K_X+\Delta)-S)\to\pi_*\mathcal{O}_X(lm(K_X+\Delta)) \\ &\to \pi_*\mathcal{O}_S(lm(K_S+\Delta_S))\to 0. \end{aligned} \tag{4.2}$$

By definition, it is obvious that every element of $H^0(X_U,\mathcal{O}_X(lm(K_X+\Delta)-S))$ is contained in $\mathrm{PA}(X_U,\mathcal{O}_X(lm(K_X+\Delta)))$. By (4.2), we can extend

$$\mathrm{A}\left(S_U,\mathcal{O}_S(lm(K_S+\Delta_S))\right)$$

to

$$\mathrm{PA}\left(X_U,\mathcal{O}_X(lm(K_X+\Delta))\right)$$

and check that $\mathrm{PA}\left(X_U,\mathcal{O}_X(lm(K_X+\Delta))\right)$ generates $\mathcal{O}_X(lm(K_X+\Delta))$ over U.

Step 4. In this step, we will treat (4).

In this case, by Proposition 4.6, we can run a $(K_X+\Delta-\varepsilon\lfloor\Delta\rfloor)$-minimal model program with ample scaling over Z around $W_Z:=\pi_Z^{-1}(W)$ (see [Fuj12, Theorem 1.2 and Lemma 9.4]) and finally get (X',Δ') and a $(K_{X'}+\Delta'-\varepsilon\lfloor\Delta'\rfloor)$-negative extremal Fano contraction $\varphi\colon X'\to V$ as in Lemma 4.5. Then we have (II) or (III) in Lemma 4.5. From now, we will freely use the notation in Lemma 4.5 and its proof. We note that $p\colon(X,\Delta)\dashrightarrow(X',\Delta')$ is B-bimeromorphic over Y. The situation is summarized in the following commutative diagram.

$$\begin{tikzcd}
D_i \arrow[rr, dashed] \arrow[d, hook] & & D'_i \arrow[d, hook] & D'^{\nu}_i \arrow[l, "\nu_i"'] \arrow[d, "\rho_i"] \\
X \arrow[rr, dashed, "p"] \arrow[drr, dashed, "q"'] \arrow[dd, "\pi"'] \arrow[ddr, "f"] & & X' \arrow[d, "\varphi"'] & D^{\dagger\nu}_i \arrow[dl] \\
& & V \arrow[dl] & \\
Y & Z \arrow[l, "\pi_Z"] & &
\end{tikzcd}$$

We take any element s of $\mathrm{A}\left(S_U,\mathcal{O}_S(m(K_S+\Delta_S))\right)$. By Remark 2.21, we note that there exist natural isomorphisms

$$H^0\left(X_U,\mathcal{O}_X(m(K_X+\Delta))\right)\simeq H^0\left(X'_U,\mathcal{O}_{X'}(m(K_{X'}+\Delta'))\right)$$

and

$$H^0\left(S_U,\mathcal{O}_S(m(K_S+\Delta_S))\right)\simeq H^0\left(S'_U,\mathcal{O}_{S'}(m(K_{S'}+\Delta_{S'}))\right)$$

induced by p, where $\pi'\colon X'\to Y$, $X'_U:=\pi'^{-1}(U)$, $S':=\lfloor\Delta'\rfloor$, and $S'_U:=S'\cap X'_U$. Hence s induces

$$s'\in H^0\left(S'_U,\mathcal{O}_{S'}(m(K_{S'}+\Delta_{S'}))\right).$$

Let m be a sufficiently large and divisible positive integer such that $\mathcal{O}_X(m(K_X+\Delta))\simeq f^*\mathcal{L}$ for some line bundle $\mathcal{L}$ on Z. The section s' induces a section

$$s''_i\in H^0(D'^{\nu}_i,\mathcal{O}_{D'^{\nu}_i}(m(K_{D'^{\nu}_i}+\Delta_{D'^{\nu}_i})))$$

