

NOTES ON ACCEPTABLE BUNDLES I

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ABSTRACT. The notion of acceptable bundles plays a fundamental role in the Simpson–Mochizuki theory. This paper presents a detailed study of acceptable bundles on a punctured disk. In addition to its expository aspects, we introduce a new invariant and provide arguments that differ from those of Simpson and Mochizuki.

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1. INTRODUCTION

The notion of acceptable bundles plays a fundamental role in the Simpson–Mochizuki theory; see, for example, [S1], [S2], [M1], [M2], [M3], [M4], [M5], and [M6]. The present paper and the subsequent paper [FFO] provide detailed studies of acceptable bundles on a punctured disk and on a partially punctured polydisk, respectively. While these papers are primarily expository in nature, they also contain new arguments that differ from those of Simpson and Mochizuki.

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Takuro Mochizuki gives a general account of acceptable bundles in a broad setting in [M4, Chapter 21, Acceptable Bundles]. However, the primary focus there is on higher-dimensional generalizations of the results of Simpson ([S1] and [S2]), and the treatment of the most basic case, namely, acceptable bundles on a punctured disk, is relatively brief. The present paper is intended to fill this gap. In particular, we give a detailed study of acceptable bundles on a punctured disk and introduce new tools that will be used in subsequent work. In the subsequent paper [FFO], we study acceptable bundles on a partially punctured polydisk, building on the results of the present paper.

Let E be a holomorphic vector bundle over $\Delta^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$, and let h be a smooth Hermitian metric on E . We denote the curvature form of the Chern connection associated with (E, h) by $\sqrt{-1}\Theta_h(E)$, which is a smooth $\text{Hom}(E, E)$ -valued $(1, 1)$ -form on Δ^* .

We consider the Poincaré metric on Δ^* given by

$$\omega_P := \frac{\sqrt{-1} dz \wedge d\bar{z}}{|z|^2(-\log |z|^2)^2}.$$

The induced metric on $\text{Hom}(E, E)$ by h is also denoted by h , whenever there is no risk of confusion.

Let us recall the definition of acceptable vector bundles on Δ^* in the sense of Mochizuki (see [M1], [M2], [M3], and [M4, Chapter 21]).

Definition 1.1 (Acceptable bundles, see Definition 2.1). Let (E, h) be a Hermitian holomorphic vector bundle on Δ^* . We say that (E, h) is an *acceptable vector bundle* (in the sense of Mochizuki) if there exists a constant $C > 0$ such that

$$|\Theta_h(E)|_{h, \omega_P} \leq C \quad \text{on } \Delta^*,$$

where $|\bullet|_{h, \omega_P}$ denotes the pointwise norm of \bullet with respect to the Hermitian metric induced by h and ω_P .

Although Simpson treats a more general setting (see [S1, Section 10] and [S2, Section 3]), in this paper we adopt the above definition of acceptable vector bundles.

Definition 1.2 (Prolongation by increasing orders, see Definition 2.3). Let (E, h) be an acceptable vector bundle on Δ^* , and let a be any real number. For any open subset $U \subset \Delta$, we define

$${}_a E(U) := \left\{ f \in E(U \setminus \{0\}) \mid |f|_h = O\left(\frac{1}{|z|^{a+\varepsilon}}\right) \text{ for every } \varepsilon > 0 \right\},$$

where $|f|_h$ denotes the norm of f with respect to the Hermitian metric h . Then we obtain a sheaf of \mathcal{O}_Δ -modules, denoted by ${}_a E$. When $a = 0$, we usually write ${}^0 E := {}_0 E$.

The following foundational result is due to Simpson (see [S1] and [S2]):

Theorem 1.3 (Simpson, see [S1] and [S2]). *Let (E, h) be an acceptable vector bundle on Δ^* . Then ${}_a E$ is a holomorphic vector bundle for every $a \in \mathbb{R}$.*

More precisely, Simpson asserts the coherence of ${}_a E$ in a slightly more general setting. Furthermore, he states that the desired coherence follows from the theory of Cornalba–Griffiths [CG], with a minor modification. For details, see the discussion on pages 909–910 of [S1].

The next corollary follows easily from the definition of ${}_a E$ and Theorem 1.3:

Corollary 1.4 (see Section 7). *In the setting of Theorem 1.3, for $a, b \in \mathbb{R}$, we have:*

- (i) ${}_a E$ is locally free;
- (ii) ${}_a E \subset j_* E$ and ${}_a E|_{\Delta^*} = E$, where $j: \Delta^* \hookrightarrow \Delta := \{z \in \mathbb{C} \mid |z| < 1\}$;
- (iii) ${}_a E \subset {}_b E$ if $a \leq b$;
- (iv) ${}_{a+1} E = {}_a E \otimes \mathcal{O}_\Delta([0])$;
- (v) ${}_{a+\varepsilon} E = {}_a E$ for all sufficiently small $\varepsilon > 0$;
- (vi) The set $\{a \in \mathbb{R} \mid {}_a E / {}_{<a} E \neq 0\}$ is discrete in \mathbb{R} , where ${}_{<a} E := \bigcup_{b < a} {}_b E \subset j_* E$.

Thus, we can regard ${}_* E := ({}_a E \mid a \in \mathbb{R})$ as a filtered bundle over E in the sense of Mochizuki (see Section 8).

To prove Theorem 1.3, we first establish the following special case, which plays a crucial role in the overall proof.

Proposition 1.5 (Proposition 4.1). *Let L be a holomorphic line bundle on Δ^* and let h be a Hermitian metric on L such that*

$$-C \cdot \omega_P \leq \sqrt{-1} \Theta_h(L) \leq C \cdot \omega_P$$

holds on Δ^ for some constant $C > 0$. That is, writing*

$$\sqrt{-1} \Theta_h(L) = f(z) \cdot \omega_P,$$

we have $|f(z)| \leq C$ on Δ^ . Then ${}_a L$ is a holomorphic line bundle for every $a \in \mathbb{R}$.*

Note that a more precise description of ${}_a L$ is provided in Theorem 4.4. The authors believe that the explicit formulation given in this paper is new.

Theorem 1.6 (Theorem 4.4). *Let (L, h) be an acceptable line bundle on Δ^* . By taking a suitable trivialization*

$$(L, h) \simeq (\mathcal{O}_{\Delta^*}, |\cdot|^2 e^{-2\varphi}),$$

we have the following properties.

- (i) *The limit*

$$\gamma := \lim_{z \rightarrow 0} \frac{\varphi(z)}{\log |z|} \in \mathbb{R}$$

exists.

- (ii) *Let f be a holomorphic function on $\Delta(0, r)^*$ for some $0 < r < 1$, where $\Delta(0, r)^* := \{z \in \mathbb{C} \mid 0 < |z| < r\}$. Then $f \in ({}_a L)_0$ holds for some $a \in \mathbb{R}$ if and only if f is meromorphic at 0, where $({}_a L)_0$ denotes the stalk of ${}_a L$ at $0 \in \Delta$.*
- (iii) *Let f be a meromorphic function on some open neighborhood of 0 and let a be any real number. Then $f \in ({}_a L)_0$ holds if and only if*

$$(1.1) \quad \lim_{z \rightarrow 0} \frac{\log(|f|e^{-\varphi})}{\log |z|} \geq -a.$$

Note that

$$\lim_{z \rightarrow 0} \frac{\log(|f|e^{-\varphi})}{\log |z|} = \text{ord}_0 f - \gamma$$

holds. Therefore, (1.1) is equivalent to

$$\text{ord}_0 f \geq -\lfloor a - \gamma \rfloor.$$

(iv) Let f be a meromorphic function on some open neighborhood of 0 and let a be any real number. Then $f \notin ({}_aL)_0$ holds if and only if

$$(1.2) \quad \lim_{z \rightarrow 0} \frac{\log(|f|e^{-\varphi})}{\log|z|} < -a.$$

Note that (1.2) implies that

$$|f|e^{-\varphi} > \frac{1}{|z|^a}$$

holds on some small open neighborhood of 0.

The following corollaries follow directly from the description of ${}_aL$ in the proof of Proposition 1.5:

Corollary 1.7 (Duality for line bundles, see Corollary 4.3). *In Proposition 1.5, we have*

$$({}_aL)^\vee = {}_{-a+1-\varepsilon}(L^\vee)$$

for all sufficiently small $\varepsilon > 0$.

Corollary 1.8. *Let L be a holomorphic line bundle on Δ^* , and let h be a flat Hermitian metric on L . Then*

$$(L, h) \simeq \left(\mathcal{O}_{\Delta^*}, \frac{|\cdot|^2}{|z|^{2c}} \right)$$

for some $c \in \mathbb{R}$.

In this paper, we introduce a new invariant $\gamma({}_aE)$ for studying the structure of ${}_aE$, and establish the following result:

Theorem 1.9 (see Definition 7.4, Corollary 7.6, Theorem 7.13, and Theorem 12.3). *Let (E, h) be an acceptable vector bundle on Δ^* with $\text{rank } E = r$, and let $\{v_1, \dots, v_r\}$ be a local frame of ${}_aE$ near the origin. Define*

$$\gamma({}_aE) := -\frac{1}{2} \liminf_{z \rightarrow 0} \frac{\log \det H(h, \mathbf{v})}{\log|z|},$$

where $H(h, \mathbf{v})$ is the $r \times r$ matrix $(h(v_i, v_j))$. Then $\gamma({}_aE)$ is a well-defined real-valued invariant of ${}_aE$.

Furthermore, if we let

$$\mathcal{P}ar_a(E, h) := \{b_1, \dots, b_r\},$$

then we have

$$\gamma({}_aE) = -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(h, \mathbf{v})}{\log|z|} = \sum_{i=1}^r b_i.$$

For the precise definition of the parabolic weights $\mathcal{P}ar_a(E, h)$, see 7.11 below.

Note that if we define

$$\{\lambda_1, \dots, \lambda_k\} := \{\lambda \in (a-1, a] \mid \lambda E / {}_{<\lambda}E \neq 0\}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$, then

$$\sum_{i=1}^r b_i = \sum_{i=1}^k \lambda_i \dim_{\mathbb{C}} (\lambda_i E / {}_{<\lambda_i}E).$$

This theorem plays a central role in our analysis. We emphasize that the most technically challenging part of this paper is the proof of the identity

$$\gamma({}_a E) = \sum_{i=1}^r b_i.$$

Theorem 1.10 (Determinant bundles, see Theorem 7.5). *Let (E, h) be an acceptable vector bundle on Δ^* . Then the determinant bundle $(\det E, \det h)$ is an acceptable line bundle on Δ^* , and*

$$\det({}_a E) = \gamma({}_a E) \det E$$

holds for every $a \in \mathbb{R}$.

The proof of Theorem 1.10 closely follows that of Proposition 1.5 (see Proposition 4.1), once the well-definedness of $\gamma({}_a E)$ is established. By using $\gamma({}_a L)$, we can reformulate Corollary 1.7 as follows.

Lemma 1.11 (Duality for line bundles, see Lemma 13.1). *Let (L, h) be an acceptable line bundle on Δ^* . Let $a \in \mathbb{R}$ be any real number. Then we have ${}_a L = \gamma({}_a L)L$, and $\text{Par}_a(L, h) = \{\gamma({}_a L)\}$.*

Moreover, if $0 < \varepsilon \ll 1$, then

$$\gamma(-_{a+1-\varepsilon}(L^\vee)) = -\gamma({}_a L).$$

In particular, the following equality holds:

$$(\gamma({}_a L)L)^\vee = -_{\gamma({}_a L)}(L^\vee).$$

In contrast, the proofs of the following theorems, namely Theorem 1.12, Theorem 1.13, and Theorem 1.14, rely on the equality $\gamma({}_a E) = \sum_{i=1}^r b_i$ in Theorem 1.9, and are therefore considerably more involved.

Theorem 1.12 (Dual bundles, see Theorem 13.2). *Let (E, h) be an acceptable vector bundle on Δ^* , and let a be any real number. Then,*

$$({}_a E)^\vee = -_{a+1-\varepsilon}(E^\vee)$$

holds for any sufficiently small $\varepsilon > 0$.

Moreover, let $\{v_1, \dots, v_r\}$ be a local frame of ${}_a E$ near the origin, compatible with the parabolic filtration, such that $v_i \in {}_{b_i} E \setminus <_{b_i} E$ for each i . For each i , define

$$v_i^\vee := (-1)^{i-1} v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r \otimes (v_1 \wedge \dots \wedge v_r)^{\otimes -1}.$$

Then $\{v_1^\vee, \dots, v_r^\vee\}$ forms a local frame of $-_{a+1-\varepsilon}(E^\vee)$ near the origin, compatible with the parabolic filtration, such that

$$v_i^\vee \in -_{b_i}(E^\vee) \setminus <_{-b_i}(E^\vee)$$

for each i . In particular, we have

$$\text{Par}_a(E, h) = \{b_1, \dots, b_r\} \quad \text{and} \quad \text{Par}_{-a+1-\varepsilon}(E^\vee, h^\vee) = \{-b_1, \dots, -b_r\}.$$

As an immediate consequence of Theorem 1.12, we have:

Theorem 1.13 (Weak norm estimate, see Theorem 13.3). *Let $\{v_1, \dots, v_r\}$ be a local frame of ${}_a E$ around the origin, compatible with the parabolic filtration, such that*

$$v_i \in {}_{b_i} E \setminus <_{b_i} E \quad \text{for every } i.$$

We define

$$H(h, \mathbf{v}') := (h(v_i \cdot |z|^{b_i}, v_j \cdot |z|^{b_j}))_{i,j}.$$

Then there exist positive constants C and M such that

$$C^{-1}(-\log |z|)^{-M} I_r \leq H(h, \mathbf{v}')(z) \leq C(-\log |z|)^M I_r$$

holds in a neighborhood of the origin, where I_r is the identity matrix of size r .

Theorem 1.14 (Tensor products, see Theorem 16.2). *Let (E_1, h_1) and (E_2, h_2) be acceptable vector bundles on Δ^* . Then the tensor product bundle $(E_1 \otimes E_2, h_1 \otimes h_2)$ is also acceptable, and*

$${}_a(E_1 \otimes E_2) = \sum_{a_1+a_2 \leq a} {}_{a_1}E_1 \otimes {}_{a_2}E_2$$

holds for any $a \in \mathbb{R}$.

Finally, we remark that significant effort has been made to ensure that this paper is as self-contained as possible.

This paper focuses solely on acceptable bundles over the punctured disk and does not address any applications. There is already extensive literature on related topics; see, for example, [B1], [B2], [BB], [SS1], and [SS2]. Our selection of references reflects the authors' preferences and perspective. We apologize for omitting many important works and refer interested readers to the broader literature.

We now outline the organization of the present paper. In Section 2, we collect some basic definitions and state a few elementary properties that follow directly from them. In Section 3, we prove some preliminary lemmas concerning harmonic and holomorphic functions on a punctured disk. Section 4 is devoted to the proof of Proposition 1.5 (see Proposition 4.1), where we describe the prolongation of acceptable line bundles by increasing orders. To the best of the authors' knowledge, this treatment is new. In Section 5, we briefly discuss $\bar{\partial}$ -equations and derive a growth estimate via the L^2 -method. In Section 6, we prove Theorem 1.3, establishing the prolongation of acceptable vector bundles by increasing orders. In Section 7, we introduce a new invariant and prove some fundamental properties of prolongations of acceptable bundles. In Section 8, we briefly review the framework of filtered bundles for later use. Section 9 collects several elementary inequalities, which will play a crucial role in the subsequent section. In Section 10, we establish Simpson's key lemma, which is one of the main ingredients in the proof of Theorem 12.3 given in Section 12. In Section 11, we study the behavior of acceptable bundles via cyclic covers. Section 12 is devoted to the proof of Theorem 12.3, which is one of the most technically involved results in this paper. In Section 13, we investigate the prolongation of dual vector bundles. In Section 14, we present some examples of filtered bundles introduced in Section 8. In Section 15, we return to the study of dual bundles, now within the framework of filtered bundles. In Section 16, we examine the prolongation of tensor products of acceptable bundles, again in the context of filtered bundles. Finally, in Section 17, we study Hom bundles from the perspective of filtered bundles.

While certain parts of the exposition may be new, and others have been simplified or clarified, we believe that all essential results are already contained, perhaps implicitly, within the substantial works of Simpson and Mochizuki (see [S1], [S2], [M1], [M2], [M3], [M4], [M5], [M6], and so on). We have cited the most relevant references to their works, though we do not aim to exhaustively list all related material. We nevertheless hope that the present paper contributes to making their profound and extensive theories more accessible.

1.15 (Convention). Let \mathcal{F} be a sheaf on a topological space X . Unless explicitly stated otherwise, we write $f \in \mathcal{F}$ to indicate that f is a local section $f \in \mathcal{F}(U)$ over some open subset $U \subset X$.

In this paper, we do not distinguish between holomorphic vector bundles on a complex manifold X and the corresponding locally free \mathcal{O}_X -modules. These are treated as equivalent unless stated otherwise.

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2. PRELIMINARIES

In this paper, we will almost always work over either the punctured disk $\Delta^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ or the unit disk $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$. Let

$$\omega_P := \frac{\sqrt{-1} dz \wedge d\bar{z}}{|z|^2(-\log|z|^2)^2}$$

denote the Poincaré metric on Δ^* . Then the pair (Δ^*, ω_P) defines a Kähler manifold.

Let us recall the definition of *acceptable bundles* on a punctured disk Δ^* in the sense of Mochizuki. As already mentioned in Section 1, Simpson treats a more general setting in [S1] and [S2].

Definition 2.1 (Acceptable bundles). Let E be a holomorphic vector bundle on the punctured disk Δ^* , and let h be a Hermitian metric on E . Then (E, h) admits a *Chern connection* $D = D' + \bar{\partial}$, whose curvature form is given by

$$\sqrt{-1}\Theta_h(E) := \sqrt{-1}D^2.$$

This is a smooth $(1, 1)$ -form on Δ^* with values in $\text{Hom}(E, E)$.

We use the same notation h to denote the induced Hermitian metric on $\text{Hom}(E, E)$, whenever there is no risk of confusion.

We say that (E, h) is an *acceptable bundle* on Δ^* if the norm of $\sqrt{-1}\Theta_h(E)$ is bounded on Δ^* , that is, there exists a constant $C > 0$ such that

$$|\Theta_h(E)|_{h, \omega_P} \leq C \quad \text{on } \Delta^*,$$

where $|\bullet|_{h, \omega_P}$ denotes the pointwise norm of \bullet with respect to the Hermitian metric h and the Poincaré metric ω_P .

Lemma 2.2 easily follows from the definition.

Lemma 2.2. *Let (E, h) be an acceptable vector bundle on Δ^* . Then the dual bundle (E^\vee, h^\vee) and the determinant line bundle $(\det E, \det h)$ are also acceptable.*

Let (E_1, h_1) and (E_2, h_2) be acceptable vector bundles on Δ^ . Then the tensor product $(E_1 \otimes E_2, h_1 \otimes h_2)$ and the Hom bundle $(\text{Hom}(E_1, E_2), h_1^\vee \otimes h_2)$ are acceptable.*

Proof of Lemma 2.2. Since $\Theta_{h^\vee}(E^\vee) = -\Theta_h(E)$ and

$$\Theta_{h_1 \otimes h_2}(E_1 \otimes E_2) = \Theta_{h_1}(E_1) \otimes \text{Id}_{E_2} + \text{Id}_{E_1} \otimes \Theta_{h_2}(E_2),$$

it follows that both (E^\vee, h^\vee) and $(E_1 \otimes E_2, h_1 \otimes h_2)$ are acceptable. Using the natural identification $\text{Hom}(E_1, E_2) = E_1^\vee \otimes E_2$, we see that the Hom bundle $\text{Hom}(E_1, E_2)$ is also acceptable. Note that $\det E$ is a direct summand of $E^{\otimes \text{rank } E}$. Hence, $(\det E, \det h)$ is acceptable. This completes the proof of Lemma 2.2. \square

The main object of this paper is the prolongation by increasing orders.

Definition 2.3 (Prolongation by increasing orders). Let (E, h) be an acceptable vector bundle on Δ^* and let a be any real number. For any open subset U of Δ , we put

$${}_a E(U) := \left\{ f \in E(U \setminus \{0\}) \mid |f|_h = O\left(\frac{1}{|z|^{a+\varepsilon}}\right) \text{ for every } \varepsilon \right\},$$

where $|f|_h$ denotes the norm of f with respect to the Hermitian metric h . Then we obtain a sheaf of \mathcal{O}_Δ -modules, denoted by ${}_a E$. When $a = 0$, we usually use ${}^\diamond E$ to denote ${}_0 E$.

Let us briefly recall the positivity of vector bundles. For details, see, for example, [Dem1, Chapter 10] and [Dem2, Chapter VII, §6 Positivity Concepts for Vector Bundles].

Definition 2.4 (Positivity of vector bundles). Let X be a complex manifold of dimension one, that is, $\dim X = 1$. Let E be a holomorphic vector bundle on X , and let h be a Hermitian metric on E . Let D denote the Chern connection of (E, h) , and define the curvature form by

$$\sqrt{-1}\Theta_h(E) := \sqrt{-1}D^2$$

as before. Then the curvature form $\sqrt{-1}\Theta_h(E)$ and the metric h induce a Hermitian form θ_E on $T_X \otimes E$.

If θ_E is positive definite, positive semi-definite, negative, or negative semi-definite, then we say that (E, h) (or equivalently, $\sqrt{-1}\Theta_h(E)$) is *Nakano positive*, *Nakano semipositive*, *Nakano negative*, or *Nakano seminegative*, respectively.

Since $\dim X = 1$, Nakano (semi)positivity and (semi)negativity are equivalent to *Griffiths* (semi)positivity and (semi)negativity, respectively.

In this paper, we sometimes omit the terms ‘‘Nakano’’ and ‘‘Griffiths’’ since we are working in dimension one.

