# ON FINITENESS OF RELATIVE LOG PLURICANONICAL REPRESENTATIONS

OSAMU FUJINO

Abstract. We prove the finiteness of relative log pluricanonical representations in the complex analytic setting. Then we discuss the abundance conjecture for semi-log canonical pairs in the complex analytic setting as an application. Roughly speaking, in the complex analytic setting, we reduce the abundance conjecture for semi-log canonical pairs to the one for log canonical pairs. Moreover, we show that we can reduce the abundance conjecture for projective morphisms of complex analytic spaces to the original abundance conjecture for projective varieties.

Contents



## 1. Introduction

This paper will fill a missing part of the minimal model program for projective morphisms between complex analytic spaces (see [Fuj11], [Fuj12], [Fuj13], [Fuj14], [Fuj15], [FF], [DHP], [EH2], [LM], and so on). Roughly speaking, this paper is a complex analytic generalization of [FG] (see also [Fuj1]). One of the main purposes of this paper is to establish the following result on the abundance conjecture.

**Theorem 1.1** (Abundance theorem for semi-log canonical pairs in the complex analytic setting, cf. [FG, Theorem 1.5])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces, let $W$ be a compact subset of $Y$, and let $(X, \Delta)$ be a semi-log canonical pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\nu\colon X^\nu \to X$ be the normalization. Assume that $K_{X^\nu} + \Theta := \nu^*(K_X + \Delta)$ is $\pi \circ \nu$-semiample over some open neighborhood of $W$. Then there exists an open neighborhood $U$ of $W$ and a divisible positive integer $m$ such that $\mathcal{O}_X(m(K_X + \Delta))$ is $\pi$-generated over $U$.*

In order to prove Theorem 1.1, we need:

*Date*: 2024/6/11, version 0.02.
2020 *Mathematics Subject Classification.* Primary 14E30; Secondary 14E07, 32C15.
*Key words and phrases.* log pluricanonical representations, minimal model program, abundance conjecture, complex analytic spaces, semi-log canonical pairs, log canonical flips.

**Theorem 1.2** (Finiteness of relative log pluricanonical representations, I, cf. [FG, Theorem 1.1])**.** *Let $\pi\colon X\to Y$ be a projective morphism from a (not necessarily connected) normal complex analytic space $X$ onto a complex variety $Y$ such that $(X,\Delta)$ is log canonical and that every irreducible component of $X$ is dominant onto $Y$. Let $m$ be a positive integer such that $m(K_X+\Delta)$ is Cartier and $\pi_*\mathcal{O}_X(m(K_X+\Delta))\neq 0$. Assume that $K_X+\Delta$ is $\pi$-semiample. Then the image of*

$$\rho_m\colon \operatorname{Bim}(X/Y,\Delta)\to \operatorname{Aut}_{\mathcal{O}_Y}\left(\pi_*\mathcal{O}_X(m(K_X+\Delta))\right)$$

*is a finite group, where $\operatorname{Bim}(X/Y,\Delta)$ is the group of all $B$-bimeromorphic maps of $(X,\Delta)$ over $Y$.*

As an easy consequence of Theorem 1.2, we have a useful corollary. We will use it in the proof of Theorem 1.1.

**Corollary 1.3** (Finiteness of relative log pluricanonical representations, II, cf. [FG, Theorem 1.1])**.** *Let $(X,\Delta)$ be an equidimensional (not necessarily connected) log canonical pair and let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces. Let $m$ be a positive integer such that $m(K_X+\Delta)$ is Cartier and $\pi_*\mathcal{O}_X(m(K_X+\Delta))\neq 0$. Assume that $K_X+\Delta$ is $\pi$-semiample. Let $W$ be a compact subset of $Y$ and let $U$ be a semianalytic Stein open subset of $Y$ with $U\subset W$. Let $\operatorname{Bim}(X/Y,\Delta;W)$ be the group of all $B$-bimeromorphic maps $g$ defined over some open neighborhood $U_g$ of $W$. In this setting, we can consider*

$$\rho_m^{WU}\colon \operatorname{Bim}(X/Y,\Delta;W)\to \operatorname{Aut}_{\mathcal{O}_U}\left(\pi_*\mathcal{O}_{\pi^{-1}(U)}(m(K_X+\Delta))\right).$$

*Then $\rho_m^{WU}(\operatorname{Bim}(X/Y,\Delta;W))$ is a finite group.*

As an application of Theorem 1.1, we have:

**Theorem 1.4** (Freeness for nef and log abundant log canonical bundles, cf. [FG, Theorem 1.6])**.** *Let $(X,\Delta)$ be a semi-log canonical pair and let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces. Assume that $K_X+\Delta$ is $\mathbb{Q}$-Cartier and is $\pi$-nef and $\pi$-log abundant with respect to $(X,\Delta)$ over $Y$. Let $W$ be a compact subset of $Y$. Then there exists a positive integer $m$ such that $\mathcal{O}_X(m(K_X+\Delta))$ is $\pi$-generated over some open neighborhood of $W$.*

Theorem 1.4 is well known when $\pi\colon X\to Y$ is algebraic (see [FG, Theorem 1.6]). As mentioned above, we prove Theorem 1.4 as an application of Theorem 1.1. In our proof of Theorem 1.4 given in this paper, we use a kind of a canonical bundle formula (see [Fuj3] and [Fuj6]). Hence, it is not so obvious. When $K_X+\Delta$ is only $\mathbb{R}$-Cartier, we have the following theorem.

**Theorem 1.5.** *Let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces and let $W$ be a Stein compact subset of $Y$ such that $\Gamma(W,\mathcal{O}_Y)$ is noetherian. Let $U$ be an open subset of $Y$ and let $L$ be a compact subset of $Y$ such that $L\subset U\subset W$. Let $(X,\Delta)$ be a log canonical pair such that $K_X+\Delta$ is $\pi$-nef and $\pi$-log abundant with respect to $(X,\Delta)$ over $Y$. Then $K_X+\Delta$ is $\pi$-semiample over some open neighborhood of $L$.*

We note that Stein compact subsets play an important role in [Fuj11].

**Remark 1.6** (Stein compact subsets)**.** A compact subset on an analytic space is said to be *Stein compact* if it admits a fundamental system of Stein open neighborhoods. Let $W$ be a Stein compact subset on a complex analytic space $Y$. Then, by Siu's theorem, $\Gamma(W,\mathcal{O}_Y)$ is noetherian if and only if $W\cap Z$ has only finitely many connected components for any analytic subset $Z$ which is defined over an open neighborhood of $W$. Hence, if $W$

is a Stein compact semianalytic subset of a complex analytic space $Y$, then $\Gamma(W, \mathcal{O}_Y)$ is always noetherian.

By combining Theorem 1.5 with [EH2, Theorem 1.2], we can prove the existence of log canonical flips in the complex analytic setting. We learned it from Kenta Hashizume.

**Theorem 1.7** (Existence of log canonical flips)**.** *Let $\varphi\colon X \to Z$ be a small projective bimeromorphic morphism of normal complex varieties such that $(X, \Delta)$ is log canonical and that $-(K_X + \Delta)$ is $\varphi$-ample. Then we have a commutative diagram*

$$\begin{array}{ccc} (X, \Delta) & \overset{\phi}{\dashrightarrow} & (X^+, \Delta^+) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi^+} & \\ & Z & \end{array}$$

*satisfying the following properties:*

(i) *$\varphi^+\colon X^+ \to Z$ is a small projective bimeromorphic morphism of normal complex varieties,*
(ii) *$(X^+, \Delta^+)$ is log canonical, where $\Delta^+$ is the strict transform of $\Delta$ on $X^+$, and*
(iii) *$K_{X^+} + \Delta^+$ is $\varphi^+$-ample.*

*We usually simply say that $\phi\colon (X, \Delta) \dashrightarrow (X^+, \Delta^+)$ is a log canonical flip.*

Let us recall the abundance conjecture for projective log canonical pairs for the reader's convenience. It is well known that the abundance conjecture is one of the most important and the deepest conjectures in the theory of minimal models.

**Conjecture 1.8** (Abundance conjecture for projective log canonical pairs)**.** *Let $(X, \Delta)$ be a projective log canonical pair such that $K_X + \Delta$ is nef. Then $K_X + \Delta$ is semiample.*

It is well known that Conjecture 1.8 has already been solved in $\dim X \leq 3$. When $\dim X \geq 4$, it is still widely open. By Theorem 1.4, we have:

**Theorem 1.9** (cf. [Fuj11, Theorem 1.30])**.** *Assume that Conjecture 1.8 holds in dimension $n$. Let $\pi\colon X \to Y$ be a projective surjective morphism of normal complex varieties with* $\dim X \leq n$ *and let $(X, \Delta)$ be a log canonical pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Assume that $K_X + \Delta$ is $\pi$-nef. Let $W$ be a compact subset of $Y$. Then there exists a positive integer $m$ such that $\mathcal{O}_X(m(K_X + \Delta))$ is $\pi$-generated over some open neighborhood of $W$.*

When $K_X + \Delta$ is only $\mathbb{R}$-Cartier, we have:

**Corollary 1.10.** *Assume that Conjecture 1.8 holds in dimension $n$. Let $\pi\colon X \to Y$ be a projective surjective morphism of normal complex varieties with* $\dim X \leq n$ *and let $(X, \Delta)$ be a log canonical pair. Assume that $K_X + \Delta$ is $\pi$-nef. Let $W$ be a Stein compact subset of $Y$ such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Let $U$ be an open subset of $Y$ and let $L$ be a compact subset of $Y$ with $L \subset U \subset W$. Then $K_X + \Delta$ is $\pi$-semiample over some open neighborhood of $L$.*

Theorem 1.9 and Corollary 1.10 says that we can reduce the abundance conjecture for projective morphisms of complex analytic spaces to the original abundance conjecture for projective varieties. Hence, for the abundance conjecture for projective morphisms between complex analytic spaces, it is sufficient to solve Conjecture 1.8. When $(X, \Delta)$ is a kawamata log terminal pair, Theorem 1.9 has already been treated in [Fuj11, Theorem 1.30].

We note that in this paper we use the minimal model program for projective morphisms between complex analytic spaces established in [Fuj11], [EH1], and [EH2]. Moreover, we use vanishing theorems proved in [Fuj12] (see also [Fuj15] and [FF]). In this paper, we do not use Kollár's gluing theory in [K]. We do not know if it works for complex analytic spaces or not.

**Remark 1.11.** We can easily check that Theorem 1.1 recovers [HX, Theorem 2], which is the original algebraic version of this problem. Hence this paper gives an alternative proof of [HX, Theorem 2] without using Kollár's gluing theory in [K].

**Remark 1.12.** By [Fuj11], [EH1], [EH2], and this paper, we think that we can formulate and prove various results of the minimal model program for log canonical pairs in the complex analytic setting. We do not treat them here. The details will be discussed elsewhere.

We look at the organization of this paper. In Section 2, we collect basic definitions and results necessary for this paper. In Section 3, we treat the finiteness of relative log pluricanonical representations. Our proof of Theorem 1.2 uses the finiteness of log pluricanoincal representations for projective log canonical pairs established in [FG]. In Section 4, we discuss the abundance conjecture for semi-log canonical pairs in the complex analytic setting. More precisely, we prove Theorem 1.1, which is one of the main results of this paper. In Section 5, we prove Theorem 1.4 as an application of Theorem 1.1. Then we prove Theorem 1.9 as an easy consequence of Theorem 1.4. We also prove Theorems 1.5, 1.7, and Corollary 1.10. In the final section, Section 6, we give some supplementary comments on [Fuj1] and [FG] for the reader's convenience. We remove some minor troubles from [Fuj1] and [FG].

**Acknowledgments.** The author was partially supported by JSPS KAKENHI Grant Numbers JP19H01787, JP20H00111, JP21H04994, JP23K20787. The author thanks Professor Shigefumi Mori for warm encouragement. He also would like to thank Makoto Enokizono, Taro Fujisawa, Yoshinori Gongyo, and Kenta Hashizume very much.

In this paper, every complex analytic space is assumed to be *Hausdorff* and *second-countable*. A reduced and irreducible complex analytic space is called a *complex variety*. We will freely use the basic results on complex analytic geometry in [BS] and [Fis]. For the minimal model program for projective morphisms between complex analytic spaces, see [Fuj11] (see also [EH1] and [EH2]). For the basic definitions and results in the theory of minimal models for higher-dimensional algebraic varieties, see [Fuj4] and [Fuj10] (see also [KM] and [K]). In this paper, we sometimes use semianalytic sets. For the basic properties of semianalytic sets, see [BieM1].

## 2. Preliminaries

In this section, we will collect some basic definitions and results necessary for this paper. Let us start with the following basic definitions.

**Definition 2.1** ([Fuj11, Definition 2.32])**.** Let $X$ be a normal complex variety and let $D=\sum_i a_iD_i$ be an $\mathbb{R}$-divisor on $X$ such that $D_i$ is a prime divisor on $X$ for every $i$ with $D_i \neq D_j$ for $i \neq j$. We put

$$\lfloor D\rfloor := \sum_i \lfloor a_i\rfloor D_i, \quad \lceil D\rceil := -\lfloor -D\rfloor, \quad \text{and} \quad \{D\} := D-\lfloor D\rfloor.$$

We also put

$$D^{=1} := \sum_{a_i=1} D_i \quad \text{and} \quad D^{<1} := \sum_{a_i<1} a_i D_i.$$

We note that $D$ is called a *boundary* $\mathbb{Q}$*-divisor* when $a_i \in \mathbb{Q}$ and $a_i \leq 1$ for every $i$.

Let us recall the definition of log canonical pairs and log canonical strata. For the details of singularities of pairs, see [Fuj4], [Fuj10], [Fuj11, Section 3], [K], and so on. Although there are some subtle problems for complex analytic singularities of pairs, we do not repeat the details here.

**Definition 2.2** (Log canonical pairs and log canonical strata, see [Fuj11, Definition 3.1])**.** Let $X$ be a normal complex analytic space and let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If $a(E, X, \Delta) \geq -1$ (resp. $> -1$) holds for any proper bimeromorphic morphism $f\colon Y \to X$ from a normal complex analytic space $Y$ and every $f$-exceptional divisor $E$, then $(X, \Delta)$ is called a *log canonical* (resp. *purely log terminal*) *pair*. If $(X, \Delta)$ is purely log terminal and $\lfloor \Delta \rfloor = 0$, then we say that $(X, \Delta)$ is a *kawamata log terminal pair*.

Let $(X, \Delta)$ be a log canonical pair. The image of $E$ with $a(E, X, \Delta) = -1$ for some $f\colon Y \to X$ is called a *log canonical center* of $(X, \Delta)$. A closed subset $S$ of $X$ is called a *log canonical stratum* of $(X, \Delta)$ if $S$ is an irreducible component of $X$ or a log canonical center of $(X, \Delta)$.

Let us recall the definition of divisorial log terminal pairs in the complex analytic setting (see [Fuj11, Definition 3.7]). Note that [KM, Definition 2.37, Proposition 2.40, Theorem 2.44] is helpful.

**Definition 2.3** (Divisorial log terminal pairs)**.** Let $X$ be a normal complex analytic space and let $\Delta$ be a boundary $\mathbb{R}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If there exists a proper bimeromorphic morphism $f\colon Y \to X$ from a smooth complex variety $Y$ such that $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup \mathrm{Supp}\, f_*^{-1}\Delta$ are simple normal crossing divisors on $Y$ and that the discrepancy coefficient $a(E, X, \Delta) > -1$ holds for every $f$-exceptional divisor $E$, then $(X, \Delta)$ is called a *divisorial log terminal pair*. We note that $\mathrm{Exc}(f)$ denotes the *exceptional locus* of $f$.

We note that Definitions 2.2 and 2.3 work for a finite disjoint union of normal complex varieties. In Definitions 2.2 and 2.3, $X$ is not necessarily connected. It is well known that a divisorial log terminal pair is a log canonical pair.