over U for $i=1,2$. In Case (III), s''_1 is ι-invariant. Hence s''_1 descends to a section t of $\mathcal{L}_V$ over U, where $\mathcal{L}_V$ is the pull-back of $\mathcal{L}$ to V. In Case (II), s''_1 also naturally descends to a section t of $\mathcal{L}_V$ over U. In Case (III), the pull-back of φ^*t to D'^{ν}_1 coincides with s''_1 by construction. In Case (II), on a small open subset $\widetilde{U}$ of U such that $\varphi^{-1}(\widetilde{U}) \simeq \mathbb{P}^1 \times \widetilde{U}$ and that $\varphi|_{\varphi^{-1}(\widetilde{U})} \colon \mathbb{P}^1 \times \widetilde{U} \to \widetilde{U}$ is the second projection, the difference between s''_2 and the pull-back of φ^*t to D'^{ν}_2 is at most $(-1)^m$ (see the proof of [AFKM, 12.3.4 Theorem]). By construction, it is obvious that the pull-back of φ^*t to D'^{ν}_1 coincides with s''_1. Hence, we have $s'|_{(\Delta')^h} = (\varphi^*t)|_{(\Delta')^h}$ holds if m is even. From now, we will see that $(\varphi^*t)|_{\lfloor \Delta' \rfloor} = s'|_{\lfloor \Delta' \rfloor}$ holds as in Case 4 in the proof of [Fuj1, Proposition 4.5]. Let $(\Delta')^v$ be the vertical part of $\lfloor \Delta' \rfloor$. We can write $(\Delta')^v = \sum_i \varphi^*P_i$ such that $Q_i := \operatorname{Supp} P_i$ is a prime divisor on V for every i and $Q_i \neq Q_j$ for $i \neq j$. We put $E_i := \varphi^*P_i$. Then it is sufficient to check that $s'|_{E_i} = (\varphi^*t)|_{E_i}$ holds for every i. Let F_i be an irreducible component of $E_i \cap (\Delta')^h$ such that $\varphi \colon F_i \to Q_i$ is dominant. Since $(\Delta')^h \cap (\Delta')^v \neq \emptyset$, we can always take such F_i. We consider the following commutative diagram:

$$\begin{tikzcd}
\pi_*\mathcal{O}_{E_i}(m(K_{X'}+\Delta')) \arrow[r] & \pi_*\mathcal{O}_{F_i}(m(K_{X'}+\Delta')) \\
\pi_{V*}\left(\mathcal{L}_V|_{Q_i}\right) \arrow[u, "\simeq"] \arrow[r, equal] & \pi_{V*}\left(\mathcal{L}_V|_{Q_i}\right), \arrow[u, hook, "j"']
\end{tikzcd}$$

where $\pi_V \colon V \to Y$ is the structure morphism. The left vertical arrow is an isomorphism by Lemma 4.2. The map j is injective since $\varphi \colon F_i \to Q_i$ is dominant. Since $s'|_{F_i} = (\varphi^*t)|_{F_i}$, we have $s'|_{E_i} = (\varphi^*t)|_{E_i}$ for every i. Thus we have $s'|_{\lfloor \Delta' \rfloor} = (\varphi^*t)|_{\lfloor \Delta' \rfloor}$. This means that s can be lifted to a member of $\mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)$. By construction, it is not difficult to see that $\mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over U.

We finish the proof. □

As a consequence of Lemma 4.8 and Proposition 4.9, we obtain the following key lemma, which plays a crucial role in the proof of Theorem 1.1.

Lemma 4.10 (Abundance for semi-divisorial log terminal pairs in the complex analytic setting)**.** *Let (X,Δ) be a semi-divisorial log terminal pair and let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces. Let U be an open subset of Y and let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian with $U \subset W$. Assume that $K_X+\Delta$ is π-semiample. Let P be an arbitrary point of U. Then there exists a semianalytic Stein open neighborhood U_P of P and a positive integer m such that admissible sections generate $\mathcal{O}_X(m(K_X+\Delta))$ over U_P.*

Proof. Let $\nu \colon X^{\nu} \to X$ be the normalization. By definition, we see that any admissible section on X^{ν} over U_P descends to an admissible section on X over U_P since X is simple normal crossing in codimension one and satisfies Serre's S_2 condition. Hence, by taking the normalization, we may assume that X is normal. By [Fuj12, Theorems 1.21 and 1.27], we take a dlt blow-up and may assume that X is $\mathbb{Q}$-factorial over W, By Lemma 4.8 and Proposition 4.9, it is sufficient to prove this lemma for (S, Δ_S), where $S := \lfloor \Delta \rfloor$ and $K_S + \Delta_S := (K_X+\Delta)|_S$. By repeating this process finitely many times, we can reduce the problem to the case where (X,Δ) is kawamata log terminal. In this case, any section is preadmissible (see Remark 2.28). Thus, by Lemma 4.8, we obtain the desired result. □

Let us prove Theorem 1.1, which is one of the main results of the present paper.