The property established in Lemma 2.5 below is a fundamental feature of acceptable bundles. In fact, it may be said that this is the only property of acceptable bundles needed in this paper.

Lemma 2.5. *Let (E, h) be an acceptable vector bundle on Δ^* such that*

$$|\Theta_h(E)|_{h, \omega_P} \leq C$$

holds on Δ^ . Then we have*

$$-C\omega_P \otimes \text{Id}_E \leq_{\text{Nak}} \sqrt{-1}\Theta_h(E) \leq_{\text{Nak}} C\omega_P \otimes \text{Id}_E.$$

Here, $A \leq_{\text{Nak}} B$ means that the Hermitian form on $T_{\Delta^} \otimes E$ induced by $B - A$ and h is Nakano semipositive.*

Proof of Lemma 2.5. For any $x \in \Delta^*$, we take a local coordinate w centered at x such that

$$\omega_P = \sqrt{-1}dw \wedge d\bar{w}$$

around x . Let $\{e_1, \dots, e_r\}$ be a local holomorphic frame of E , which is orthonormal at x . Let $\{e^1, \dots, e^r\}$ be its dual in E^\vee . We write

$$\sqrt{-1}\Theta_h(E) = R_\alpha^\beta dw \wedge d\bar{w} \otimes e^\alpha \otimes e_\beta$$

around x . We put $R_{\alpha\bar{\beta}} = \sum_\gamma h_{\gamma\bar{\beta}} R_\alpha^\gamma$, where $h_{\gamma\bar{\beta}} := h(e_\gamma, e_\beta)$. Since $(h_{\gamma\bar{\beta}})$ is the identity matrix at x , $R_{\alpha\bar{\beta}}(x) = R_\alpha^\beta(x)$ holds. By assumption,

$$\sum_{\alpha, \beta} |R_{\alpha\bar{\beta}}(x)|^2 = |\Theta_h(E)(x)|_{h, \omega_P}^2 \leq C^2.$$

For any $u = \sum_\alpha u^\alpha \frac{\partial}{\partial w} \otimes e_\alpha$, by using the Cauchy–Schwarz inequality twice,

$$\begin{aligned} \left| \sum_{\alpha, \beta} R_{\alpha\bar{\beta}}(x) u^\alpha \bar{u}^\beta \right|^2 &\leq \left(\sum_\beta \left| \sum_\alpha R_{\alpha\bar{\beta}}(x) u^\alpha \right|^2 \right) \left(\sum_\beta |\bar{u}^\beta|^2 \right) \\ &\leq \left(\sum_\beta \left(\sum_\alpha |R_{\alpha\bar{\beta}}(x)|^2 \right) \left(\sum_\alpha |u^\alpha|^2 \right) \right) \left(\sum_\beta |\bar{u}^\beta|^2 \right) \\ &= |u|_{h, \omega_P}^4 \cdot \sum_{\alpha, \beta} |R_{\alpha\bar{\beta}}(x)|^2 \\ &\leq |u|_{h, \omega_P}^4 \cdot C^2. \end{aligned}$$

This implies that

$$-C|u|_{h, \omega_P}^2 \leq \sum_{\alpha, \beta} R_{\alpha\bar{\beta}}(x) u^\alpha \bar{u}^\beta \leq C|u|_{h, \omega_P}^2.$$

This is what we wanted. We finish the proof of Lemma 2.5. \square

Remark 2.6. Although in Lemma 2.5 we considered only the case over the punctured disk, the same statement holds over Kähler manifolds of arbitrary dimension. For details, see [DH, Lemma 2.10].

We need the following well-known result in this paper.

Lemma 2.7. *Let E be a holomorphic vector bundle on a complex manifold X with $\dim X = 1$ and let h be a smooth Hermitian metric on E such that $\sqrt{-1}\Theta_h(E)$ is semi-negative. Let s be any holomorphic section of E on X . Then $\log |s|_h^2$ is subharmonic.*

We give a proof of Lemma 2.7 for the sake of completeness although it is well known.

Proof of Lemma 2.7. Let $\{\bullet, \bullet\}$ denote the sesquilinear pairing

$$C^\infty(X, \wedge^p T_X^\vee \otimes E) \times C^\infty(X, \wedge^q T_X^\vee \otimes E) \rightarrow C^\infty(X, \wedge^{p+q} T_X^\vee \otimes \mathbb{C})$$

induced by the Hermitian metric h .

Let Ω be an open subset of X , and assume that $E|_\Omega$ is trivialized as $\Omega \times \mathbb{C}^r$ by a C^∞ frame $\{e_\lambda\}$. Then for any sections

$$u = \sum_\lambda u_\lambda \otimes e_\lambda, \quad v = \sum_\mu v_\mu \otimes e_\mu,$$

we have

$$\{u, v\} = \sum_{\lambda, \mu} u_\lambda \wedge \bar{v}_\mu \cdot h(e_\lambda, e_\mu).$$

Let $D = D' + \bar{\partial}$ denote the Chern connection associated with (E, h) . Outside the zero set of s , we have

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\log|s|_h^2 &= \sqrt{-1}\frac{\{D's, D's\}}{|s|_h^2} - \sqrt{-1}\frac{\{D's, s\} \wedge \{s, D's\}}{|s|_h^4} - \frac{\{\sqrt{-1}\Theta_h(E)s, s\}}{|s|_h^2} \\ &\geq -\frac{\{\sqrt{-1}\Theta_h(E)s, s\}}{|s|_h^2} \end{aligned}$$

by the Cauchy–Schwarz inequality.

Since $\sqrt{-1}\Theta_h(E)$ is assumed to be seminegative, it follows that

$$\sqrt{-1}\partial\bar{\partial}\log|s|_h^2 \geq 0$$

outside the zero set of s . That is, $\log|s|_h^2$ is subharmonic on $X \setminus \{s = 0\}$.

Moreover, since $\log|s|_h^2$ is locally bounded from above, it extends to a subharmonic function on all of X (see [NO, (3.3.25) Theorem]).

This completes the proof of Lemma 2.7. \square

3. LEMMAS FOR FUNCTIONS ON A PUNCTURED DISK

In this section, we present several elementary lemmas used in the proof of Proposition 1.5 (see also Proposition 4.1). We begin with a result concerning the Lelong number. The following lemma is well known; for details, see, for example, [Dem1, 2.B. Lelong Numbers] and [Dem2, Chapter III, (6.9) Example].

Lemma 3.1 (Lelong number). *Let u be a subharmonic function on Δ . Then we have*

$$(3.1) \quad \lim_{r \rightarrow 0} \int_{\Delta(0,r)} \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}u = \liminf_{z \rightarrow 0} \frac{u(z)}{\log|z|}.$$

We define

$$\nu(u, 0) := \liminf_{z \rightarrow 0} \frac{u(z)}{\log|z|}$$

and call it the Lelong number of u at 0. Note that the expression $\partial\bar{\partial}u$ is understood in the sense of currents.

Let u_1 and u_2 be subharmonic functions on Δ . Then $u_1 + u_2$ is also subharmonic on Δ . By (3.1), we have the identity

$$\nu(u_1 + u_2, 0) = \nu(u_1, 0) + \nu(u_2, 0).$$

We recall the following elementary lemma.

Lemma 3.2 (Harmonic functions on Δ^*). *Let f be a harmonic function on Δ^* . Then there exist a holomorphic function g on Δ^* and a real constant $c \in \mathbb{R}$ such that*

$$f(z) = \operatorname{Re} g(z) + c \log|z|.$$

We include a detailed proof of Lemma 3.2 for completeness.

Proof of Lemma 3.2. Consider the universal covering

$$\pi: H := \{w \in \mathbb{C} \mid \operatorname{Re} w < 0\} \rightarrow \Delta^*$$

given by $\pi(w) = e^w$. Then $f \circ \pi$ is a harmonic function on the simply connected domain H , so there exists a holomorphic function $p(w)$ on H such that

$$\operatorname{Re} p(w) = f \circ \pi(w).$$

Define

$$q(w) := p(w + 2\pi\sqrt{-1}) - p(w).$$

Then $q(w)$ is holomorphic on H , and since $\pi(w + 2\pi\sqrt{-1}) = \pi(w)$, we have

$$\operatorname{Re} q(w) = \operatorname{Re} p(w + 2\pi\sqrt{-1}) - \operatorname{Re} p(w) = f \circ \pi(w + 2\pi\sqrt{-1}) - f \circ \pi(w) = 0.$$

Hence, $q(w)$ is a purely imaginary constant, i.e.,

$$q(w) = 2\pi\sqrt{-1}c$$

for some real constant $c \in \mathbb{R}$.

Set

$$r(w) := p(w) - cw.$$

Then $r(w)$ is holomorphic and satisfies

$$\begin{aligned} r(w + 2\pi\sqrt{-1}) &= p(w + 2\pi\sqrt{-1}) - c(w + 2\pi\sqrt{-1}) \\ &= p(w) + 2\pi\sqrt{-1}c - cw - 2\pi\sqrt{-1}c \\ &= r(w). \end{aligned}$$

Thus, r is $2\pi\sqrt{-1}$ -periodic and descends to a holomorphic function $g(z)$ on Δ^* such that

$$g \circ \pi(w) = r(w).$$

Therefore,

$$f(z) = \operatorname{Re} p(w) = \operatorname{Re}(r(w) + cw) = \operatorname{Re} g(z) + c \log |z|,$$

where we used that $w = \log z$ for $z \in \Delta^*$. This completes the proof. \square

We next state another elementary lemma.

Lemma 3.3. *Let g be a holomorphic function on Δ^* . Assume that*

$$\operatorname{Re} g(z) \leq C(-\log |z|)$$

holds on Δ^ for some constant $C > 0$. Then g extends holomorphically to the origin; that is, the origin is a removable singularity of g .*

We also provide a proof of Lemma 3.3 for the reader's convenience.

Proof of Lemma 3.3. By the Casorati–Weierstrass theorem or Picard's big theorem, g is meromorphic at 0. So we may write

$$g(z) = \frac{p(z)}{z^m},$$

where $p(z)$ is holomorphic on Δ with $p(0) \neq 0$ and m is an integer.

Let $z = re^{\sqrt{-1}\theta}$. Suppose, for contradiction, that $m > 0$. Then we can choose $\theta_0 \in [0, 2\pi)$ such that

$$\frac{p(0)}{e^{\sqrt{-1}m\theta_0}} \in \mathbb{R}_{>0}.$$

Since p is continuous and $p(0) \neq 0$, there exists $0 < r_0 \ll 1$ such that for all $0 < r < r_0$, the real part of

$$\frac{p(re^{\sqrt{-1}\theta_0})}{e^{\sqrt{-1}m\theta_0}}$$

is greater than some constant $a > 0$. It follows that

$$\operatorname{Re} g(re^{\sqrt{-1}\theta_0}) = \operatorname{Re} \left(\frac{p(re^{\sqrt{-1}\theta_0})}{r^m e^{\sqrt{-1}m\theta_0}} \right) > \frac{a}{r^m}.$$

But the assumption gives

$$\operatorname{Re} g(re^{\sqrt{-1}\theta_0}) \leq C(-\log r).$$

This is a contradiction for sufficiently small r , since r^{-m} grows much faster than $-\log r$ as $r \rightarrow 0$. Hence, $m \leq 0$. This implies that g is holomorphic at 0. \square

4. PROLONGATION OF ACCEPTABLE LINE BUNDLES

In this section, we prove Proposition 1.5, along with Corollaries 1.7 and 1.8. We recall that for any $a \in \mathbb{R}$,

$$[a] := \min\{n \in \mathbb{Z} \mid n \geq a\} \quad \text{and} \quad \lfloor a \rfloor := \max\{n \in \mathbb{Z} \mid n \leq a\}.$$

Proposition 4.1 (Proposition 1.5). *Let (L, h) be an acceptable line bundle on Δ^* . Then ${}_\alpha L$ is a holomorphic line bundle on Δ for every $\alpha \in \mathbb{R}$.*

A more precise description of ${}_\alpha L$ is given in Theorem 4.4 below.

Proof of Proposition 4.1. We will see the behavior of the metric h around the origin by taking a suitable trivialization of L on Δ^* concretely.

Step 1. We put

$$\omega_P := \frac{\sqrt{-1}dz \wedge d\bar{z}}{|z|^2(-\log|z|^2)^2}$$

and

$$\chi(N) := -N \log(-\log|z|^2).$$

We can check that

$$\sqrt{-1}\partial\bar{\partial}\chi(N) = N\omega_P.$$

Since (L, h) is an acceptable line bundle on Δ^* , there exists $C > 0$ such that

$$(4.1) \quad -C \cdot \omega_P \leq \sqrt{-1}\Theta_h(L) \leq C \cdot \omega_P$$

holds on Δ^* . We fix some positive number N with $N \geq C$. We consider Hermitian metrics $he^{-\chi(N)}$ and $he^{-\chi(-N)}$ on L . Then we obtain

$$\begin{aligned} \sqrt{-1}\Theta_{he^{-\chi(N)}}(L) &= \sqrt{-1}\Theta_h(L) + \sqrt{-1}\partial\bar{\partial}\chi(N) \\ &= \sqrt{-1}\Theta_h(L) + N\omega_P \geq 0 \end{aligned}$$

and

$$\begin{aligned} \sqrt{-1}\Theta_{he^{-\chi(-N)}}(L) &= \sqrt{-1}\Theta_h(L) + \sqrt{-1}\partial\bar{\partial}\chi(-N) \\ &= \sqrt{-1}\Theta_h(L) - N\omega_P \leq 0 \end{aligned}$$

by (4.1).

Step 2. Since L is a holomorphic line bundle on Δ^* , we can trivialize L on Δ^* (see, for example, [F, 30.3. Theorem]). Hence, from now, we assume $L = \mathcal{O}_{\Delta^*}$. Then we can write

$$h = |\cdot|^2 e^{-2\varphi}$$

with some smooth function φ on Δ^* . We note that

$$\sqrt{-1}\Theta_h(L) = \sqrt{-1}\partial\bar{\partial}2\varphi$$

on Δ^* .

Step 3. Since

$$\int_{\Delta(0, r_0)} \omega_P < \infty$$

for every $0 < r_0 < 1$, we can see ω_P a closed positive $(1, 1)$ -current on Δ . By (4.1), $\sqrt{-1}\Theta_h(L)$ can be seen as a $(1, 1)$ -current on Δ . Since $\dim_{\mathbb{C}} \Delta = 1$, $\sqrt{-1}\Theta_h(L)$ is obviously d -closed. Hence $\sqrt{-1}\Theta_h(L)$ defines a closed $(1, 1)$ -current on Δ . By Step 1,

$$\sqrt{-1}\Theta_{he^{-\chi(N)}}(L) \quad \text{and} \quad -\sqrt{-1}\Theta_{he^{-\chi(-N)}}(L)$$

are closed positive $(1, 1)$ -current on Δ . This is a very special case of the Skoda–El Mir extension theorem (see [Dem1, (1.18) Theorem] and [Dem2, Chapter III, §2.A]).

Step 4. Since we are working on Δ , we can find subharmonic functions ψ_1 and ψ_2 on Δ such that

$$\sqrt{-1}\Theta_{he^{-\chi(N)}}(L) = \sqrt{-1}\partial\bar{\partial}^2\psi_1$$

and

$$-\sqrt{-1}\Theta_{he^{-\chi(-N)}}(L) = \sqrt{-1}\partial\bar{\partial}^2\psi_2.$$

Since $2\varphi + \chi(N) - 2\psi_1$ is harmonic on Δ^* , by Lemma 3.2, we can write

$$(4.2) \quad 2\varphi + \chi(N) = 2\psi_1 + c_1 \log |z|^2 + 2 \operatorname{Re} g_1(z)$$

for some holomorphic function g_1 on Δ^* and some $c_1 \in \mathbb{R}$. Similarly, we can write

$$(4.3) \quad -2\varphi + \chi(N) = 2\psi_2 + c_2 \log |z|^2 + 2 \operatorname{Re} g_2(z)$$

for some holomorphic function g_2 on Δ^* and some $c_2 \in \mathbb{R}$. For the details, see, for example, [Dem2, Chapter III, §1.C].

Step 5. By multiplying $e^{g_1(z)}$, we take a different trivialization of L . Then h becomes $|\cdot|^2 e^{-2\varphi + 2 \operatorname{Re} g_1}$. Hence, by considering this new trivialization of L on Δ^* , that is, by replacing $-2\varphi + 2 \operatorname{Re} g_1$ with -2φ , we may assume that

$$(4.4) \quad 2\varphi + \chi(N) = 2\psi_1 + c_1 \log |z|^2$$

holds. In this case,

$$2\chi(N) = 2\psi_1 + 2\psi_2 + (c_1 + c_2) \log |z|^2 + 2 \operatorname{Re} g_2(z)$$

holds by (4.3) and (4.4). Note that $\chi(N)$, ψ_1 , ψ_2 , and $\log |z|^2$ are subharmonic functions on Δ . We have

$$\frac{-\operatorname{Re} g_2(z)}{\log |z|} = \frac{\psi_1(z)}{\log |z|} + \frac{\psi_2(z)}{\log |z|} + c_1 + c_2 + \frac{-\chi(N)}{\log |z|}.$$

Therefore, we obtain

$$\begin{aligned} \liminf_{z \rightarrow 0} \frac{-\operatorname{Re} g_2(z)}{\log |z|} &\geq \liminf_{z \rightarrow 0} \frac{\psi_1(z)}{\log |z|} + \liminf_{z \rightarrow 0} \frac{\psi_2(z)}{\log |z|} + c_1 + c_2 + \liminf_{z \rightarrow 0} \frac{-\chi(N)}{\log |z|} \\ &= \nu(\psi_1, 0) + \nu(\psi_2, 0) + c_1 + c_2. \end{aligned}$$

Thus there exists some $C > 0$ such that

$$\frac{-\operatorname{Re} g_2(z)}{\log |z|} \geq -C$$

holds over some open neighborhood of 0. This implies that

$$\operatorname{Re}(-g_2(z)) \leq C(-\log |z|)$$

holds around 0. By Lemma 3.3, we see that g_2 is holomorphic on Δ . Therefore, $\operatorname{Re} g_2(z)$ is a harmonic function on Δ . Hence, by replacing ψ_2 with $\psi_2 - \operatorname{Re} g_2(z)$, we may further assume that

$$(4.5) \quad -2\varphi + \chi(N) = 2\psi_2 + c_2 \log |z|^2$$

holds. By (4.4) and (4.5), we have

$$2\chi(N) = 2\psi_1 + 2\psi_2 + (c_1 + c_2) \log |z|^2.$$

Thus, we obtain

$$(4.6) \quad \nu_1 + \nu_2 + (c_1 + c_2) = 0$$

by Lemma 3.1, where $\nu_1 := \nu(\psi_1, 0)$ and $\nu_2 := \nu(\psi_2, 0)$. We put $\gamma := \nu_1 + c_1$. Then $\nu_2 + c_2 = -\gamma$ by (4.6).

By (4.4), we have

$$\liminf_{z \rightarrow 0} \frac{\varphi(z)}{\log |z|} = \nu_1 + c_1 = \gamma.$$

By (4.5), we have

$$\liminf_{z \rightarrow 0} \frac{-\varphi(z)}{\log |z|} = \nu_2 + c_2 = -\gamma.$$

Therefore, we obtain

$$\gamma = \liminf_{z \rightarrow 0} \frac{\varphi(z)}{\log |z|} \leq \limsup_{z \rightarrow 0} \frac{\varphi(z)}{\log |z|} = \gamma.$$

Hence, we finally obtain

$$(4.7) \quad \gamma = \lim_{z \rightarrow 0} \frac{\varphi(z)}{\log |z|}.$$

Step 6. In this final step, we will prove the following claim.

Claim. *The following equality*

$${}_{\alpha}L = \mathcal{O}_{\Delta} \cdot z^{-\lfloor \alpha - \gamma \rfloor}$$

holds, that is, ${}_{\alpha}L$ is generated by $z^{-\lfloor \alpha - \gamma \rfloor}$.

We give a detailed proof of Claim for the sake of completeness.

Proof of Claim. We put $m_{\alpha} := \lfloor \alpha - \gamma \rfloor$. Then we have

$$m_{\alpha} \leq \alpha - \gamma < m_{\alpha} + 1.$$

Throughout this proof, we will freely shrink Δ around 0.

First, we will prove the inclusion $\mathcal{O}_{\Delta} \cdot z^{-m_{\alpha}} \subset {}_{\alpha}L$. Let f be any local section of $\mathcal{O}_{\Delta} \cdot z^{-m_{\alpha}}$. By (4.7), for any $\varepsilon > 0$, we have

$$\gamma - \varepsilon \leq \frac{\varphi(z)}{\log |z|} = \frac{-\varphi(z)}{-\log |z|} \leq \gamma + \varepsilon$$

around 0. Therefore, we obtain

$$(-\gamma + \varepsilon) \log |z| \leq -\varphi(z) \leq (-\gamma - \varepsilon) \log |z|$$

on some open neighborhood of 0. Thus, we have

$$|z|^{-\gamma + \varepsilon} \leq e^{-\varphi(z)} \leq |z|^{-\gamma - \varepsilon}$$

around 0. Hence we have

$$\begin{aligned} |f|_h |z|^{\alpha+\varepsilon} &= |f| e^{-\varphi(z)} |z|^{\alpha+\varepsilon} \\ &\leq |f| |z|^{-\gamma-\varepsilon} |z|^{\alpha+\varepsilon} \\ &= |f| |z|^{\alpha-\gamma} \\ &\leq |f| |z|^{m_\alpha} \\ &\leq C \end{aligned}$$

around 0 since f is a local section of $\mathcal{O}_\Delta \cdot z^{-m_\alpha}$. Here we used $|z|^{m_\alpha} \geq |z|^{\alpha-\gamma}$ since $m_\alpha \leq \alpha - \gamma$ and $|z| < 1$. Thus, we see that f is in ${}_\alpha L$. This is what we wanted.

