ADJUNCTION FOR PURELY LOG TERMINAL PAIRS

OSAMU FUJINO

Abstract. We study adjunction for purely log terminal pairs in the complex analytic setting.

Contents

1. Dlt blow-ups revisited

The following theorem complements [F1, Theorem 1.21 (Dlt blow-ups, I)] and [F1, Theorem 1.27 (Dlt blow-ups, II)]. This result is well known for algebraic varieties and may be useful in geometric applications. Since we work in the complex analytic setting, the formulation is slightly more involved.

We note that (X, Δ) is not assumed to be log canonical in [F1, Theorem 1.27]. In contrast, (X, Δ) is assumed to be log canonical in Theorem 1.1.

Theorem 1.1 (Dlt blow-ups, III). *Let (X, Δ) be a log canonical pair. Let $\pi\colon X \to Y$ be a projective morphism of complex analytic spaces, and let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian.*

Let $f\colon Z \to X$ be a projective bimeromorphic morphism from a smooth variety Z such that $\operatorname{Exc}(f)$, the exceptional locus of f, is a simple normal crossing divisor on Z. We further assume that the support of $\operatorname{Exc}(f) + f^{-1}_*\Delta$ is a simple normal crossing divisor on Z.

Let $\mathcal{E}$ be a subset of the f-exceptional divisors $\{E_j\}$ satisfying the following two conditions.

(i) *If $a(E_j, X, \Delta) = -1$, then $E_j \in \mathcal{E}$.*
(ii) *If $E_j \in \mathcal{E}$, then $a(E_j, X, \Delta) \leq 0$.*

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Then, after shrinking Y around W, we can construct the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & Z' \arrow[dl, "f'"] \\
& X \arrow[d, "\pi"] & \\
& Y &
\end{tikzcd}$$

such that

(1) *f' is a projective bimeromorphic morphism,*
(2) *ϕ does not extract any divisors,*
(3) *ϕ is an isomorphism at the general points of E_j for $E_j \in \mathcal{E}$, and*
(4) *ϕ contracts every f-exceptional divisor E_j with $E_j \notin \mathcal{E}$.*

Furthermore, we set

$$\Delta_{Z'} := f'^{-1}_* \Delta - \sum_{E_j \in \mathcal{E}} a(E_j, X, \Delta)\phi_* E_j.$$

Then the following holds:

(5) *Z' is $\mathbb{Q}$-factorial over W, and $(Z', \Delta_{Z'})$ is divisorial log terminal, and*
$$K_{Z'} + \Delta_{Z'} = f'^*(K_X + \Delta).$$

The proof of Theorem 1.1 is an easy application of the minimal model program for projective morphisms of complex analytic spaces established in [F1]. We give the proof for the sake of completeness.

Proof of Theorem 1.1. By [F1, Lemma 2.16], we can take an open neighborhood U of W and a Stein compact subset W' of Y such that $U \subset W'$ and $\Gamma(W', \mathcal{O}_Y)$ is noetherian. As usual, we freely shrink Y around W without explicit mention.

Let A be a general π-ample $\mathbb{Q}$-divisor on X such that $A \cdot C > 2 \dim X$ for every projective curve C on X with $\pi(C)$ a point. Let $\varepsilon > 0$ be sufficiently small. We define

$$d(E_j) = \begin{cases} -a(E_j, X, \Delta) & \text{if} \quad E_j \in \mathcal{E}, \\ \max\{-a(E_j, X, \Delta) + \varepsilon, 0\} & \text{if} \quad E_j \notin \mathcal{E}. \end{cases}$$

We set

$$\Theta := f^{-1}_* \Delta + \sum d(E_j) E_j.$$

By definition, we have

$$K_Z + \Theta = f^*(K_X + \Delta) + \sum_{E_j \notin \mathcal{E}} \left(d(E_j) + a(E_j, X, \Delta)\right) E_j.$$

Note that

$$F := \sum_{E_j \notin \mathcal{E}} \left(d(E_j) + a(E_j, X, \Delta)\right) E_j$$

is effective and f-exceptional by construction. Since the support of Θ is a simple normal crossing divisor and its coefficients lie in $[0, 1]$, the pair (Z, Θ) is divisorial log terminal.