Proof of Theorem 1.1. We take an arbitrary point $P \in W$. Since W is compact, it is sufficient to prove that there exists a positive integer m_P such that $\mathcal{O}_X(m_P(K_X+\Delta))$ is π-generated over some open neighborhood of P. We take a semianalytic Stein open neighborhood U_P of P and a Stein compact subset W_P of Y with $U_P \subset W_P$ such that $\Gamma(W_P, \mathcal{O}_Y)$ is noetherian. Let $\nu\colon X^\nu \to X$ be the normalization with $K_{X^\nu}+\Theta := \nu^*(K_X+\Delta)$. By [Fuj12, Theorems 1.21 and 1.27], after shrinking Y around W_P suitably, we take a dlt blow-up $\alpha\colon \widetilde{X} \to X^\nu$ with $K_{\widetilde{X}}+\widetilde{\Delta} := \alpha^*(K_{X^\nu}+\Theta)$ such that $\widetilde{X}$ is $\mathbb{Q}$-factorial over W_P and $(\widetilde{X}, \widetilde{\Delta})$ is divisorial log terminal. We consider $\widetilde{\pi} := \pi\circ\nu\circ\alpha\colon \widetilde{X} \to Y$. If necessary, we replace U_P with a smaller semianalytic Stein open neighborhood of P. Then, by Lemma 4.10, there exists a semianalytic Stein open neighborhood U_P and a positive integer m_P such that admissible sections generate $\mathcal{O}_{\widetilde{X}}(m_P(K_{\widetilde{X}}+\widetilde{\Delta}))$ over U_P. Note that X is normal crossing in codimension one and satisfies Serre's S_2 condition since (X,Δ) is semi-log canonical. Hence any admissible section over U_P descends to a section of $\mathcal{O}_X(m_P(K_X+\Delta))$ over U_P. Thus $\mathcal{O}_X(m_P(K_X+\Delta))$ is π-generated over U_P. As we mentioned above, since W is compact, we can take an open neighborhood U of W and a divisible positive integer m such that $\mathcal{O}_X(m(K_X+\Delta))$ is π-generated over U. We finish the proof of Theorem 1.1. $\square$

The following elementary lemma shows that [HX, Theorem 2] can be deduced from Theorem 1.1. Consequently, Kollár's gluing theory in [K] is not required for the proof of [HX, Theorem 2].

Lemma 4.11. *Let $\pi\colon X \to Y$ be a proper morphism of algebraic schemes defined over $\mathbb{C}$ and let $\mathcal{L}$ be a line bundle on X. Let U be a nonempty open subset of Y in the classical topology. Assume that $\mathcal{L}$ is π-generated over U. Then there exists a Zariski open subset V of Y such that $\mathcal{L}$ is π-generated over V with $U \subset V$.*

Proof. Let $\mathcal{C}$ be the cokernel of $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$. We put $V := Y \setminus \pi(\operatorname{Supp}\mathcal{C})$. Then, by definition, V is a Zariski open subset with $U \subset V$ and $\mathcal{L}$ is π-generated over V. $\square$

5. Freeness for nef and log abundant log canonical bundles

In this section, we first prove Theorem 1.4. As a straightforward application of Theorem 1.4, we then establish Theorem 1.10. We also give proofs of Theorem 1.5 and Corollary 1.11. The proof of Theorem 1.4 begins with Theorem 5.1, which is well known in the algebraic setting (see [Fuj7]). Once Theorem 5.1 is established, the proof of Theorem 1.4 follows without much difficulty.

Theorem 5.1. *Let (X,Δ) be an irreducible divisorial log terminal pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Assume that $K_X+\Delta$ is $\mathbb{Q}$-Cartier and is π-nef and π-abundant over Y. We further assume that $K_S+\Delta_S$ is π-semiample, where $S := \lfloor \Delta \rfloor$ and $K_S+\Delta_S := (K_X+\Delta)|_S$. Let W be a compact subset of Y. Then there exists a positive integer m such that $\mathcal{O}_X(m(K_X+\Delta))$ is π-generated over some open neighborhood of W.*

Proof. We can modify the argument in [Fuj7, Section 6] for our complex analytic setting. Since the Kawamata–Viehweg vanishing theorem holds for projective morphisms of complex analytic spaces, we can generalize [Fuj7, Theorem 6.1], which is a slight generalization of the Kawamata–Shokurov basepoint-free theorem, for our complex analytic setting. By [Fuj12, Theorem 21.4], which is a kind of canonical bundle formula, and the argument in Step 2 in the proof of [Fuj12, Theorem 23.2], we can prove a complex analytic generalization of [Fuj7, Theorem 6.2]. Therefore, we see that the desired statement holds (see also [Fuj7, Theorem 1.1]). □

Let us prove Theorem 1.4.