From now, we will prove the opposite inclusion ${}_\alpha L \subset \mathcal{O}_\Delta \cdot z^{-m_\alpha}$. Let f be any local section of ${}_\alpha L$. Since $\alpha - \gamma < m_\alpha + 1$, we have $m_\alpha + 1 - (\alpha - \gamma) > 0$. We put

$$\varepsilon := \frac{m_\alpha + 1 - (\alpha - \gamma)}{3} > 0.$$

Then $\alpha - \gamma + 2\varepsilon = m_\alpha + 1 - \varepsilon$. By shrinking Δ suitably, there exists some constant $C > 0$ such that

$$|f|_h |z|^{\alpha+\varepsilon} = |f| e^{-\varphi(z)} |z|^{\alpha+\varepsilon} \leq C$$

holds for $z \in \Delta^*$. As before, we may assume that

$$|z|^{-\gamma+\varepsilon} \leq e^{-\varphi(z)} \leq |z|^{-\gamma-\varepsilon}$$

holds around 0. Therefore, we obtain

$$\begin{aligned} |f| |z|^{m_\alpha+1-\varepsilon} &= |f| |z|^{\alpha-\gamma+2\varepsilon} \\ &= |f| |z|^{-\gamma+\varepsilon} |z|^{\alpha+\varepsilon} \\ &\leq |f| e^{-\varphi(z)} |z|^{\alpha+\varepsilon} \\ &\leq C \end{aligned}$$

around 0. This means that $z^{m_\alpha} f$ is holomorphic at 0, that is, f is in $\mathcal{O}_\Delta \cdot z^{-m_\alpha}$. This is what we wanted.

Hence we have ${}_\alpha L = \mathcal{O}_\Delta \cdot z^{-m_\alpha}$. We finish the proof of Claim. \square

In particular, ${}_\alpha L$ is a holomorphic line bundle on Δ . This completes the proof of Proposition 4.1. \square

Although we do not use the following observation in this paper, we record it here for possible future use.

Remark 4.2. By (4.1) and Lemma 3.1, we can easily verify that $\nu_1 = \nu_2 = 0$ in the proof of Proposition 4.1, since the Lelong number $\nu(\chi(N), 0)$ of $\chi(N)$ at 0 is zero. Therefore, we have $\gamma = c_1 = -c_2$. Hence,

$$2\varphi + \chi(N) = 2\psi_1 + \gamma \log |z|^2 \quad \text{and} \quad -2\varphi + \chi(N) = 2\psi_2 - \gamma \log |z|^2.$$

In particular, we obtain $\psi_1 + \psi_2 = \chi(N)$. Thus,

$$h = |\cdot|^2 e^{-2\varphi} = \frac{|\cdot|^2}{|z|^{2\gamma}} e^{-2\psi_1 + \chi(N)} = \frac{|\cdot|^2}{|z|^{2\gamma}} e^{-\psi_1 + \psi_2},$$

and

$$h^\vee = |\cdot|^2 e^{2\varphi} = |\cdot|^2 |z|^{2\gamma} e^{-2\psi_2 + \chi(N)} = |\cdot|^2 |z|^{2\gamma} e^{-\psi_2 + \psi_1}.$$

Let us prove Corollary 1.7.

Corollary 4.3 (Corollary 1.7). *Let (L, h) be an acceptable line bundle on Δ^* . Then, for every $\alpha \in \mathbb{R}$,*

$$({}_\alpha L)^\vee = {}_{-\alpha+1-\varepsilon}(L^\vee)$$

holds for all sufficiently small $\varepsilon > 0$.

Proof of Corollary 4.3. In the proof of Proposition 4.1, the metric of L^\vee is $|\cdot|^2 e^{2\varphi}$. We replace φ with $-\varphi$ and use the same argument as in the proof of Proposition 4.1. More precisely, for L , we used (4.5) in the proof of Proposition 4.1. For L^\vee , it is sufficient to use (4.4). Then ${}_\beta(L^\vee)$ is generated by $z^{-\lfloor \beta + \gamma \rfloor}$ for every $\beta \in \mathbb{R}$. We put $\beta = -\alpha + 1 - \varepsilon$. If $0 < \varepsilon \ll 1$, then

$$\begin{aligned} -\lfloor \beta + \gamma \rfloor &= -\lfloor -\alpha + 1 - \varepsilon + \gamma \rfloor \\ &= \lceil \alpha - 1 + \varepsilon - \gamma \rceil \\ &= \lfloor \alpha - \gamma \rfloor. \end{aligned}$$

Hence we obtain the desired equality ${}_{-\alpha+1-\varepsilon}(L^\vee) = ({}_\alpha L)^\vee$ for $0 < \varepsilon \ll 1$. \square

By the proof of Proposition 4.1, Corollary 1.8 is almost obvious.

Proof of Corollary 1.8. In (4.4) in the proof of Proposition 4.1, we can make $N = 0$, $\psi_1 = 0$, and $c_1 = c$. Then $e^{-2\varphi} = \frac{1}{|z|^{2c}}$. This is what we wanted. \square

For the reader's convenience, we summarize Proposition 4.1 along with its proof. To the best of the authors' knowledge, the following explicit description appears to be new.

Theorem 4.4 (Theorem 1.6). *Let (L, h) be an acceptable line bundle on Δ^* . By taking a suitable trivialization*

$$(L, h) \simeq (\mathcal{O}_{\Delta^*}, |\cdot|^2 e^{-2\varphi}),$$

we have the following properties.

(i) *The limit*

$$\gamma := \lim_{z \rightarrow 0} \frac{\varphi(z)}{\log |z|} \in \mathbb{R}$$

exists.

- (ii) *Let f be a holomorphic function on $\Delta(0, r)^*$ for some $0 < r < 1$, where $\Delta(0, r)^* := \{z \in \mathbb{C} \mid 0 < |z| < r\}$. Then $f \in ({}_\alpha L)_0$ holds for some $\alpha \in \mathbb{R}$ if and only if f is meromorphic at 0, where $({}_\alpha L)_0$ denotes the stalk of ${}_\alpha L$ at $0 \in \Delta$.*
- (iii) *Let f be a meromorphic function on some open neighborhood of 0 and let α be any real number. Then $f \in ({}_\alpha L)_0$ holds if and only if*

$$(4.8) \quad \lim_{z \rightarrow 0} \frac{\log(|f|e^{-\varphi})}{\log |z|} \geq -\alpha.$$

Note that

$$\lim_{z \rightarrow 0} \frac{\log(|f|e^{-\varphi})}{\log |z|} = \text{ord}_0 f - \gamma$$

holds. Therefore, (4.8) is equivalent to

$$\text{ord}_0 f \geq -\lfloor \alpha - \gamma \rfloor.$$

(iv) Let f be a meromorphic function on some open neighborhood of 0 and let α be any real number. Then $f \notin (\alpha L)_0$ holds if and only if

$$(4.9) \quad \lim_{z \rightarrow 0} \frac{\log(|f|e^{-\varphi})}{\log|z|} < -\alpha.$$

Note that (4.9) implies that

$$|f|e^{-\varphi} > \frac{1}{|z|^\alpha}$$

holds on some small open neighborhood of 0.

From the above description of ${}_\alpha L$, the following result is immediate. We state it explicitly for later use.

Corollary 4.5. *Let (L, h) be an acceptable line bundle on Δ^* . If $f \in (\alpha L)_0$, then there exists $\beta < \alpha$ such that $f \notin (\beta L)_0$.*

We close this section with an important remark.

Remark 4.6. We consider

$$(L, h) = \left(\mathcal{O}_{\Delta^*}, \frac{|\cdot|^2}{|z|^{2c}} \right),$$

that is, $h = |\cdot|^2 e^{-2\varphi}$ with $\varphi = c \log|z|$. In this case, we have

$$\sqrt{-1}\Theta_h(L) = \sqrt{-1}\partial\bar{\partial}(2\varphi) = 0$$

on Δ^* . Note that we can see $\log|z|$ as a subharmonic function on Δ and that $\sqrt{-1}\partial\bar{\partial}(2\varphi)$ is not zero as a current on Δ .

5. ON GROWTH ESTIMATES

In this section, we present the minimal analytic results needed in later sections, for the reader's convenience. We begin with a discussion of the $\bar{\partial}$ -equation, from which we derive a growth estimate via L^2 -methods.

5.1 (Setting). Let g be the Kähler metric on Δ^* defined by ω_P . Note that Δ^* is a complete Kähler manifold, even though g itself is not complete. Moreover, Δ^* is a Stein manifold.

Let (E, h) be an acceptable vector bundle on Δ^* . Then, by Lemma 2.5, there exists a positive real number N_0 such that for every $N \geq N_0$, $\sqrt{-1}\Theta_{he^{-\chi(N)}}(E)$ is Nakano semipositive, and $\sqrt{-1}\Theta_{he^{-\chi(-N)}}(E)$ is Griffiths seminegative.

For simplicity, we also denote by g the metric on $K_{\Delta^*}^{\otimes -1}$ induced by ω_P , whenever no confusion arises. Note that the line bundle $(K_{\Delta^*}^{\otimes -1}, g)$ is acceptable. Therefore, by Lemma 2.2, the vector bundle $E \otimes K_{\Delta^*}^{\otimes -1}$ is also acceptable.

Hence, we can choose a sufficiently large positive integer $N \geq N_0$ such that

$$\sqrt{-1}\Theta_{hge^{-\chi(N)}}(E \otimes K_{\Delta^*}^{\otimes -1}) - \omega_P \otimes \text{Id}_E$$

is Nakano semipositive, again by Lemma 2.5.

Lemma 5.2 is a straightforward application of the $\bar{\partial}$ -equation.

Lemma 5.2. *Let (E, h) be an acceptable vector bundle on Δ^* . Let e be any element of E_{z_0} for some point $z_0 \in \Delta^*$. Assume that*

$$\sqrt{-1}\Theta_{hge^{-\chi(N)}}(E \otimes K_{\Delta^*}^{\otimes -1}) - \omega_P \otimes \text{Id}_E$$

is Nakano semipositive. Then there exists a holomorphic section $v(z)$ of E on Δ^* such that $v(z_0) = e$ and

$$\|v\|_{he^{-\chi(N)}}^2 := \int_{\Delta^*} |v|_{he^{-\chi(N)}}^2 \omega_P < \infty.$$

Proof of Lemma 5.2. Take a local holomorphic section $u(z)$ of E defined near z_0 such that $u(z_0) = e$. More precisely, $u(z) \in \Gamma(U, E)$ for some open neighborhood U of z_0 . Choose a smooth function ρ on Δ^* such that $\rho \geq 0$, $\text{supp } \rho \Subset U$, and $\rho = 1$ on some open neighborhood of z_0 .

Consider the smooth E -valued $(0, 1)$ -form with compact support:

$$\frac{\bar{\partial}(\rho(z)u(z))}{z - z_0}.$$

It is clearly $\bar{\partial}$ -closed and can be regarded as a smooth $(0, 1)$ -form on Δ^* with values in

$$K_{\Delta^*} \otimes (E \otimes K_{\Delta^*}^{\otimes -1}).$$

Since

$$\sqrt{-1}\Theta_{hge^{-\chi(N)}}(E \otimes K_{\Delta^*}^{\otimes -1}) - \omega_P \otimes \text{Id}_E$$

is Nakano semipositive by assumption, the $\bar{\partial}$ -equation can be solved in the L^2 sense.

Thus, we can find a measurable E -valued function $w(z)$ such that

$$\int_{\Delta^*} |w(z)|_{he^{-\chi(N)}}^2 \omega_P < \infty$$

and that

$$\bar{\partial}w(z) = \frac{\bar{\partial}(\rho(z)u(z))}{z - z_0}.$$

For details, see for example [Dem2, Chapter VIII, (6.1) Theorem].

Define

$$v(z) := \rho(z)u(z) - (z - z_0)w(z).$$

Then $v(z)$ is holomorphic on Δ^* , satisfies $v(z_0) = e$, and

$$\|v\|_{he^{-\chi(N)}}^2 = \int_{\Delta^*} |v(z)|_{he^{-\chi(N)}}^2 \omega_P < \infty.$$

This completes the proof of Lemma 5.2. \square

Lemma 5.3 is a straightforward consequence of the mean value inequality for subharmonic functions.

Lemma 5.3. *Let (E, h) be an acceptable vector bundle on Δ^* . Let N be a positive integer such that $\sqrt{-1}\Theta_{he^{-\chi(N)}}(E)$ is Nakano semipositive and $\sqrt{-1}\Theta_{he^{-\chi(-N)}}(E)$ is Griffiths seminegative.*

Suppose that a holomorphic section v of E satisfies

$$\|v\|_{he^{-\chi(N)}}^2 := \int_{\Delta^*} |v|_{he^{-\chi(N)}}^2 \omega_P < \infty.$$

Then, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|v(z)|_h \leq C_\varepsilon \cdot \frac{1}{|z|^\varepsilon}$$

holds on $\Delta(0, r)^ := \{z \in \mathbb{C} \mid 0 < |z| < r\}$ for some sufficiently small $r > 0$.*

We include a proof of Lemma 5.3 for completeness.

Proof of Lemma 5.3. In this proof, each C_i denotes a positive constant for every i .

Step 1. By assumption, the bundle $(E, he^{-\chi(-N)})$ is Griffiths seminegative. Hence, by Lemma 2.7, the function

$$\log |s|_{he^{-\chi(-N)}}$$

is subharmonic for any holomorphic section s of E on Δ^* . In particular, this applies to v , so we may use the mean value inequality for $\log |v|_{he^{-\chi(-N)}}^2$.

Step 2. Fix $z \in \Delta^*$ with $0 < |z| < \frac{1}{4}$. Then:

$$\begin{aligned} \log |v(z)|_{he^{-\chi(-N)}}^2 &\leq \frac{4}{\pi |z|^2} \int_{|w-z| \leq \frac{|z|}{2}} \log |v(w)|_{he^{-\chi(-N)}}^2 dw \wedge d\bar{w} \\ &\leq \log \left(\frac{4}{\pi |z|^2} \int_{|w-z| \leq \frac{|z|}{2}} |v(w)|_{he^{-\chi(-N)}}^2 dw \wedge d\bar{w} \right) \\ (5.1) \quad &\leq \log \left(C_1 \int_{|w-z| \leq \frac{|z|}{2}} \frac{1}{|w|^2} |v(w)|_{he^{-\chi(-N)}}^2 dw \wedge d\bar{w} \right) \\ &\leq \log C_2 + \log \int_{|w-z| \leq \frac{|z|}{2}} |v(w)|_{he^{-\chi(N)}}^2 \omega_P \\ &\leq C_3 + \log \|v\|_{he^{-\chi(N)}}^2. \end{aligned}$$

Here, the first inequality is the mean value inequality for subharmonic functions, and the second follows from Jensen's inequality.

Step 3. Using the estimate (5.1), we obtain:

$$\begin{aligned} |v(z)|_h &= |v(z)|_{he^{-\chi(-N)}} \cdot (-\log |z|^2)^{N/2} \\ &\leq C_4 \|v\|_{he^{-\chi(N)}} \cdot \frac{1}{|z|^\varepsilon} \end{aligned}$$

for some constant $C_4 > 0$ and any given $\varepsilon > 0$. This completes the desired estimate.

The proof of Lemma 5.3 is now complete. \square

The following lemma is also a consequence of subharmonicity. We will repeatedly use it in subsequent sections.

Lemma 5.4 (cf. [M4, Lemma 21.2.7]). *Let (E, h) be an acceptable vector bundle on Δ^* . Let f be a holomorphic section of E on Δ^* such that*

$$|f|_h = O\left(\frac{1}{|z|^{a+\varepsilon}}\right)$$

for any $\varepsilon > 0$. We assume that $(E, he^{-\chi(-N)})$ is Griffiths seminegative. We put

$$H(z) := |f|_{he^{-\chi(-N)}}^2 |z|^{2a} = |f|_h^2 |z|^{2a} (-\log |z|^2)^{-N}.$$

Then $H(z)$ is bounded on $\Delta(0, r_0)$ for any $0 < r_0 < 1$.

Proof of Lemma 5.4. We put $H_\varepsilon(z) := H(z)|z|^{2\varepsilon}$ for any $\varepsilon > 0$. Note that $\log H_\varepsilon(z)$ is subharmonic on Δ^* by Lemma 2.7. By assumption, we have $\lim_{z \rightarrow 0} \log H_\varepsilon(z) = -\infty$. Hence $\log H_\varepsilon(z)$ is subharmonic on Δ (see [NO, (3.3.25) Theorem]). Therefore, we have

$$(5.2) \quad \max_{|z| \leq r_0} |H_\varepsilon(z)| = \max_{|z|=r_0} H_\varepsilon(z).$$

Note that $H(z)$ is a continuous function on $|z| = r_0$ and that $H_{\varepsilon_1}(z) \leq H_{\varepsilon_2}(z)$ holds on $|z| = r_0$ for $0 \leq \varepsilon_2 \leq \varepsilon_1 \leq 1$. By taking the limit for $\varepsilon \rightarrow 0$, we obtain that $H(z)$ is bounded on $\Delta(0, r_0)$ by (5.2). We finish the proof of Lemma 5.4. \square

6. PROLONGATION OF ACCEPTABLE VECTOR BUNDLES

In this section, we establish Theorem 1.3.

Proof of Theorem 1.3. In the following proof, we will use Proposition 1.5 (see Proposition 4.1), which is a special case of Theorem 1.3.

Step 1. Let (E, h) be the given acceptable vector bundle on Δ^* and let α be any real number. We put $E' := E$ and $h' := h \cdot |z|^{2\alpha}$ and consider (E', h') . Then we have

$$\sqrt{-1}\Theta_h(E) = \sqrt{-1}\Theta_{h'}(E')$$

on Δ^* . Hence (E', h') is also an acceptable vector bundle on Δ^* . By the definition of (E', h') , ${}_{\alpha}E = {}_0E'$ obviously holds. Therefore, it is sufficient to prove that ${}^{\circ}E = {}_0E$ is a holomorphic vector bundle on Δ . By definition, ${}^{\circ}E$ is a torsion-free sheaf on Δ . Thus, it is sufficient to prove that ${}^{\circ}E$ is coherent since ${}^{\circ}E$ is a sheaf on Δ .

Step 2. Let $z_0 \in \Delta^*$ be any point. Let $\{e_1, \dots, e_k\}$ be a basis of the fiber E_{z_0} , where $k = \dim_{\mathbb{C}} E_{z_0} = \text{rank } E$.

From now on, we allow ourselves to shrink the unit disk Δ and replace it with a smaller disk of the form

$$\Delta(0, r) := \{z \in \mathbb{C} \mid |z| < r\}$$

for some $0 < r < 1$, without explicitly mentioning it.

By Lemmas 5.2 and 5.3, for each i , we can find a holomorphic section $v_i(z)$ of E on Δ^* such that $v_i(z_0) = e_i$ and

$$|v_i(z)|_h = O\left(\frac{1}{|z|^\varepsilon}\right)$$

for every $\varepsilon > 0$. In other words, $v_i \in \Gamma(\Delta, {}^{\circ}E)$ for all i .

Step 3. We put $L := \det(E)$. Then L is an acceptable line bundle on Δ^* by Lemma 2.2. Since

$$(v_1 \wedge \cdots \wedge v_k)(z_0) \neq 0,$$

$v_1 \wedge \cdots \wedge v_k$ is a nontrivial holomorphic section of ${}^{\circ}L$. We fix a trivialization

$${}^{\circ}L = \mathcal{O}_{\Delta} \cdot \mathbf{e}.$$

Then we can write

$$v_1 \wedge \cdots \wedge v_k = a(z)\mathbf{e}$$

for some holomorphic function $a(z)$ on Δ . We put $l := \text{ord}_0 a(z) \geq 0$.

Step 4. Since $a(z)$ is a holomorphic function on Δ , we may assume that $a(z) \neq 0$ for all $z \in \Delta^*$ by shrinking Δ around 0. Then $(v_1 \wedge \cdots \wedge v_k)(z) \neq 0$ for all $z \in \Delta^*$. Therefore the morphism

$$(6.1) \quad \mathcal{O}_{\Delta}^{\oplus k} \rightarrow {}^{\circ}E$$

defined by v_1, \dots, v_k is isomorphic over Δ^* .

Step 5. Let s be any local section of ${}^\diamond E$ around 0. Since the morphism (6.1) is isomorphic over Δ^* , we can write

$$s(z) = \sum_{i=1}^k s_i(z)v_i(z)$$

such that $s_i(z)$ is a holomorphic function on Δ^* for every i . Since

$$\begin{aligned} s \wedge v_2 \wedge \cdots \wedge v_k &= s_1(z)v_1 \wedge \cdots \wedge v_k \\ &= s_1(z)a(z)\mathbf{e} \end{aligned}$$

is a holomorphic section of ${}^\diamond L$, we have $\text{ord}_0 s_1(z) \geq -l$. Similarly, we obtain $\text{ord}_0 s_i(z) \geq -l$ for every i . This implies that

$$(6.2) \quad {}^\diamond E \subset \bigoplus_{i=1}^k \mathcal{O}_\Delta \cdot \frac{v_i}{z_i^l} \subset j_* E,$$

where $j: \Delta^* \hookrightarrow \Delta$. By definition, $({}^\diamond E)|_{\Delta^*} = E$ holds. Since we have

$${}^\diamond E \subset \bigoplus_{i=1}^k \mathcal{O}_\Delta(l[0])$$

by (6.2), the stalk $({}^\diamond E)_0$ is a finitely generated $\mathcal{O}_{\Delta,0}$ -module. Then, by shrinking Δ around 0 if necessary, we obtain a morphism

$$\mathcal{O}_\Delta^{\oplus n} \rightarrow {}^\diamond E$$

for some positive integer n , which induces a surjection on the stalk at 0. The direct sum of this morphism with the morphism (6.1) is surjective over the entire disk Δ . Hence, ${}^\diamond E$ is locally finitely generated over Δ . This implies that ${}^\diamond E$ is a coherent \mathcal{O}_Δ -module.