**Remark 2.4.** If we shrink $X$ to a relatively compact open subset of $X$ in Definition 2.3, then we can assume that $f$ is a composite of a finite sequence of blow-ups. In particular, $f$ is projective. For the details, see [Fuj11, Lemma 3.9] and [BieM2].

Let us define semi-log canonical pairs and semi-divisorial log terminal pairs in the complex analytic setting.

**Definition 2.5** (Semi-log canonical pairs and semi-divisorial log terminal pairs)**.** Let $X$ be an equidimensional reduced complex analytic space that is normal crossing in codimension one and satisfies Serre's $S_2$ condition. Let $\Delta$ be an effective $\mathbb{R}$-divisor on $X$ such that the singular locus of $X$ does not contain any irreducible components of $\mathrm{Supp}\,\Delta$. In this situation, the pair $(X, \Delta)$ is called a *semi-log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and

(2) $(X^\nu, \Theta)$ is log canonical, where $\nu\colon X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta := \nu^*(K_X + \Delta)$.

Let $(X, \Delta)$ be a semi-log canonical pair in the above sense. If each irreducible component of $X$ is normal and $(X^\nu, \Theta)$ is divisorial log terminal, then we say that $(X, \Delta)$ is a *semi-divisorial log terminal pair* (an *sdlt pair*, for short). Let $S$ be a closed subset of $X$. We say that $S$ is a *semi-log canonical stratum* of $(X, \Delta)$ if and only if $S$ is an irreducible component of $X$ or the $\nu$-image of some log canonical center of $(X^\nu, \Theta)$. When $(X, \Delta)$ is log canonical, then a semi-log canonical stratum $S$ is called a *log canonical stratum* of $(X, \Delta)$ (see Definition 2.2).

For various results on algebraic (resp. complex analytic) semi-log canonical pairs, see [Fuj9] (resp. [Fuj14]).

**Remark 2.6.** Note that the definition of semi-divisorial log terminal pairs in Definition 2.5 is different from [K, Definition 5.19]. Our definition is a direct analytic generalization of the one in [Fuj1] (see [Fuj1, Definition 1.1]).

The following lemma is well known when $X$ is an algebraic variety. We state it here explicitly for the sake of completeness.

**Lemma 2.7.** *Let $(X, \Delta)$ be a divisorial log terminal pair. We put $S := \lfloor \Delta \rfloor$ and $K_S + \Delta_S := (K_X + \Delta)|_S$ by adjunction. Then $(S, \Delta_S)$ is semi-divisorial log terminal in the sense of Definition 2.5. More precisely, let $S = S_1 + \cdots + S_l$ be the irreducible decomposition. We put $T := S_1 + \cdots + S_l$ for some $l$ with $1 \leq l \leq k$. Then $T$ is Cohen–Macaulay and is simple normal crossing in codimension one. In particular, every irreducible component of $S$ is normal. We put $K_{S_i} + \Delta_{S_i} := (K_X + \Delta)|_{S_i}$ for every $i$. Then $(S_i, \Delta_{S_i})$ is divisorial log terminal. Thus we see that $(T, \Delta_T)$, where $K_T + \Delta_T := (K_X + \Delta)|_T$, is semi-divisorial log terminal.*

*Proof.* By [RRV], we can apply the proof of [Fuj10, Theorem 3.13.6] to our setting with some suitable modifications (see also Remark 2.4). Then we obtain that $T$ is Cohen–Macaulay. It is obvious that $T$ is simple normal crossing in codimension one. Hence we can easily check all the other statements. □

We will repeatedly use Lemma 2.8 in subsequent sections.

**Lemma 2.8.** *Let $(X, \Delta)$ be a log canonical pair such that $(X, \Delta - \lfloor \Delta \rfloor)$ is kawamata log terminal. We put $S := \lfloor \Delta \rfloor$ and $K_S + \Delta_S := (K_X + \Delta)|_S$. Then $S$ is Cohen–Macaulay and is semi-log canonical.*

*Proof.* Since $(X, \Delta - S)$ is kawamata log terminal, $X$ has only rational singularities. Therefore, $X$ is Cohen–Macaulay. Since $S$ is $\mathbb{Q}$-Cartier, $\mathcal{O}_X(-S)$ is Cohen–Macaulay. This implies that $\mathcal{O}_S$ is Cohen–Macaulay. For the details, see [KM, Corollary 5.25], [K, Corollaries 2.62, 2.63, and 2.88], and so on. By adjunction, we see that $(S, \Delta_S)$ is semi-log canonical. □

We need nef and log abundant divisors in Theorem 1.4.

**Definition 2.9** (Nef and abundant line bundles)**.** Let $\pi\colon X \to Y$ be a projective surjective morphism from a normal complex variety $X$ onto a complex variety $Y$. Let $\mathcal{L}$ be a $\pi$-nef line bundle on $X$. If $\kappa(F, \mathcal{L}|_F) = \nu(F, \mathcal{L}|_F)$ holds for analytically sufficiently general fibers $F$, then $\mathcal{L}$ is said to be *$\pi$-nef and $\pi$-abundant over $Y$*. Similarly, we can define $\pi$-nef and $\pi$-abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

**Remark 2.10.** In Definition 2.9, if $\mathcal{L}$ is $\pi$-semiample, then it is easy to see that $\mathcal{L}$ is $\pi$-nef and $\pi$-abundant over $Y$.

We will freely use the following elementary lemma.

**Lemma 2.11.** *Let $\pi\colon X \to Y$ be a projective surjective morphism from a normal complex variety $X$ onto a complex variety $Y$ and let $\mathcal{L}$ be a $\pi$-nef and $\pi$-abundant line bundle on $X$. Let $p\colon Z \to X$ be a projective surjective morphism from a normal complex variety $Z$. Then $p^*\mathcal{L}$ is $(\pi \circ p)$-nef and $(\pi \circ p)$-abundant over $Y$.*

**Definition 2.12** (Nef and log abundant line bundles)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let $(X, \Delta)$ be a semi-log canonical pair. Let $\mathcal{L}$ be a line bundle on $X$. We say that $\mathcal{L}$ is *$\pi$-nef and $\pi$-log abundant with respect to* $(X, \Delta)$ *over* $Y$ if and only if $\mathcal{L}|_{S^\nu}$ is nef and abundant over $\pi(S)$ for every semi-log canonical stratum $S$ of $(X, \Delta)$, where $\mathcal{L}|_{S^\nu}$ denotes the pull-back of $\mathcal{L}$ to the normalization of $S$. Similarly, we can define $\pi$-nef and $\pi$-log abundant $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors with respect to $(X, \Delta)$.

For $\mathbb{R}$-Cartier $\mathbb{R}$-divisors, we need the following definitions. In this paper, we will use $\mathbb{R}$-Cartier $\mathbb{R}$-divisors only in Theorems 1.5, 1.7, and Corollary 1.10.

**Definition 2.13** (Relatively abundant $\mathbb{R}$-Cartier $\mathbb{R}$-divisors)**.** Let $\pi\colon X \to Y$ be a projective morphism from a normal complex variety $X$ onto a complex variety $Y$. Let $D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. If $\kappa_\sigma(F, D|_F) = \kappa_\iota(F, D|_F)$ holds for analytically sufficiently general fibers $F$, then $D$ is said to be $\pi$-abundant over $Y$.

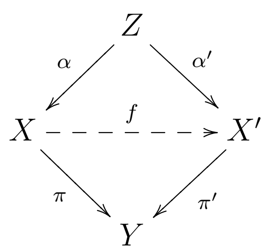
For the details of $\kappa_\sigma$ and $\kappa_\iota$, see [N, Chapter V, §2] and [Fuj10, Section 2.5], respectively.

**Definition 2.14** (Nef and log abundant $\mathbb{R}$-Cartier $\mathbb{R}$-divisors)**.** Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces and let $(X, \Delta)$ be a log canonical pair. Let $D$ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$. We say that $D$ is *$\pi$-nef and $\pi$-log abundant with respect to* $(X, \Delta)$ *over* $Y$ if and only if $D|_{S^\nu}$ is nef and abundant over $\pi(S)$ for every log canonical stratum of $(X, \Delta)$, where $D|_{S^\nu}$ denotes the pull-back of $D$ to the normalization of $S$.

**Remark 2.15.** A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is $\pi$-nef and $\pi$-log abundant with respect to $(X, \Delta)$ over $Y$ in the sense of Definition 2.14 if and only if it is $\pi$-nef and $\pi$-log abundant with respect to $(X, \Delta)$ over $Y$ in the sense of Definition 2.12.

Let us introduce the notion of $B$-bimeromorphic maps, which is obviously a generalization of the notion of $B$-birational maps.

**Definition 2.16** ($B$-bimeromorphic maps)**.** Let $\pi\colon X \to Y$ and $\pi'\colon X' \to Y$ be projective morphisms of complex analytic spaces and let $(X, \Delta)$ and $(X', \Delta')$ be log canonical pairs. We say that a bimeromorphic map $f\colon X \dashrightarrow X'$ over $Y$ is *$B$-bimeromorphic over $Y$* if there exists a commutative diagram



such that $Z$ is a normal complex analytic space, $\alpha$ and $\alpha'$ are proper bimeromorphic morphisms, and

$$\alpha^*(K_X+\Delta)=\alpha'^*(K_{X'}+\Delta')$$

holds. Let $m$ be a positive integer such that $m(K_X+\Delta)$ and $m(K_{X'}+\Delta')$ are Cartier. Then we have

$$\begin{aligned} f^*\colon \pi'_*\mathcal{O}_{X'}(m(K_{X'}+\Delta')) &\xrightarrow{\alpha'^*} \pi'_*\alpha'_*\mathcal{O}_Z(\alpha'^*(m(K_{X'}+\Delta'))) \\ &\simeq \pi_*\alpha_*\mathcal{O}_Z(\alpha^*(m(K_X+\Delta))) \xrightarrow{(\alpha^*)^{-1}} \pi_*\mathcal{O}_X(m(K_X+\Delta)). \end{aligned}$$

We put

$$\mathrm{Bim}(X/Y,\Delta):=\{f \mid f\colon (X,\Delta) \dashrightarrow (X,\Delta) \text{ is } B\text{-bimeromorphic over } Y\}.$$

Then it is obvious that $\mathrm{Bim}(X/Y,\Delta)$ has a natural group structure.

Let $W$ be a compact subset of $Y$. Then we put

$$\mathrm{Bim}(X/Y,\Delta;W):=\left\{ g \;\middle|\; \begin{array}{l} g\in \mathrm{Bim}\left(\pi^{-1}(U_g)/U_g, \Delta|_{\pi^{-1}(U_g)}\right) \text{ such that} \\ U_g \text{ is an open neighborhood of } W \end{array} \right\}.$$

Note that $\mathrm{Bim}(X/Y,\Delta;W)$ also has a natural group structure.

We make small remarks on Definition 2.16.

**Remark 2.17.** If $Y$ is a point in Definition 2.16, then $(X,\Delta)$ is a projective log canonical pair and $\mathrm{Bim}(X/Y,\Delta)$ is nothing but $\mathrm{Bir}(X,\Delta)$ in [Fuj1] and [FG].

**Remark 2.18.** In Definition 2.16, $X$ and $X'$ are not necessarily irreducible. In the proof of Theorem 1.2, we have to treat $\mathrm{Bir}(X,\Delta)$ in the case where $X$ is a disjoint union of normal projective varieties.

**Remark 2.19.** Let $(X,\Delta)=:\bigsqcup_i(X_i,\Delta_i)$ and $(X',\Delta')=:\bigsqcup_i(X'_i,\Delta'_i)$ be the irreducible decompositions. Let $f\colon X\dashrightarrow X'$ be a $B$-bimeromrophic map over $Y$ as in Definition 2.16. Then, there exists a permutation $\sigma$ such that

$$f_i:=f|_{X_i}\colon X_i\dashrightarrow X'_{\sigma(i)}$$

is a $B$-bimeromorphic map over $Y$ between irreducible log canonical pairs $(X_i,\Delta_i)$ and $(X'_{\sigma(i)},\Delta'_{\sigma(i)})$. We note that $\pi(X_i)=\pi'(X'_{\sigma(i)})$ holds for every $i$.

**Remark 2.20** (see [FG, Remark 2.15])**.** Let $(X,\Delta)$ and $(X',\Delta')$ be log canonical pairs. Let $f\colon (X,\Delta)\dashrightarrow (X',\Delta')$ be a $B$-bimeromorphic map over $Y$ as in Definition 2.16. We assume that $(X,\Delta-\lfloor\Delta\rfloor)$ and $(X',\Delta'-\lfloor\Delta'\rfloor)$ are kawamata log terminal. We put $S:=\lfloor\Delta\rfloor$ and $S':=\lfloor\Delta'\rfloor$. By replacing $Y$ with a relatively compact open subset, we may assume that $Z$ in Definition 2.16 is smooth and

$$\alpha^*(K_X+\Delta)=:K_{Z'}+\Delta_{Z'}:=\alpha'^*(K_{X'}+\Delta')$$

such that $\mathrm{Supp}\,\Delta_Z$ is a simple normal crossing divisor on $Z$. We may further assume that $\alpha$ and $\alpha'$ are projective in Definition 2.16. We put $K_S+\Delta_S:=(K_X+\Delta)|_S$ and $K_{S'}+\Delta_{S'}:=(K_{X'}+\Delta')|_{S'}$. By applying $\alpha_*$ and $\alpha'_*$ to

$$0\to\mathcal{O}_Z(\lceil-(\Delta_Z^{<1})\rceil-\Delta_Z^{=1})\to\mathcal{O}_Z(\lceil-(\Delta_Z^{<1})\rceil)\to\mathcal{O}_{\Delta_Z^{=1}}(\lceil-(\Delta_Z^{<1})\rceil)\to 0,$$

we have $\alpha_*\mathcal{O}_{\Delta_Z^{=1}}\simeq\mathcal{O}_S$ and $\alpha'_*\mathcal{O}_{\Delta_Z^{=1}}\simeq\mathcal{O}_{S'}$. Here we used

$$R^1\alpha_*\mathcal{O}_Z(\lceil-(\Delta_Z^{<1})\rceil-\Delta_Z^{=1})=R^1\alpha'_*\mathcal{O}_Z(\lceil-(\Delta_Z^{<1})\rceil-\Delta_Z^{=1})=0,$$

which is nothing but the relative Kawamata–Viehweg vanishing theorem. Thus $f$ induces an isomorphism

$$(\alpha^*)^{-1}\circ(\alpha')^*\colon \pi'_*\mathcal{O}_{S'}(m(K_{S'}+\Delta_{S'}))\xrightarrow{\sim}\pi_*\mathcal{O}_S(m(K_S+\Delta_S)).$$

We note that $f$ does not necessarily induce a bimeromorphic map $S \dashrightarrow S'$ in the above setting.

Let us introduce the notion of $B$-pluricanonical representations in the relative complex analytic setting.

**Definition 2.21** ($B$-pluricanonical representations)**.** Let $X$ be a normal complex analytic space such that $(X,\Delta)$ is log canonical and let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces. Let $m$ be a positive integer such that $m(K_X+\Delta)$ is Cartier. Then we have a group homomorphism

$$\rho_m\colon \operatorname{Bim}(X/Y,\Delta)\to \operatorname{Aut}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X(m(K_X+\Delta)))$$

given by $\rho_m(g)=g^*$ for $g\in\operatorname{Bim}(X/Y,\Delta)$. It is called the *$B$-pluricanonical representation* or *log pluricanonical representation* for $(X,\Delta)$ over $Y$. When $Y$ is a point, we have

$$\rho_m\colon \operatorname{Bir}(X,\Delta)\to \operatorname{Aut}_{\mathbb{C}}\left(H^0(X,\mathcal{O}_X(m(K_X+\Delta)))\right).$$

Theorem 1.2 is a generalization of the following theorem, which is one of the main results of [FG]. We note that we need it in the proof of Theorem 1.2. In [HX], Hacon and Xu independently proved a slightly weaker theorem (see [HX, Theorem 1]), which seems to be insufficient for the purpose of this paper.