Proof of Theorem 1.4. Let P be an arbitrary point of W. Since W is compact, it is sufficient to prove that there exist a positive integer m_P and an open neighborhood U_P of P such that $\mathcal{O}_X(m_P(K_X+\Delta))$ is π-generated over U_P. By Theorem 1.1, we may assume that X is normal. By taking a Stein compact subset W_P such that $P \in W_P$ and $\Gamma(W_P, \mathcal{O}_Y)$ is noetherian. By [Fuj12, Theorems 1.21 and 1.27], after shrinking Y around W_P suitably, we take a dlt blow-up and may assume that (X,Δ) is divisorial log terminal. By induction on dimension, we may assume that $K_S+\Delta_S$ is π-semiample over some open neighborhood of P, where $S := \lfloor \Delta \rfloor$ and $K_S+\Delta_S := (K_X+\Delta)|_S$. Hence, by Theorem 5.1, we obtain the desired statement. We finish the proof. □

The following proof is essentially due to Kenta Hashizume (see [H2, Lemma 3.4]).

Proof of Theorem 1.5. We can freely shrink Y around W suitably and always assume that Y is Stein. By taking a dlt blow-up (see [Fuj12, Theorems 1.21 and 1.27]), we may assume that (X,Δ) is divisorial log terminal and is $\mathbb{Q}$-factorial over W. By induction, we may assume that $K_S+\Delta_S := (K_X+\Delta)|_S$ is π-semiample over some open neighborhood of L for every log canonical center S of (X,Δ). By applying the argument in the proof of [H2, Lemma 3.4], we can write $K_X+\Delta = \sum_i r_i(K_X+\Delta_i)$ such that (X,Δ_i) is divisorial log terminal, $K_X+\Delta_i$ is $\mathbb{Q}$-Cartier, r_i is a positive real number, and $K_X+\Delta_i$ is π-nef and π-log abundant over some open neighborhood of L for every i. Hence, by Theorem 1.4, there exists a positive integer m_i such that $\mathcal{O}_X(m_i(K_X+\Delta_i))$ is π-generated over some open neighborhood of L. Hence, $K_X+\Delta$ is π-semiample over some open neighborhood of L. This is what we wanted. □

Theorem 1.7 is almost obvious by Theorem 1.5 and [EH2, Theorem 1.2].

Proof of Theorem 1.7. We take an arbitrary point $P \in Z$. Then it is sufficient to prove the existence of a log canonical model of (X,Δ) over some open neighborhood of P. We take $P \in U_1 \subset W_1 \subset U_2 \subset W_2$, where U_i is a Stein open subset of Z for $i=1,2$ and W_i is a Stein compact subset of Z such that $\Gamma(W_i,\mathcal{O}_Z)$ is noetherian for $i=1,2$. Throughout this proof, we can freely shrink Z around W_2 suitably. Since $-(K_X+\Delta)$ is φ-ample, we can take an effective $\mathbb{R}$-divisor A on X such that $K_X+\Delta+A \sim_{\mathbb{R},\varphi} 0$ and that $(X,\Delta+A)$ is log canonical. By [Fuj12, Theorems 1.21 and 1.27], we take a dlt blow-up $p\colon (X',\Delta') \to (X,\Delta)$ over some open neighborhood of W_2. We note that $(X',\Delta'+A')$ is log canonical with $K_{X'}+\Delta'+A' \sim_{\mathbb{R},\varphi'} 0$, where $A' := p^*A$ and $\varphi' := \varphi\circ p\colon X' \to Z$. It is sufficient to construct a log canonical model of (X',Δ') over some open neighborhood of P. By [EH2, Theorem 1.2], after finitely many flips and divisorial contractions, we finally obtain (X'',Δ'') over some open neighborhood of W_2 such that $K_{X''}+\Delta''$ is nef over W_2. By construction, $K_{X''}+\Delta''+A'' \sim_{\mathbb{R},\varphi''} 0$ holds, where A'' is the pushforward of A' on X'' and $\varphi''\colon X'' \to Z$ is the structure morphism. Thus, by [G1, Theorem 6.1], we can check that $K_{X''}+\Delta''$ is φ''-nef and φ''-log abundant with respect to (X'',Δ'') over U_2 (see also [H3, Remark 3.7]). Therefore, by Theorem 1.5, $K_{X''}+\Delta''$ is φ''-semiample over some open neighborhood of P. This means that (X',Δ') has a log canonical model over some open neighborhood of P. This is what we wanted. We finish the proof. □

We prove Theorem 1.10 as an application of Theorem 1.4.

Proof of Theorem 1.10. Let P be an arbitrary point of W. Since W is compact, it is sufficient to prove that there exist a positive integer m_P and an open neighborhood U_P of P such that $\mathcal{O}_X(m_P(K_X+\Delta))$ is π-generated over U_P. From now, we will freely shrink Y around P. By [Fuj12, Theorems 1.21 and 1.27], we take a dlt blow-up. Thus we may assume that (X,Δ) is divisorial log terminal. Let S be a log canonical stratum of (X,Δ) with $K_S+\Delta_S := (K_X+\Delta)|_S$. It is obvious that $K_S+\Delta_S$ is π-nef. By applying Conjecture 1.9 to an analytically sufficiently general fiber F of $S \to \pi(S)$, we see that $K_S+\Delta_S$ is π-nef and π-abundant. This means that $K_X+\Delta$ is π-nef and π-log abundant with respect to (X,Δ). Hence, by Theorem 1.4, we obtain m_P such that $\mathcal{O}_X(m_P(K_X+\Delta))$ is π-generated over some open neighborhood of P. We finish the proof. □

Let us prove Corollary 1.11, which is an easy application of Theorem 1.10.