We finish the proof of Theorem 1.3. \square

7. BASIC PROPERTIES

In this section, we introduce a new invariant $\gamma({}_\alpha E)$ and discuss basic properties of ${}_\alpha E$ and $\gamma({}_\alpha E)$.

7.1 (Setting). Let (E, h) be an acceptable vector bundle over Δ^* with $\text{rank } E = r$. Let $\mathbf{v} := \{v_1, \dots, v_r\}$ be a local frame of ${}_\alpha E$ defined on some open neighborhood of 0. We consider the $r \times r$ matrix

$$H(h, \mathbf{v}) := (h(v_i, v_j))_{i,j}.$$

More precisely, $H(h, \mathbf{v})$ is an $r \times r$ Hermitian matrix-valued function on Δ^* . Hence, we sometimes write $H(h, \mathbf{v})(z)$ to denote the value of $H(h, \mathbf{v})$ at $z \in \Delta^*$. If there is no risk of confusion, we may simplify the notation by writing $H(\mathbf{v})$ and $H(\mathbf{v})(z)$ in place of $H(h, \mathbf{v})$ and $H(h, \mathbf{v})(z)$, respectively.

We have

$$|h(v_i, v_j)| \leq |v_i|_h |v_j|_h = O\left(\frac{1}{|z|^{2\alpha+\varepsilon}}\right)$$

for any $\varepsilon > 0$. This means that for any $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ such that

$$\det H(\mathbf{v}) = |\det H(\mathbf{v})| \leq C_\varepsilon |z|^{-2\alpha r - \varepsilon}.$$

Thus we obtain

$$\log \det H(\mathbf{v}) \leq \log C_\varepsilon - (2\alpha r + \varepsilon) \log |z|.$$

Hence we have

$$\frac{\log \det H(\mathbf{v})}{\log |z|} \geq \frac{\log C_\varepsilon}{\log |z|} - (2\alpha r + \varepsilon).$$

Therefore,

$$\liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|}$$

satisfies

$$\liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|} \geq -2\alpha r - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$(7.1) \quad \liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|} \geq -2\alpha r > -\infty.$$

Lemma 7.2. *In the above definition,*

$$\liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|}$$

is independent of the choice of the frame $\mathbf{v} = \{v_1, \dots, v_r\}$ of ${}_\alpha E$.

Proof of Lemma 7.2. Let $\mathbf{w} := \{w_1, \dots, w_r\}$ be another frame of ${}_\alpha E$ around 0. Then we can write

$$(w_1, \dots, w_r) = (v_1, \dots, v_r)A(z)$$

around 0, where $A(z)$ is an invertible $r \times r$ matrix. Thus we have

$$\det H(\mathbf{w}) = |\det A(z)|^2 \det H(\mathbf{v}).$$

Hence

$$\frac{\log \det H(\mathbf{w})}{\log |z|} = \frac{2|\det A(z)|}{\log |z|} + \frac{\log \det H(\mathbf{v})}{\log |z|}.$$

Since $\det A(0) \neq 0$, we obtain

$$\liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{w})}{\log |z|} = \liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|}.$$

This is what we wanted. We finish the proof of Lemma 7.2. □

We can prove the following lemma.

Lemma 7.3.

$$-\infty < \liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|} < \infty.$$

Proof of Lemma 7.3. By (7.1), we have already checked the left inequality. Hence it is sufficient to prove the right inequality. Since $(\det E, \det h)$ is an acceptable line bundle on Δ^* by Lemma 2.2, we can freely use Theorem 4.4.

By the above observation, we have

$$v_1 \wedge \dots \wedge v_r \in {}_{\alpha r} \det E.$$

By Corollary 4.5, there exists some real number $\beta < \alpha r$ such that $v_1 \wedge \dots \wedge v_r \notin {}_\beta \det E$. Hence, by Theorem 4.4 (iv), we can take d and $C > 0$ such that

$$|\det H(\mathbf{v})|^{1/2} = |v_1 \wedge \dots \wedge v_r| \geq \frac{C}{|z|^d}$$

holds around 0. This implies

$$\frac{1}{2} \log \det H(\mathbf{v}) \geq -d \cdot \log |z| + \log C$$

Hence

$$\liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|} \leq -2d < \infty.$$

This is what we wanted. We finish the proof of Lemma 7.3. \square

Having completed the necessary preparations, we now define $\gamma(\alpha E)$.

Definition 7.4. We put

$$\gamma(\alpha E) := -\frac{1}{2} \liminf_{z \rightarrow 0} \frac{\log \det H(h, \mathbf{v})}{\log |z|} \in \mathbb{R}.$$

The following theorem is the main theorem of this section.

Theorem 7.5 (Determinant bundles, see Theorem 1.10). *Let (E, h) be an acceptable vector bundle on Δ^* . Let α be any real number. Then*

$$\det(\alpha E) = \gamma(\alpha E) \det E$$

holds.

We give a detailed proof of Theorem 7.5, which is essentially the same as the proof of Proposition 4.1.

Proof of Theorem 7.5. Let $\mathbf{v} := \{v_1, \dots, v_r\}$ be a frame of αE on Δ . We put

$$H(h, \mathbf{v})(z) := (h(v_i, v_j)(z))_{i,j}.$$

We note that

$$\det(\alpha E) = \mathcal{O}_\Delta v_1 \wedge \dots \wedge v_r$$

and

$$(7.2) \quad \det E = \mathcal{O}_{\Delta^*} v_1 \wedge \dots \wedge v_r \simeq \mathcal{O}_{\Delta^*}.$$

We consider $L := \det E$. Let h_L be the induced metric on L . Note that (L, h_L) is an acceptable line bundle on Δ^* by Lemma 2.2 since (E, h) is acceptable. By using the trivialization (7.2), we argue as in the proof of Proposition 4.1. In this setting,

$$h_L = |\cdot|^2 e^{-2\varphi_\alpha}$$

with

$$(7.3) \quad \begin{aligned} e^{-2\varphi_\alpha} &= h_L(v_1 \wedge \dots \wedge v_r, v_1 \wedge \dots \wedge v_r) \\ &= \det H(\mathbf{v}). \end{aligned}$$

Thus

$$\varphi_\alpha = -\frac{1}{2} \log \det H(\mathbf{v}).$$

Therefore,

$$(7.4) \quad \liminf_{z \rightarrow 0} \frac{-\varphi_\alpha(z)}{\log |z|} = \frac{1}{2} \liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|} = -\gamma(\alpha E).$$

Note that (L, h_L) is an acceptable line bundle on Δ^* . As in the proof of Proposition 4.1, we can write

$$(7.5) \quad 2\varphi_\alpha + \chi(N) = 2\psi_1 + c_1 \log |z|^2 + 2 \operatorname{Re} g_1(z)$$

and

$$(7.6) \quad -2\varphi_\alpha + \chi(N) = 2\psi_2 + c_2 \log |z|^2 + 2 \operatorname{Re} g_2(z),$$

where ψ_1 and ψ_2 are subharmonic functions on Δ and g_1 and g_2 are holomorphic functions on Δ^* . By (7.5), we have

$$-\operatorname{Re} g_1(z) = \psi_1 + c_1 \log |z| - \varphi_\alpha - \chi(N).$$

Hence we have

$$\frac{-\operatorname{Re} g_1(z)}{\log |z|} = \frac{\psi_1(z)}{\log |z|} + c_1 + \frac{-\varphi_\alpha(z)}{\log |z|} + \frac{-\chi(N)}{\log |z|}.$$

Therefore, we obtain

$$\begin{aligned} \liminf_{z \rightarrow 0} \frac{-\operatorname{Re} g_1(z)}{\log |z|} &\geq \liminf_{z \rightarrow 0} \frac{\psi_1(z)}{\log |z|} + c_1 + \liminf_{z \rightarrow 0} \frac{-\varphi_\alpha(z)}{\log |z|} + \liminf_{z \rightarrow 0} \frac{-\chi(N)}{\log |z|} \\ &= \nu(\psi_1, 0) + c_1 - \gamma(\alpha E) \\ &> -\infty. \end{aligned}$$

Here we used Lemma 7.3. Thus there exists some $C > 0$ such that

$$\frac{-\operatorname{Re} g_1(z)}{\log |z|} \geq -C$$

holds over some open neighborhood of 0. This implies that

$$\operatorname{Re}(-g_1(z)) \leq C(-\log |z|)$$

holds around 0. By Lemma 3.3, we see that g_1 is holomorphic on Δ . Hence we have

$$\nu_1 + c_1 - \gamma(\alpha E) \leq 0 = \liminf_{z \rightarrow 0} \frac{-\operatorname{Re} g_1(z)}{\log |z|},$$

where $\nu_1 := \nu(\psi_1, 0)$. Note that $e^{g_1(z)}$ is a nowhere vanishing holomorphic function on Δ . By replacing v_1 with $e^{g_1(z)}v_1$, we may assume that

$$(7.7) \quad 2\varphi_\alpha + \chi(N) = 2\psi_1 + c_1 \log |z|^2$$

and

$$(7.8) \quad -2\varphi_\alpha + \chi(N) = 2\psi_2 + c_2 \log |z|^2 + 2 \operatorname{Re} g_2(z)$$

hold, after replacing g_2 accordingly. By (7.7) and (7.8), we have

$$2\chi(N) = 2\psi_1 + 2\psi_2 + (c_1 + c_2) \log |z|^2 + 2 \operatorname{Re} g_2(z)$$

holds. Note that $\chi(N)$, ψ_1 , ψ_2 , and $\log |z|^2$ are subharmonic functions on Δ . We have

$$\frac{-\operatorname{Re} g_2(z)}{\log |z|} = \frac{\psi_1(z)}{\log |z|} + \frac{\psi_2(z)}{\log |z|} + c_1 + c_2 + \frac{-\chi(N)}{\log |z|}.$$

Therefore, we obtain

$$\begin{aligned} \liminf_{z \rightarrow 0} \frac{-\operatorname{Re} g_2(z)}{\log |z|} &\geq \liminf_{z \rightarrow 0} \frac{\psi_1(z)}{\log |z|} + \liminf_{z \rightarrow 0} \frac{\psi_2(z)}{\log |z|} + c_1 + c_2 + \liminf_{z \rightarrow 0} \frac{-\chi(N)}{\log |z|} \\ &= \nu(\psi_1, 0) + \nu(\psi_2, 0) + c_1 + c_2. \end{aligned}$$

Thus there exists some $C > 0$ such that

$$\frac{-\operatorname{Re} g_2(z)}{\log |z|} \geq -C$$

holds over some open neighborhood of 0. This implies that

$$\operatorname{Re}(-g_2(z)) \leq C(-\log|z|)$$

holds around 0. By Lemma 3.3, we see that g_2 is holomorphic on Δ . In particular, $\operatorname{Re} g_2(z)$ is harmonic on Δ . By replacing ψ_2 with $\psi_2 + \operatorname{Re} g_2(z)$, we can finally assume that

$$(7.9) \quad 2\varphi_\alpha + \chi(N) = 2\psi_1 + c_1 \log|z|^2$$

and

$$(7.10) \quad -2\varphi_\alpha + \chi(N) = 2\psi_2 + c_2 \log|z|^2$$

hold. By (7.10), we obtain

$$-\gamma(\alpha E) = \nu_2 + c_2,$$

where $\nu_2 := \nu(\psi_2, 0)$. By (7.9) and (7.10), we have

$$\chi(N) = \psi_1 + \psi_2 + (c_1 + c_2) \log|z|.$$

Therefore, by Lemma 3.1, we obtain

$$0 = \nu_1 + \nu_2 + c_1 + c_2.$$

This means that

$$\nu_1 + c_1 = -(\nu_2 + c_2) = \gamma(\alpha E).$$

By (7.9), we have

$$(7.11) \quad \liminf_{z \rightarrow 0} \frac{\varphi_\alpha(z)}{\log|z|} = \nu_1 + c_1 = \gamma(\alpha E).$$

Thus, by (7.4) and (7.11), we get

$$\lim_{z \rightarrow 0} \frac{\varphi_\alpha(z)}{\log|z|} = \gamma(\alpha E), \quad \lim_{z \rightarrow 0} \frac{\psi_1(z)}{\log|z|} = \nu_1, \quad \text{and} \quad \lim_{z \rightarrow 0} \frac{\psi_2(z)}{\log|z|} = \nu_2.$$

As in the proof of Proposition 4.1, we obtain

$$(7.12) \quad {}_\beta L = {}_\beta(\det E) = \mathcal{O}_\Delta \cdot z^{-\lfloor \beta - \gamma(\alpha E) \rfloor} v_1 \wedge \cdots \wedge v_r.$$

In particular,

$$\det(\alpha E) = \gamma(\alpha E) \det E.$$

We finish the proof of Theorem 7.5. \square

As a byproduct of the proof of Theorem 7.5, we have the following useful result, that is, we can replace \liminf with \lim in the definition of $\gamma(\alpha E)$.

Corollary 7.6. *In the same setting as in 7.1, we have*

$$\gamma(\alpha E) = -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log|z|}.$$

Proof of Corollary 7.6. By the proof of Lemma 7.2, we may assume that \mathbf{v} is a frame of αE on Δ . In the proof of Theorem 7.5, we have

$$\gamma(\alpha E) = \lim_{z \rightarrow 0} \frac{\varphi_\alpha(z)}{\log|z|} = -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log|z|}.$$

This is what we wanted. \square

For later use, we explicitly state the following result, which is an immediate consequence of Theorem 7.5 and its proof.

Corollary 7.7. *Let (E, h) be an acceptable vector bundle over Δ^* with $\text{rank } E = r$. Let $\mathbf{v} := \{v_1, \dots, v_r\}$ be a local frame of ${}_{\alpha}E$, defined over an open neighborhood of 0, such that $v_i \in {}_{b_i}E$ for each i . Then we have*

$$(7.13) \quad \gamma({}_{\alpha}E) \leq \sum_{i=1}^r b_i,$$

and

$$(7.14) \quad \gamma(\gamma({}_{\alpha}E) \det E) = \gamma({}_{\alpha}E).$$

Proof of Corollary 7.7. Since $v_i \in {}_{b_i}E$ for each i , we have

$$v_1 \wedge \cdots \wedge v_r \in \sum_{i=1}^r b_i \det E.$$

On the other hand, from (7.12) in the proof of Theorem 7.5, we obtain

$$\begin{aligned} \gamma({}_{\alpha}E) \det E &= \mathcal{O}_{\Delta} \cdot v_1 \wedge \cdots \wedge v_r, \quad \text{and} \\ \sum_{i=1}^r b_i \det E &= \mathcal{O}_{\Delta} \cdot z^{-\lfloor \sum_{i=1}^r b_i - \gamma({}_{\alpha}E) \rfloor} v_1 \wedge \cdots \wedge v_r. \end{aligned}$$

Therefore, it follows that

$$\sum_{i=1}^r b_i \geq \gamma({}_{\alpha}E) \quad \text{and} \quad \gamma({}_{\alpha}E) \det E \subset \sum_{i=1}^r b_i \det E.$$

This yields (7.13).

We note that $\gamma({}_{\alpha}E) \det E = \det({}_{\alpha}E)$ by Theorem 7.5. From (7.3) in the proof of Theorem 7.5, we obtain

$$\gamma(\gamma({}_{\alpha}E) \det E) = \liminf_{z \rightarrow 0} \frac{\varphi_{\alpha}(z)}{\log |z|} = -\frac{1}{2} \liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|} = \gamma({}_{\alpha}E).$$

This proves (7.14).

This completes the proof of Corollary 7.7. \square

From now, we discuss some basic properties of $\gamma({}_{\alpha}E)$ and ${}_{\alpha}E$. We note that ${}_{\alpha}E = E$ holds on Δ^* by definition.

Lemma 7.8. *For $\alpha \leq \beta$, we have the following properties.*

- (i) ${}_{\alpha}E \subset {}_{\beta}E$ holds.
- (ii) $\gamma({}_{\alpha}E) \leq \gamma({}_{\beta}E)$, and $\gamma({}_{\beta}E) - \gamma({}_{\alpha}E) \in \mathbb{Z}_{\geq 0}$.
- (iii) ${}_{\alpha}E = {}_{\beta}E$ if and only if $\gamma({}_{\alpha}E) = \gamma({}_{\beta}E)$.
- (iv) ${}_{\alpha+1}E = {}_{\alpha}E \otimes \mathcal{O}_{\Delta}([0])$.
- (v) $\gamma({}_{\alpha+1}E) - \gamma({}_{\alpha}E) = \text{rank } E$.

Proof of Lemma 7.8. It is obvious that (i) holds by definition. Let $\mathbf{w} := \{w_1, \dots, w_r\}$ be a frame of ${}_{\alpha}E$ on Δ and let $\mathbf{v} := \{v_1, \dots, v_r\}$ be a frame of ${}_{\beta}E$ on Δ . Thus we can write

$$(w_1, \dots, w_r) = (v_1, \dots, v_r)A(z)$$

where $A(z)$ is an $r \times r$ matrix. By definition, $A(z)$ is invertible on Δ^* . Hence $\det A(z) \neq 0$ for every $z \in \Delta^*$. Then we can write

$$\det A(z) = z^m f(z)$$

for some $m \in \mathbb{Z}_{\geq 0}$ such that $f(0) \neq 0$. In this setting, we obtain

$$\begin{aligned} \det H(\mathbf{w})(z) &= \det H(\mathbf{v})(z) |\det A(z)|^2 \\ &= \det H(\mathbf{v})(z) |z|^{2m} |f(z)|^2. \end{aligned}$$

Therefore, we have

$$\frac{\log \det H(\mathbf{w})(z)}{\log |z|} = \frac{\log \det H(\mathbf{v})(z)}{\log |z|} + 2m + 2 \frac{\log |f(z)|}{\log |z|}.$$

Thus we obtain

$$\begin{aligned} \gamma_{(\alpha}E) &= -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(\mathbf{w})(z)}{\log |z|} \\ &= -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})(z)}{\log |z|} - m - \lim_{z \rightarrow 0} \frac{\log |f(z)|}{\log |z|} \\ &= \gamma_{(\beta}E) - m. \end{aligned}$$

This implies

$$\gamma_{(\beta}E) - \gamma_{(\alpha}E) = m \in \mathbb{Z}_{\geq 0}.$$

Thus we have (ii). By the above argument, ${}_{\alpha}E = {}_{\beta}E$ if and only if $\det A(0) \neq 0$. Moreover, $\det A(0) \neq 0$ if and only if $m = 0$. Thus, ${}_{\alpha}E = {}_{\beta}E$ if and only if $\gamma_{(\alpha}E) = \gamma_{(\beta}E)$. This is (iii). Since $\mathbf{w} = \{w_1, \dots, w_r\}$ is a frame of ${}_{\alpha}E$, we can easily check that

$$\frac{\mathbf{w}}{z} = \left\{ \frac{w_1}{z}, \dots, \frac{w_r}{z} \right\}$$

is a frame of ${}_{\alpha+1}E$ on Δ^* . Thus we can directly check that

$${}_{\alpha+1}E = {}_{\alpha}E \otimes \mathcal{O}_{\Delta}([0])$$

and

$$\gamma_{(\alpha+1}E) = \gamma_{(\alpha}E) + r.$$

Thus we obtain (iv) and (v). We finish the proof of Lemma 7.8. \square

Lemma 7.9. *Let (E, h) be an acceptable vector bundle on Δ^* . Then, for every $\alpha \in \mathbb{R}$,*

$${}_{\alpha}E = \bigcap_{\beta > \alpha} {}_{\beta}E$$

holds.

Proof of Lemma 7.9. Since ${}_{\alpha}E \subset {}_{\beta}E$ for $\beta > \alpha$ by Lemma 7.8 (i), we have

$${}_{\alpha}E \subset \bigcap_{\beta > \alpha} {}_{\beta}E.$$

From now, we will prove the opposite inclusion. We take $v \in \bigcap_{\beta > \alpha} {}_{\beta}E$. Let ε be any positive real number. We can take β' such that $\alpha < \beta' < \alpha + \varepsilon$ and ε' such that $0 < \varepsilon' < \alpha + \varepsilon - \beta'$. There exists some open neighborhood U of 0 such that $v \in \Gamma(U, {}_{\beta'}E)$. Then

$$|v|_h |z|^{\beta' + \varepsilon'} < C$$

holds for some positive real number C . Hence we have

$$|v|_h |z|^{\alpha + \varepsilon} = |v|_h |z|^{\beta' + \varepsilon'} |z|^{\alpha + \varepsilon - \beta' - \varepsilon'} < C$$

since $\alpha + \varepsilon - \beta' - \varepsilon' > 0$ and $|z| < 1$. This implies

$$|v|_h = O\left(\frac{1}{|z|^{\alpha + \varepsilon}}\right).$$

Thus we obtain $v \in {}_{\alpha}E$. We finish the proof of Lemma 7.9. \square

Lemma 7.10. *Let (E, h) be an acceptable vector bundle on Δ^* . Then, for every $\alpha \in \mathbb{R}$, there exists $\delta > 0$ such that*

$${}_{\alpha}E = {}_{\alpha+\varepsilon}E$$

holds for every $\varepsilon \in [0, \delta)$.