We take a general $(\pi \circ f)$-ample $\mathbb{Q}$-divisor H on Z such that $K_Z + \Theta + f^*A + H$ is nef over W'. We run a $(K_Z + \Theta + f^*A)$-minimal model program over Y around W' with scaling of H. We note that by [F1, Lemma 9.4] this minimal model program can be regarded as a $(K_Z+\Theta)$-minimal model program over X. Then we obtain a sequence of flips and divisorial contractions over X starting from $(Z_0,\Theta_0):=(Z,\Theta)$:

$$(Z_0,\Theta_0)\overset{\phi_0}{\dashrightarrow}(Z_1,\Theta_1)\overset{\phi_1}{\dashrightarrow}\cdots\overset{\phi_{i-1}}{\dashrightarrow}(Z_i,\Theta_i)\overset{\phi_i}{\dashrightarrow}\cdots,$$

where $\Theta_{i+1}:=(\phi_i)_*\Theta_i$, $H_{i+1}:=(\phi_i)_*H_i$, and $F_{i+1}:=(\phi_i)_*F_i$, for every i. We also have a sequence of real numbers

$$1\geq\lambda_0\geq\lambda_1\geq\cdots\geq\lambda_i\geq\cdots\geq 0$$

such that $K_{Z_i}+\Theta_i+f_i^*A+\lambda_iH_i$ is nef over W', where $f_i\colon Z_i\to X$ for every i. Each (Z_i,Θ_i) is divisorial log terminal and Z_i is $\mathbb{Q}$-factorial over W'. If ϕ_i is a divisorial contraction, then it contracts an irreducible component of F_i, since F_i is effective. By [F1, Lemma 13.7] and its proof, we can show that $K_{Z_m}+\Theta_m$ lies in $\overline{\mathrm{Mov}}(Z_m/X;\pi^{-1}(W'))$ for some m. By the negativity lemma (see [F1, Lemma 4.6]) applied to $f_m\colon Z_m\to X$, we deduce that F_m vanishes on $(\pi\circ f_m)^{-1}(U)$. Although Z_m is $\mathbb{Q}$-factorial over W', it is not necessarily $\mathbb{Q}$-factorial over W. If Z_m is $\mathbb{Q}$-factorial over W, we set $(Z',\Delta_{Z'}):=(Z_m,\Theta_m)$. Otherwise, we take a small projective $\mathbb{Q}$-factorialization $\psi\colon Z'\to Z_m$ by [F1, Theorem 1.24] and set $(Z',\Delta_{Z'}):=(Z',\psi^*\Theta_m)$. Then $(Z',\Delta_{Z'})$ is divisorial log terminal and Z' is $\mathbb{Q}$-factorial over W. By construction, the induced bimeromorphic map $\phi\colon Z\dashrightarrow Z'$ contracts F and is an isomorphism in codimension one outside $\operatorname{Supp}F$. Hence $f'\colon Z'\to X$ satisfies all the desired properties. ☐

The following lemma is well known to experts in the algebraic setting. We will use it in the proof of Theorem 2.1.

Lemma 1.2. *Let $(X,S+B)$ be a purely log terminal pair such that $\lfloor S+B\rfloor=S$ is irreducible. Let W be a Stein compact subset of X such that $W\cap S\neq\emptyset$ and $\Gamma(W,\mathcal{O}_X)$ is noetherian.*

Then, after shrinking X around W if necessary, there exists a small projective bimeromorphic morphism $\pi\colon X'\to X$ such that X' is $\mathbb{Q}$-factorial over W and $S':=\pi_*^{-1}S$ is π-anti-nef.

In particular, if C is a curve on X' such that $C\cap S'\neq\emptyset$ and $\pi(C)$ is a point, then $C\subset S'$.