Proof of Corollary 1.11. We can freely shrink Y around W. By using Shokurov's polytope (see [Fuj12]), we can write $K_X+\Delta=\sum_i r_i(K_X+\Delta_i)$ such that (X,Δ_i) is log canonical, $K_X+\Delta_i$ is $\mathbb{Q}$-Cartier, r_i is a positive real number, and $K_X+\Delta_i$ is π-nef over W for every i. In particular, $K_X+\Delta_i$ is π-nef over U for every i. Then, by Theorem 1.10, there exists a positive integer m_i such that $\mathcal{O}_X(m_i(K_X+\Delta_i))$ is π-generated over some open neighborhood of L for every i. This implies that $K_X+\Delta$ is π-semiample over some open neighborhood of L. We finish the proof. □

Anyway, by Theorem 1.10 and Corollary 1.11, we are released from the abundance conjecture for projective morphisms of complex analytic spaces. We close this section with an important conjecture.

Conjecture 5.2. *Let $\pi\colon X\to Y$ be a projective surjective morphism of normal complex varieties and let (X,Δ) be a log canonical pair. Let W be a compact subset of Y. Assume that $K_X+\Delta$ is π-nef over W. Then $K_X+\Delta$ is π-nef over some open neighborhood of W.*

If Conjecture 5.2 holds, then we can prove that $K_X+\Delta$ is π-semiample over some open neighborhood of W in Theorem 1.5 and Corollary 1.11. In the case $\dim X=2$, Conjecture 5.2 has been completely resolved in [M].

6. Dlt blow-ups and some applications

In this section, we discuss *dlt blow-ups* (see also [Fuj12, Theorems 1.21 and 1.27]) and some applications. This section heavily relies on the minimal model program established in [EH2] and hence in [Fuj12].

We begin with the following statement. In the original algebraic setting, the reader can find it in [HMX, Proposition 3.3.1].

Theorem 6.1 (Dlt blow-ups for log canonical pairs)**.** *Let X be a normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X such that (X,Δ) is log canonical. Let W be a Stein compact subset of X such that $\Gamma(W,\mathcal{O}_X)$ is noetherian. Then, after shrinking X around W suitably, we can construct a projective bimeromorphic morphism $f\colon Z\to X$ from a normal complex variety Z with the following properties:*

(i) *Z is $\mathbb{Q}$-factorial over W,*
(ii) *$a(E,X,\Delta)=-1$ for every f-exceptional divisor E on Z, and*
(iii) *(Z,Δ_Z) is divisorial log terminal, where $K_Z+\Delta_Z=f^*(K_X+\Delta)$.*

Moreover, let S be an irreducible component of Δ and let T be the strict transform of S on Z. Then we can make $f\colon Z\to X$ satisfy:

(iv) *there exists an effective f-exceptional $\mathbb{Q}$-divisor F on Z with $f(F) \subset S$ such that $-T - F$ is f-nef over W.*

The following proof is well known for algebraic varieties. In the complex analytic setting, it is sufficient to use [EH2, Theorem 1.2].

Proof of Theorem 6.1. By [Fuj12, Theorem 1.27], after shrinking X around W suitably, we can take a projective bimeromorphic morphism $g\colon V \to X$ from a normal complex variety V with $K_V+\Delta_V = g^*(K_X+\Delta)$ satisfying (i), (ii), and (iii). Let T_V be the strict transform of S on V. By [EH2, Theorem 1.2], after running a suitable minimal model program with ample scaling over X around W, we obtain a minimal model $(V', \Delta_{V'}-T_{V'})$ of (V, Δ_V-T_V) over some open neighborhood of W. We note that $\phi\colon V \dashrightarrow V'$, $\phi_*\Delta_V = \Delta_{V'}$, and $\phi_*T_V = T_{V'}$. We put $g_{V'}\colon V' \to X$. Then we obtain that $K_{V'} + \Delta_{V'} \sim_{\mathbb{R}, g_{V'}} 0$, $(V', \Delta_{V'})$ is log canonical, and $K_{V'}+(\Delta_{V'}-T_{V'}) \sim_{\mathbb{R}, g_{V'}} -T_{V'}$ is $g_{V'}$-nef over W. By [Fuj12, Theorem 1.27] again, we can take a dlt blow-up $h\colon (Z, \Delta_Z) \to (V', \Delta_{V'})$ with $K_Z+\Delta_Z = h^*(K_{V'}+\Delta_{V'})$ after shrinking X around W. Then we can write $h^*(-T_{V'}) = -T - F$. By construction, F is an effective f-exceptional $\mathbb{Q}$-divisor with $f(F) \subset S$, where $f := g_{V'} \circ h\colon Z \to X$. It is not difficult to see that (Z, Δ_Z) satisfies (i), (ii), (iii), and (iv). Hence we finish the proof. □