**Theorem 2.22** ([FG, Theorem 1.1])**.** *Let $(X,\Delta)$ be a projective log canonical pair. Suppose that $m(K_X+\Delta)$ is Cartier and that $K_X+\Delta$ is semiample. Then $\rho_m(\operatorname{Bir}(X,\Delta))$ is a finite group.*

In the proof of Theorem 1.2, Burnside's theorem plays a crucial role. Hence we state it explicitly for the sake of completeness. For the proof, see, for example, [CR, (36.1) Theorem].

**Theorem 2.23** (Burnside)**.** *Let $G$ be a subgroup of* $\mathrm{GL}(n,\mathbb{C})$*. If the order of any element $g$ of $G$ is uniformly bounded, then $G$ is a finite group.*

In order to prove Theorem 1.1, we need the notion of *admissible* and *preadmissible* sections, which are first introduced in [Fuj1].

**Definition 2.24** (Admissible and preadmissible sections, see [Fuj1, Definition 4.1])**.** Let $(X,\Delta)$ be a semi-divisorial log terminal pair and let $\pi\colon X\to Y$ be a projective morphism of complex analytic spaces. Let $W$ be a Stein compact subset of $Y$ such that $\Gamma(W,\mathcal{O}_Y)$ is noetherian. Let $X=\bigcup_i X_i$ be the irreducible decomposition. As usual,

$$\nu\colon X^\nu=\bigsqcup_i X_i\to\bigcup_i X_i=X$$

is the normalization with

$$\nu^*(K_X+\Delta)=K_{X^\nu}+\Theta=:\bigsqcup_i(K_{X_i}+\Theta_i).$$

Let $S$ be the disjoint union of $\lfloor\Theta_i\rfloor$'s. We put

$$K_S+\Delta_S:=(K_{X^\nu}+\Theta)|_S.$$

Then, by adjunction, $(S, \Delta_S)$ is semi-divisorial log terminal. Let $m$ be a positive integer such that $m(K_X + \Delta)$ is Cartier. Let $U$ be a semianalytic Stein open subset of $Y$ with $U \subset W$. In particular, the number of the connected components of $U$ is finite (see, for example, [BieM1, Corollary 2.7]). We put $X_U := \pi^{-1}(U)$ and $S_U := (\pi \circ \nu)^{-1}(U)$. Then we define *preadmissible* and *admissible* sections inductively as follows.

(1) $s \in H^0(X_U, \mathcal{O}_X(m(K_X+\Delta))) \simeq H^0(U, \pi_*\mathcal{O}_X(m(K_X+\Delta)))$ is *preadmissible* if the restriction $\nu^*s|_{S_U} \in H^0(S_U, \mathcal{O}_S(m(K_S+\Delta_S)))$ is admissible.

(2) $s \in H^0(X_U, \mathcal{O}_X(m(K_X+\Delta)))$ is *admissible* if $s$ is preadmissible and $g^*(s|_{X_j}) = s|_{X_i}$ holds for every $B$-bimeromorphic map $g\colon (X_i, \Theta_i) \dashrightarrow (X_j, \Theta_j)$ defined over some open neighborhood $U_g$ of $W$ for every $i$, $j$.

Then we put

$$\begin{aligned}&\mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)\\ &:= \{s \mid s \in H^0(X_U, \mathcal{O}_X(m(K_X+\Delta))) \text{ is preadmissible}\}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{A}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)\\ &:= \{s \mid s \in H^0(X_U, \mathcal{O}_X(m(K_X+\Delta))) \text{ is admissible}\}.\end{aligned}$$

We note that if $Z$ is any analytic subset defined over some open neighborhood of $W$ then $U \cap Z$ is a semianalytic Stein open subset of $Z$ contained in $W \cap Z$. Thus the number of the connected components of $U \cap Z$ is finite (see, for example, [BieM1, Corollary 2.7]).

Let $U'$ be a semianalytic Stein open subset of $Y$ such that $U' \subset U$. We put $X_{U'} := \pi^{-1}(U')$. Then there exist natural restriction maps

$$\mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right) \to \mathrm{PA}\left(X_{U'}, \mathcal{O}_X(m(K_X+\Delta))\right)$$

and

$$\mathrm{A}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right) \to \mathrm{A}\left(X_{U'}, \mathcal{O}_X(m(K_X+\Delta))\right).$$

**Remark 2.25.** In Definition 2.24, the natural map

$$H^0(X_U, \mathcal{O}_X(m(K_X+\Delta))) \to H^0(U, \pi_*\mathcal{O}_X(m(K_X+\Delta)))$$

is an isomorphism of topological vector spaces since $U$ is Stein (see, for example, [P, Lemma II.1]).

The following remark is almost obvious by definition. We state it explicitly for the sake of completeness.

**Remark 2.26.** In Definition 2.24, if

$$s \in \mathrm{A}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right) \quad (\text{resp. } \mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)),$$

then

$$s^l \in \mathrm{A}\left(X_U, \mathcal{O}_X(lm(K_X+\Delta))\right) \quad (\text{resp. } \mathrm{PA}\left(X_U, \mathcal{O}_X(lm(K_X+\Delta))\right))$$

for every positive integer $l$. Moreover, if $\mathrm{A}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over $U$, then $\mathrm{A}\left(X_U, \mathcal{O}_X(lm(K_X+\Delta))\right)$ generates $\mathcal{O}_X(lm(K_X+\Delta))$ over $U$ for every positive integer $l$. Similarly, if $\mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over $U$, then $\mathrm{PA}\left(X_U, \mathcal{O}_X(lm(K_X+\Delta))\right)$ generates $\mathcal{O}_X(lm(K_X+\Delta))$ over $U$ for every positive integer $l$.

The following remark is obvious by definition.

**Remark 2.27.** In Definition 2.24, if $(X, \Delta)$ is kawamarta log terminal, then any section $s \in H^0(X_U, \mathcal{O}_X(m(K_X + \Delta)))$ is preadmissible by definition.

In our complex analytic setting, we can reformulate Claim ($\mathrm{A}_n$) and Claim ($\mathrm{B}_n$) in the proof of [Fuj1, Lemma 4.9] as follows. We note that $(X, \Delta_X)$ is *sub log canonical* when $X$ is smooth and $\Delta_X$ is a subboundary $\mathbb{Q}$-divisor such that $\operatorname{Supp} \Delta_X$ is a simple normal crossing divisor. For sub log canonical pairs, we can define log canonical centers as in Definition 2.2.

**Lemma 2.28.** *Let $\pi \colon X \to Y$ be a projective bimeromorphic morphism of smooth complex varieties and let $\Delta_X$ (resp. $\Delta_Y$) be a boundary $\mathbb{Q}$-divisor on $X$ (resp. $Y$) such that $\operatorname{Supp} \Delta_X$ (resp. $\operatorname{Supp} \Delta_Y$) is a simple normal crossing divisor on $X$ (resp. $Y$). We assume that $K_X + \Delta_X = \pi^*(K_Y + \Delta_Y)$.*

(i) *If $T$ is a log canonical center of $(Y, \Delta_Y)$, then there exists a log canonical center $S$ of $(X, \Delta_X)$ such that $\pi \colon S \to T$ is bimeromorphic.*

(ii) *Let $m$ be a positive integer such that $m(K_Y + \Delta_Y)$ is Cartier. If $S$ is a log canonical center of $(X, \Delta_X)$ such that $\pi \colon S \to \pi(S)$ is not bimeromorphic, then there exists a log canonical center $S'$ of $(X, \Delta_X)$ with $S' \subset S$ such that $\pi \colon S' \to \pi(S)$ is bimeromorphic and the natural restriction map*

$$\pi_*\mathcal{O}_S(m(K_S + \Delta_S)) \to \pi_*\mathcal{O}_{S'}(m(K_{S'} + \Delta_{S'}))$$

*induced by the inclusion $S' \hookrightarrow S$ is an isomorphism, where $K_S + \Delta_S := (K_X + \Delta_X)|_S$ and $K_{S'} + \Delta_{S'} := (K_X + \Delta_X)|_{S'}$.*

*Proof.* The proof of Claim ($\mathrm{A}_n$) and Claim ($\mathrm{B}_n$) in the proof of [Fuj1, Lemma 4.9] works in our setting (see also [Fuj2, Lemma 7.2]). □

We will freely use Lemma 2.28 in subsequent sections.

## 3. Finiteness of relative log pluricanonical representations

In this section, we will prove Theorem 1.2 and Corollary 1.3. We note that we use Theorem 2.22 for the proof of Theorem 1.2. Let us start with an elementary lemma.

**Lemma 3.1.** *Let $Y$ be a complex manifold, which is connected. Let*

$$\rho \colon G \to \mathrm{GL}(r, \mathcal{O}_Y)$$

*be a group homomorphism. We further consider*

$$\rho_y := \mathrm{ev}_y \circ \rho \colon G \to \mathrm{GL}(r, \mathcal{O}_Y) \to \mathrm{GL}(r, \mathbb{C}),$$

*where $\mathrm{ev}_y$ is the evaluation map at $y \in Y$. We assume that $\operatorname{Im} \rho_y = \rho_y(G)$ is a finite group for every $y \in Y$. Then $\mathrm{ev}_y \colon \rho(G) \to \rho_y(G)$ is an isomorphism for every $y \in Y$. In particular, $\operatorname{Im} \rho = \rho(G)$ is a finite group.*

*Proof.* It is obvious that $\mathrm{ev}_y \colon \rho(G) \to \rho_y(G)$ is surjective for every $y \in Y$. We take an arbitrary point $y_0 \in Y$. It is sufficient to prove that $\mathrm{ev}_{y_0} \colon \rho(G) \to \rho_{y_0}(G)$ is injective. We take $g \in \rho(G)$ such that $\mathrm{ev}_{y_0}(g) = E_r$, where $E_r$ is the $r \times r$ identity matrix. Note that $\mathrm{ev}_y(g)$ is semisimple and every eigenvalue of $\mathrm{ev}_y(g)$ is a root of unity for every $y \in Y$ since $\rho_y(G)$ is a finite group by assumption. We consider the characteristic polynomial $\chi(t) := \det(tE_r - g)$. The coefficients of $\chi(t)$ are holomorphic and take values in $K$, where $K$ is the subfield of $\mathbb{C}$ generated by all roots of unity. Hence they are constant. Since $\mathrm{ev}_{y_0}(g) = E_r$, we see that every eigenvalue of $\mathrm{ev}_y(g)$ is 1 for every $y \in Y$. This implies that $\mathrm{ev}_y(g) = E_r$ holds for every $y \in Y$ because $\mathrm{ev}_y(g)$ is semisimple. Hence we have $g = E_r$, that is, $\mathrm{ev}_{y_0}\colon \rho(G) \to \rho_{y_0}(G)$ is injective. We finish the proof. □

Theorem 3.2 is one of the most important results in this paper.

**Theorem 3.2.** *Let $\pi\colon X \to Y$ be a projective morphism from a normal complex analytic space $X$ onto a polydisc $Y$ such that $(X, \Delta)$ is divisorial log terminal and that $K_X + \Delta$ is $\pi$-semiample. Let $\varphi\colon Z \to X$ be a projective bimeromorphic morphism from a smooth complex analytic space $Z$ with $K_Z + \Delta_Z := \varphi^*(K_X + \Delta)$ such that $\pi \circ \varphi\colon Z \to Y$ is smooth and projective and that $\operatorname{Supp} \Delta_Z$ is a simple normal crossing divisor on $Z$ and is relatively normal crossing over $Y$. Let $m$ be a positive integer such that $m(K_X + \Delta)$ is Cartier. We assume that $R^i\pi_*\mathcal{O}_X(m(K_X+\Delta))$ is locally free for every $i$ and $\pi_*\mathcal{O}_X(m(K_X+\Delta)) \simeq \mathcal{O}_Y^{\oplus r}$ for some positive integer $r$. We consider*

$$\rho_m\colon \operatorname{Bim}(X/Y, \Delta) \to \operatorname{GL}(r, \mathcal{O}_Y) \simeq \operatorname{Aut}_{\mathcal{O}_Y}(\pi_*\mathcal{O}_X(m(K_X + \Delta)))$$

*and*

$$\rho_{m,y} := \mathrm{ev}_y \circ \rho_m\colon \operatorname{Bim}(X/Y, \Delta) \to \operatorname{GL}(r, \mathcal{O}_Y) \to \operatorname{GL}(r, \mathbb{C}),$$

*where $\mathrm{ev}_y$ is the evaluation map at $y \in Y$. Then $\operatorname{Im} \rho_{m,y}$ is a finite group for every $y \in Y$. Moreover,*

$$\mathrm{ev}_y\colon \operatorname{Im} \rho_m \to \operatorname{Im} \rho_{m,y}$$

*is an isomorphism for every $y \in Y$. In particular, $\operatorname{Im} \rho_m$ is a finite group.*

*We note that, in the above setting, $X_y := \pi^{-1}(y)$ is a normal projective scheme, $(X_y, \Delta_y)$ is divisorial log terminal, where $K_{X_y} + \Delta_y := (K_X + \Delta)|_{X_y}$, and*

$$\mathrm{ev}_y\colon \pi_*\mathcal{O}_X(m(K_X + \Delta)) \to H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \tag{3.1}$$

*by the base change theorem.*

**Remark 3.3.** In Theorem 3.2, $X$ is not necessarily connected.

Let us prove Theorem 3.2.

*Proof of Theorem 3.2.* By Lemma 3.1, it is sufficient to prove the finiteness of $\operatorname{Im} \rho_{m,y}$ for every $y \in Y$. In Step 1, we will prove the description (3.1) of the evaluation map $\mathrm{ev}_y$. Then, in Step 2, we will prove the finiteness of $\operatorname{Im} \rho_{m,y}$.

**Step 1.** We put $d := \dim Y$. We note that $Y$ is a polydisc by assumption. We take general hyperplanes $H_1, \cdots, H_d$ on $Y$ passing through $y$. Then $\left(X, \Delta + \sum_{i=1}^d \pi^*H_i\right)$ is a divisorial log terminal pair. We note that $(\pi \circ \varphi)^*\left(\sum_{i=1}^d H_i\right)$ and $\operatorname{Supp}\left(\Delta_Z + (\pi \circ \varphi)^*\left(\sum_{i=1}^d H_i\right)\right)$ are simple normal crossing divisors on $Z$. By construction, $X_y$ is a log canonical center of $\left(X, \Delta + \sum_{i=1}^d \pi^*H_i\right)$. This implies that $X_y$ is normal and $(X_y, \Delta_y)$ is divisorial log terminal. Since $R^i\pi_*\mathcal{O}_X(m(K_X + \Delta))$ is locally free for every $i$ by assumption, we have

$$\pi_*\mathcal{O}_X(m(K_X + \Delta)) \otimes \mathbb{C}(y) \simeq H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y)))$$

by the base change theorem. Hence we have the desired description (3.1) of the evaluation map $\mathrm{ev}_y$.

**Step 2.** We take an arbitrary element $g$ of $\operatorname{Bim}(X/Y, \Delta)$. By Theorem 2.23, it is sufficient to prove that the order of $\rho_{m,y}(g) = \mathrm{ev}_y \circ \rho_m(g)$ is uniformly bounded. We make $H_1$ general in Step 1 and put $Y' := H_1$, $X' := \pi^* H_1$, and $K_{X'} + \Delta' := (K_X + X' + \Delta)|_{X'}$. Then the above $g$ induces $g' \in \mathrm{Bim}(X'/Y', \Delta')$ such that $\mathrm{ev}_y \circ \rho_m(g) = \mathrm{ev}_y \circ \rho'_m(g')$ holds, where

$$\rho'_m \colon \mathrm{Bim}(X'/Y', \Delta') \to \mathrm{Aut}_{\mathcal{O}_{Y'}}\left(\pi_* \mathcal{O}_{X'}(m(K_{X'} + \Delta'))\right).$$

By repeating this process finitely many times, we may assume that $Y$ is a disc. Hence $X_y$ is a divisor on $X$.