Proof of Lemma 7.10. Note that $\gamma({}_{\alpha+\varepsilon}E) - \gamma({}_{\alpha}E)$ is a \mathbb{Z} -valued non-decreasing function for $\varepsilon \in \mathbb{R}$. As we already proved in Theorem 7.8 (v), $\gamma({}_{\alpha+1}E) - \gamma({}_{\alpha}E) = \text{rank } E$. Hence there exists $\delta > 0$ such that $\gamma({}_{\alpha+\varepsilon}E)$ is constant for every $\varepsilon \in (0, \delta)$. Since

$$\bigcap_{\varepsilon \in (0, \delta)} {}_{\alpha+\varepsilon}E = \bigcap_{\beta > \alpha} {}_{\beta}E = {}_{\alpha}E$$

by Lemma 7.9, we have ${}_{\alpha+\varepsilon}E = {}_{\alpha}E$ holds for $\varepsilon \in (0, \delta)$. We finish the proof of Lemma 7.10. \square

7.11 (Parabolic weights). We set

$$\mathcal{P}ar_{\alpha}^{\text{red}}(E, h) := \{\lambda \in (\alpha - 1, \alpha] \mid {}_{\lambda}E / {}_{<\lambda}E \neq 0\},$$

where

$${}_{<\lambda}E := \bigcup_{\mu < \lambda} {}_{\mu}E \subset {}_{\lambda}E.$$

Then we obtain

$$\mathcal{P}ar_{\alpha}^{\text{red}}(E, h) = \{\lambda_1, \dots, \lambda_k\},$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$.

If there is no risk of confusion, we simply write $\mathcal{P}ar_{\alpha}^{\text{red}}(E)$ or $\mathcal{P}ar_{(\alpha}^{\text{red}}E)$ instead of $\mathcal{P}ar_{\alpha}^{\text{red}}(E, h)$.

We set

$$l_i := \dim_{\mathbb{C}}({}_{\lambda_i}E / {}_{<\lambda_i}E).$$

Then we can verify, by Lemma 7.8 (v), that

$$(7.15) \quad \sum_{i=1}^k l_i = r = \text{rank } E.$$

Thus, we define

$$\mathcal{P}ar_{\alpha}(E, h) := \underbrace{\{\lambda_1, \dots, \lambda_1\}}_{l_1 \text{ times}}, \dots, \underbrace{\{\lambda_k, \dots, \lambda_k\}}_{l_k \text{ times}}.$$

If there is no risk of confusion, we write $\mathcal{P}ar_{\alpha}(E)$ or $\mathcal{P}ar_{(\alpha}E)$ for $\mathcal{P}ar_{\alpha}(E, h)$.

Furthermore, if the multiplicity of λ_i is not important in the context, we may also use $\mathcal{P}ar_{\alpha}(E, h)$ to denote $\mathcal{P}ar_{\alpha}^{\text{red}}(E, h)$.

Lemma 7.12. *For every i , we have*

$$\gamma({}_{\lambda_i}E) - \gamma({}_{\lambda_i-\varepsilon}E) = l_i$$

for $0 < \varepsilon \ll 1$. Therefore, we have

$$\gamma({}_{\beta}E) - \gamma({}_{\alpha}E) = \dim_{\mathbb{C}}({}_{\beta}E / {}_{\alpha}E)$$

for every $\beta \geq \alpha$.

Proof of Lemma 7.12. We fix a sufficiently small positive real number ε such that

$$\dim_{\mathbb{C}}(\lambda_i E / \lambda_{i-\varepsilon} E) = l_i.$$

Let $\mathbf{v} := \{v_1, \dots, v_r\}$ be a frame of $\lambda_i E$ around 0 and let $\mathbf{w} := \{w_1, \dots, w_r\}$ be a frame of $\lambda_{i-\varepsilon} E$ around 0. Then we can write

$$(w_1, \dots, w_r) = (v_1, \dots, v_r)A(z)$$

for some $r \times r$ matrix $A(z)$ around 0. Since $\dim_{\mathbb{C}}(\lambda_i E / \lambda_{i-\varepsilon} E) = l_i$, we obtain $\text{rank } A(0) = r - l_i$. According to the theory of elementary divisors in the ring $\mathbb{C}\{z\}$, we can write

$$\det A(z) = z^{m_i} f(z)$$

such that $m_i \geq l_i$ and $f(z)$ is holomorphic with $f(0) \neq 0$. Hence we can prove that

$$\gamma(\lambda_i E) - \gamma(\lambda_{i-\varepsilon} E) = m_i \geq l_i.$$

Thus, by Lemma 7.8 (v) and (7.15), we obtain

$$r = \gamma(\alpha E) - \gamma(\alpha-1 E) = \sum_{i=1}^k m_i \geq \sum_{i=1}^k l_i = r.$$

This implies that $m_i = l_i$ for every i , that is,

$$\gamma(\lambda_i E) - \gamma(\lambda_{i-\varepsilon} E) = l_i.$$

This is what we wanted. We finish the proof of Lemma 7.12. \square

We will prove the following important formula in Section 12, which plays a crucial role for the study of αE . The proof of Theorem 7.13 is much more difficult than the argument in this section.

Theorem 7.13 (see Theorem 1.9 and Theorem 12.3 below). *Let E be an acceptable vector bundle on Δ^* . Then the following equality*

$$\gamma(\alpha E) = \sum_{\lambda_i \in \text{Par}_{\text{red}}(\alpha E)} \lambda_i \dim_{\mathbb{C}}(\lambda_i E / \prec_{\lambda_i} E)$$

holds.

Sections 9 through 12 will be devoted to the proof of Theorem 7.13.

Definition 7.14. Let $\mathbf{v} = \{v_1, \dots, v_r\}$ be a frame of αE , that is,

$$\alpha E = \bigoplus_{i=1}^r \mathcal{O}_{\Delta} \cdot v_i.$$

If there exists a decomposition

$$\mathbf{v} = \bigsqcup_{\alpha-1 < \beta \leq \alpha} \mathbf{v}_{\beta}$$

such that \mathbf{v}_{β} is a tuple of sections of βE and that \mathbf{v}_{β} induces a basis of $\beta E / \prec_{\beta} E$, then \mathbf{v} is called a frame of αE compatible with the parabolic filtration. Note that $\mathbf{v}_{\beta} = \emptyset$ if $\beta \notin \text{Par}_{\alpha}(E, h)$.

Remark 7.15. Since ${}_{\alpha}E$ is a holomorphic vector bundle on Δ , we can always take a trivialization (see, for example, [F, 30.4. Theorem]). Therefore, there exists a frame $\{e_1, \dots, e_r\}$ of ${}_{\alpha}E$ on Δ , that is,

$${}_{\alpha}E = \bigoplus_{i=1}^r \mathcal{O}_{\Delta} \cdot e_i$$

holds. Note that $\{e_1, \dots, e_r\}$ gives a basis of the quotient vector space ${}_{\alpha}E/{}_{\alpha-1}E$. Thus, we can take $(a_{ij}) \in \mathrm{GL}(r, \mathbb{C})$ such that $\{v_1, \dots, v_r\}$, where $v_j := \sum_{i=1}^r e_i a_{ij}$ for every j , gives a frame of ${}_{\alpha}E$ compatible with the parabolic filtration.

Lemma 7.16. *Let $\mathbf{v} = \{v_1, \dots, v_r\}$ be a local frame of ${}_{\alpha}E$ defined over some open neighborhood of 0, such that $v_i \in {}_{\beta_i}E \setminus <_{\beta_i}E$ for every i . Then $\beta_i \in (\alpha - 1, \alpha]$ for every i .*

Proof of Lemma 7.16. Since $v_i \in {}_{\alpha}E$, it follows that $\beta_i \leq \alpha$ for all i . Moreover, since $\{v_1, \dots, v_r\}$ forms a local frame of ${}_{\alpha}E$ near 0, it induces a basis of the quotient vector space ${}_{\alpha}E/{}_{\alpha-1}E$. This implies that each β_i lies in the interval $(\alpha - 1, \alpha]$. \square

Lemma 7.17. *Let $\mathbf{v} = \{v_1, \dots, v_r\}$ be a frame of ${}_{\alpha}E$ compatible with the parabolic filtration such that*

$$v_i \in {}_{\beta_i}E \setminus <_{\beta_i}E$$

for every i . In particular,

$$\mathrm{Par}_{\alpha}(E, h) = \{\beta_1, \dots, \beta_r\}.$$

Let α' be any real number. Let m_i be the smallest integer satisfying $\beta_i - m_i \leq \alpha'$ for every i . Then

$$\mathbf{v}' := \{z^{m_1}v_1, \dots, z^{m_r}v_r\}$$

is a frame of ${}_{\alpha'}E$ compatible with the parabolic filtration.

Proof of Lemma 7.17. We put $w_i := z^{m_i}v_i$ for every i . We note that the map

$$(7.16) \quad z^m \times : {}_{\lambda}E \rightarrow {}_{\lambda-m}E$$

is an isomorphism for every $m \in \mathbb{Z}$ and every real number λ . By definition, we see that $w_i \in {}_{\alpha'}E$ for every i . By the isomorphism (7.16), we can check that ${}_{\alpha'}E$ is spanned by \mathbf{v}' . Thus we have

$${}_{\alpha'}E = \bigoplus_{i=1}^r \mathcal{O}_{\Delta} \cdot w_i,$$

that is, \mathbf{v}' is a frame of ${}_{\alpha'}E$. By (7.16) again, we can check that \mathbf{v}' is a frame of ${}_{\alpha'}E$ compatible with the parabolic filtration. We finish the proof of Lemma 7.17 \square

We conclude this section with the following remark.

Remark 7.18. The acceptability near the origin is preserved under the coordinate rescaling $z \mapsto z/C$, where C is a positive constant. Note that the condition $v \in {}_{\alpha}E \setminus <_{\alpha}E$ and the quantity

$$\gamma({}_{\alpha}E) = -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})}{\log |z|}$$

are invariant under this rescaling. Therefore, such rescaling can be employed when we are concerned only with the behavior near the origin.

8. ON FILTERED PROLONGATION OF ACCEPTABLE BUNDLES

In this short section, we recall the framework of filtered bundles as introduced by Mochizuki. His notation turns out to be particularly convenient in various contexts.

We have already verified the following properties of ${}_aE$.

8.1 (Filtered prolongation of acceptable bundles). Let (E, h) be an acceptable vector bundle on Δ^* . We define

$$\mathcal{P}_a^h E := {}_aE$$

for every $a \in \mathbb{R}$, and set

$$\mathcal{P}^h E := \bigcup_{a \in \mathbb{R}} \mathcal{P}_a^h E \subset j_* E,$$

where $j: \Delta^* \hookrightarrow \Delta$ is the natural inclusion.

Then, $\mathcal{P}^h E$ is a locally free $\mathcal{O}_\Delta(*[0])$ -module of finite rank, where $\mathcal{O}_\Delta(*[0])$ is the sheaf of meromorphic functions on Δ with poles only at 0. The following properties hold:

- (i) For every $a \in \mathbb{R}$, $\mathcal{P}_a^h E$ is a locally free \mathcal{O}_Δ -submodule of $\mathcal{P}^h E$.
- (ii) $\mathcal{P}_a^h E(*[0]) = \mathcal{P}^h E$ for every $a \in \mathbb{R}$.
- (iii) For any $a \leq b$, we have $\mathcal{P}_a^h E \subset \mathcal{P}_b^h E$.
- (iv) For any $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have

$$\mathcal{P}_{a+n}^h E = \mathcal{P}_a^h E(n[0]).$$

- (v) For any $a \in \mathbb{R}$, there exists $\varepsilon > 0$ such that

$$\mathcal{P}_{a+\varepsilon}^h E = \mathcal{P}_a^h E.$$

Therefore, it is natural to introduce the notion of filtered bundles as follows.

Definition 8.2 (Filtered bundles). We denote by \mathcal{O}_Δ the sheaf of holomorphic functions on Δ , and by $\mathcal{O}_\Delta(*[0])$ the sheaf of meromorphic functions on Δ with poles only at 0.

Let \mathcal{E} be a locally free $\mathcal{O}_\Delta(*[0])$ -module. A *filtered bundle over \mathcal{E}* is an increasing family of locally free \mathcal{O}_Δ -modules $\mathcal{P}_a \mathcal{E} \subset \mathcal{E}$ indexed by $a \in \mathbb{R}$, satisfying the following conditions:

- (1) Each $\mathcal{P}_a \mathcal{E}$ is a lattice in \mathcal{E} , i.e.,

$$\mathcal{P}_a \mathcal{E} \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta(*[0]) = \mathcal{E}.$$

- (2) For any $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have

$$\mathcal{P}_{a+n} \mathcal{E} = \mathcal{P}_a \mathcal{E} \otimes_{\mathcal{O}_\Delta} \mathcal{O}_\Delta(n[0]).$$

- (3) For any $a \in \mathbb{R}$, there exists $\varepsilon > 0$ such that

$$\mathcal{P}_{a+\varepsilon} \mathcal{E} = \mathcal{P}_a \mathcal{E}.$$

In this case, we also say that $\mathcal{P}_* \mathcal{E}$ is a filtered bundle on $(\Delta, 0)$ for simplicity.

For any $a \in \mathbb{R}$, define

$$\mathcal{P}_{<a} \mathcal{E} := \sum_{b < a} \mathcal{P}_b \mathcal{E}, \quad \text{and} \quad \text{Gr}_a^{\mathcal{P}}(\mathcal{E}) := \mathcal{P}_a \mathcal{E} / \mathcal{P}_{<a} \mathcal{E}.$$

We may naturally regard $\text{Gr}_a^{\mathcal{P}}(\mathcal{E})$ as a finite-dimensional \mathbb{C} -vector space.

A frame $\mathbf{v} = \{v_1, \dots, v_{\text{rank } \mathcal{E}}\}$ of $\mathcal{P}_a \mathcal{E}$ is said to be *compatible* with the parabolic structure if there exists a decomposition

$$\mathbf{v} = \bigsqcup_{a-1 < b \leq a} \mathbf{v}_b$$

such that the following holds:

- For each b , \mathbf{v}_b is a tuple of sections of $\mathcal{P}_b\mathcal{E}$, and it induces a basis of $\mathrm{Gr}_b^{\mathcal{P}}(\mathcal{E})$.

For any non-zero section s of \mathcal{E} , the number

$$\mathrm{deg}^{\mathcal{P}}(s) := \min \{c \in \mathbb{R} \mid s \in \mathcal{P}_c\mathcal{E}\}$$

is called the *parabolic degree* of s . If $s = 0$, we set $\mathrm{deg}^{\mathcal{P}}(s) := -\infty$.

By Definition 8.2, we can say that $\mathcal{P}_*^h E = (\mathcal{P}_a^h E \mid a \in \mathbb{R})$ is a *filtered bundle over $\mathcal{P}^h E$* .

Remark 8.3. Definition 8.2 is essentially the same as [M6, 2.11.1, Filtered Bundles on a Neighborhood of 0 in \mathbb{C}]. It is a local definition. For the global setting, see [M6, 2.11.3, Global Case]. In this paper, we are only concerned with the one-dimensional case. For the higher-dimensional case, we refer the reader to Section 2 of [M5] (see also [FFO, Section 4]).

In the following sections, we will use whichever of the notations ${}_a E$ and $\mathcal{P}_a^h E$ is more convenient in context. In particular, when discussing tensor products in Section 16 and Hom bundles in Section 17, the notation $\mathcal{P}_a^h E$ appears to be more suitable.

9. SOME ELEMENTARY INEQUALITIES

In this section, we present some elementary facts that will be used later. The arguments in this section are essentially due to Simpson [S3].

We denote

$$B(a, r) := \{z \in \mathbb{C} \mid |z - a| < r\}, \quad \bar{B}(a, r) := \{z \in \mathbb{C} \mid |z - a| \leq r\},$$

and let $\mathrm{Area}(\Omega)$ denote the area of a set Ω .

Definition 9.1. Fix a positive real number r . We define

$$B_r(w) := \inf \left\{ \frac{1}{r^3} \int_{\Omega} \log |w - z| d\lambda(z) \mid \Omega \subset B(0, r), \mathrm{Area}(\Omega) = r^3 \right\},$$

for $w \in \mathbb{C}$, where Ω is an open subset of \mathbb{C} , and $d\lambda$ denotes the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$.

The following estimate is straightforward.

Lemma 9.2. *For any $w \in \mathbb{C}$ and any positive real number r , we have*

$$B_r(w) \geq \frac{3}{2} \log r - \frac{1}{2} \log \pi - \frac{1}{2}.$$

Proof of Lemma 9.2. By definition, we have

$$\begin{aligned} B_r(w) &\geq \inf \left\{ \frac{1}{r^3} \int_{\Omega} \log |w - z| d\lambda(z) \mid \mathrm{Area}(\Omega) = r^3 \right\} \\ &= \inf \left\{ \frac{1}{r^3} \int_{\Omega} \log |z| d\lambda(z) \mid \mathrm{Area}(\Omega) = r^3 \right\}, \end{aligned}$$

where the second equality follows by translation invariance of Lebesgue measure.

It is easy to see that the minimum is attained when $\Omega = B(0, a)$ with $a = \pi^{-1/2}r^{3/2}$. Therefore,

$$\begin{aligned} B_r(w) &\geq \frac{1}{r^3} \int_{B(0,a)} \log |z| \, d\lambda(z) \\ &= \frac{1}{r^3} \int_0^{2\pi} d\theta \int_0^a t \log t \, dt \\ &= \frac{2\pi}{r^3} \left(\left[\frac{1}{2} t^2 \log t \right]_0^a - \frac{1}{2} \int_0^a t \, dt \right) \\ &= \frac{2\pi}{r^3} \left(\frac{1}{2} a^2 \log a - \frac{1}{4} a^2 \right) \\ &= \frac{3}{2} \log r - \frac{1}{2} \log \pi - \frac{1}{2}, \end{aligned}$$

as claimed. This completes the proof of Lemma 9.2. \square

Lemma 9.3. *Let $r > 0$. Then for every $w \in \overline{B}(0, 2)$, the following inequality holds:*

$$B_r(w) \geq \frac{3}{2} \log \left(\frac{|w|}{2} \right) - \frac{1}{2} \log \pi - \frac{1}{2}.$$

Proof of Lemma 9.3. If $|w| \leq 2r$, then the conclusion follows directly from Lemma 9.2 and the inequality $\log(|w|/2) \leq \log r$.

If $|w| \geq 2r$, then for any $z \in B(0, r)$,

$$|w - z| \geq |w| - |z| \geq |w| - r \geq \frac{|w|}{2}.$$

Thus,

$$\begin{aligned} B_r(w) &\geq \inf \left\{ \frac{1}{r^3} \int_{\Omega} \log \left(\frac{|w|}{2} \right) \, d\lambda(z) \mid \Omega \subset B(0, r), \text{Area}(\Omega) = r^3 \right\} \\ &= \log \left(\frac{|w|}{2} \right). \end{aligned}$$

Combining both cases, we obtain the claimed inequality using the fact that $\log(|w|/2) \leq 0$ for $w \in \overline{B}(0, 2)$ and

$$\frac{1}{2} \log \pi + \frac{1}{2} > 0.$$

This completes the proof of Lemma 9.3. \square

Lemma 9.4. *Let $r \in \mathbb{R}$ with $0 < r < 1$. Then for all $z, w \in \mathbb{C}$ with $r \leq |z| < 1$ and $|w| \leq 2$, the following inequality holds:*

$$\log |w - z| \leq \frac{2}{3} \cdot \frac{\log |z|}{\log r} \cdot B_r(w) + \frac{1}{3} \log \pi + \frac{1}{3} + 2 \log 2.$$

Proof of Lemma 9.4. Since $r \leq |z| < 1$, we have $\log r \leq \log |z| < 0$, hence

$$0 < \frac{\log |z|}{\log r} \leq 1.$$

If $|w| \leq |z|$, then

$$\log |w - z| \leq \log(2|z|) = \log |z| + \log 2 = \frac{\log |z|}{\log r} \log r + \log 2.$$

Applying Lemma 9.2, we obtain

$$\begin{aligned} \log |w - z| &\leq \frac{\log |z|}{\log r} \cdot \frac{2}{3} \left(B_r(w) + \frac{1}{2} \log \pi + \frac{1}{2} \right) + \log 2 \\ &\leq \frac{2}{3} \cdot \frac{\log |z|}{\log r} \cdot B_r(w) + \frac{1}{3} \log \pi + \frac{1}{3} + \log 2. \end{aligned}$$

If $|w| \geq |z|$, then

$$\log |w - z| \leq \log(2|w|) = \log \left(\frac{|w|}{2} \right) + 2 \log 2,$$

and since $\log(|w|/2) \leq 0$ and $\log |z|/\log r \leq 1$, we get

$$\log |w - z| \leq \frac{\log |z|}{\log r} \cdot \log \left(\frac{|w|}{2} \right) + 2 \log 2.$$

Applying Lemma 9.3, we obtain

$$\begin{aligned} \log |w - z| &\leq \frac{\log |z|}{\log r} \cdot \frac{2}{3} \left(B_r(w) + \frac{1}{2} \log \pi + \frac{1}{2} \right) + 2 \log 2 \\ &\leq \frac{2}{3} \cdot \frac{\log |z|}{\log r} \cdot B_r(w) + \frac{1}{3} \log \pi + \frac{1}{3} + 2 \log 2. \end{aligned}$$

This completes the proof of Lemma 9.4. \square

10. SIMPSON'S KEY LEMMA

The main goal of this section is to establish the following key lemma (see Lemma 10.1), which is essentially due to Simpson. Note that our version is slightly different from the original statement (see [S1, Lemma 10.2]). However, our formulation of Lemma 10.1 is sufficient for the proof of Theorem 12.3.

In this section, we frequently use the following notation:

$$S(a, r) := \{z \in \mathbb{C} \mid |z - a| = r\},$$

and

$$B(a, r)^* := \{z \in \mathbb{C} \mid 0 < |z - a| < r\} = B(a, r) \setminus \{a\}.$$

Lemma 10.1 ([S1, Lemma 10.2]). *Let δ be a positive real number with $\delta < 1$. Suppose that h is a smooth Hermitian metric on the trivial holomorphic vector bundle $\mathcal{O}_{B(0, 1+2\delta)^*}^{\oplus k}$ over the punctured disk $B(0, 1+2\delta)^* := B(0, 1+2\delta) \setminus \{0\}$, and that h has negative curvature. Assume further that the eigenvalues of h are less than or equal to 1, and that*

$$|\det h| \leq C|z|$$

holds for some positive constant C . Then there exist a positive constant C' and a constant section $e \in \mathbb{C}^k$ of $\mathcal{O}_{B(0, 1+2\delta)^}^{\oplus k}$ such that*

$$|e(z)|_h \leq C'|z|^{\frac{1}{3k}}$$

for all $z \in B(0, 1)^$.*

Before starting the proof of the lemma above, we need to prove several preliminary results.