Proof of Lemma 1.2. Throughout the proof, we freely shrink X around W without explicit mention. By [F1, Theorem 1.24], there exists a small projective bimeromorphic morphism $f\colon Z\to X$ such that Z is $\mathbb{Q}$-factorial over W. We set $S_Z:=f_*^{-1}S$ and $B_Z:=f_*^{-1}B$. Then

$$K_Z+S_Z+B_Z=f^*(K_X+S+B),$$

and (Z,S_Z+B_Z) is purely log terminal. In particular, (Z,B_Z) is kawamata log terminal and

$$K_Z+B_Z\sim_{\mathbb{R},f}-S_Z.$$

By [F1, Theorem 1.8], we obtain a minimal model $\phi\colon(Z,B_Z)\dashrightarrow(X',B')$ over X:

$$\begin{tikzcd}
Z \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X' \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

where $B':=\phi_*B_Z=\pi_*^{-1}B$. Then

$$K_{X'}+B'\sim_{\mathbb{R},\pi}-S',$$

where $S' := \phi_* S_Z = \pi_*^{-1} S$. Hence $-S'$ is π-nef. □

2. ADJUNCTION FOR PURELY LOG TERMINAL PAIRS

In this section, we show that a precise version of adjunction holds for purely log terminal pairs. This is an application of Theorem 1.1. This result is also well known in the case of algebraic varieties.

Let X be a normal complex variety and let $S+B$ be an effective $\mathbb{R}$-divisor on X such that K_X+S+B is $\mathbb{R}$-Cartier, S is reduced and irreducible, and S and B have no common irreducible components. Let $\nu\colon S^\nu \to S$ be the normalization, and write $K_{S^\nu}+B_{S^\nu} = \nu^*(K_X+S+B)$. By inversion of adjunction for log canonicity, we know that (S^ν, B_{S^ν}) is log canonical if and only if $(X, S+B)$ is log canonical in a neighborhood of S. For details, see [F2]. By the connectedness lemma of Shokurov–Kollár, (S^ν, B_{S^ν}) is kawamata log terminal if and only if $(X, S+B)$ is purely log terminal in a neighborhood of S. We note that the connectedness lemma of Shokurov–Kollár is an easy consequence of the Kawamata–Viehweg vanishing theorem for projective bimeromorphic morphisms of complex analytic spaces. We also note that if $(X, S+B)$ is purely log terminal in a neighborhood of S, then S is automatically normal. We do not prove these results here, since the proof for algebraic varieties applies with only minor modifications.

From now on, we assume that $(X, S+B)$ is purely log terminal and set $K_S+B_S := (K_X+S+B)|_S$ by adjunction. Let W be a compact subset of X such that $W\cap S \neq \emptyset$. Let E be a divisor over an open neighborhood U_E of W such that the center of E intersects W. We define

$$a(E, X, S+B)_W := a(E, U_E, S|_{U_E} + B|_{U_E}).$$

In this situation,

$$\mathrm{discrep}(\mathrm{center}\cap S \neq \emptyset, X, S+B)_W$$

denotes the infimum of $a(E, X, S+B)_W$, where E runs through all divisors over some open neighborhood of W which are exceptional and whose center has non-empty intersection with $S\cap W$. Similarly,

$$\mathrm{totaldiscrep}(S, B_S)_{W\cap S}$$

denotes the infimum of $a(F, S, B_S)_{W\cap S}$, where F runs through all divisors over some open neighborhood of $W\cap S$ in S whose center intersects $W\cap S$.

Theorem 2.1 (Adjunction for purely log terminal pairs). *Let $(X, S+B)$ be a purely log terminal pair such that $\lfloor S+B \rfloor = S$ is irreducible. Let W be a Stein compact subset of X such that $\Gamma(W, \mathcal{O}_X)$ is noetherian. Then*

$$\mathrm{totaldiscrep}(S, B_S)_{W\cap S} = \mathrm{discrep}(\mathrm{center}\cap S \neq \emptyset, X, S+B)_W$$

holds.

Remark 2.2. Let $P\in W\cap S$ be a point. After shrinking X around W if necessary, we take the blow-up of X along a general closed analytic subspace $C\subset S$ of codimension two such that $P\in C$. Then we obtain

$$\mathrm{discrep}(\mathrm{center}\cap S \neq \emptyset, X, S+B)_W \leq 0.$$

Before proving Theorem 2.1, we explain why we adopt the above formulation.