We first assume that $X_y$ is connected. Let $l$ be the number of the log canonical strata of $(X_y, \Delta_y)$. We consider

$$\rho_m \colon \mathrm{Bir}(V, \Delta_V) \to \mathrm{Aut}_{\mathbb{C}}\left(H^0(V, \mathcal{O}_V(m(K_V + \Delta_V)))\right),$$

where $(V, \Delta_V)$ is a log canonical stratum of $(X_y, \Delta_y)$. Since $K_V + \Delta_V$ is semiample, $\rho_m\left(\mathrm{Bir}(V, \Delta_V)\right)$ is a finite group by Theorem 2.22. Then we put

$$k := \mathrm{lcm}\left\{\#\rho_m\left(\mathrm{Bir}(V, \Delta_V)\right) \mid (V, \Delta_V) \text{ is a log canonical stratum of } (X_y, \Delta_y)\right\}$$

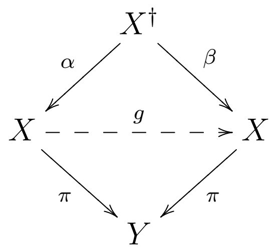
**Claim.** $\rho_{m,y}(g)^{l!k} = E_r$ holds.

*Proof of Claim.* We consider log canonical strata $(T, \Delta_T)$ of $(X_y, \Delta_y)$ satisfying that the natural restriction map

$$H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \to H^0(T, \mathcal{O}_T(m(K_T + \Delta_T))) \tag{3.2}$$

is an isomorphism. We put $t := \min \dim T$.

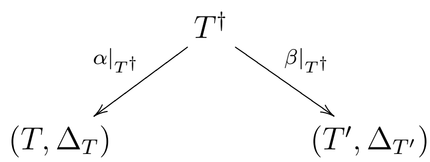
Let $(T, \Delta_T)$ be a $t$-dimensional log canonical stratum of $(X_y, \Delta_y)$ such that the natural restriction map (3.2) is an isomorphism. We consider the following commutative diagram as in Definition 2.16




where $g$ is a $B$-bimeromorphic map of $(X, \Delta)$ over $Y$ taken above. By shrinking $Y$ around $y$, we may assume that $X^\dagger$ is smooth, $\alpha$ and $\beta$ are projective, and

$$\alpha^*(K_X + \Delta) =: K_{X^\dagger} + \Delta_{X^\dagger} := \beta^*(K_X + \Delta)$$

such that $\mathrm{Supp}\, \Delta_{X^\dagger} \cup \mathrm{Supp}(\pi \circ \alpha)^* y$ is a simple normal crossing divisor on $X^\dagger$. We take $X^\dagger$ suitably. Then, by Lemma 2.28 (see also the proof of [Fuj1, Lemma 4.9] and [FG, Lemma 2.16]), we can find a log canonical stratum $(T', \Delta_{T'})$ of $(X_y, \Delta_y)$ and a commutative diagram




such that $\alpha|_{T^\dagger}$ and $\beta|_{T^\dagger}$ are proper birational and that

$$(\beta|_{T^\dagger}) \circ (\alpha|_{T^\dagger})^{-1} \colon (T, \Delta_T) \dashrightarrow (T', \Delta_{T'})$$

is a $B$-birational map of projective divisorial log terminal pairs. Note that there are only finitely many log canonical strata contained in $X_y$. Thus we can find $t$-dimensional log canonical strata $(S_i, \Delta_{S_i})$ of $(X_y, \Delta_y)$ for $1 \leq i \leq p$ and a natural embedding

$$H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \hookrightarrow \bigoplus_i H^0(S_i, \mathcal{O}_{S_i}(m(K_{S_i} + \Delta_{S_i})))$$

such that $g$ induces $\widetilde{g} \in \mathrm{Bir}(S, \Delta_S)$, where $(S, \Delta_S) := \bigsqcup_i (S_i, \Delta_{S_i})$, satisfying the following commutative diagram:

$$\begin{array}{ccccc} 0 \longrightarrow & H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) & \longrightarrow & \bigoplus_i H^0(S_i, \mathcal{O}_{S_i}(m(K_{S_i} + \Delta_{S_i}))) \\ & \downarrow {\scriptstyle \rho_{m,y}(g)} & & \downarrow {\scriptstyle \rho_m(\widetilde{g})} \\ 0 \longrightarrow & H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) & \longrightarrow & \bigoplus_i H^0(S_i, \mathcal{O}_{S_i}(m(K_{S_i} + \Delta_{S_i}))). \end{array}$$

We note the following description of $\rho_{m,y}(g)$. Let $V$ be the union of the irreducible components of $(\Delta_{X^\dagger} + (\pi \circ \alpha)^* y)^{=1}$ mapped to $y$. We put

$$K_V + \Delta_V := (K_{X^\dagger} + \Delta_{X^\dagger} + (\pi \circ \alpha)^* y)|_V.$$

Then we can check that $\alpha_* \mathcal{O}_V \simeq \mathcal{O}_{X_y} \simeq \beta_* \mathcal{O}_V$ holds, which is an easy consequence of the strict support condition established in [Fuj12, Theorem 1.1 (i)] (see, for example, the proof of Lemma 4.2 below). Thus $\rho_{m,y}(g)$ can be written as

$$\rho_{m,y} \colon H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))) \xrightarrow{\beta^*} H^0(V, \mathcal{O}_V(m(K_V + \Delta_V)))$$
$$\xrightarrow{(\alpha^*)^{-1}} H^0(X_y, \mathcal{O}_{X_y}(m(K_{X_y} + \Delta_y))).$$

Since $\rho_m(\widetilde{g})^{l!k} = \mathrm{id}$ on $\bigoplus_i H^0(S_i, \mathcal{O}_{S_i}(m(K_{S_i} + \Delta_{S_i})))$ by the definitions of $l$ and $k$, we have $\rho_{m,y}(g)^{l!k} = E_r$. This is what we wanted. □

We note that $l!k$ is independent of $g$. Therefore, Claim implies that $\mathrm{Im}\, \rho_{m,y}$, which is a subgroup of $\mathrm{GL}(r, \mathbb{C})$, is a finite group by Burnside's theorem (see Theorem 2.23). Thus we finish the proof under the assumption that $X_y$ is connected.

From now, we assume that $X_y$ is not connected. Let $a$ denote the number of the connected components of $X_y$. Then $g^{a!}$ preserves each connected component of $X_y$. Thus, by the above argument, we can take a positive integer $b$ such that $\rho_{m,y}(g)^b = E_r$ holds for every $g \in \mathrm{Bim}(X/Y, \Delta)$. Thus, by Burnside's theorem (see Theorem 2.23), we see that $\mathrm{Im}\, \rho_{m,y}$ is a finite group.

We finish the proof. □

We can prove Theorem 1.2 as an easy application of Theorem 3.2.

*Proof of Theorem 1.2.* Let $U$ be a nonempty open subset of $Y$. We consider the following commutative diagram

$$\begin{array}{ccc} \rho_m \colon \mathrm{Bim}(X/Y, \Delta) & \longrightarrow & \mathrm{Aut}_{\mathcal{O}_Y}\left(\pi_* \mathcal{O}_X(m(K_X + \Delta))\right) \\ \downarrow & & \downarrow \\ \rho_m \colon \mathrm{Bim}(\pi^{-1}(U)/U, \Delta|_{\pi^{-1}(U)}) & \longrightarrow & \mathrm{Aut}_{\mathcal{O}_U}\left(\pi_* \mathcal{O}_{\pi^{-1}(U)}(m(K_X + \Delta))\right). \end{array}$$

Note that the vertical arrows are natural restriction maps. It is obvious that the restriction map

$$\mathrm{Aut}_{\mathcal{O}_Y}\left(\pi_* \mathcal{O}_X(m(K_X + \Delta))\right) \to \mathrm{Aut}_{\mathcal{O}_U}\left(\pi_* \mathcal{O}_{\pi^{-1}(U)}(m(K_X + \Delta))\right)$$

is injective since $Y$ is irreducible. Hence, in order to prove Theorem 1.2, we can freely replace $Y$ with a small nonempty open subset of $Y$. We take a Stein compact subset $W$ of $Y$ such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Then, by [Fuj11, Theorems 1.21 and 1.27], we can take a dlt blow-up $\psi\colon (X', \Delta') \to (X, \Delta)$. By replacing $\pi\colon (X, \Delta) \to Y$ with $\pi' := \pi \circ \psi\colon (X', \Delta') \to Y$, we may further assume that $(X, \Delta)$ is divisorial log terminal. By taking a resolution of singularities of $X$ (see, for example, [BieM2]) and shrinking $Y$ suitably, we may assume that $\pi\colon (X, \Delta) \to Y$ satisfies all the conditions in Theorem 3.2. Then, by Theorem 3.2, $\rho_m(\mathrm{Bim}(X/Y, \Delta))$ is a finite group. This is what we wanted. We finish the proof. $\square$

Let us prove Corollary 1.3, which is almost obvious by Theorem 1.2. We will use it in the proof of Theorem 1.1.

*Proof of Corollary 1.3.* We decompose $(X, \Delta) =: \bigsqcup_i (X_i, \Delta_i)$ such that $\pi_i := \pi|_{X_i}\colon X_i \to Y_i := \pi(X_i)$ is surjective and every irreducible component of $X_i$ is dominant onto $Y_i$ for every $i$. We may assume that $Y_i \neq Y_j$ for $i \neq j$. Since $U$ is a semianalytic Stein open subset of $Y$, $Y_i \cap U$ is a finite disjoint union of semianalytic Stein open subsets of $Y_i$ (see, for example, [BieM1, Corollary 2.7]). Let $U'$ be a connected component of $Y_i \cap U$. Then, by Theorem 1.3, the image of

$$\rho_m\colon \mathrm{Bim}\left(\pi_i^{-1}(U')/U', \Delta_i|_{\pi_i^{-1}(U')}\right) \to \mathrm{Aut}_{\mathcal{O}_{U'}}\left(\pi_{i*}\mathcal{O}_{\pi_i^{-1}(U')}(m(K_{X_i} + \Delta_i))\right) \tag{3.3}$$

is a finite group. Note that there exists a natural restriction map

$$\mathrm{Bim}(X/Y, \Delta; W) \to \mathrm{Bim}\left(\pi_i^{-1}(U')/U', \Delta_i|_{\pi_i^{-1}(U')}\right). \tag{3.4}$$

By the natural restriction map (3.4),

$$\rho_m^{WU'}\colon \mathrm{Bim}(X/Y, \Delta; W) \to \mathrm{Aut}_{\mathcal{O}_{U'}}\left(\pi_{i*}\mathcal{O}_{\pi_i^{-1}(U')}(m(K_{X_i} + \Delta_i))\right)$$

factors through $\rho_m$ in (3.3). Thus, we have

$$\rho_m^{WU'}(\mathrm{Bim}(X/Y, \Delta; W)) \subset \rho_m\left(\mathrm{Bim}\left(\pi_i^{-1}(U')/U', \Delta_i|_{\pi_i^{-1}(U')}\right)\right).$$

Since $\rho_m^{WU}(\mathrm{Bim}(X/Y, \Delta; W))$ is contained in

$$\prod_{U'} \rho_m^{WU'}(\mathrm{Bim}(X/Y, \Delta; W)),$$

where $U'$ runs over all connected components of $Y_i \cap U$ for all $i$. Hence we see that $\rho_m^{WU}(\mathrm{Bim}(X/Y, \Delta; W))$ is a finite group. We finish the proof. $\square$

## 4. Abundance for semi-log canonical pairs

In this section, we will prove Theorem 1.1. This section is essentially the same as [Fuj1] although we need the minimal model program for projective morphisms between complex analytic spaces established in [Fuj11] (see also [EH1] and [EH2]).

The following lemma is well known. It is an easy application of the relative Kawamata–Viehweg vanishing theorem.

**Lemma 4.1** (Connectedness lemma)**.** *Let $(X, \Delta)$ be a log canonical pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$. Assume that $-(K_X + \Delta)$ is $\pi$-nef and $\pi$-big. Then* $\mathrm{Nklt}(X, \Delta) \cap \pi^{-1}(y)$ *is connected for every $y \in Y$, where* $\mathrm{Nklt}(X, \Delta)$ *denotes the non-kawamata log terminal locus of $(X, \Delta)$. In particular, if $(X, \Delta - \lfloor \Delta \rfloor)$ is kawamata log terminal, $\lfloor \Delta \rfloor \cap \pi^{-1}(y)$ is connected for every $y \in Y$.*

*Proof.* The usual proof in the algebraic setting can work with only some suitable modifications. This is because the Kawamata–Viehweg vanishing theorem holds for projective morphisms between complex analytic spaces. In this proof, we can freely shrink $Y$ around $y$. We consider the following short exact sequence:

$$0 \to \mathcal{J}(X, \Delta) \to \mathcal{O}_X \to \mathcal{O}_{\mathrm{Nklt}(X,\Delta)} \to 0,$$

where $\mathcal{J}(X, \Delta)$ denotes the multiplier ideal sheaf of $(X, \Delta)$. By the relative Kawamata–Viehweg–Nadel vanishing theorem, we have

$$0 \to \pi_*\mathcal{J}(X, \Delta) \to \mathcal{O}_Y \to \pi_*\mathcal{O}_{\mathrm{Nklt}(X,\Delta)} \to 0.$$

This implies that $\mathrm{Nklt}(X, \Delta) \cap \pi^{-1}(y)$ is connected. □

The following lemma also claims that the union of log canonical centers is connected in some suitable setting. Lemma 4.2 is much harder than Lemma 4.1. The proof heavily depends on the strict support condition established in [Fuj12, Theorem 1.1 (i)] (see also [Fuj15] and [FF]).