Lemma 10.2. *Let r_1 and δ_1 be positive real numbers. Let u be a subharmonic function defined on $B(0, r_1 + \delta_1)$ such that u is smooth outside the origin. Let $f(z)$ be a smooth function on $B(0, r_1 + \delta_1)$. Define*

$$\sigma(z) := \partial\bar{\partial}u = \frac{\partial^2 u}{\partial z \partial \bar{z}} dz \wedge d\bar{z} = \frac{1}{4} \Delta u dz \wedge d\bar{z}.$$

Note that

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

denotes the Laplacian with respect to z , understood in the sense of distributions. Since u is subharmonic, $\sqrt{-1}\partial\bar{\partial}u$ is a closed positive $(1, 1)$ -current. Hence, $\sqrt{-1}\sigma(z)$ defines a positive Radon measure (see, for example, [NO, (3.1.14) Lemma]). In this setting, we have

$$(10.1) \quad \begin{aligned} & \int_{\overline{B(0, r_1)}} f(z) \sigma(z) - \frac{1}{4} \int_{\overline{B(0, r_1)}} (\Delta f) u(z) dz \wedge d\bar{z} \\ &= \int_{S(0, r_1)} \left(\frac{\partial u}{\partial \bar{z}} f(z) d\bar{z} + \frac{\partial f}{\partial z} u(z) dz \right). \end{aligned}$$

For the sake of completeness, we provide a detailed proof of Lemma 10.2, although it is more or less standard.

Proof of Lemma 10.2. Take a smooth function $\varphi(z)$ on \mathbb{C} such that $\text{supp } \varphi \subset B(0, r_1)$, and $\varphi(z) = 1$ on $B(0, \frac{1}{2}r_1 + \delta_2)$ for some small constant $0 < \delta_2 \ll 1$. Define $g(z) := \varphi(z)f(z)$ and $h(z) := f(z) - g(z)$. Then $f(z) = g(z) + h(z)$, where $\text{supp } g \subset B(0, r_1)$ and $h(z) = 0$ on $B(0, \frac{1}{2}r_1 + \delta_2)$.

Let ρ be a smooth function on \mathbb{C} , supported in $B(0, 1)$, radial (i.e., $\rho(z)$ depends only on $|z|$), non-negative, and normalized so that

$$\int_{\mathbb{C}} \rho(z) d\lambda(z) = 1,$$

where $d\lambda(z) := \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$ denotes the Lebesgue measure on \mathbb{C} . Define the family of smoothing kernels

$$\rho_\varepsilon(z) := \frac{1}{\varepsilon^2} \rho\left(\frac{z}{\varepsilon}\right).$$

Set $u_\varepsilon := u * \rho_\varepsilon$. Then u_ε is a smooth subharmonic function on a neighborhood of $\overline{B(0, r_1)}$ for sufficiently small $\varepsilon > 0$.

By applying Stokes' theorem to $g(z)$ and u_ε , we obtain:

$$(10.2) \quad \begin{aligned} & \int_{\overline{B(0, r_1)}} g(z) \partial\bar{\partial}u_\varepsilon - \frac{1}{4} \int_{\overline{B(0, r_1)}} (\Delta g) u_\varepsilon(z) dz \wedge d\bar{z} \\ &= \int_{S(0, r_1)} \left(\frac{\partial u_\varepsilon}{\partial \bar{z}} g(z) d\bar{z} + \frac{\partial g}{\partial z} u_\varepsilon(z) dz \right). \end{aligned}$$

We note that we have

$$\frac{\partial u_\varepsilon}{\partial \bar{z}} = \left(\frac{\partial u}{\partial \bar{z}} \right) * \rho_\varepsilon,$$

where $\frac{\partial u}{\partial \bar{z}}$ is taken in the sense of distributions. Since u is smooth outside the origin, both $u_\varepsilon(z)$ and $\frac{\partial u_\varepsilon}{\partial \bar{z}}$ converge uniformly to $u(z)$ and $\frac{\partial u}{\partial \bar{z}}$, respectively, on an open neighborhood of $S(0, r_1)$ as $\varepsilon \rightarrow +0$. It is well known that $\partial\bar{\partial}u_\varepsilon \rightarrow \partial\bar{\partial}u$ in the sense of currents, and $u_\varepsilon \rightarrow u$ in the sense of distributions.

Since g is smooth with compact support, we may let $\varepsilon \rightarrow +0$ in (10.2) to obtain:

$$(10.3) \quad \begin{aligned} & \int_{\overline{B}(0,r_1)} g(z) \partial \bar{\partial} u - \frac{1}{4} \int_{\overline{B}(0,r_1)} (\Delta g) u(z) dz \wedge d\bar{z} \\ &= \int_{S(0,r_1)} \left(\frac{\partial u}{\partial \bar{z}} g(z) d\bar{z} + \frac{\partial g}{\partial z} u(z) dz \right). \end{aligned}$$

Next, since $h(z) = 0$ and $\Delta h = 0$ on $B(0, \frac{1}{2}r_1 + \delta_2)$, we can apply Stokes' theorem to get:

$$\begin{aligned} & \int_{\overline{B}(0,r_1)} h(z) \sigma(z) - \frac{1}{4} \int_{\overline{B}(0,r_1)} (\Delta h) u(z) dz \wedge d\bar{z} \\ &= \frac{1}{4} \int_{\overline{B}(0,r_1) \setminus B(0, \frac{1}{2}r_1)} (h(z) \Delta u - u(z) \Delta h) dz \wedge d\bar{z} \\ &= \int_{S(0,r_1)} \left(\frac{\partial u}{\partial \bar{z}} h(z) d\bar{z} + \frac{\partial h}{\partial z} u(z) dz \right) - \int_{S(0, \frac{1}{2}r_1)} \left(\frac{\partial u}{\partial \bar{z}} h(z) d\bar{z} + \frac{\partial h}{\partial z} u(z) dz \right). \end{aligned}$$

Since $h(z) = 0$ and $\partial h / \partial z = 0$ on a neighborhood of $S(0, \frac{1}{2}r_1)$, the second boundary integral vanishes. Hence,

$$(10.4) \quad \begin{aligned} & \int_{\overline{B}(0,r_1)} h(z) \sigma(z) - \frac{1}{4} \int_{\overline{B}(0,r_1)} (\Delta h) u(z) dz \wedge d\bar{z} \\ &= \int_{S(0,r_1)} \left(\frac{\partial u}{\partial \bar{z}} h(z) d\bar{z} + \frac{\partial h}{\partial z} u(z) dz \right). \end{aligned}$$

By adding (10.3) and (10.4), we obtain the desired equality (10.1).

This completes the proof of Lemma 10.2. \square

Lemma 10.3. *Let $\delta > 0$ be a real number, and let u be a subharmonic function on $B(0, 1 + 2\delta)$ that is smooth on $B(0, 1 + 2\delta)^* := B(0, 1 + 2\delta) \setminus \{0\}$. Then for every $a \in B(0, 1)^*$, we have*

$$(10.5) \quad \begin{aligned} u(a) &= \frac{\sqrt{-1}}{\pi} \int_{B(0,1+\delta)} \log |z - a| \sigma(z) \\ &+ \frac{1}{\pi \sqrt{-1}} \int_{S(0,1+\delta)} \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{2\pi \sqrt{-1}} \int_{S(0,1+\delta)} \frac{1}{z - a} u(z) dz. \end{aligned}$$

Here, $\sqrt{-1} \sigma(z) := \sqrt{-1} \partial \bar{\partial} u$, computed in the sense of currents, defines a positive Radon measure on $B(0, 1 + 2\delta)$ because u is subharmonic (see, for example, [NO, (3.1.14) Lemma]).

Proof of Lemma 10.3. Let r_1 and r_2 be small positive real numbers such that $\overline{B}(0, r_1) \cap \overline{B}(a, r_2) = \emptyset$ and $\overline{B}(a, r_2) \subset B(0, 1 + \delta)$. Define

$$\Omega_1 := \overline{B}(0, 1 + \delta) \setminus (B(0, r_1) \cup B(a, r_2)).$$

On an open neighborhood of Ω_1 , both u and $\log |z - a|$ are smooth. Thus, we have

$$d \left(\frac{\partial u}{\partial \bar{z}} \log |z - a|^2 d\bar{z} + \frac{1}{z - a} u dz \right) = 2 \log |z - a| \partial \bar{\partial} u$$

on an open neighborhood of Ω_1 . Applying Stokes' theorem yields

$$\begin{aligned}
 2 \int_{\Omega_1} \log |z - a| \partial \bar{\partial} u &= \int_{\partial \Omega_1} \left(\frac{\partial u}{\partial \bar{z}} \log |z - a|^2 d\bar{z} + \frac{1}{z - a} u dz \right) \\
 &= \int_{S(0,1+\delta)} \left(2 \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{z - a} u dz \right) \\
 (10.6) \quad &\quad - \int_{S(0,r_1)} \left(2 \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{z - a} u dz \right) \\
 &\quad - \int_{S(a,r_2)} \left(2 \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{z - a} u dz \right).
 \end{aligned}$$

As $r_2 \rightarrow +0$, elementary computations give

$$\int_{S(a,r_2)} \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} \rightarrow 0, \quad \int_{S(a,r_2)} \frac{1}{z - a} u dz \rightarrow 2\pi \sqrt{-1} u(a).$$

Taking the limit as $r_2 \rightarrow +0$ in (10.6), we obtain

$$\begin{aligned}
 2 \int_{\Omega_2} \log |z - a| \partial \bar{\partial} u &= \int_{S(0,1+\delta)} \left(2 \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{z - a} u dz \right) \\
 &\quad - \int_{S(0,r_1)} \left(2 \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{z - a} u dz \right) \\
 &\quad - 2\pi \sqrt{-1} u(a),
 \end{aligned}$$

where

$$\Omega_2 := \overline{B}(0, 1 + \delta) \setminus B(0, r_1).$$

Solving for $u(a)$, we obtain

$$\begin{aligned}
 u(a) &= \frac{\sqrt{-1}}{\pi} \int_{\Omega_2} \log |z - a| \partial \bar{\partial} u \\
 (10.7) \quad &\quad + \frac{1}{2\pi \sqrt{-1}} \int_{S(0,1+\delta)} \left(2 \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{z - a} u dz \right) \\
 &\quad - \frac{1}{2\pi \sqrt{-1}} \int_{S(0,r_1)} \left(2 \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{z - a} u dz \right).
 \end{aligned}$$

We put $f(z) := \log |z - a|^2$. Then $\Delta f = 0$ on a neighborhood of $\overline{B}(0, r_1)$. Thus, by Lemma 10.2, we have

$$(10.8) \quad 2 \int_{\overline{B}(0,r_1)} \log |z - a| \sigma(z) = \int_{S(0,r_1)} \left(2 \frac{\partial u}{\partial \bar{z}} \log |z - a| d\bar{z} + \frac{1}{z - a} u dz \right).$$

Combining (10.7) and (10.8), we obtain the desired identity (10.5) since $\overline{B}(0, 1 + \delta) \setminus \Omega_2 = B(0, r_1)$.

This completes the proof of Lemma 10.3. \square

10.4 (Setting). We now proceed to prove Lemma 10.1. First, we clarify the setting of the lemma. Define the function

$$f(z, e) := \log |e(z)|_h,$$

where $z \in B(0, 1 + 2\delta)^*$ and $e \in S^{2k-1} := \{v \in \mathbb{C}^k \mid |v| = 1\} \subset \mathbb{C}^k$. Then f is a smooth function on $B(0, 1 + 2\delta)^* \times S^{2k-1}$, and satisfies $f(z, e) \leq 0$, since the eigenvalues of h are less than or equal to 1.

By Lemma 2.7, and since the curvature of h is negative, it follows that $f(z, e)$ is a smooth subharmonic function on $B(0, 1 + 2\delta)^*$ for every $e \in S^{2k-1}$. Therefore, $f(z, e)$ extends to a locally integrable subharmonic function on $B(0, 1 + 2\delta)$ for every $e \in S^{2k-1}$ (see [NO, (3.3.25) Theorem]).

We define

$$\mu(z, e) := \Delta f(z, e),$$

where

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$$

is the Laplacian in the sense of distributions, taken with respect to the variable z .

Then, for each fixed $e \in S^{2k-1}$, the function $f(z, e)$ being subharmonic implies that $\mu(z, e)$ defines a positive Radon measure on $B(0, 1 + 2\delta)$ (see, for example, [NO, (3.1.14) Lemma]).

Lemma 10.5. *In the setting of 10.4, there exists a positive constant C such that the following inequality holds:*

$$\left| f(z, e) - \frac{1}{2\pi} \int_{B(0, 1+\delta)} \log |w - z| \mu(w, e) d\lambda(w) \right| \leq C$$

for all $(z, e) \in B(0, 1)^* \times S^{2k-1}$, where $d\lambda(w) := \frac{\sqrt{-1}}{2} dw \wedge d\bar{w}$.

Proof of Lemma 10.5. Consider the function

$$(z, e) \mapsto \left| \frac{1}{\pi\sqrt{-1}} \int_{S(0, 1+\delta)} \frac{\partial f}{\partial \bar{w}}(w, e) \log |w - z| d\bar{w} + \frac{1}{2\pi\sqrt{-1}} \int_{S(0, 1+\delta)} \frac{f(w, e)}{w - z} dw \right|.$$

This function is continuous on the compact set $\bar{B}(0, 1) \times S^{2k-1}$. Therefore, there exists a constant $C > 0$ such that

$$\left| \frac{1}{\pi\sqrt{-1}} \int_{S(0, 1+\delta)} \frac{\partial f}{\partial \bar{w}}(w, e) \log |w - z| d\bar{w} + \frac{1}{2\pi\sqrt{-1}} \int_{S(0, 1+\delta)} \frac{f(w, e)}{w - z} dw \right| \leq C$$

for all $(z, e) \in \bar{B}(0, 1) \times S^{2k-1}$.

On the other hand, by Lemma 10.3 and the identity

$$4\partial\bar{\partial}f(w, e) = \mu(w, e)dw \wedge d\bar{w},$$

we have

$$\begin{aligned} f(z, e) - \frac{1}{2\pi} \int_{B(0, 1+\delta)} \log |w - z| \mu(w, e) d\lambda(w) \\ = \frac{1}{\pi\sqrt{-1}} \int_{S(0, 1+\delta)} \frac{\partial f}{\partial \bar{w}}(w, e) \log |w - z| d\bar{w} + \frac{1}{2\pi\sqrt{-1}} \int_{S(0, 1+\delta)} \frac{f(w, e)}{w - z} dw. \end{aligned}$$

This proves the desired estimate. \square

Lemma 10.6. *In the setting of 10.4, there exists a positive constant C such that the inequality*

$$\int_{B(0, 1+\delta)} \mu(z, e) d\lambda(z) \leq C$$

holds for every $e \in S^{2k-1}$.

Proof of Lemma 10.6. Fix a smooth function $\varphi(z)$ on $B(0, 1 + 2\delta)$ with the following properties:

- $0 \leq \varphi(z) \leq 1$ for all $z \in B(0, 1 + 2\delta)$,
- $\varphi(z) = 1$ for all $z \in B(0, 1 + \delta)$, and
- the support of φ is compact and contained in $B(0, 1 + 2\delta)$.

Since

$$e \mapsto \int_{B(0,1+2\delta)} \Delta\varphi(z)f(z,e)d\lambda(z)$$

is a smooth function on S^{2k-1} , there exists a positive constant C such that

$$\int_{B(0,1+2\delta)} \Delta\varphi(z)f(z,e)d\lambda(z) \leq C$$

for all $e \in S^{2k-1}$.

On the other hand, by the definition of the Laplacian in the sense of distributions, we have

$$\int_{B(0,1+2\delta)} \varphi(z)\mu(z,e)d\lambda(z) = \int_{B(0,1+2\delta)} \Delta\varphi(z)f(z,e)d\lambda(z).$$

Therefore,

$$\begin{aligned} \int_{B(0,1+\delta)} \mu(z,e)d\lambda(z) &= \int_{B(0,1+\delta)} \varphi(z)\mu(z,e)d\lambda(z) \\ &\leq \int_{B(0,1+2\delta)} \varphi(z)\mu(z,e)d\lambda(z) \\ &= \int_{B(0,1+2\delta)} \Delta\varphi(z)f(z,e)d\lambda(z) \\ &\leq C \end{aligned}$$

for all $e \in S^{2k-1}$, as claimed. \square

Now, we begin the proof of Lemma 10.1.

Proof of Lemma 10.1. If $C \leq 1$, then the inequality $|\det h| \leq |z|$ clearly holds. In the case $C > 1$, replacing h by $C^{-1/k}h$ allows us to assume $C = 1$ without loss of generality.

Let C_1 and C_2 be the positive constants obtained in Lemmas 10.5 and 10.6, respectively. That is, C_1 satisfies

$$(10.9) \quad \left| f(z,e) - \frac{1}{2\pi} \int_{B(0,1+\delta)} \log|w-z| \mu(w,e) d\lambda(w) \right| \leq C_1$$

for all $(z,e) \in B(0,1)^* \times S^{2k-1}$, and C_2 satisfies

$$(10.10) \quad \int_{B(0,1+\delta)} \mu(z,e) d\lambda(z) \leq C_2$$

for all $e \in S^{2k-1}$.

By Lemma 10.7 below, there exists $0 < r_0 < 1$ such that for every $0 < r < r_0$ there exist $e \in S^{2k-1}$ and an open subset Ω_r satisfying

$$\Omega_r \subset \left\{ z \in \overline{B}(0,r) \mid f(z,e) \leq \frac{1}{2k} \log r + \log 2 \right\}$$

and $\text{Area}(\Omega_r) = r^3$, where $\text{Area}(\Omega_r)$ is the area of Ω_r . Then, from (10.9), we obtain

$$\frac{1}{2\pi} \int_{B(0,1+\delta)} \log|w-z| \mu(w,e) d\lambda(w) - C_1 \leq f(z,e) \leq \frac{1}{2k} \log r + \log 2$$

for all $z \in \Omega_r$. By (10.9),

$$\frac{1}{2\pi} \int_{B(0,1+\delta)} \log |w - z| \mu(w, e) d\lambda(w)$$

is integrable over Ω_r . Applying the averaging operator

$$\frac{1}{r^3} \int_{\Omega_r} \bullet d\lambda(z),$$

we obtain

$$\begin{aligned} \frac{1}{2k} \log r + \log 2 &\geq \frac{1}{r^3} \int_{\Omega_r} f(z, e) d\lambda(z) \\ &\geq \frac{1}{r^3} \int_{\Omega_r} \left(\frac{1}{2\pi} \int_{B(0,1+\delta)} \log |w - z| \mu(w, e) d\lambda(w) - C_1 \right) d\lambda(z) \\ (10.11) \quad &= \frac{1}{2\pi} \int_{B(0,1+\delta)} \left(\frac{1}{r^3} \int_{\Omega_r} \log |w - z| d\lambda(z) \right) \mu(w, e) d\lambda(w) - C_1 \\ &\geq \frac{1}{2\pi} \int_{B(0,1+\delta)} B_r(w) \mu(w, e) d\lambda(w) - C_1, \end{aligned}$$

by the definition of $B_r(w)$ in Definition 9.1.

On the other hand, Lemma 9.4 implies that for all $z \in B(0, 1) \setminus B(0, r)$,

$$\begin{aligned} &\int_{B(0,1+\delta)} \log |w - z| \mu(w, e) d\lambda(w) \\ &\leq \int_{B(0,1+\delta)} \left(\frac{2}{3} \cdot \frac{\log |z|}{\log r} \cdot B_r(w) + \frac{1}{3} \log \pi + \frac{1}{3} + 2 \log 2 \right) \mu(w, e) d\lambda(w) \\ (10.12) \quad &= \frac{2 \log |z|}{3 \log r} \int_{B(0,1+\delta)} B_r(w) \mu(w, e) d\lambda(w) \\ &\quad + \left(\frac{1}{3} \log \pi + \frac{1}{3} + 2 \log 2 \right) \int_{B(0,1+\delta)} \mu(w, e) d\lambda(w) \end{aligned}$$

for all $z \in B(0, 1) \setminus B(0, r)$.

Combining these inequalities (10.11) and (10.12) with (10.10), we obtain

$$\begin{aligned} &\int_{B(0,1+\delta)} \log |w - z| \mu(w, e) d\lambda(w) \\ &\leq \frac{4\pi \log |z|}{3 \log r} \left(\frac{1}{2k} \log r + \log 2 + C_1 \right) + \left(\frac{1}{3} \log \pi + \frac{1}{3} + 2 \log 2 \right) C_2 \\ &\leq \frac{2\pi}{3k} \log |z| + \frac{4\pi}{3} \left(\log 2 + C_1 \right) + \left(\frac{1}{3} \log \pi + \frac{1}{3} + 2 \log 2 \right) C_2 \end{aligned}$$

Using (10.9) again, we deduce

$$f(z, e) \leq \frac{1}{3k} \log |z| + \frac{2}{3} \log 2 + \frac{5}{3} C_1 + \frac{1}{2\pi} \left(\frac{1}{3} \log \pi + \frac{1}{3} + 2 \log 2 \right) C_2$$

for all $z \in B(0, 1) \setminus B(0, r)$. Setting

$$C' := \exp \left(\frac{2}{3} \log 2 + \frac{5}{3} C_1 + \frac{1}{2\pi} \left(\frac{1}{3} \log \pi + \frac{1}{3} + 2 \log 2 \right) C_2 \right),$$

we obtain the inequality

$$f(z, e) \leq \frac{1}{3k} \log |z| + \log C',$$

or equivalently,

$$|e(z)|_h \leq C' |z|^{\frac{1}{3k}}$$

for all $z \in B(0, 1) \setminus B(0, r)$.