As a straightforward application of Theorem 6.1, we consider Theorem 6.2, which is a slight generalization of [Fuj18, Theorem 2.1.6]. Note that [Fuj18, Theorem 2.1.6] follows easily from Theorem 6.2. For further details, see [HMX, Proof of (1.1)]. Notably, Theorem 6.2 can be viewed as a complex analytic generalization of [HMX, Theorem 1.4].

Theorem 6.2 (ACC for the log canonical thresholds for complex analytic spaces)**.** *Fix a positive integer n and a set $I \subset [0, 1]$ which satisfies the* DCC*. Then there is a finite subset $I_0 \subset I$ with the following properties:*

If X is a normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and that

(1) $\dim X = n$*,*
(2) *(X, Δ) is log canonical,*
(3) *the coefficients of Δ belong to I, and*
(4) *there exists a log canonical center $C \subset X$ which is contained in every component of Δ,*

then the coefficients of Δ belong to I_0.

Proof of Theorem 6.2. We note that [HMX, Lemma 5.1] also holds in the complex analytic setting by virtue of Theorem 6.1. For details, we refer the reader to the proof of [HMX, Lemma 5.1]. Consequently, the arguments in [HMX, Section 5] apply equally well in the complex analytic setting. Therefore, Theorem 6.2 follows from the ACC for numerically trivial pairs (see [HMX, Theorem 1.5]). □

Theorem 1.8 is significantly more profound than Theorem 6.1. It can be regarded as a complex analytic version of [FH, Lemma 3.5]. For related results in the original algebraic setting, see [FH]. Let us prove Theorem 1.8.

Proof of Theorem 1.8. Without mentioning it explicitly, we will freely shrink X around W throughout this proof. By [Fuj12, Theorem 1.27], we can take a projective bimeromorphic morphism $g\colon V \to X$ with $K_V+\Delta_V = g^*(K_X+\Delta)$ satisfying (i), (ii), and (iii). We consider

$K_V+\Delta^\dagger_V=g^*(K_X+\Delta)-G_V$, where $\Delta^\dagger_V:=\Delta^{\leq 1}_V+\operatorname{Supp}\Delta^{>1}_V$. We take a rational number ε with $0<\varepsilon\ll 1$. Then we consider

$$K_V+\Delta^\dagger_V-\varepsilon G_V=g^*(K_X+\Delta)-(1+\varepsilon)G_V.$$

Note that $(V,\Delta^\dagger_V-\varepsilon G_V)$ is divisorial log terminal and $K_V+\Delta^\dagger_V-\varepsilon G_V$ is g-log abundant with respect to $(V,\Delta^\dagger_V-\varepsilon G_V)$ (see [G1, Theorem 6.1]). We take a general g-ample effective $\mathbb{Q}$-divisor A on V such that $(V,\Delta^\dagger_V-\varepsilon G_V+A)$ is log canonical and $K_V+\Delta^\dagger_V-\varepsilon G_V+A$ is g-nef over W. We run a $(K_V+\Delta^\dagger_V-\varepsilon G_V)$-minimal model program over X around W with scaling of A starting from $(V_0,\Delta^\dagger_{V_0}-\varepsilon G_{V_0}):=(V,\Delta^\dagger_V-\varepsilon G_V)$. Then we have a sequence of flips and divisorial contractions:

$$V_0\overset{\phi_0}{\dashrightarrow}V_1\overset{\phi_1}{\dashrightarrow}\cdots\overset{\phi_{i-1}}{\dashrightarrow}V_i\overset{\phi_i}{\dashrightarrow}\cdots$$