**Lemma 4.2.** *Let $(X, \Delta)$ be a log canonical pair and let $\pi\colon X \to Y$ be a projective morphism of normal complex varieties with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$. Let $W$ be a compact subset of $Y$. We assume that $K_X + \Delta \sim_{\mathbb{Q},\pi} 0$ holds. We put $Y' := \bigcup_i \pi(C_i) \subsetneq Y$, where $\{C_i\}$ is a set of some log canonical centers of $(X, \Delta)$. Let $X'$ be the union of the log canonical centers of $(X, \Delta)$ mapped to $Y'$ by $\pi$. Then, after shrinking $Y$ around $W$ suitably, $\pi_*\mathcal{O}_{X'} \simeq \mathcal{O}_{Y'}$ holds. In particular, $\pi_*\mathcal{O}_{X'} \simeq \mathcal{O}_{Y'}$ holds on an open subset $U$ contained in $W$.*

*Proof.* Throughout this proof, we will freely shrink $Y$ around $W$ without mentioning it explicitly. Let $p\colon Z \to X$ be a projective bimeromorphic morphism from a smooth complex variety $Z$ with $K_Z + \Delta_Z := p^*(K_X + \Delta)$ (see [BieM2]). We may assume that $(\pi \circ p)^{-1}(Y')$ and $p^{-1}(X')$ are simple normal crossing divisors on $Z$. We may further assume that the union of $(\pi \circ p)^{-1}(Y')$, $p^{-1}(X')$, and $\mathrm{Supp}\,\Delta_Z$ is contained in a simple normal crossing divisor on $Z$. Let $V$ be the union of the irreducible components of $\Delta_Z^{=1}$ mapped to $Y'$ by $\pi \circ p$. We put $A := \lceil -(\Delta_Z^{<1}) \rceil$, which is a $p$-exceptional effective divisor on $Z$. By assumption, we have

$$A - V - (K_Z + \Delta_Z^{=1} - V + \{\Delta_Z\}) \sim_{\mathbb{Q},\pi\circ p} 0.$$

We consider the following short exact sequence

$$0 \to \mathcal{O}_Z(A - V) \to \mathcal{O}_Z(A) \to \mathcal{O}_V(A) \to 0.$$

We note that no log canonical centers of $(Z, \Delta_Z^{=1} - V + \{\Delta_Z\})$ map to $Y'$ by construction. Then we have

$$0 \to (\pi \circ p)_*\mathcal{O}_Z(A - V) \to \mathcal{O}_Y \to (\pi \circ p)_*\mathcal{O}_V(A) \to 0.$$

Here we used the strict support condition for $R^1(\pi \circ p)_*\mathcal{O}_Z(A - V)$ (see [Fuj12, Theorem 1.1 (i)]) in order to prove the connecting homomorphism

$$\delta\colon (\pi \circ p)_*\mathcal{O}_V(A) \to R^1(\pi \circ p)_*\mathcal{O}_Z(A - V)$$

is zero. This implies that $(\pi \circ p)_*\mathcal{O}_V(A) \simeq \mathcal{O}_{Y'}$ holds. Similarly, we have the short exact sequence

$$0 \to p_*\mathcal{O}_Z(A - V) \to \mathcal{O}_X \to p_*\mathcal{O}_V(A) \to 0$$

since no log canonical centers of $(Z, \Delta_Z^{=1} - V + \{\Delta_Z\})$ map to $X'$ by construction. This implies that $p_*\mathcal{O}_V(A) \simeq \mathcal{O}_{X'}$. Hence we have $\pi_*\mathcal{O}_{X'} \simeq \mathcal{O}_{Y'}$. We finish the proof of Lemma 4.2. □

As an easy corollary of Lemma 4.2, we have:

**Corollary 4.3** ([Fuj1, Lemma 4.2]). *Let $(X, \Delta)$ be a divisorial log terminal pair and let $\pi\colon X \to Y$ be a projective morphism of normal complex varieties with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_Y$. Let $W$ be a compact subset of $Y$. We assume that $K_X + \Delta \sim_{\mathbb{Q},\pi} 0$ holds. If $Y' := \pi(\lfloor\Delta\rfloor) \subsetneq Y$, then, after shrinking $Y$ around $W$ suitably, we have $\pi_*\mathcal{O}_{\lfloor\Delta\rfloor} \simeq \mathcal{O}_{Y'}$.*

*Proof.* Since $(X, \Delta)$ is divisorial log terminal, $\lfloor\Delta\rfloor$ is the union of all log canonical centers of $(X, \Delta)$. Therefore, by Lemma 4.2, we have $\pi_*\mathcal{O}_{\lfloor\Delta\rfloor} \simeq \mathcal{O}_{Y'}$. □

The following lemma, which is a toy model of Lemma 4.5 and Proposition 4.6 below, is sufficient for [Fuj2], [Fuj5], [Fuj7], and [G2]. Therefore, we do not treat any subtle problems when $K_X + \Delta$ is numerically trivial.

**Lemma 4.4.** *Let $(X, \Delta)$ be a projective $\mathbb{Q}$-factorial divisorial log terminal pair such that $K_X + \Delta \sim_{\mathbb{Q}} 0$. Assume that $\lfloor\Delta\rfloor$ is not connected. Then $\lfloor\Delta\rfloor = S_1 + S_2$ such that $(S_i, \Delta_{S_i})$ is kawamata log terminal with $K_{S_i} + \Delta_{S_i} := (K_X + \Delta)|_{S_i}$ for $i = 1, 2$ and that $(S_1, \Delta_1)$ is $B$-birationally equivalent to $(S_2, \Delta_{S_2})$. In particular, $(X, \Delta)$ is purely log terminal.*

*Proof.* Note that $K_X + \Delta - \varepsilon\lfloor\Delta\rfloor$ is not pseudo-effective for a small positive rational number $\varepsilon$. By running a $(K_X + \Delta - \varepsilon\lfloor\Delta\rfloor)$-minimal model program with ample scaling (see [BCHM]), we finally get an extremal Fano contraction morphism, which is generically a $\mathbb{P}^1$-bundle with two disjoint sections. More precisely, we have

$$\begin{array}{ccc} p\colon X & \dashrightarrow & X' \\ & & \downarrow{\scriptstyle \varphi} \\ & & V \end{array}$$

where $p\colon X \dashrightarrow X'$ is a finite sequence of flips and divisorial contractions and $\varphi\colon X' \to V$ is a $(K_{X'} + \Delta' - \varepsilon\lfloor\Delta'\rfloor)$-negative extremal Fano contraction with $\dim V = \dim X - 1$. We can check that the number of the connected components of $\lfloor\Delta\rfloor$ is preserved by the above minimal model program by applying Lemma 4.1 to each step. Hence we obtain that $\lfloor\Delta'\rfloor = S'_1 + S'_2$, $\varphi\colon S'_i \to V$ is an isomorphism for $i = 1, 2$, and $S'_1 \cap S'_2 = \emptyset$. By using [AFKM, 12.3.4 Theorem], we can check that $\varphi\colon (S'_i, \Delta_{S'_i}) \to (V, P)$ is a $B$-bimeromorphic isomorphism for some effective $\mathbb{Q}$-divisor $P$ on $V$, where $K_{S'_i} + \Delta_{S'_i} := (K_{X'} + \Delta')|_{S'_i}$. Then, by Lemma 4.2, we see that there are no log canonical centers except $\lfloor\Delta'\rfloor$. This implies that $(X', \Delta')$ is purely log terminal. Hence, $(X, \Delta)$ is purely log terminal and $(S_1, \Delta_{S_1})$ is $B$-birationally equivalent to $(S_2, \Delta_{S_2})$. This is what we wanted. □

The following lemma is very important.

**Lemma 4.5.** *Let $(X', \Delta')$ be a log canonical pair and let $\pi'\colon X' \to Y$ be a projective surjective morphism of normal complex varieties. Let $W$ be a Stein compact subset of $Y$ such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Assume that $X'$ is $\mathbb{Q}$-factorial over $W$. Let $f'\colon X' \to Z$ be a projective surjective morphism of normal complex varieties over $Y$ such that $K_{X'} + \Delta' \sim_{\mathbb{Q},f'} 0$, and $\pi_Z\colon Z \to Y$ is projective, where $\pi_Z$ is the structure morphism. Assume that $(X', \Delta' - \varepsilon\lfloor\Delta'\rfloor)$ is kawamata log terminal for some small positive rational number $\varepsilon$ and there exists a $(K_{X'} + \Delta' - \varepsilon\lfloor\Delta'\rfloor)$-negative extremal Fano contraction $\varphi := \varphi_R\colon X' \to V$ over $Z$ associated to an extremal ray $R$ of $\overline{\mathrm{NE}}(X'/Z; \pi_Z^{-1}(W))$ with $\dim V = \dim X' - 1$.*

*Note that $V$ is $\mathbb{Q}$-factorial over $W$ and has only kawamata log terminal singularities.*

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & V \\ {\scriptstyle \pi'}\downarrow & \searrow^{f'} & \downarrow \\ Y & \xleftarrow[\pi_Z]{} & Z \end{array}$$

*Then the horizontal part $(\Delta')^h$ of $\lfloor \Delta' \rfloor$ with respect to $\varphi$ satisfies one of the following conditions.*

(I) $(\Delta')^h = D'_1$, *which is irreducible, and* $\deg[D'_1 : V] = 1$.
(II) $(\Delta')^h = D'_1 + D'_2$ *such that $D'_i$ is irreducible and* $\deg[D'_i : V] = 1$ *for* $i = 1, 2$.
(III) $(\Delta')^h = D'_1$, *which is irreducible, and* $\deg[D'_1 : V] = 2$.

*We define $\Delta_{D'_i}$ by*

$$K_{D'_i} + \Delta_{D'_i} = (K_{X'} + \Delta')|_{D'_i}$$

*for $i = 1, 2$. Let $\nu_i \colon D'^{\nu}_i \to D'_i$ be the normalization for $i = 1, 2$. We put*

$$K_{D'^{\nu}_i} + \Delta_{D'^{\nu}_i} := \nu_i^*(K_{D'_i} + \Delta_{D'_i})$$

*for $i = 1, 2$. After shrinking $Y$ around $W$ suitably, we have the following statements.*

**Case** (I)**.** $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is connected for every $v \in V$.

**Case** (II)**.** The number of the connected components of $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is at most two for every $v \in V$ and

$$(\varphi \circ \nu_2)^{-1} \circ (\varphi \circ \nu_1) : (D'^{\nu}_1, \Delta_{D'^{\nu}_1}) \dashrightarrow (D'^{\nu}_2, \Delta_{D'^{\nu}_2})$$

is a $B$-bimeromorphic map over $V$.

**Case** (III)**.** The number of the connected components of $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is at most two for every $v \in V$ and there exists a $B$-bimeromorphic map

$$\iota \colon (D'^{\nu}_1, \Delta_{D'^{\nu}_1}) \dashrightarrow (D'^{\nu}_1, \Delta_{D'^{\nu}_1})$$

over $V$ with $\iota \neq \mathrm{id}$ and $\iota^2 = \mathrm{id}$.

*Moreover, in* (II) *and* (III)*, if $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is not connected for some $v \in V$, then $(X', \Delta')$ is purely log terminal in a neighborhood of $\varphi^{-1}(v)$.*

More details on Cases (II) and (III) will be discussed in the following proof.

*Proof of Theorem 4.5.* We have $R^i\varphi_*\mathcal{O}_{X'} = 0$ by the relative Kawamata–Viehweg vanishing theorem. Therefore, we see that general fibers of $\varphi \colon X' \to V$ are $\mathbb{P}^1$. Hence the mapping degree of $(\Delta')^h$, the horizontal part of $\lfloor \Delta' \rfloor$, is at most two. Therefore, we have (I), (II), and (III).

In Case (I), $(\Delta')^h = D'_1$ is irreducible and $\varphi$-ample. Since $\varphi$ is an extremal Fano contraction, the vertical part of $\lfloor \Delta' \rfloor$ is the pull-back of some effective $\mathbb{Q}$-divisor on $V$. Hence $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is connected for every $v \in V$.

In Case (II), we consider the following commutative diagram

$$\begin{array}{ccc} D'_i & \xleftarrow{\nu_i} & D'^{\nu}_i \\ \downarrow & \swarrow_{\rho_i} & \\ D^{\dagger}_i & & \\ {\scriptstyle \psi_i}\downarrow & & \\ V & & \end{array}$$

where $D'_i \to D^{\dagger}_i \to V$ is the Stein factorization for $i = 1, 2$. Since the mapping degree $\deg[D'_i : V] = 1$, $\psi_i\colon D^{\dagger}_i \to V$ is an isomorphism for $i = 1, 2$. We put

$$K_{D^{\dagger}_i} + \Delta_{D^{\dagger}_i} := \rho_{i*}(K_{D^{\nu}_i} + \Delta_{D^{\nu}_i})$$

for $i = 1, 2$. Then we can check that

$$\psi_2^{-1} \circ \psi_1\colon (D^{\dagger}_1, \Delta_{D^{\dagger}_1}) \to (D^{\dagger}_2, \Delta_{D^{\dagger}_2})$$

is a $B$-bimeromorphic isomorphism. More precisely, by taking general hyperplane cuts and applying [AFKM, 12.3.4 Theorem] to our setting, we see that there exists an effective $\mathbb{Q}$-divisor $P$ on $V$ such that $\psi_i\colon (D^{\dagger}_i, \Delta_{D^{\dagger}_i}) \to (V, P)$ is a $B$-bimeromorphic isomorphism for $i = 1, 2$. Hence

$$(\varphi \circ \nu_2)^{-1} \circ (\varphi \circ \nu_1) : (D'^{\nu}_1, \Delta_{D'^{\nu}_1}) \dashrightarrow (D'^{\nu}_2, \Delta_{D'^{\nu}_2})$$

is a $B$-bimeromorphic map over $V$.

In Case (III), we consider the following commutative diagram

$$\begin{array}{ccc} D'_1 & \xleftarrow{\nu_1} & D'^{\nu}_1 \\ \downarrow & & \downarrow{\scriptstyle \rho_1} \\ D^{\dagger}_1 & \xleftarrow{\nu^{\dagger}_1} & D^{\dagger\nu}_1 \\ \downarrow & \swarrow & \\ V & & \end{array}$$

where $D'_1 \to D^{\dagger}_1 \to V$ is the Stein factorization and $\nu^{\dagger}_1\colon D^{\dagger\nu}_1 \to D^{\dagger}_1$ is the normalization. We put

$$K_{D^{\dagger\nu}_1} + \Delta_{D^{\dagger\nu}_1} := \rho_*(K_{D'^{\nu}_1} + \Delta_{D'^{\nu}_1}).$$

Then there exists an isomorphism $\iota^{\dagger}\colon D^{\dagger\nu}_1 \to D^{\dagger\nu}_1$ over $V$ such that $\iota^{\dagger} \neq \mathrm{id}$ and $(\iota^{\dagger})^2 = \mathrm{id}$. Over a nonempty open subset of $V$ over which $D^{\dagger}_1$ is a union of two sections, the situation is the same as in Case (II). Where $D^{\dagger}_1 \to V$ is a ramified double cover of smooth varieties, $D^{\dagger\nu}_1 \to D^{\dagger}_1$ is an isomorphism and the ramification locus is $\iota^{\dagger}$-invariant. Hence we can check that $\iota^{\dagger}$ preserves $\Delta_{D^{\dagger\nu}_1}$. Therefore, we obtain a $B$-bimeromorphic involution map

$$\iota\colon (D'^{\nu}_1, \Delta_{D'^{\nu}_1}) \dashrightarrow (D'^{\nu}_1, \Delta_{D'^{\nu}_1})$$

over $V$.

We assume that $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is not connected in (II) and (III). Then $D'_i$ is finite over some open neighborhood of $v$. Therefore, $D'_i \to D^{\dagger}_i$ is an isomorphism for $i = 1, 2$. In particular, $D'_i$ is normal for $i = 1, 2$. By Lemma 4.2, we can prove that there are no log canonical centers except $(\Delta')^h$ over some open neighborhood of $v$. This means that $(X', \Delta')$ is purely log terminal in a neighborhood of $\varphi^{-1}(v)$. This is what we wanted.

We finish the proof of Lemma 4.5. □

By Lemma 4.5, we have:

**Proposition 4.6.** *Let $(X, \Delta)$ be a divisorial log terminal pair and let $\pi\colon X \to Y$ be a projective surjective morphism of normal complex varieties. Let $W$ be a Stein compact subset of $Y$ such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Assume that $X$ is $\mathbb{Q}$-factorial over $W$. Let $f\colon X \to Z$ be a projective surjective morphism of normal complex varieties over $Y$ such that $f_*\mathcal{O}_X \simeq \mathcal{O}_Z$, $K_X + \Delta \sim_{\mathbb{Q}, f} 0$, and $\pi_Z\colon Z \to Y$ is projective, where $\pi_Z$ is the structure morphism. We further assume that $\lfloor \Delta \rfloor \cap f^{-1}(z)$ is not connected for some $z \in \pi_Z^{-1}(W)$. Then, after shrinking $Y$ around $W$ suitably, the number of the connected components of $\lfloor \Delta \rfloor \cap f^{-1}(z)$ is at most two for every $z \in Z$. There exists a meromorphic map $q\colon X \dashrightarrow V$ over $Z$ whose general fiber is $\mathbb{P}^1$ such that $V$ is $\mathbb{Q}$-factorial over $W$ and has only kawamata log terminal singularities. The horizontal part $\Delta^h$ of $\lfloor \Delta \rfloor$ with respect to $q$ satisfies one of the following conditions.*

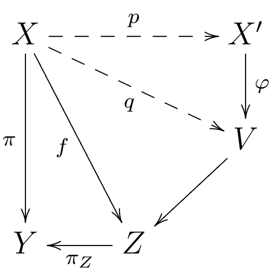
(i) *$\Delta^h = D_1$, which is irreducible, the mapping degree* $\deg[D_1 : V] = 2$, *and there is a $B$-bimermorphic involution on $(D_1, \Delta_{D_1})$ over $Z$.*

(ii) *$\Delta^h = D_1 + D_2$ such that $D_i$ is irreducible for $i = 1, 2$ and*

$$(q|_{D_2})^{-1} \circ (q|_{D_1})\colon (D_1, \Delta_{D_1}) \dashrightarrow (D_2, \Delta_{D_2})$$

*is a $B$-bimeromorphic map over $Z$.*

*We note that $K_{D_i} + \Delta_{D_i} := (K_X + \Delta)|_{D_i}$ and $(D_i, \Delta_{D_i})$ is divisorial log terminal for $i = 1, 2$. More precisely, by a $(K_X + \Delta - \varepsilon\lfloor \Delta \rfloor)$-minimal model program with ample scaling over $Z$ around $\pi_Z^{-1}(W)$, after shrinking $Y$ around $W$ suitably, we have $p\colon (X, \Delta) \dashrightarrow (X', \Delta')$ over $Z$ and $(X', \Delta')$ satisfies* (II) *and* (III) *in Lemma 4.5.*




The reader can find more details in the following proof.