Let $r_i \rightarrow 0$ be any decreasing sequence with $0 < r_i < r_0$. Then by the above argument, we can find $e_i \in S^{2k-1}$ such that

$$(10.13) \quad |e_i(z)|_h \leq C' |z|^{\frac{1}{3k}}$$

for all $z \in B(0, 1) \setminus B(0, r_i)$. Since S^{2k-1} is compact, we may, after passing to a subsequence, assume that

$$\lim_{i \rightarrow \infty} e_i = e \in S^{2k-1}.$$

Then from (10.13), it follows that

$$|e(z)|_h \leq C' |z|^{\frac{1}{3k}}$$

holds for all $z \in B(0, 1)^*$. This completes the proof of Lemma 10.1. \square

The following lemma is used in the proof of Lemma 10.1 above.

Lemma 10.7. *In the setting of Lemma 10.1, assume that $|\det h| \leq |z|$. Then there exists a constant $0 < r_0 < 1$ such that for every $0 < r < r_0$, there exists a vector $e \in S^{2k-1}$ for which*

$$\left\{ z \in \overline{B}(0, r) \mid f(z, e) < \frac{1}{2k} \log(2^{2k} r) \right\}$$

contains an open subset Ω_r with $\text{Area}(\Omega_r) = r^3$.

Proof of Lemma 10.7. By assumption, namely $|\det h| \leq |z|$ on $\overline{B}(0, r)$, we can choose a vector $e^\dagger \in S^{2k-1}$ such that

$$f(z, e^\dagger) = \log |e^\dagger(z)|_h \leq \frac{1}{k} \log r < \frac{1}{2k} \log r < 0.$$

Let v be any vector with Euclidean norm $|v|_{\text{Euclid}} \leq r^{1/(2k)}$. Then

$$|e^\dagger + v|_h \leq |e^\dagger|_h + |v|_{\text{Euclid}} < 2r^{1/(2k)},$$

since all eigenvalues of h are less than or equal to 1.

Without loss of generality, we may assume that $0 < r < r'_0$ for some sufficiently small constant r'_0 .

Then, for each $z \in \overline{B}(0, r)$, the volume of the set of $e \in S^{2k-1}$ for which the above bound holds is at least

$$\alpha \left(r^{1/(2k)} \right)^{2k-1} = \alpha r^{1-(1/(2k))},$$

for some positive constant α .

This implies that the volume of the subset of $S^{2k-1} \times \overline{B}(0, r)$ where the bound holds is at least

$$\alpha r^{1-(1/(2k))} \cdot \pi r^2 = \alpha \pi r^{-1/(2k)} r^3.$$

Suppose, for contradiction, that for every $e \in S^{2k-1}$, the area of the region in $\overline{B}(0, r)$ where the bound holds is less than r^3 . Then the total volume in $S^{2k-1} \times \overline{B}(0, r)$ would be less than

$$\sigma_{2k-1} \cdot r^3,$$

where σ_{2k-1} denotes the volume of the unit sphere S^{2k-1} in \mathbb{C}^k .

However, if r is sufficiently small, we have

$$\alpha\pi r^{-1/(2k)}r^3 > \sigma_{2k-1} \cdot r^3,$$

which is a contradiction.

Therefore, there exists a sufficiently small constant $0 < r_0 < 1$ such that for every $0 < r < r_0$, there exists at least one $e' \in S^{2k-1}$ such that the set

$$\left\{ z \in \overline{B}(0, r) \mid f(z, e') < \frac{1}{2k} \log(2^{2k}r) \right\}$$

contains an open subset Ω_r with $\text{Area}(\Omega_r) = r^3$. \square

11. ON CYCLIC COVERS

In what follows, we briefly discuss cyclic covers, which will be used in later arguments. Let us recall the following elementary fact for the reader's convenience.

Lemma 11.1. *Let m be any positive integer with $m \geq 2$, and let ϵ be a complex number such that $\epsilon^m = 1$ and $\epsilon \neq 1$. Then*

$$\sum_{i=0}^{m-1} \epsilon^i = 0.$$

Proof of Lemma 11.1. Since

$$1 - \epsilon^m = (1 - \epsilon)(1 + \epsilon + \cdots + \epsilon^{m-1}) = 0,$$

and $1 - \epsilon \neq 0$, it follows that

$$\sum_{i=0}^{m-1} \epsilon^i = 0.$$

\square

The following lemma is the main result of this section.

Lemma 11.2 (cf. [S1, Lemma 10.3]). *Let (E, h) be an acceptable vector bundle on $X = \Delta^*$ with $\text{rank } E = r$. Let $\pi: W := \Delta^* \rightarrow X$ be the m -fold cyclic cover of Δ^* given by $\pi(w) = z^m$, where z is the coordinate on X and w is the coordinate on W . Let*

$$\{v_1, \dots, v_r\}$$

be a frame of ${}^\circ E = {}_0E$ compatible with the parabolic filtration, such that $v_i \in {}_{b_i}E \setminus {}_{<b_i}E$ for each i . Let α be any real number, and let $m_{\alpha,i}$ be the smallest integer such that

$$mb_i - m_{\alpha,i} \leq \alpha$$

for each i . Then

$$\{w^{m_{\alpha,1}}\pi^*v_1, \dots, w^{m_{\alpha,r}}\pi^*v_r\}$$

*is a frame of ${}_\alpha(\pi^*E)$ compatible with the parabolic filtration.*

Proof of Lemma 11.2. By direct calculation, $\pi^*\omega_P$ is the Poincaré metric on W . Therefore, it is straightforward to verify that π^*E is an acceptable vector bundle on W . By definition, we can readily see that $w^{m_{\alpha,i}}\pi^*v_i$ is a section of ${}_\alpha(\pi^*E)$ for each i .

Step 1. Let $G = \mathbb{Z}/m\mathbb{Z} = \langle g \rangle$ be the Galois group of $\pi: W \rightarrow X$. Then G acts naturally on π^*E , and this action preserves the metric. Let U be any open subset of X . Then we have

$$H^0(\pi^{-1}(U), \pi^*E) = H^0(U, \pi_*\pi^*E).$$

We also have the decomposition

$$(11.1) \quad \pi_*\pi^*E = \bigoplus_{j=0}^{m-1} w^j E,$$

i.e., the \mathcal{O}_W -module π^*E decomposes into a direct sum of \mathcal{O}_X -modules as in (11.1), under the action of G . The action of G on the right-hand side is given by $g^*w = \zeta w$, where ζ is an m th root of unity.

Step 2. In this step, we prove that

$$\{w^{m_{\alpha,1}}\pi^*v_1, \dots, w^{m_{\alpha,r}}\pi^*v_r\}$$

is a frame of ${}_{\alpha}(\pi^*E)$.

Take any $u \in H^0(\pi^{-1}(U), \pi^*E)$. By (11.1), we can write

$$u = \sum_{j=0}^{m-1} w^j u_j,$$

where $u_j \in H^0(U, E)$ for each j . Assume that

$$|u|_{\pi^*h} \leq \frac{C}{|w|^\lambda}$$

holds for some $C > 0$ and $\lambda \in \mathbb{R}$. Then, by considering

$$\sum_{l=0}^{m-1} \zeta^{-lj} (g^l)^* u$$

for each j , we obtain the same estimate:

$$|w^j u_j|_{\pi^*h} \leq \frac{C}{|w|^\lambda}$$

for every j . Here we used Lemma 11.1 and the fact that the G -action preserves the metric.

This implies that ${}_{\alpha}(\pi^*E)$ is generated by

$$\{w^{m_{\alpha,1}}\pi^*v_1, \dots, w^{m_{\alpha,r}}\pi^*v_r\}$$

for every α . Hence this set forms a frame of ${}_{\alpha}(\pi^*E)$, as desired.

Step 3. In this final step, we verify that the frame

$$\{w^{m_{\alpha,1}}\pi^*v_1, \dots, w^{m_{\alpha,r}}\pi^*v_r\}$$

is compatible with the parabolic filtration.

Assume that

$$\beta := mb_1 - m_{\alpha,1} = \dots = mb_l - m_{\alpha,l}$$

for some $l \geq 1$. Under this assumption, it suffices to show that

$$\{w^{m_{\alpha,1}}\pi^*v_1, \dots, w^{m_{\alpha,l}}\pi^*v_l\}$$

is linearly independent in the quotient space ${}_{\beta}(\pi^*E)/{}_{<\beta}(\pi^*E)$.

Suppose that

$$(11.2) \quad a_1 w^{m_{\alpha,1}} \pi^* v_1 + \cdots + a_l w^{m_{\alpha,l}} \pi^* v_l \in {}_{<\beta}(\pi^* E),$$

for some $a_i \in \mathbb{C}$.

Note that if $mb_i - m_{\alpha,i} = mb_j - m_{\alpha,j}$ and $b_i \neq b_j$, then $m_{\alpha,i} \neq m_{\alpha,j}$ and $|m_{\alpha,i} - m_{\alpha,j}| < m$. Therefore, using the decomposition (11.1) as in Step 2, we may assume that $b_1 = \cdots = b_l$. In this case, (11.2) implies

$$a_1 v_1 + \cdots + a_l v_l \in {}_{<b_1} E.$$

Since $\{v_1, \dots, v_r\}$ is compatible with the parabolic filtration, this implies that $a_1 = \cdots = a_l = 0$.

Thus,

$$\{w^{m_{\alpha,1}} \pi^* v_1, \dots, w^{m_{\alpha,r}} \pi^* v_r\}$$

is compatible with the parabolic filtration, as claimed.

This completes the proof of Lemma 11.2. \square

The converse of the above lemma also holds, as shown below.

Lemma 11.3. *Let (E, h) be an acceptable vector bundle on $X = \Delta^*$ with $\text{rank } E = r$. Let $\pi: W := \Delta^* \rightarrow X$ be the m -fold cyclic cover of Δ^* given by $\pi(w) = z^m$, where z is the coordinate on X and w is the coordinate on W . Let*

$$\{v_1, \dots, v_r\}$$

be a frame of ${}^\diamond E = {}_0 E$ such that $v_i \in {}_{b_i} E \setminus {}_{<b_i} E$ for each i . Let α be any real number, and let $m_{\alpha,i}$ be the smallest integer such that

$$mb_i - m_{\alpha,i} \leq \alpha$$

for each i . If

$$\{w^{m_{\alpha,1}} \pi^* v_1, \dots, w^{m_{\alpha,r}} \pi^* v_r\}$$

is a frame of ${}_\alpha(\pi^ E)$ compatible with the parabolic filtration for some α , then*

$$\{v_1, \dots, v_r\}$$

is a frame of ${}^\diamond E = {}_0 E$ compatible with the parabolic filtration.

Proof of Lemma 11.3. Assume that $\beta := b_1 = \cdots = b_l$ for some $l \geq 1$. Under this assumption, it suffices to show that $\{v_1, \dots, v_l\}$ is linearly independent in the quotient space ${}_\beta E / {}_{<\beta} E$.

Suppose that

$$a_1 v_1 + \cdots + a_l v_l \in {}_{<\beta} E$$

for some $a_1, \dots, a_l \in \mathbb{C}$. Let $n := m_{\alpha,1} = \cdots = m_{\alpha,l}$. Then we have

$$a_1 w^n \pi^* v_1 + \cdots + a_l w^n \pi^* v_l \in {}_{<m\beta-n}(\pi^* E).$$

But by assumption, the set

$$\{w^n \pi^* v_1, \dots, w^n \pi^* v_l\}$$

is part of a frame of ${}_\alpha(\pi^* E)$ that is compatible with the parabolic filtration. This implies that the above linear combination lies in a lower filtration step only if all coefficients vanish, i.e.,

$$a_1 = \cdots = a_l = 0.$$

Therefore, $\{v_1, \dots, v_r\}$ is compatible with the parabolic filtration, as claimed. \square

12. ON DETERMINANT BUNDLES

The main purpose of this section is to establish Theorem 12.3 and Corollary 12.4, which will play crucial roles in the subsequent sections. We begin with an elementary lemma from Diophantine approximation.

Definition 12.1. Let $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{R}^l$ be a vector. We define

$$R_m(\alpha) := (m\alpha_1 - n_1, \dots, m\alpha_l - n_l),$$

where each n_i is the integer that minimizes $|m\alpha_i - n_i|$. We also define

$$\delta_m(\alpha) := \max_i |m\alpha_i - n_i|.$$

Lemma 12.2. Let $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{R}^l$ be a vector such that $\alpha_i \notin \mathbb{Q}$ for every i . Then, for any real number $q > 1$, there exists a positive integer $m \leq q$ such that

$$\delta_m(\alpha) \leq q^{-1/l}.$$

Proof of Lemma 12.2. This follows easily from [C, Chapter I, Theorem VI]. It is essentially a consequence of Minkowski's theorem. We omit the details. \square

The following theorem is the main result of this section.

Theorem 12.3. Let $\{v_1, \dots, v_r\}$ be a frame of ${}_a E$ around the origin, compatible with the parabolic filtration, such that $v_i \in {}_{b_i} E \setminus <_{b_i} E$ for every i . Then we have the following equality:

$$\gamma({}_a E) = \sum_{i=1}^r b_i.$$

By combining Theorem 7.5 with Theorem 12.3, we obtain the following important result on determinant bundles, which will also play a crucial role in the subsequent sections.

Corollary 12.4. We use the same notation as in Theorem 12.3. Then we have

$$\det({}_a E) = \sum_{i=1}^r b_i (\det E).$$

The following two remarks are straightforward, but we include them for completeness.

Remark 12.5. Let (E, h) be an acceptable vector bundle on Δ^* . Let $\{v_1, \dots, v_r\}$ be a frame of ${}_a E$. We consider

$$(E^\dagger, h^\dagger) := (E, h \cdot |z|^{2c}),$$

where c is a real number. Then (E^\dagger, h^\dagger) is also an acceptable vector bundle on Δ^* since $\partial\bar{\partial} \log |z|^{2c} = 0$ on Δ^* . It is easy to see that

$${}_{a-c} E^\dagger = {}_a E$$

holds and that $\{v_1, \dots, v_r\}$ is a frame of ${}_{a-c} E^\dagger$. By definition, we have

$$\gamma({}_{a-c} E^\dagger) = \gamma({}_a E) - rc.$$

We further assume that $\{v_1, \dots, v_r\}$ is compatible with the parabolic filtration such that $v_i \in {}_{b_i} E \setminus <_{b_i} E$ for every i . Then it is obvious that $\{v_1, \dots, v_r\}$ is a frame of ${}_{a-c} E^\dagger$ compatible with the parabolic filtration such that $v_i \in {}_{b_i-c} E^\dagger \setminus <_{b_i-c} E^\dagger$ for every i . We note that

$$\gamma({}_a E) - \sum_{i=1}^r b_i = \gamma({}_{a-c} E^\dagger) - \sum_{i=1}^r (b_i - c)$$

holds. Hence, in the proof of Theorem 12.3, we can freely replace h with $h \cdot |z|^{2c}$ for any real number c .

Remark 12.6. Let (E, h) be an acceptable vector bundle on Δ^* , and let $\{v_1, \dots, v_r\}$ be a frame of ${}_a E$. Consider the pair

$$(\tilde{E}, \tilde{h}) := (E, h e^{-\chi(-N)}),$$

where N is a real number. It is straightforward to verify that (\tilde{E}, \tilde{h}) is also an acceptable vector bundle on Δ^* , and that ${}_\alpha \tilde{E} = {}_\alpha E$ for every $\alpha \in \mathbb{R}$. Note that $\{v_1, \dots, v_r\}$ is a frame of ${}_a \tilde{E}$, and that $v_i \in {}_{b_i} \tilde{E} \setminus <_{b_i} \tilde{E}$ if and only if $v_i \in {}_{b_i} E \setminus <_{b_i} E$. By definition, we have $\gamma({}_a \tilde{E}) = \gamma({}_a E)$. It is also clear that $\{v_1, \dots, v_r\}$ is a frame of ${}_a \tilde{E}$ compatible with the parabolic filtration if and only if it is so for ${}_a E$. Therefore, in the proof of Theorem 12.3, we can freely replace h with $h e^{-\chi(-N)}$ for any real number N .

Let us now prove Theorem 12.3.

Proof of Theorem 12.3. Although the inequality

$$\gamma({}_a E) \leq \sum_{i=1}^r b_i$$

was already established in Corollary 7.7, we provide an alternative proof in Step 1. It should be noted that the assumption that the frame $\{v_1, \dots, v_r\}$ is compatible with the parabolic filtration is not required in this step. The discussion in Step 1 will be needed in Step 2. In Step 2, we will establish the reverse inequality, where Lemma 10.1 will play a crucial role.

Step 1. In this step, we prove that

$$\gamma({}_a E) \leq \sum_{i=1}^r b_i.$$

As noted above, the assumption that $\{v_1, \dots, v_r\}$ is compatible with the parabolic filtration is not needed here.

By Lemma 5.4, we can write

$$|v_i|_h = \frac{v_i^\dagger(z)}{|z|^{b_i}} (-\log |z|)^{M_i}$$

around the origin, where each $M_i \in \mathbb{Z}_{>0}$ and $v_i^\dagger(z)$ is bounded for all i .

Since $v_i \in {}_{b_i} E \setminus <_{b_i} E$, we know that

$$\frac{v_i^\dagger(z)}{|z|^c}$$

is unbounded for any $c > 0$.

Let $\mathbf{v} := \{v_1, \dots, v_r\}$, and consider $\det H(\mathbf{v})(z)$ as in 7.1. Then we can write

$$\det H(\mathbf{v})(z) = |z|^{-2 \sum b_i} (-\log |z|)^{2r \sum M_i} u(z),$$

where $u(z)$ is bounded.

Therefore, by Corollary 7.6, we have

$$\begin{aligned} \gamma({}_a E) &= -\frac{1}{2} \liminf_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})(z)}{\log |z|} \\ &= -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(\mathbf{v})(z)}{\log |z|} \\ &= \sum_{i=1}^r b_i - \frac{1}{2} \lim_{z \rightarrow 0} \frac{\log u(z)}{\log |z|} \\ &\leq \sum_{i=1}^r b_i, \end{aligned}$$

since

$$\lim_{z \rightarrow 0} \frac{\log u(z)}{\log |z|} \geq 0.$$

This completes the proof of the inequality.

Step 2. In this step, we prove the reverse inequality:

$$\gamma({}_a E) \geq \sum_{i=1}^r b_i.$$

We emphasize that the assumption that $\{v_1, \dots, v_r\}$ is compatible with the parabolic filtration is essential in this step.

Suppose, to the contrary, that

$$\gamma({}_a E) < \sum_{i=1}^r b_i.$$

This implies that

$$(12.1) \quad \lim_{z \rightarrow 0} \frac{\log u(z)}{\log |z|} > 0$$

in Step 1.

By replacing h with $h \cdot |z|^{2 \max_i \{b_i\} + \varepsilon}$ for some small $\varepsilon > 0$, we may assume that $a = 0$ and $b_i \in (-1, 0)$ for all i (see Remark 12.5). Next, by replacing h with $h \cdot e^{-\chi(-N)}$ for some sufficiently large $N \gg 0$, we may further assume that the curvature of h is negative (see Remark 12.6).

Note that all entries of the matrix h are bounded, since each $b_i \in (-1, 0)$. Rescaling the coordinate via $z \mapsto z/C$ for some constant $C > 0$ does not affect the values of $\gamma({}_a E)$ and b_i (see Remark 7.18). Therefore, by choosing an appropriate rescaling, we may assume that the frame $\{v_1, \dots, v_r\}$ is defined on the unit disk $B(0, 1)$. Applying the rescaling $z \mapsto z/2$ once more, we may further assume that the pair (E, h) is defined and trivialized on $B(0, 2)$. Then, by further replacing h with $(1/C)h$ for some sufficiently large constant $C \gg 0$, we may assume that all eigenvalues of h are ≤ 1 on some open neighborhood of the closed disk $\overline{B}(0, 1 + \delta)$ for sufficiently small $\delta > 0$.

By Lemma 12.2, we can choose a sufficiently large positive integer m such that $mb_i \in \mathbb{Z}$ for all rational b_i , and

$$\delta_m(b) < \frac{1}{6 \operatorname{rank} E}.$$

Note that m can be taken arbitrarily large.

Set $b'_i := mb_i - n_i$ for each i , where n_i is the integer minimizing $|mb_i - n_i|$ (see Definition 12.1).

Now consider the m -fold cyclic cover

$$\pi: B(0, 1 + \delta') \rightarrow B(0, 1 + \delta), \quad \pi(w) = z^m$$

for some $\delta' > 0$. Define

$$(E', h') := \left(\pi^* E, \pi^* h \cdot |w|^{2 \max_i \{b'_i\}} \right).$$

Then, since $\gamma({}_a E) < \sum_{i=1}^r b_i$ (i.e., (12.1)) and m is sufficiently large, there exists a constant $C' > 0$ such that

$$|\det h'| \leq C'|w|$$

on some open neighborhood of the closed disk $\overline{B}(0, 1 + \delta'')$ for sufficiently small $\delta'' > 0$.

By Lemma 11.2, the set

$$\{w^{n_1} \pi^* v_1, \dots, w^{n_r} \pi^* v_r\}$$

forms a frame of ${}_0 E'$ that is compatible with the parabolic filtration.

Applying Lemma 10.1, we obtain a constant section e' of E' and a constant $C'' > 0$ such that

$$|e'(w)|_{h'} \leq C'' |w|^{\frac{1}{3 \operatorname{rank} E}}$$

holds near the origin.