with

$$1\geq\lambda_0\geq\lambda_1\geq\cdots$$

such that $K_{V_i}+\Delta^\dagger_{V_i}-\varepsilon G_{V_i}+\lambda_iA_i$ is g_{V_i}-nef over W, where $g_{V_i}\colon V_i\to X$, $\Delta^\dagger_{V_i}:=(\phi_{i-1})_*\Delta^\dagger_{V_{i-1}}$, $G_{V_i}:=(\phi_{i-1})_*G_{V_{i-1}}$, and $A_i:=(\phi_{i-1})_*A_{i-1}$ for every $i\geq 1$. We put $\lambda_\infty:=\lim_{i\to\infty}\lambda_i$. Then we obtain $\lambda_\infty=0$. For the details, see, for example, the proof of [Fuj12, Lemma 13.7]. On the other hand, since $K_{V_i}+\Delta^\dagger_{V_i}+G_{V_i}\sim_{\mathbb{R},g_{V_i}}0$, $K_{V_i}+\Delta^\dagger_{V_i}-\varepsilon G_{V_i}$ is g_{V_i}-log abundant with respect to $(V_i,\Delta^\dagger_{V_i}-\varepsilon G_{V_i})$ for every $i\geq 0$ by [G1, Theorem 6.1]. Then, by [EH2, Theorem 1.3], it terminates at a minimal model $(V',\Delta^\dagger_{V'}-\varepsilon G_{V'})$. We note that $\phi\colon V\dashrightarrow V'$, $\phi_*\Delta^\dagger_V=\Delta^\dagger_{V'}$, and $\phi_*G_V=G_{V'}$. We put $g_{V'}\colon V'\to X$. Since $K_V+\Delta^\dagger_V\sim_{\mathbb{R},g}-G_V$ and $K_V+\Delta^\dagger_V-\varepsilon G_V\sim_{\mathbb{R},g}-(1+\varepsilon)G_V$, the above minimal model program is also a $(K_V+\Delta^\dagger_V)$-minimal model program. In particular, $(V',\Delta^\dagger_{V'})$ is a divisorial log terminal pair. We put $(Z,\Delta_Z):=(V',\Delta_{V'})$ and $f:=g_{V'}$. Then it is easy to see that it satisfies (i), (ii), (iii), and (iv). Since

$$-(1+\varepsilon)G\sim_{\mathbb{R},f}K_Z+\Delta^\dagger_Z-\varepsilon G$$

and $K_Z+\Delta^\dagger_Z-\varepsilon G$ is f-nef and f-log abundant with respect to $(Z,\Delta^\dagger_Z-\varepsilon G)$, it is f-semiample over some open neighborhood of L. This implies that $-G$ is f-semiample over some open neighborhood of L by Theorem 1.5. This is (v). We finish the proof of Theorem 1.8. □

We can quickly recover the log canonical inversion of adjunction (see [Fuj16, Theorem 1.1]) from Theorem 1.8.

Theorem 6.3 (Inversion of adjunction for log canonicity, see [Fuj16, Theorem 1.1])**.** *Let X be a normal complex variety and let $S+B$ be an effective $\mathbb{R}$-divisor on X such that K_X+S+B is $\mathbb{R}$-Cartier, S is reduced, and S and B have no common irreducible components. Let $\nu\colon S^\nu\to S$ be the normalization with $K_{S^\nu}+B_{S^\nu}=\nu^*(K_X+S+B)$, where B_{S^ν} denotes Shokurov's different. Then $(X,S+B)$ is log canonical in a neighborhood of S if and only if (S^ν,B_{S^ν}) is log canonical.*

For the sake of completeness, we provide a proof of Theorem 6.3 based on Theorem 1.8. In the proof of Theorem 6.3 below, we make use of (iv) of Theorem 1.8, but (v) of Theorem 1.8 is not needed.

Proof of Theorem 6.3. If $(X,S+B)$ is log canonical in a neighborhood of S, then we can easily check that (S^ν,B_{S^ν}) is log canonical. From now, we will prove that $(X,S+B)$ is log canonical in a neighborhood of S under the assumption that (S^ν,B_{S^ν}) is log canonical.

We take an arbitrary point $P \in S$. It is sufficient to prove that $(X, S+B)$ is log canonical around P. We take a Stein compact subset W of X such that $\Gamma(W, \mathcal{O}_X)$ is noetherian and an open neighborhood U of P such that $P \in U \subset W$. Let $f\colon Z \to X$ be a dlt blow-up after shrinking X around W with $K_Z + \Delta_Z = f^*(K_X + \Delta)$ as in Theorem 1.8. Note that $K_Z + \Delta^{\dagger}_Z = f^*(K_X + \Delta) - G$ is f-nef over W. We also note that $\mathrm{Nlc}(Z, \Delta_Z) = \mathrm{Supp}\, G$. Since $-G$ is f-nef over W, $\mathrm{Nlc}(Z, \Delta_Z) = f^{-1}(\mathrm{Nlc}(X, \Delta))$ holds over U. Let T be the strict transform of S on Z. Since $(S^{\nu}, B_{S^{\nu}})$ is log canonical, $T \cap \mathrm{Supp}\, G = \emptyset$. This implies that $S \cap \mathrm{Nlc}(X, \Delta) = \emptyset$ on U. Hence (X, Δ) is log canonical around S on U. We finish the proof. $\square$

7. Supplementary comments

In this final section, we provide some supplementary comments on [Fuj1] and [FG] for the reader's convenience.