Remark 2.3. Let $X := \mathbb{C}$, and let $\{P_n\}_{n\in\mathbb{Z}_{>0}}$ be a discrete set of pairwise distinct points of X. We consider the divisor

$$\Delta := \sum_{n\in\mathbb{Z}_{>0}} \frac{n-1}{n} P_n.$$

Then the pair (X, Δ) is kawamata log terminal. We note that

$$a(P_n, X, \Delta) = -\frac{n-1}{n}.$$

Hence,

$$\inf_{n\in\mathbb{Z}_{>0}} \{a(P_n, X, \Delta)\} = -1.$$

This shows that the *discrepancy*

$$\operatorname{discrep}(X, \Delta) := \inf_{E} \{a(E, X, \Delta) \mid E \text{ is an exceptional divisor over } X\}$$

and the *total discrepancy*

$$\operatorname{totaldiscrep}(X, \Delta) := \inf_{E} \{a(E, X, \Delta) \mid E \text{ is a divisor over } X\}$$

are not well behaved when X is a non-compact complex analytic space.

We now prove Theorem 2.1.

Proof of Theorem 2.1. After shrinking X around W if necessary, we take a projective bimeromorphic morphism $g\colon Z \to X$ from a smooth complex variety Z such that $\operatorname{Exc}(g)$ and the support of $\operatorname{Exc}(g) + g_*^{-1}(S+B)$ are simple normal crossing divisors on Z. By this resolution and the basic properties of discrepancy coefficients, we obtain the inequality

$$\operatorname{totaldiscrep}(S, B_S)_{W\cap S} \geq \operatorname{discrep}(\text{center} \cap S \neq \emptyset, X, S+B)_W.$$

Thus it suffices to prove the reverse inequality.

Since $(X, S+B)$ is purely log terminal, there exists a small projective bimeromorphic morphism $\pi\colon X' \to X$ such that X' is $\mathbb{Q}$-factorial over W after shrinking X around W (see [F1, Theorem 1.24]). We set $S' := \pi_*^{-1}S$ and $B' := \pi_*^{-1}B$. By Lemma 1.2, we may assume that $-S'$ is π-nef. We may further assume that $g\colon Z \to X$ factors through X', and that there exists a divisor E on Z such that the center of E intersects $S \cap W$ and

$$a(E, X, S+B)_W = \operatorname{discrep}(\text{center} \cap S = \emptyset, X, S+B)_W \leq 0.$$

We set $\mathcal{E} := \{E\}$ and apply Theorem 1.1 to the morphisms $Z \to X' \to X$ and the compact set W. Then we obtain a commutative diagram

$$\begin{tikzcd}
Z \arrow[rr, dashed, "\phi"] \arrow[dr, "f"] \arrow[ddr, swap, "g"] & & Z' \arrow[dl, swap, "f'"] \arrow[ddl, "g'"] \\
& X' \arrow[d, "\pi"] & \\
& X &
\end{tikzcd}$$

satisfying the properties in Theorem 1.1. Since X' is $\mathbb{Q}$-factorial over W, the exceptional locus $\operatorname{Exc}(f')$ is a divisor after further shrinking X around W. Hence we have $\operatorname{Exc}(f') = E' := \phi_* E$. Since $-S'$ is π-nef and $\pi \circ f'(E') = g'(E') = g(E)$ intersects S, it follows that $f'(E') \cap S' \neq \emptyset$. Therefore, $E' \cap S_{Z'} \neq \emptyset$, since $\operatorname{Exc}(f') = E'$, where $S_{Z'}$ denotes the strict transform of S on Z'. We note that $S_{Z'}$ is normal since $(Z', S_{Z'})$ is divisorial log terminal.

Moreover, $E' \cap S_{Z'}$ is a divisor on $S_{Z'}$ over some open neighborhood of W, since Z' is $\mathbb{Q}$-factorial over W by construction. We have

$$K_{Z'} + S_{Z'} + B_{Z'} - a(E, X, S+B)_W E' = g'^*(K_X + S + B),$$

where $B_{Z'}$ is the strict transform of B on Z'. By adjunction, it follows that

$$\operatorname{totaldiscrep}(S, B_S)_{W \cap S} \leq a(E, X, S+B)_W,$$

since $E' \cap S_{Z'} \neq \emptyset$ is a divisor on $S_{Z'}$ over an open neighborhood of W.

This completes the proof. □

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Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: `fujino@math.kyoto-u.ac.jp`