*Proof of Proposition 4.6.* The idea of the proof is very simple. By running a suitable minimal model program, we reduce the problem to Lemma 4.5. We note that we need the minimal model program established in [EH2] in Step 1. The minimal model program treated in [Fuj11] is sufficient for Step 2.

**Step 1.** In this step, we assume that $\lfloor \Delta \rfloor$ is not dominant onto $Z$. Under this assumption, we will prove that $\lfloor \Delta \rfloor \cap f^{-1}(z)$ is connected for every $z \in \pi_Z^{-1}(W)$.

We take an arbitrary point $z \in \pi_Z^{-1}(W)$ and a Stein compact subset $W_z$ of $Z$ such that $\Gamma(W_z, \mathcal{O}_Z)$ is noetherian and $z \in W_z$. By [EH2, Theorem 1.2], we can run a $(K_X + \Delta - \varepsilon\lfloor \Delta \rfloor)$-minimal model program with ample scaling over $Z$ around $W_z$. We finally get a commutative diagram

$$\begin{array}{ccc} X & \overset{p}{\dashrightarrow} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Z & \end{array}$$

around $W_z$ such that $p$ is a finite composite of flips and divisorial contrations and that $K_{X'} + \Delta' - \varepsilon\lfloor \Delta' \rfloor$ is nef over $W_z$. This implies that $\lfloor \Delta' \rfloor \cap f'^{-1}(z)$ is connected, where $f' \colon X' \to Z$ is the structure morphism. Since the number of the connected components of $\lfloor \Delta \rfloor \cap f^{-1}(z)$ is preserved by the above minimal model program by Lemma 4.1, $\lfloor \Delta \rfloor \cap f^{-1}(z)$ is connected.

**Step 2.** In this step, we assume that $\lfloor \Delta \rfloor$ is dominant onto $Z$. Then $K_X + \Delta - \varepsilon\lfloor \Delta \rfloor$ is not pseudo-effective over $Z$. By [Fuj11, Theorem 1.1 and Lemma 9.4], we can run a $(K_X + \Delta - \varepsilon\lfloor \Delta \rfloor)$-minimal model program with ample scaling over $Z$ around $W_Z := \pi_Z^{-1}(W)$. Then we obtain a finite sequence of divisorial contractions and flips

$$p \colon X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m =: X'$$

such that there exists a $(K_{X'} + \Delta' - \varepsilon\lfloor \Delta' \rfloor)$-negative extremal Fano contraction $\varphi \colon X' \to V$ over $Z$. If $\dim V \leq \dim X - 2$, then $\lfloor \Delta' \rfloor \cap \varphi^{-1}(v)$ is connected for every $v \in V$ since $\lfloor \Delta' \rfloor$ is $\varphi$-ample. This implies that $\lfloor \Delta' \rfloor \cap f'^{-1}(z)$ is connected for every $z \in \pi_Z^{-1}(W)$. Since the above minimal model program preserves the number of the connected components of $\lfloor \Delta \rfloor \cap f^{-1}(z)$ by Lemma 4.1, $\lfloor \Delta \rfloor \cap f^{-1}(z)$ is connected for every $z \in W$. Hence, from now, we may assume that $\dim V = \dim X - 1$. In this case, we have already described the situation in Lemma 4.5. Case (III) (resp. (II)) in Lemma 4.5 implies (i) (resp. (ii)).

We finish the proof of Proposition 4.6. □

Before we explain our gluing argument, we prepare an elementary but important lemma. Here, we need the finiteness of relative log pluricanonical representations (see Corollary 1.3).

**Lemma 4.7** ([Fuj1, Lemma 4.6])**.** *Let $(X, \Delta)$ be an equidimensional (not necessarily connected) divisorial log terminal pair and let $\pi \colon X \to Y$ be a projective morphism of complex analytic spaces. Let $W$ be a compact subset of $Y$ and let $U$ be a Stein open subset of $Y$ with $U \subset W$. We further assume that $U$ is semianalytic. We put $G := \rho_m^{WU}(\mathrm{Bim}(X/Y, \Delta; W))$. Then $G$ is a finite group. We put $X_U := \pi^{-1}(U)$. If*

$$s \in \mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X + \Delta))\right),$$

*then $g^*s|_{\lfloor \Delta \rfloor} = s|_{\lfloor \Delta \rfloor}$ and*

$$g^*s \in \mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X + \Delta))\right)$$

*for every $g \in G$. In particular,*

$$\sum_{g \in G} g^*s \in A\left(X_U, \mathcal{O}_X(m(K_X + \Delta))\right),$$

$$\prod_{g \in G} g^*s \in A\left(X_U, \mathcal{O}_X(m|G|(K_X + \Delta))\right),$$

*and*

$$\prod_{g \in G} g^*s|_{\lfloor \Delta \rfloor} = \left(s|_{\lfloor \Delta \rfloor}\right)^{|G|}.$$

*Of course,*

$$\frac{1}{|G|}\sum_{g\in G} g^*s|_{\lfloor \Delta \rfloor} = s|_{\lfloor \Delta \rfloor}$$

*holds.*

*Proof.* By Corollary 1.3, $G$ is a finite group. Then, by Lemma 2.28, it is not difficult to see that the proof of [Fuj1, Lemma 4.9] works in our complex analytic setting. Hence we omit the details here. □

Proposition 4.8, which is essentially the same as [Fuj1, Proposition 4.5], is a key step of our gluing argument.

**Proposition 4.8** ([Fuj1, Proposition 4.5])**.** *Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces such that $(X, \Delta)$ is divisorial log terminal. Let $U$ be a semianalytic Stein open subset of $Y$ and let $W$ be a Stein compact subset of $Y$ with $U \subset W$ such that $X$ is $\mathbb{Q}$-factorial over $W$ and that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. We put $S := \lfloor \Delta \rfloor$, $X_U := \pi^{-1}(U)$, and $S_U := S|_{\pi^{-1}(U)}$. Assume that*

1. *$K_X + \Delta$ is $\pi$-semiample, and*
2. *$\mathrm{A}\,(S_U, \mathcal{O}_S(m_0(K_X+\Delta)))$ generates $\mathcal{O}_S(m_0(K_X+\Delta))$ over $U$ for some positive integer $m_0$.*

*If necessary, we replace $U$ with a smaller semianalytic Stein open subset of $Y$. Then there exists a positive integer $m_1$ such that $m_1m_0 \in 2\mathbb{Z}$, the natural restriction map*

$$\mathrm{PA}\,(X_U, \mathcal{O}_X(m_1m_0(K_X+\Delta))) \to \mathrm{A}\,(S_U, \mathcal{O}_S(m_1m_0(K_X+\Delta)))$$

*is surjective, and $\mathrm{PA}\,(X_U, \mathcal{O}_X(m_1m_0(K_X+\Delta)))$ generates $\mathcal{O}_X(m_1m_0(K_X+\Delta))$ over $U$.*

The proof below is essentially the same as that of [Fuj1, Proposition 4.5]. We describe it for the reader's convenience.

*Proof of Proposition 4.8.* It is sufficient to prove this proposition for each connected component of $X$. Hence we may assume that $X$ is irreducible. Throughout this proof, we will freely shrink $Y$ around $W$ without mentioning it explicitly. We first take a relative Iitaka fibration $f\colon X \to Z$ over $Y$, that is, $f\colon X \to Z$ is a projective surjective morphism of normal complex analytic varieties such that $f_*\mathcal{O}_X \simeq \mathcal{O}_Z$ and that $\mathcal{O}_X(m(K_X+\Delta)) \simeq f^*\mathcal{L}$ holds for some positive integer $m$ and a $\pi_Z$-ample line bundle $\mathcal{L}$ on $Z$, where $\pi_Z\colon Z \to Y$ is the structure morphism.

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Z \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_Z} & \\ & Y & \end{array}$$

If $S = \lfloor \Delta \rfloor = 0$, then there is nothing to prove. Therefore, we may assume that $S = \lfloor \Delta \rfloor \neq 0$. Then we have the following four cases:

1. $Z$ is a point and $S$ is connected,
2. $\dim Z \geq 1$, $S \cap f^{-1}(z)$ is connected for every $z \in Z$, and $f(S) = Z$,
3. $\dim Z \geq 1$, $S \cap f^{-1}(z)$ is connected for every $z \in Z$, and $f(S) \subsetneq Z$, and
4. $S \cap f^{-1}(z)$ is not connected for some $z \in Z$.

**Step 1.** In this step, we will treat (1).

When $Z$ is a point, $X$ is projective and $K_X + \Delta \sim_{\mathbb{Q}} 0$. We consider the following long exact sequence:

$$\begin{aligned} 0 &\to H^0(X, \mathcal{O}_X(m_0(K_X+\Delta) - S)) \to H^0(X, \mathcal{O}_X(m_0(K_X+\Delta))) \\ &\to H^0(S, \mathcal{O}_S(m_0(K_X+\Delta))) \to \cdots . \end{aligned}$$

Since $K_X + \Delta \sim_{\mathbb{Q}} 0$ and $S \neq 0$, we obtain that $H^0(X, \mathcal{O}_X(m_0(K_X+\Delta) - S)) = 0$ and that the second and the third terms are one-dimensional. Hence we obtain the desired statement.

**Step 2.** In this step, we will treat (2).

By taking a divisible positive integer $m$ such that $\mathrm{A}(S_U, \mathcal{O}_S(m(K_X+\Delta)))$ generates $\mathcal{O}_S(m(K_X+\Delta))$ over $U$ and that $\mathcal{O}_X(m(K_X+\Delta)) \simeq f^*\mathcal{L}$ holds for some $\pi_Z$-ample line bundle $\mathcal{L}$ on $Z$. If necessary, we replace $U$ with a smaller relatively compact semianalytic Stein open subset of $Y$. By $\mathrm{A}\,(S_U, \mathcal{O}_S(m(K_X+\Delta)))$, we can construct a morphism $\Phi\colon S \to Z'$ over $U$. Since every curve in any fiber of $f|_S$ over $U$ is mapped to a point by $\Phi$, there exists a morphism $\Psi\colon Z \to Z'$ over $U$ such that $\Psi \circ (f|_S) = \Phi$. Over $U$, there exists the following commutative diagram.

$$\begin{array}{ccc} X & \hookleftarrow & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Phi} \\ Z & \xrightarrow[\Psi]{} & Z' \end{array}$$

We note that

$$\Phi\colon S \xrightarrow{f|_S} Z \xrightarrow{\Psi} Z'$$

and that $f|_S$ is surjective with connected fibers. For any

$$s \in \mathrm{A}\left(S_U, \mathcal{O}_S(m(K_X+\Delta))\right),$$

we can take $t$ such that $s = \Phi^* t$. We put $u := f^*\Psi^* t$. Then

$$u \in \mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)$$

such that $u|_S = s$. By construction, $\mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)$ generates $\mathcal{O}_X(m(K_X + \Delta))$ over $U$.

**Step 3.** In this step, we will treat (3).

This step is a relative version of [Fuj1, Lemma 4.3]. We take a divisible positive integer $m$ such that $\mathcal{O}_X(m(K_X+\Delta)) \simeq f^*\mathcal{L}$ for some $\pi_Z$-ample line bundle $\mathcal{L}$ on $Z$.

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & Z \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_Z} & \\ & Y & \end{array}$$

We put $T := f(S) \subsetneq Z$. Then $f_*\mathcal{O}_S \simeq \mathcal{O}_T$ by Corollary 4.3. Therefore, we have the following commutative diagram:

(4.1)
$$\begin{array}{ccc} \pi_*\mathcal{O}_X(lm(K_X+\Delta)) & \longrightarrow & \pi_*\mathcal{O}_S(lm(K_S+\Delta_S)) \\ \simeq\uparrow & & \uparrow\simeq \\ \pi_{Z*}\mathcal{L}^{\otimes l} & \longrightarrow & \pi_{Z*}\left(\mathcal{L}^{\otimes l}|_T\right). \end{array}$$

Note that the vertical arrows are isomorphisms. If we replace $U$ with a relatively compact semianalytic Stein open subset and make $l$ sufficiently large, then $\mathcal{L}^{\otimes l}\otimes\mathcal{I}_T$ is $\pi_Z$-generated over $U$, where $\mathcal{I}_T$ is the defining ideal sheaf of $T$ on $Z$, and $R^1\pi_{Z*}\left(\mathcal{L}^{\otimes l}\otimes\mathcal{I}_T\right)=0$ since $\mathcal{L}$ is $\pi_Z$-ample. Thus, by (4.1), we have the following short exact sequence:

$$(4.2)\quad 0\to\pi_*\mathcal{O}_X(lm(K_X+\Delta)-S)\to\pi_*\mathcal{O}_X(lm(K_X+\Delta))\to\pi_*\mathcal{O}_S(lm(K_S+\Delta_S))\to 0.$$

By definition, it is obvious that every element of $H^0(X_U,\mathcal{O}_X(lm(K_X+\Delta)-S))$ is contained in $\mathrm{PA}(X_U,\mathcal{O}_X(lm(K_X+\Delta)))$. By (4.2), we can extend

$$\mathrm{A}\left(S_U,\mathcal{O}_S(lm(K_S+\Delta_S))\right)$$

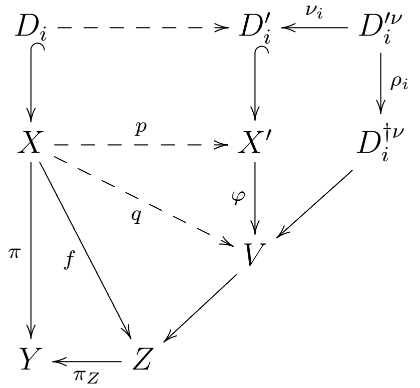
to

$$\mathrm{PA}\left(X_U,\mathcal{O}_X(lm(K_X+\Delta))\right)$$

and check that $\mathrm{PA}\left(X_U,\mathcal{O}_X(lm(K_X+\Delta))\right)$ generates $\mathcal{O}_X(lm(K_X+\Delta))$ over $U$.

**Step 4.** In this step, we will treat (4).

In this case, we can run a $(K_X+\Delta-\varepsilon\lfloor\Delta\rfloor)$-minimal model program with ample scaling over $Z$ around $W_Z:=\pi_Z^{-1}(W)$ (see [Fuj11, Theorem 1.2 and Lemma 9.4]) and finally get $(X',\Delta')$ and a $(K_{X'}+\Delta'-\varepsilon\lfloor\Delta'\rfloor)$-negative extremal Fano contraction $\varphi\colon X'\to V$ as in Lemma 4.5. Then we have (II) or (III) in Lemma 4.5. From now, we will freely use the notation in Lemma 4.5 and its proof. We note that $p\colon X\dashrightarrow X'$ is $B$-bimeromorphic over $Y$. The situation is summarized in the following commutative diagram.