7.1. In [FG, 2.20] and the proof of [FG, Theorem 4.3], we claim that the results in [Fuj1, Section 2] can be freely used based on [BCHM]. However, in order to prove [Fuj1, Proposition 2.1] in dimension $n \geq 4$ (see also [Fuj1, Remark 2.2]), the minimal model program with scaling established in [BCHM] is not sufficient. The following result is required.

Theorem 7.2 (cf. [Bir, Theorem 5.2])**.** *Let $\pi\colon X \to Y$ be a projective surjective morphism of normal quasi-projective varieties and let (X, Δ) be a $\mathbb{Q}$-factorial divisorial log terminal pair such that $K_X + \Delta \sim_{\mathbb{Q}, \pi} 0$. Assume that $\pi(\lfloor \Delta \rfloor) \subsetneq Y$, that is, $\lfloor \Delta \rfloor$ is vertical with respect to π. Then $(X, \Delta - \varepsilon \lfloor \Delta \rfloor)$ has a good minimal model over Y for every rational number ε with $0 < \varepsilon \leq 1$. In particular, every $(K_X + \Delta - \varepsilon \lfloor \Delta \rfloor)$-minimal model program with ample scaling over Y always terminates.*

If $\pi(\lfloor \Delta \rfloor) = Y$, then $K_X + \Delta - \varepsilon \lfloor \Delta \rfloor$ is not π-pseudo-effective for any rational number ε with $0 < \varepsilon \leq 1$. In this case, the minimal model program established in [BCHM] is sufficient to prove [Fuj1, Proposition 2.1] in dimension $n \geq 4$. Theorem 7.2 follows from the results in [Bir]. We note that the pair $(X, \Delta - \varepsilon \lfloor \Delta \rfloor)$ is kawamata log terminal for every rational number ε with $0 < \varepsilon \leq 1$. Hence, to prove Theorem 7.2, we do not require any deep results related to the abundance conjecture for log canonical pairs. There is no circular reasoning even if one uses [Bir, Theorem 5.2] in the context of [FG]. For further details, see [Bir, Theorem 5.2]. We also point out that the most general result in this direction is treated in [H1]. By combining [BCHM] and Theorem 7.2, we are free to apply the results in [Fuj1, Section 2] for dimension $n \geq 4$. Therefore, there are no significant issues in the arguments of [FG]. In the present paper, in Step 1 of the proof of Proposition 4.6, we use [EH2, Theorem 1.2] instead of Theorem 7.2. The minimal model program developed in [Fuj12] is insufficient for the proof of Proposition 4.6.

7.3. We make a small remark on [Fuj1, Lemma 2.3] for the reader's convenience. In the proof of [Fuj1, Lemma 2.3], we claim that there exists a $\mathbb{Q}$-divisor P on V satisfying $K_{D_i} + \mathrm{Diff}(\Delta - D_i) = u|^*_{D_i}(K_V + P)$. However, it is not clear when D_1 is irreducible and the mapping degree $\deg[D_1 : V] = 2$. In that case, we can not apply [AFKM, 12.3.4 Theorem].

Example 7.4. We put $Z := \mathbb{P}^1 \times \mathbb{P}^1$. Let Δ be a general member of $|p_1^*\mathcal{O}_{\mathbb{P}^1}(2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(2)|$, where p_i is the ith projection for $i = 1, 2$. Then Δ is a smooth elliptic curve and $K_Z + \Delta \sim 0$. We consider the first projection $h\colon Z \to R := \mathbb{P}^1$. In this setting, $u := h\colon Z \to V := R$

is a $(K_Z+\Delta-\varepsilon\lfloor\Delta\rfloor)$-negative extremal Fano contraction over R. Of course, the horizontal part $\Delta^h =: D_1$ of $\lfloor\Delta\rfloor$ is irreducible and the mapping degree $\deg[D_1 : V]$ is two. In [Fuj1, Lemma 2.3], we claim that there exists an effective $\mathbb{Q}$-divisor P on V such that $K_{D_1} = u|_{D_1}^*(K_V + P)$ holds without explaining it explicitly. It is somewhat misleading when D_1 is irreducible with $\deg[D_1 : V] = 2$.

Fortunately, as shown in the proof of Lemma 4.5 in the present paper, it is not necessary to construct a $\mathbb{Q}$-divisor P on V in Case (III). Therefore, there are no significant difficulties in the proof of [Fuj1, Lemma 2.3].

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Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: `fujino@math.kyoto-u.ac.jp`