We take any element $s$ of $\mathrm{PA}\left(X_U,\mathcal{O}_X(m(K_X+\Delta))\right)$. By Remark 2.20, we note that there exists a natural isomorphism

$$H^0\left(X_U,\mathcal{O}_X(m(K_X+\Delta))\right)\simeq H^0\left(X'_U,\mathcal{O}_{X'}(m(K_{X'}+\Delta'))\right)$$

induced by $p$, where $\pi'\colon X'\to Y$ and $X'_U:=\pi'^{-1}(U)$. Hence $s$ induces

$$s'\in H^0\left(X'_U,\mathcal{O}_{X'}(m(K_{X'}+\Delta'))\right).$$

Let $m$ be a sufficiently large and divisible positive integer such that $\mathcal{O}_X(m(K_X+\Delta))\simeq f^*\mathcal{L}$ for some line bundle $\mathcal{L}$ on $Z$. The section $s'$ induces a section

$$s''_i\in H^0(D'^{\nu}_i,\mathcal{O}_{D'^{\nu}_i}(m(K_{D'^{\nu}_i}+\Delta_{D'^{\nu}_i})))$$

over $U$ for $i=1,2$. In Case (III), $s''_1$ is $\iota$-invariant. Hence $s''_1$ descends to a section $t$ of $\mathcal{L}_V$ over $U$, where $\mathcal{L}_V$ is the pull-back of $\mathcal{L}$ to $V$. In Case (II), $s''_1$ also naturally descends to a section $t$ of $\mathcal{L}_V$ over $U$. In Case (III), the pull-back of $\varphi^*t$ to $D'^{\nu}_1$ coincides with $s''_1$ by construction. In Case (II), on a small open subset $\widetilde{U}$ of $U$ such that $\varphi^{-1}(\widetilde{U})\simeq\mathbb{P}^1\times\widetilde{U}$ and that $\varphi|_{\varphi^{-1}(\widetilde{U})}\colon\mathbb{P}^1\times\widetilde{U}\to\widetilde{U}$ is the second projection, the difference between $s''_2$ and the pull-back of $\varphi^*t$ to $D'^{\nu}_2$ is at most $(-1)^m$ (see the proof of [AFKM, 12.3.4 Theorem]). By construction, it is obvious that the pull-back of $\varphi^*t$ to $D_1'^{\nu}$ coincides with $s_1''$. Hence, we have $s'|_{(\Delta')^h} = (\varphi^*t)|_{(\Delta')^h}$ holds if $m$ is even. From now, we will see that $(\varphi^*t)|_{\lfloor\Delta'\rfloor} = s'|_{\lfloor\Delta'\rfloor}$ holds as in Case 4 in the proof of [Fuj1, Proposition 4.5]. Let $(\Delta')^v$ be the vertical part of $\lfloor\Delta'\rfloor$. We can write $(\Delta')^v = \sum_i \varphi^*P_i$ such that $Q_i := \operatorname{Supp} P_i$ is a prime divisor on $V$ for every $i$ and $Q_i \neq Q_j$ for $i \neq j$. We put $E_i := \varphi^*P_i$. Then it is sufficient to check that $s'|_{E_i} = (\varphi^*t)|_{E_i}$ holds for every $i$. Let $F_i$ be an irreducible component of $E_i \cap (\Delta')^h$ such that $\varphi\colon F_i \to Q_i$ is dominant. Since $(\Delta')^h \cap (\Delta')^v \neq \emptyset$, we can always take such $F_i$. We consider the following commutative diagram:

$$\begin{array}{ccc}
\pi_*\mathcal{O}_{E_i}(m(K_{X'}+\Delta')) & \longrightarrow & \pi_*\mathcal{O}_{F_i}(m(K_{X'}+\Delta')) \\
\simeq \big\uparrow & & \big\uparrow j \\
\pi_{V*}\left(\mathcal{L}_V|_{Q_i}\right) & = & \pi_{V*}\left(\mathcal{L}_V|_{Q_i}\right),
\end{array}$$

where $\pi_V\colon V \to Y$ is the structure morphism. The left vertical arrow is an isomorphism by Lemma 4.2. The map $j$ is injective since $\varphi\colon F_i \to Q_i$ is dominant. Since $s'|_{F_i} = (\varphi^*t)|_{F_i}$, we have $s'|_{E_i} = (\varphi^*t)|_{E_i}$ for every $i$. Thus we have $s'|_{\lfloor\Delta'\rfloor} = (\varphi^*t)|_{\lfloor\Delta'\rfloor}$. This means that $s$ can be lifted to a member of $\mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)$. By construction, it is not difficult to see that $\mathrm{PA}\left(X_U, \mathcal{O}_X(m(K_X+\Delta))\right)$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over $U$.

We finish the proof. □

We need the following lemma for inductive gluing arguments.

**Lemma 4.9** ([Fuj1, Lemma 4.7])**.** *In Proposition 4.8, we can replace*

$$\mathrm{PA}\left(X_U, \mathcal{O}_X(m_1m_0(K_X+\Delta))\right)$$

*with*

$$\mathrm{A}\left(X_U, \mathcal{O}_X(m_1m_0(K_X+\Delta))\right)$$

*if we make $m_1$ sufficiently divisible.*

*Proof.* We put $G := \rho_m^{WU}\left(\mathrm{Bim}(X/Y, \Delta; W)\right)$. Then $G$ is a finite group. For any $s \in \mathrm{PA}(X_U, \mathcal{O}_X(m(K_X+\Delta)))$, we define

$$t := \frac{1}{|G|}\sum_{g\in G} g^*s.$$

Then $t \in \mathrm{A}(X_U, \mathcal{O}_X(m(K_X+\Delta)))$ such that $t|_S = s|_S$ by Lemma 4.7. We put $G := \{g_1, \cdots, g_N\}$ with $N := |G|$. Let $\sigma_i$ be the $i$th elementary symmetric polynomial for $1 \leq i \leq N$. Then we have

$$\{s=0\} \supset \bigcap_{j=1}^{N}\{g_j^*s=0\} = \bigcap_{i=1}^{N}\{\sigma_i(g_1^*s, \cdots, g_N^*s)=0\}.$$

Therefore, by considering

$$\sigma_i^{N!/i}(g_1^*s, \cdots, g_N^*s) \in \mathrm{A}(X_U, \mathcal{O}_X(N!m(K_X+\Delta)))$$

for $s \in \mathrm{PA}(X_U, \mathcal{O}_X(m(K_X+\Delta)))$, we can check that $\mathrm{A}(X_U, \mathcal{O}_X(N!m(K_X+\Delta)))$ generates $\mathcal{O}_X(N!m(K_X+\Delta))$ over $U$ under the assumption that $\mathrm{PA}(X_U, \mathcal{O}_X(m(K_X+\Delta)))$ generates $\mathcal{O}_X(m(K_X+\Delta))$ over $U$. Thus we can obtain the desired statement of Lemma 4.9. We finish the proof. □

By Proposition 4.8 and Lemma 4.9, we have:

**Lemma 4.10** (Abundance for semi-divisorial log terminal pairs in the complex analytic setting)**.** *Let $(X, \Delta)$ be a semi-divisorial log terminal pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Let $W$ be a Stein compact subset of $Y$ such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Assume that $K_X + \Delta$ is $\pi$-semiample. Let $P$ be an arbitrary point of $W$. Then there exists a semianalytic Stein open neighborhood $U_P$ of $P$ and a positive integer $m$ such that admissible sections generate $\mathcal{O}_X(m(K_X + \Delta))$ over $U_P$.*

*Proof.* Let $\nu\colon X^\nu \to X$ be the normalization. By definition, we see that any admissible section on $X^\nu$ descends to an admissible section on $X$ since $X$ is simple normal crossing in codimension one and satisfies Serre's $S_2$ condition. Hence, by taking the normalization, we may assume that $X$ is normal. By [Fuj11, Theorems 1.21 and 1.27], we take a dlt blow-up and may assume that $X$ is $\mathbb{Q}$-factorial over $W$, By Proposition 4.8 and Lemma 4.9, it is sufficient to prove this lemma for $(S, \Delta_S)$, where $S := \lfloor \Delta \rfloor$ and $K_S + \Delta_S := (K_X + \Delta)|_S$. By repeating this process finitely many times, we can reduce the problem to the case where $(X, \Delta)$ is kawamata log terminal. In this case, any section is preadmissible (see Remark 2.27). Thus, by Lemma 4.9, we obtain the desired result. □

Let us prove Theorem 1.1, which is one of the main results of this paper.

*Proof of Theorem 1.1.* We take an arbitrary point $P \in W$. Since $W$ is compact, it is sufficient to prove that there exists a positive integer $m_P$ such that $\mathcal{O}_X(m_P(K_X + \Delta))$ is $\pi$-generated over some open neighborhood of $P$. We take a semianalytic Stein open neighborhood $U_P$ of $P$ and a Stein compact subset $W_P$ of $Y$ with $U_P \subset W_P$ such that $\Gamma(W_P, \mathcal{O}_Y)$ is noetherian. Let $\nu\colon X^\nu \to X$ be the normalization with $K_{X^\nu} + \Theta := \nu^*(K_X + \Delta)$. By [Fuj11, Theorems 1.21 and 1.27], after shrinking $Y$ around $W_P$ suitably, we take a dlt blow-up $\alpha\colon \widetilde{X} \to X$ with $K_{\widetilde{X}} + \widetilde{\Delta} := \alpha^*(K_{X^\nu} + \Theta)$ such that $\widetilde{X}$ is $\mathbb{Q}$-factorial over $W_P$ and $(\widetilde{X}, \widetilde{\Delta})$ is divisorial log terminal. We consider $\widetilde{\pi} := \pi \circ \nu \circ \alpha\colon \widetilde{X} \to Y$. If necessary, we replace $U_P$ with a smaller semianalytic Stein open neighborhood of $P$. Then, by Lemma 4.10, there exists a semianalytic Stein open neighborhood $U_P$ and a positive integer $m_P$ such that admissible sections generate $\mathcal{O}_{\widetilde{X}}(m_P(K_{\widetilde{X}} + \widetilde{\Delta}))$ over $U_P$. Note that $X$ is normal crossing in codimension one and satisfies Serre's $S_2$ condition since $(X, \Delta)$ is semi-log canonical. Hence any admissible section descends to a section of $\mathcal{O}_X(m_P(K_X + \Delta))$. Thus $\mathcal{O}_X(m_P(K_X + \Delta))$ is $\pi$-generated over $U_P$. As we mentioned above, since $W$ is compact, we can take an open neighborhood $U$ of $W$ and a divisible positive integer $m$ such that $\mathcal{O}_X(m(K_X + \Delta))$ is $\pi$-generated over $U$. We finish the proof of Theorem 1.1. □

By using the following easy lemma, we can check that [HX, Theorem 2] follows from Theorem 1.1. This means that we do not need Kollár's gluing theory in [K] for the proof of [HX, Theorem 2].

**Lemma 4.11.** *Let $\pi\colon X \to Y$ be a proper morphism of algebraic schemes defined over $\mathbb{C}$ and let $\mathcal{L}$ be a line bundle on $X$. Let $U$ be a nonempty open subset of $Y$ in the classical topology. Assume that $\mathcal{L}$ is $\pi$-generated over $U$. Then there exists a Zariski open subset $V$ of $Y$ such that $\mathcal{L}$ is $\pi$-generated over $V$ with $U \subset V$.*

*Proof.* Let $\mathcal{C}$ be the cokernel of $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$. We put $V := Y \setminus \pi(\operatorname{Supp} \mathcal{C})$. Then, by definition, $V$ is a Zariski open subset with $U \subset V$ and $\mathcal{L}$ is $\pi$-generated over $V$. □

## 5. Freeness for nef and log abundant log canonical bundles

In this section, we will prove Theorem 1.4. Then we will prove Theorem 1.9 as an easy application of Theorem 1.4. We will also prove Theorem 1.5 and Corollary 1.10. For the proof of Theorem 1.4, we first treat the following theorem. In the algebraic setting, it is well known (see [Fuj6]). Once we know Theorem 5.1, it is not difficult to prove Theorem 1.4.

**Theorem 5.1.** *Let $(X, \Delta)$ be an irreducible divisorial log terminal pair and let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces. Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and is $\pi$-nef and $\pi$-abundant over $Y$. We further assume that $K_S + \Delta_S$ is $\pi$-semiample, where $S := \lfloor \Delta \rfloor$ and $K_S + \Delta_S := (K_X + \Delta)|_S$. Let $W$ be a compact subset of $Y$. Then there exists a positive integer $m$ such that $\mathcal{O}_X(m(K_X + \Delta))$ is $\pi$-generated over some open neighborhood of $W$.*

*Proof.* We can modify the argument in [Fuj6, Section 6] for our complex analytic setting. Since the Kawamata–Viehweg vanishing theorem holds for projective morphisms of complex analytic spaces, we can generalize [Fuj6, Theorem 6.1], which is a slight generalization of the Kawamata–Shokurov basepoint-free theorem, for our complex analytic setting. By [Fuj11, Theorem 21.4], which is a kind of a canonical bundle formula, and the argument in Step 2 in the proof of [Fuj11, Theorem 23.2], we can prove a complex analytic generalization of [Fuj6, Theorem 6.2]. Therefore, we see that the desired statement holds (see also [Fuj6, Theorem 1.1]). □

Let us prove Theorem 1.4.

*Proof of Theorem 1.4.* Let $P$ be an arbitrary point of $W$. Since $W$ is compact, it is sufficient to prove that there exist a positive integer $m_P$ and an open neighborhood $U_P$ of $P$ such that $\mathcal{O}_X(m_P(K_X + \Delta))$ is $\pi$-generated over $U_P$. By Theorem 1.1, we may assume that $X$ is normal. By taking a Stein compact subset $W_P$ such that $P \in W_P$ and $\Gamma(W_P, \mathcal{O}_Y)$ is noetherian. By [Fuj11, Theorems 1.21 and 1.27], after shrinking $Y$ around $W_P$ suitably, we take a dlt blow-up and may assume that $(X, \Delta)$ is divisorial log terminal. By induction on dimension, we may assume that $K_S + \Delta_S$ is $\pi$-semiample over some open neighborhood of $P$, where $S := \lfloor \Delta \rfloor$ and $K_S + \Delta_S := (K_X + \Delta)|_S$. Hence, by Theorem 5.1, we obtain the desired statement. We finish the proof. □

The following proof is essentially due to Kenta Hashizume (see [H2, Lemma 3.4]).

*Proof of Theorem 1.5.* We can freely shrink $Y$ around $W$ suitably and always assume that $Y$ is Stein. By taking a dlt blow-up (see [BCHM, Theorems 1.21 and 1.27]), we may assume that $(X, \Delta)$ is divisorial log terminal and is $\mathbb{Q}$-factorial over $W$. By induction, we may assume that $K_S + \Delta_S := (K_X + \Delta)|_S$ is $\pi$-semiample over some open neighborhood of $L$ for every log canonical center $S$ of $(X, \Delta)$. By applying the argument in the proof of [H2, Lemma 3.4], we can write $K_X + \Delta = \sum_i r_i(K_X + \Delta_i)$ such that $(X, \Delta_i)$ is divisorial log terminal, $K_X + \Delta_i$ is $\mathbb{Q}$-Cartier, $r_i$ is a positive real number, and $K_X + \Delta_i$ is $\pi$-nef and $\pi$-log abundant over some open neighborhood of $L$ for every $i$. Hence, by Theorem 1.4, there exists a positive integer $m_i$ such that $\mathcal{O}_X(m_i(K_X + \Delta_i))$ is $\pi$-generated over some open neighborhood of $L$. Hence, $K_X + \Delta$ is $\pi$-semiample over some open neighborhood of $L$. This is what we wanted. □

Theorem 1.7 is almost obvious by Theorem 1.5 and [EH2, Theorem 1.2].

*Proof of Theorem 1.7.* We take an arbitrary point $P \in Z$. Then it is sufficient to prove the existence of a log canonical model of $(X, \Delta)$ over some open neighborhood of $P$. We take $P \in U_1 \subset W_1 \subset U_2 \subset W_2$, where $U_i$ is a Stein open subset of $Z$ for $i = 1, 2$ and $W_i$ is a Stein compact subset of $Z$ such that $\Gamma(W_i, \mathcal{O}_Z)$ is noetherian for $i = 1, 2$. Throughout this proof, we can freely shrink $Z$ around $W_2$ suitably. Since $-(K_X + \Delta)$ is $\varphi$-ample, we can take an effective $\mathbb{R}$-divisor $A$ on $X$ such that $K_X + \Delta + A \sim_{\mathbb{R},\varphi} 0$ and that $(X, \Delta + A)$ is log canonical. By [Fuj11, Theorems 1.21 and 1.27], we take a dlt blow-up $p\colon (X', \Delta') \to (X, \Delta)$ over some open neighborhood of $W_2$. We note that $(X', \Delta' + A')$ is log canonical with $K_{X'} + \Delta' + A' \sim_{\mathbb{R},\varphi'} 0$, where $A' := p^*A$ and $\varphi' := \varphi \circ p\colon X' \to Z$. It is sufficient to construct a log canonical model of $(X', \Delta')$ over some open neighborhood of $P$. By [EH2, Theorem 1.2], after finitely many flips and divisorial contractions, we finally obtain $(X'', \Delta'')$ over some open neighborhood of $W_2$ such that $K_{X''} + \Delta''$ is nef over $W_2$. By construction, $K_{X''} + \Delta'' + A'' \sim_{\mathbb{R},\varphi''} 0$ holds, where $A''$ is the pushforward of $A'$ on $X''$ and $\varphi''\colon X'' \to Z$ is the structure morphism. Thus, by [G1, Theorem 6.1], we can check that $K_{X''} + \Delta''$ is $\varphi''$-nef and $\varphi''$-log abundant with respect to $(X'', \Delta'')$ over $U_2$ (see also [H3, Remark 3.7]). Therefore, by Theorem 1.5, $K_{X''} + \Delta''$ is $\varphi''$-semiample over some open neighborhood of $P$. This means that $(X', \Delta')$ has a log canonical model over some open neighborhood of $P$. This is what we wanted. We finish the proof. □

We prove Theorem 1.9 as an application of Theorem 1.4.

*Proof of Theorem 1.9.* Let $P$ be an arbitrary point of $W$. Since $W$ is compact, it is sufficient to prove that there exist a positive integer $m_P$ and an open neighborhood $U_P$ of $P$ such that $\mathcal{O}_X(m_P(K_X + \Delta))$ is $\pi$-generated over $U_P$. From now, we will freely shrink $Y$ around $P$. By [Fuj11, Theorems 1.21 and 1.27], we take a dlt blow-up. Thus we may assume that $(X, \Delta)$ is divisorial log terminal. Let $S$ be a log canonical stratum of $(X, \Delta)$ with $K_S + \Delta_S := (K_X + \Delta)|_S$. It is obvious that $K_S + \Delta_S$ is $\pi$-nef. By applying Conjecture 1.8 to an analytically sufficiently general fiber $F$ of $S \to \pi(S)$, we see that $K_S + \Delta_S$ is $\pi$-nef and $\pi$-abundant. This means that $K_X + \Delta$ is $\pi$-nef and $\pi$-log abundant with respect to $(X, \Delta)$. Hence, by Theorem 1.4, we obtain $m_P$ such that $\mathcal{O}_X(m_P(K_X + \Delta))$ is $\pi$-generated over some open neighborhood of $P$. We finish the proof. □

Let us prove Corollary 1.10, which is an easy application of Theorem 1.9.

*Proof of Corollary 1.10.* We can freely shrink $Y$ around $W$. By using Shokurov's polytope (see [Fuj11]), we can write $K_X + \Delta = \sum_i r_i(K_X + \Delta_i)$ such that $(X, \Delta_i)$ is log canonical, $K_X + \Delta_i$ is $\mathbb{Q}$-Cartier, $r_i$ is a positive real number, and $K_X + \Delta_i$ is $\pi$-nef over $W$ for every $i$. In particular, $K_X + \Delta_i$ is $\pi$-nef over $U$ for every $i$. Then, by Theorem 1.9, there exists a positive integer $m_i$ such that $\mathcal{O}_X(m_i(K_X + \Delta_i))$ is $\pi$-generated over some open neighborhood of $L$ for every $i$. This implies that $K_X + \Delta$ is $\pi$-semiample over some open neighborhood of $L$. We finish the proof. □

Anyway, by Theorem 1.9 and Corollary 1.10, we are released from the abundance conjecture for projective morphisms of complex analytic spaces. We close this section with an important conjecture.

**Conjecture 5.2.** *Let $\pi\colon X \to Y$ be a projective surjective morphism of normal complex varieties and let $(X, \Delta)$ be a log canonical pair. Let $W$ be a compact subset of $Y$. Assume that $K_X + \Delta$ is $\pi$-nef over $W$. Then $K_X + \Delta$ is $\pi$-nef over some open neighborhood of $W$.*

If Conjecture 5.2 holds true, then we can prove that $K_X+\Delta$ is $\pi$-semiample over some open neighborhood of $W$ in Theorem 1.5 and Corollary 1.10.

## 6. Supplementary comments

In this final section, we will make some supplementary comments on [Fuj1] and [FG] for the reader's convenience.

**6.1.** In [FG, 2.20] and the proof of [FG, Theorem 4.3], we claim that we can freely use the results in [Fuj1, Section 2] by [BCHM]. However, in order to prove [Fuj1, Proposition 2.1] in dimension $n \geq 4$ (see also [Fuj1, Remark 2.2]), the minimal model program with scaling established in [BCHM] is not sufficient. We need the following result.

**Theorem 6.2** (cf. [Bir, Theorem 5.2])**.** *Let $\pi\colon X \to Y$ be a projective surjective morphism of normal quasi-projective varieties and let $(X, \Delta)$ be a $\mathbb{Q}$-factorial divisorial log terminal pair such that $K_X + \Delta \sim_{\mathbb{Q},\pi} 0$. Assume that $\pi(\lfloor \Delta \rfloor) \subsetneq Y$, that is, $\lfloor \Delta \rfloor$ is vertical with respect to $\pi$. Then $(X, \Delta - \varepsilon\lfloor \Delta \rfloor)$ has a good minimal model over $Y$ for every rational number $\varepsilon$ with $0 < \varepsilon \leq 1$. In particular, every $(K_X + \Delta - \varepsilon\lfloor \Delta \rfloor)$-minimal model program with ample scaling over $Y$ always terminates.*

If $\pi(\lfloor \Delta \rfloor) = Y$, then $K_X+\Delta-\varepsilon\lfloor \Delta \rfloor$ is not $\pi$-pseudo-effective for every rational number $\varepsilon$ with $0 < \varepsilon \leq 1$. In this case, the minimal model program proved in [BCHM] is sufficient for the proof of [Fuj1, Proposition 2.1] in dimension $n \geq 4$. Theorem 6.2 follows from [Bir]. We note that $(X, \Delta - \varepsilon\lfloor \Delta \rfloor)$ is kawamata log terminal for every rational number $\varepsilon$ with $0 < \varepsilon \leq 1$. Hence, for the proof of Theorem 6.2, we need no deep results on the abundance conjecture for log canonical pairs. There is no circular reasoning even if we use [Bir, Theorem 5.2] in [FG]. For the details, see [Bir, Theorem 5.2]. We also note that the most general result in this direction is treated in [H1]. By [BCHM] and Theorem 6.2 above, we can freely use the results in [Fuj1, Section 2] in dimension $n \geq 4$. Therefore, there are no serious troubles in [FG]. In this paper, in Step 1 in the proof of Proposition 4.6, we use [EH2, Theorem 1.2] instead of Theorem 6.2 above. The minimal model program established in [Fuj11] is insufficient for the proof of Proposition 4.6.

**6.3.** We make a small remark on [Fuj1, Lemma 2.3] for the reader's convenience. In the proof of [Fuj1, Lemma 2.3], we claim that there exists a $\mathbb{Q}$-divisor $P$ on $V$ satisfying $K_{D_i} + \mathrm{Diff}(\Delta - D_i) = u|^*_{D_i}(K_V + P)$. However, it is not clear when $D_1$ is irreducible and the mapping degree $\deg[D_1 : V] = 2$. In that case, we can not apply [AFKM, 12.3.4 Theorem].

**Example 6.4.** We put $Z := \mathbb{P}^1 \times \mathbb{P}^1$. Let $\Delta$ be a general member of $|p_1^*\mathcal{O}_{\mathbb{P}^1}(2) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(2)|$, where $p_i$ is the $i$th projection for $i = 1, 2$. Then $\Delta$ is a smooth elliptic curve and $K_Z+\Delta \sim 0$. We consider the first projection $h\colon Z \to R := \mathbb{P}^1$. In this setting, $u := h\colon Z \to V := R$ is a $(K_Z+\Delta-\varepsilon\lfloor \Delta \rfloor)$-negative extremal Fano contraction over $R$. Of course, the horizontal part $\Delta^h =: D_1$ of $\lfloor \Delta \rfloor$ is irreducible and the mapping degree $\deg[D_1 : V]$ is two. In [Fuj1, Lemma 2.3], we claim that there exists an effective $\mathbb{Q}$-divisor $P$ on $V$ such that $K_{D_1} = u|^*_{D_1}(K_V + P)$ holds without explaining it explicitly. It is somewhat misleading when $D_1$ is irreducible with $\deg[D_1 : V] = 2$.

Fortunately, as we see in the proof of Lemma 4.5 in this paper, we do not have to construct a $\mathbb{Q}$-divisor $P$ on $V$ in Case (III). Hence, there are no serious troubles in the proof of [Fuj1, Lemma 2.3].

## References


[AFKM] D. Abramovich, L.-Y. Fong, J. Kollár, J. M$^c$Kernan, Semi log canonical surface, *Flips and Abundance for Algebraic Threefolds*, Astérisque **211**, Soc. Math. France, Montrouge, 1992, 139–154.

[BS] C. Bănică, O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*, Translated from the Romanian. Editura Academiei, Bucharest; John Wiley & Sons, London–New York–Sydney, 1976.

[BieM1] E. Bierstone, P. D. Milman, Semianalytic and subanalytic sets, Inst. Hautes Études Sci. Publ. Math. No. **67** (1988), 5–42.

[BieM2] E. Bierstone, P. D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. **128** (1997), no. 2, 207–302.

[Bir] C. Birkar, Existence of log canonical flips and a special LMMP, Publ. Math. Inst. Hautes Études Sci. **115** (2012), 325–368.

[BCHM] C. Birkar, P. Cascini, C. D. Hacon, J. M$^c$Kernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[CR] C. W. Curtis, I. Reiner, *Representation theory of finite groups and associative algebras*, Reprint of the 1962 original. AMS Chelsea Publishing, Providence, RI, 2006.

[DHP] O. Das, C. Hacon, M. Păun, On the 4-dimensional minimal model program for Kähler varieties, Adv. Math. **443** (2024), Paper No. 109615.

[EH1] M. Enokizono, K. Hashizume, Semistable reduction for complex analytic spaces, preprint (2024). arXiv:2404.05124 [math.AG]

[EH2] M. Enokizono, K. Hashizume, Minimal model program for log canonical pairs on complex analytic spaces, preprint (2024). arXiv:2404.05126 [math.AG]

[Fis] G. Fischer, *Complex analytic geometry*, Lecture Notes in Mathematics, Vol. **538**. Springer-Verlag, Berlin–New York, 1976.

[Fuj1] O. Fujino, Abundance theorem for semi log canonical threefolds, Duke Math. J. **102** (2000), no. 3, 513–532.

[Fuj2] O. Fujino, The indices of log canonical singularities, Amer. J. Math. **123** (2001), no. 2, 229–253.

[Fuj3] O. Fujino, On Kawamata's theorem, *Classification of algebraic varieties*, 305–315, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011.

[Fuj4] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[Fuj5] O. Fujino, On isolated log canonical singularities with index one, J. Math. Sci. Univ. Tokyo **18** (2011), no. 3, 299–323.

[Fuj6] O. Fujino, Basepoint-free theorems: saturation, b-divisors, and canonical bundle formula, Algebra Number Theory **6** (2012), no. 4, 797–823.

[Fuj7] O. Fujino, Addendum to "On isolated log canonical singularities with index one", J. Math. Sci. Univ. Tokyo **19** (2012), no. 1, 131–133.

[Fuj8] O. Fujino, A transcendental approach to Kollár's injectivity theorem II, J. Reine Angew. Math. **681** (2013), 149–174.

[Fuj9] O. Fujino, Fundamental theorems for semi log canonical pairs, Algebr. Geom. **1** (2014), no. 2, 194–228.

[Fuj10] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.

[Fuj11] O. Fujino, Minimal model program for projective morphisms between complex analytic spaces, preprint (2022). arXiv:2201.11315 [math.AG]

[Fuj12] O. Fujino, Vanishing theorems for projective morphisms between complex analytic spaces, to appear in Math. Res. Lett.

[Fuj13] O. Fujino, Cone and contraction theorem for projective morphisms between complex analytic spaces, preprint (2022).

[Fuj14] O. Fujino, Quasi-log structures on complex analytic spaces, preprint (2022). arXiv:2209.11401 [math.AG]

[Fuj15] O. Fujino, On vanishing theorems for analytic spaces, Proc. Japan Acad. Ser. A Math. Sci. **100** (2024), no. 4, 21–25.

[FF] O. Fujino, T. Fujisawa, Variation of mixed Hodge structure and its applications, preprint (2023). arXiv:2304.00672 [math.AG]

[FG] O. Fujino, Y. Gongyo, Log pluricanonical representations and the abundance conjecture, Compos. Math. **150** (2014), no. 4, 593–620.

[G1] Y. Gongyo, On the minimal model theory for dlt pairs of numerical log Kodaira dimension zero, Math. Res. Lett. **18** (2011), no. 5, 991–1000.

[G2] Y. Gongyo, Abundance theorem for numerically trivial log canonical divisors of semi-log canonical pairs, J. Algebraic Geom. **22** (2013), no. 3, 549–564.

[HX] C. D. Hacon, C. Xu, On finiteness of **B**-representations and semi-log canonical abundance, *Minimal models and extremal rays (Kyoto, 2011)*, 361–377, Adv. Stud. Pure Math., **70**, Math. Soc. Japan, [Tokyo], 2016.

[H1] K. Hashizume, Remarks on special kinds of the relative log minimal model program, Manuscripta Math. **160** (2019), no. 3-4, 285–314.

[H2] K. Hashizume, A class of singularity of arbitrary pairs and log canonicalizations, Asian J. Math. **24** (2020), no. 2, 207–238.

[H3] K. Hashizume, Finiteness of log abundant log canonical pairs in log minimal model program with scaling, to appear in Michigan Math. J.

[K] J. Kollár, *Singularities of the minimal model program.* With a collaboration of Sándor Kovács, Cambridge Tracts in Mathematics, **200**. Cambridge University Press, Cambridge, 2013.

[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties.* With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original, Cambridge Tracts in Mathematics, **134**. Cambridge University Press, Cambridge, 1998.

[LM] S. Lyu, T. Murayama, The relative minimal model program for excellent algebraic spaces and analytic spaces in equal characteristic zero, preprint (2022). arXiv:2209.08732 [math.AG]

[N] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, **14**. Mathematical Society of Japan, Tokyo, 2004.

[P] D. Prill, The divisor class groups of some rings of holomorphic functions, Math. Z. **121** (1971), 58–80.

[RRV] J. P. Ramis, G. Ruget, J. L. Verdier, Dualité relative en géométrie analytique complexe, Invent. Math. **13** (1971), 261–283.

[U] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Notes written in collaboration with P. Cherenack, Lecture Notes in Mathematics, Vol. **439**. Springer-Verlag, Berlin-New York, 1975.



Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

*Email address*: fujino@math.kyoto-u.ac.jp