By construction, we have

$$\mathcal{P}ar({}_0 E') \subset \left(-\frac{1}{3 \operatorname{rank} E}, 0 \right],$$

which contradicts the existence of such a section e' with the above estimate.

Therefore, our assumption must be false, and we conclude that

$$\gamma({}_a E) \geq \sum_{i=1}^r b_i.$$

By Steps 1 and 2, we obtain the desired equality:

$$\gamma({}_a E) = \sum_{i=1}^r b_i.$$

This completes the proof of Theorem 12.3. \square

Proposition 12.7. *Let $\{w_1, \dots, w_r\}$ be a local frame of ${}_a E$ defined over some open neighborhood of 0, such that $w_i \in {}_{c_i} E$ for every i . Assume that*

$$\sum_{i=1}^r c_i \leq \gamma({}_a E).$$

Then the following assertions hold:

- (i) $w_i \in {}_{c_i} E \setminus <_{c_i} E$ for every i ;
- (ii) $\gamma({}_a E) = \sum_{i=1}^r c_i$;
- (iii) $c_i \in (a - 1, a]$ for every i ;
- (iv) $\{w_1, \dots, w_r\}$ is a local frame of ${}_a E$ compatible with the parabolic filtration.

Proof of Proposition 12.7. For each i , take a real number c'_i such that $w_i \in c'_i E \setminus <_{c'_i} E$. By definition, we have $c'_i \leq c_i$ for all i . By Corollary 7.7, it follows that

$$\gamma({}_a E) \leq \sum_{i=1}^r c'_i.$$

Therefore,

$$\gamma({}_a E) \leq \sum_{i=1}^r c'_i \leq \sum_{i=1}^r c_i \leq \gamma({}_a E).$$

Thus, all inequalities must be equalities, and we conclude that $c'_i = c_i$ for all i . In particular, $w_i \in c_i E \setminus <_{c_i} E$ for all i and $\gamma({}_a E) = \sum_{i=1}^r c_i$, which proves (i) and (ii).

Next, since $\{w_1, \dots, w_r\}$ forms a local frame of ${}_a E$ near 0, it follows from (i) and Lemma 7.16 that $c_i \in (a-1, a]$ for all i . Hence, (iii) follows.

Finally, consider the quotient vector space ${}_a E / {}_{a-1} E$. Suppose that $\{w_1, \dots, w_r\}$ is not compatible with the parabolic filtration. Then there exists $(a_{ij}) \in \text{GL}(r, \mathbb{C})$ such that the new frame $\{w'_1, \dots, w'_r\}$, defined by

$$w'_j = \sum_{i=1}^r w_i a_{ij},$$

satisfies $w'_j \in c'_j E \setminus <_{c'_j} E$ for each j , and

$$\sum_{j=1}^r c'_j < \sum_{i=1}^r c_i.$$

However, by Corollary 7.7 again, we have

$$\gamma({}_a E) \leq \sum_{j=1}^r c'_j,$$

which contradicts the assumption $\sum_{i=1}^r c_i \leq \gamma({}_a E)$. Hence, $\{w_1, \dots, w_r\}$ must be compatible with the parabolic filtration. This proves (iv), and completes the proof of Proposition 12.7. \square

By Corollary 7.7, Theorem 12.3, and Proposition 12.7, we obtain the following useful statement, which will be used in subsequent sections.

Corollary 12.8. *Let $\{u_1, \dots, u_r\}$ be a local frame of ${}_a E$ defined over some open neighborhood of 0, such that $u_i \in d_i E$ for every i . Then the inequality*

$$\gamma({}_a E) \leq \sum_{i=1}^r d_i$$

holds. Equality holds if and only if the following three conditions are satisfied:

- (i) $\{u_1, \dots, u_r\}$ is a local frame of ${}_a E$ compatible with the parabolic filtration;
- (ii) $u_i \in d_i E \setminus <_{d_i} E$ for every i ;
- (iii) $\text{Par}({}_a E) = \{d_1, \dots, d_r\}$.

We will use this corollary when showing that a given frame is compatible with the parabolic filtration.

13. ON DUAL BUNDLES

In this section, we investigate the prolongation of dual vector bundles. We begin by reformulating Corollary 1.7 (see also Corollary 4.3).

Lemma 13.1 (Duality for line bundles, see Lemma 1.11). *Let (L, h) be an acceptable line bundle on Δ^* . Let $\alpha \in \mathbb{R}$ be any real number. Then we have ${}_{\alpha}L = \gamma({}_{\alpha}L)$, and $\text{Par}_{\alpha}(L, h) = \{\gamma({}_{\alpha}L)\}$.*

Moreover, if $0 < \varepsilon \ll 1$, then

$$\gamma(-\alpha+1-\varepsilon(L^{\vee})) = -\gamma({}_{\alpha}L).$$

In particular, the following equality holds:

$$(\gamma({}_{\alpha}L)L)^{\vee} = -\gamma({}_{\alpha}L)(L^{\vee}).$$

Proof of Lemma 13.1. We use the same notation as in the proof of Proposition 4.1 (see also Theorem 4.4).

Since ${}_{\alpha}L = \mathcal{O}_{\Delta} \cdot z^{-\lfloor \alpha - \gamma \rfloor}$, we may take the following trivialization:

$$({}_{\alpha}L, h) \simeq (\mathcal{O}_{\Delta}, |\cdot|^2 e^{-2\varphi_{\alpha}}),$$

where $\varphi_{\alpha} := \varphi + \lfloor \alpha - \gamma \rfloor \log |z|$. Note that

$$\lim_{z \rightarrow 0} \frac{\varphi(z)}{\log |z|} = \gamma.$$

Therefore, we obtain

$$\gamma({}_{\alpha}L) = \lim_{z \rightarrow 0} \frac{\varphi_{\alpha}(z)}{\log |z|} = \lfloor \alpha - \gamma \rfloor + \gamma.$$

This immediately implies that ${}_{\alpha}L = \gamma({}_{\alpha}L)L$, and that $\text{Par}_{\alpha}(L, h) = \{\gamma({}_{\alpha}L)\}$.

Similarly, we compute

$$\gamma(-\alpha+1-\varepsilon(L^{\vee})) = \lfloor -\alpha + 1 - \varepsilon + \gamma \rfloor - \gamma.$$

For details, see the proof of Corollary 4.3.

If $0 < \varepsilon \ll 1$, then

$$\begin{aligned} \gamma(-\alpha+1-\varepsilon(L^{\vee})) &= \lfloor -\alpha + 1 - \varepsilon + \gamma \rfloor - \gamma \\ &= -\lceil \alpha - 1 + \varepsilon - \gamma \rceil - \gamma \\ &= -\lfloor \alpha - \gamma \rfloor - \gamma \\ &= -\gamma({}_{\alpha}L). \end{aligned}$$

Hence, we have

$$(\gamma({}_{\alpha}L)L)^{\vee} = ({}_{\alpha}L)^{\vee} = -\alpha+1-\varepsilon(L^{\vee}) = -\gamma({}_{\alpha}L)(L^{\vee}),$$

by Corollary 1.7 (see also Corollary 4.3).

This completes the proof of Lemma 13.1. \square

The main result of this section is the following theorem. One of the main ingredients in the proof of Theorem 13.2 is Theorem 12.3.

Theorem 13.2 (Dual bundles, see Theorem 1.12). *Let (E, h) be an acceptable vector bundle on Δ^* , and let a be any real number. Then,*

$$({}_a E)^{\vee} = -a+1-\varepsilon(E^{\vee})$$

holds for any sufficiently small $\varepsilon > 0$.

Moreover, let $\{v_1, \dots, v_r\}$ be a local frame of ${}_aE$ near the origin, compatible with the parabolic filtration, such that $v_i \in {}_{b_i}E \setminus <_{b_i}E$ for each i . For each i , define

$$(13.1) \quad v_i^\vee := (-1)^{i-1} v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r \otimes (v_1 \wedge \dots \wedge v_r)^{\otimes -1}.$$

Then $\{v_1^\vee, \dots, v_r^\vee\}$ forms a local frame of ${}_{-a+1-\varepsilon}(E^\vee)$ near the origin, compatible with the parabolic filtration, such that

$$v_i^\vee \in {}_{-b_i}(E^\vee) \setminus <_{-b_i}(E^\vee)$$

for each i . In particular, we have

$$\text{Par}_a(E, h) = \{b_1, \dots, b_r\} \quad \text{and} \quad \text{Par}_{-a+1-\varepsilon}(E^\vee, h^\vee) = \{-b_1, \dots, -b_r\}.$$

Proof of Theorem 13.2. By Corollary 7.7, we have

$$\gamma(\gamma({}_aE) \det E) = \gamma({}_aE).$$

Therefore, by Lemma 13.1, we obtain

$$(13.2) \quad (\gamma({}_aE) \det E)^\vee = {}_{-\gamma({}_aE)}(\det E)^\vee.$$

As a result, we have

$$(13.3) \quad \begin{aligned} ({}_aE)^\vee &= \bigwedge_{i=1}^{r-1} ({}_aE) \otimes (\det({}_aE))^\vee \\ &= \bigwedge_{i=1}^{r-1} ({}_aE) \otimes (\gamma({}_aE) \det E)^\vee \\ &= \bigwedge_{i=1}^{r-1} ({}_aE) \otimes {}_{-\gamma({}_aE)}(\det E)^\vee, \end{aligned}$$

where we have used Theorem 7.5 and (13.2).

By (13.1) and (13.3), it follows that

$$v_i^\vee \in {}_{-b_i}(E^\vee)$$

since $\gamma({}_aE) = \sum_{i=1}^r b_i$ by Theorem 12.3.

Since $\mathbf{v} := \{v_1, \dots, v_r\}$ is a local frame of ${}_aE$, we have $b_i \in (a-1, a]$ for every i by Lemma 7.16. Therefore, $-b_i \leq -a+1-\varepsilon$ for every i when $0 < \varepsilon \ll 1$. Thus, $v_i^\vee \in {}_{-a+1-\varepsilon}(E^\vee)$ for every i .

By definition, $\mathbf{v}' := \{v_1^\vee, \dots, v_r^\vee\}$ is the dual frame of $\{v_1, \dots, v_r\}$. Thus, it gives a local frame of $({}_aE)^\vee$ near the origin. This implies that

$$(13.4) \quad ({}_aE)^\vee \subset {}_{-a+1-\varepsilon}(E^\vee).$$

Claim. We have the inclusion

$${}_{-a+1-\varepsilon}(E^\vee) \subset ({}_aE)^\vee$$

for any sufficiently small $\varepsilon > 0$.

Proof of Claim. Let $f \in {}_{-a+1-\varepsilon}(E^\vee)$. Then locally near 0, we can write

$$f = \sum_{i=1}^r f_i(z) v_i^\vee(z),$$

where each f_i is holomorphic outside 0. Since

$$|f_i(z)| = |v_i(f)| \leq |v_i(z)|_h \cdot |f(z)|_{h^\vee} = O\left(\frac{1}{|z|^{1-\delta}}\right)$$

for small $\delta > 0$, we conclude that each $f_i(z)$ is holomorphic near 0.

Hence, f extends holomorphically, and we obtain the desired inclusion. This completes the proof of Claim. \square

Therefore, by combining (13.4) with Claim, we obtain the equality:

$$({}_a E)^\vee = {}_{-a+1-\varepsilon}(E^\vee).$$

As shown above, $\mathbf{v}^\vee = \{v_1^\vee, \dots, v_r^\vee\}$ is a local frame of ${}_{-a+1-\varepsilon}(E^\vee)$ near the origin. We have already proved that $v_i^\vee \in {}_{-b_i}(E^\vee)$ for every i . By definition, it is easy to verify that

$$H(h^\vee, \mathbf{v}^\vee) = H(h, \mathbf{v})^{-1}.$$

Therefore,

$$\begin{aligned} \gamma({}_{-a+1-\varepsilon}(E^\vee)) &= -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(\mathbf{v}^\vee)}{\log |z|} \\ &= -\frac{1}{2} \lim_{z \rightarrow 0} \frac{-\log \det H(\mathbf{v})}{\log |z|} \\ &= -\gamma({}_a E) = \sum_{i=1}^r (-b_i), \end{aligned}$$

by Theorem 12.3. Finally, by Corollary 12.8, the frame \mathbf{v}^\vee satisfies all the desired properties.

This completes the proof of Theorem 13.2. \square

For each i , we set

$$v'_i := v_i \cdot |z|^{b_i}.$$

We denote $\mathbf{v}' := \{v'_1, \dots, v'_r\}$. As in 7.1, we define

$$H(h, \mathbf{v}') := (h(v'_i, v'_j))_{i,j}.$$

Now we are ready to prove the following theorem. Although this property does not play a role in the present work, it is of independent interest.

Theorem 13.3 (Weak norm estimate, see Theorem 1.13). *Let $\{v_1, \dots, v_r\}$ be a local frame of ${}_a E$ around the origin, compatible with the parabolic filtration, such that*

$$v_i \in {}_{b_i} E \setminus {}_{<b_i} E \quad \text{for every } i.$$

Then there exist positive constants C and M such that

$$C^{-1}(-\log |z|)^{-M} I_r \leq H(h, \mathbf{v}')(z) \leq C(-\log |z|)^M I_r$$

holds in a neighborhood of the origin, where I_r is the identity matrix of size r .

This means that both

$$C(-\log |z|)^M I_r - H(h, \mathbf{v}')(z) \quad \text{and} \quad H(h, \mathbf{v}')(z) - C^{-1}(-\log |z|)^{-M} I_r$$

are positive semidefinite around the origin.

Proof of Theorem 13.3. For each i , set

$$(v_i^\vee)' := v_i^\vee \cdot |z|^{-b_i}.$$

We denote $(\mathbf{v}^\vee)' := \{(v_1^\vee)', \dots, (v_r^\vee)'\}$. By Lemma 5.4, there exist positive constants C' , M' such that

$$H(h, \mathbf{v}^\vee)'(z) \leq C'(-\log |z|)^{M'} I_r$$

holds near the origin. Similarly, applying Lemma 5.4 to the dual bundle, we obtain positive constants C'' and M'' such that

$$H(h^\vee, (\mathbf{v}^\vee)')(z) \leq C''(-\log |z|)^{M''} I_r$$

holds near the origin. By definition, it is easy to verify that

$$H(h, \mathbf{v}')(z) = (H(h^\vee, (\mathbf{v}^\vee)')(z))^{-1}.$$

Combining these inequalities, we conclude that there exist positive constants C and M such that

$$C^{-1}(-\log |z|)^{-M} I_r \leq H(h, \mathbf{v}')(z) \leq C(-\log |z|)^M I_r$$

holds around the origin. This completes the proof of Theorem 13.3. \square

As a direct consequence of Theorem 13.3, we obtain the following useful estimate.

Corollary 13.4. *Let $\{v_1, \dots, v_r\}$ be a local frame of ${}_a E$ around the origin, compatible with the parabolic filtration, such that*

$$v_i \in {}_{b_i} E \setminus <_{b_i} E \quad \text{for every } i.$$

Then there exist positive constants C_0 and M_0 such that

$$\frac{C_0^{-1}}{|z|^{b_i}} (-\log |z|)^{-M_0} \leq |v_i|_h \leq \frac{C_0}{|z|^{b_i}} (-\log |z|)^{M_0}$$

holds for every i in a neighborhood of the origin.

Proof of Corollary 13.4. This follows directly from Theorem 13.3. \square

14. EXAMPLES OF FILTERED BUNDLES

Before discussing the prolongation of tensor products and Hom bundles of acceptable vector bundles on Δ^* , we set up the framework of filtered bundles. We use the same notation as in Section 8. Let us begin with a simple example, which we will use again in Section 15.

Example 14.1. Note that $\mathcal{O}_\Delta(*[0])$ is itself a locally free $\mathcal{O}_\Delta(*[0])$ -module of rank one. Let $\mathcal{P}_*^{(c)}(\mathcal{O}_\Delta(*[0]))$ denote the filtered bundle over $\mathcal{O}_\Delta(*[0])$ defined by

$$\mathcal{P}_a^{(c)}(\mathcal{O}_\Delta(*[0])) = \mathcal{O}_\Delta([a - c][0]).$$

Remark 14.2. Let $(\mathcal{O}_{\Delta^*}, h_c)$ be a flat line bundle on Δ^* , where

$$h_c := \frac{|\cdot|^2}{|z|^{2c}} = |\cdot|^2 \cdot e^{-2c \log |z|}.$$

Then we can verify that

$$\mathcal{P}_a^{h_c} \mathcal{O}_{\Delta^*} = \mathcal{O}_\Delta([a - c][0])$$

holds for every $a \in \mathbb{R}$. Hence, the filtered bundle $\mathcal{P}_*^{(c)}(\mathcal{O}_\Delta(*[0]))$ in Example 14.1 can be realized as the filtered prolongation of the acceptable line bundle $(\mathcal{O}_{\Delta^*}, h_c)$ over Δ^* . In particular, we can view $\mathcal{P}_*^{(0)}(\mathcal{O}_\Delta(*[0]))$ as the filtered prolongation of the trivial Hermitian line bundle $(\mathcal{O}_{\Delta^*}, |\cdot|^2)$. Note that $\mathcal{P}_*^{(0)}(\mathcal{O}_\Delta(*[0]))$ will be used in Proposition 15.1.

Let \mathcal{P}_*E_1 and \mathcal{P}_*E_2 be filtered bundles of rank r_1 and r_2 on $(\Delta, 0)$, respectively. Then

$$E_1 \otimes E_2 \quad \text{and} \quad \text{Hom}_{\mathcal{O}_\Delta(*[0])}(E_1, E_2)$$

are locally free $\mathcal{O}_\Delta(*[0])$ -modules of rank $r_1 r_2$.

Let $a \in \mathbb{R}$. We define

$$\mathcal{P}_a(E_1 \otimes E_2) := \sum_{b+c \leq a} \mathcal{P}_b E_1 \otimes \mathcal{P}_c E_2,$$

$$\mathcal{P}_a \text{Hom}(E_1, E_2) := \left\{ f \in \text{Hom}_{\mathcal{O}_\Delta(*[0])}(E_1, E_2) \mid f(\mathcal{P}_k E_1) \subset \mathcal{P}_{a+k} E_2 \text{ for all } k \in \mathbb{R} \right\}.$$

Suppose $\text{Par}(\mathcal{P}_0 E_1) = \{b_1, \dots, b_{r_1}\}$ and $\text{Par}(\mathcal{P}_0 E_2) = \{c_1, \dots, c_{r_2}\}$. Let $\{v_i\}$ and $\{w_j\}$ be frames of $\mathcal{P}_0 E_1$ and $\mathcal{P}_0 E_2$, respectively, which are compatible with the filtrations.

By the definition of filtered bundles, we have

$$E_1 \otimes E_2 = \sum_{i,j} \mathcal{O}_\Delta(*[0]) \cdot v_i \otimes w_j,$$

$$\text{Hom}(E_1, E_2) = \sum_{i,j} \mathcal{O}_\Delta(*[0]) \cdot v_i^\vee \otimes w_j.$$

We used condition (1) in the definition of a filtered bundle (see Definition 8.2).

Proposition 14.3. *The family $\mathcal{P}_a(E_1 \otimes E_2)$ ($a \in \mathbb{R}$) defines a filtered bundle structure over $E_1 \otimes E_2$. We denote this filtered bundle by $\mathcal{P}_*E_1 \otimes \mathcal{P}_*E_2$.*

Proof of Proposition 14.3. Fix $a \in \mathbb{R}$. By definition, we have $v_i \otimes w_j \in \mathcal{P}_{b_i} E_1 \otimes \mathcal{P}_{c_j} E_2$. Set

$$n_{ij,a} := \max\{n \in \mathbb{Z} \mid n + b_i + c_j \leq a\}.$$

Then

$$z^{-n_{ij,a}} v_i \otimes w_j \in \mathcal{P}_{b_i+n_{ij,a}} E_1 \otimes \mathcal{P}_{c_j} E_2 \subset \mathcal{P}_a(E_1 \otimes E_2).$$

Hence,

$$\sum_{i,j} \mathcal{O}_\Delta \cdot z^{-n_{ij,a}} v_i \otimes w_j \subset \mathcal{P}_a(E_1 \otimes E_2).$$

Let $b, c \in \mathbb{R}$. Set

$$n_{i,b} := \max\{n \in \mathbb{Z} \mid n + b_i \leq b\}, \quad m_{j,c} := \max\{n \in \mathbb{Z} \mid n + c_j \leq c\}.$$

Then $\{z^{-n_{i,b}} v_i\}$ and $\{z^{-m_{j,c}} w_j\}$ are frames of $\mathcal{P}_b E_1$ and $\mathcal{P}_c E_2$, respectively. Therefore,

$$\mathcal{P}_a(E_1 \otimes E_2) = \sum_{b+c \leq a} \mathcal{O}_\Delta \cdot z^{-n_{i,b}} v_i \otimes z^{-m_{j,c}} w_j.$$

By the maximality of $n_{ij,a}$, we obtain

$$\mathcal{P}_a(E_1 \otimes E_2) = \sum_{i,j} \mathcal{O}_\Delta \cdot z^{-n_{ij,a}} v_i \otimes w_j.$$

It is clear that condition (1) in the definition of filtered bundles is satisfied (see Definition 8.2). Condition (2) in Definition 8.2 follows from the identity $n_{ij,a+n} = n_{ij,a} + n$. Choose $\epsilon_{ij,a} > 0$ small enough such that $n_{ij,a+\epsilon_{ij,a}} = n_{ij,a}$, and set $\epsilon := \min_{i,j} \epsilon_{ij,a}$. Then,

$$\mathcal{P}_{a+\epsilon}(E_1 \otimes E_2) = \sum_{i,j} \mathcal{O}_\Delta \cdot z^{-n_{ij,a}} v_i \otimes w_j = \mathcal{P}_a(E_1 \otimes E_2).$$

Therefore, $\mathcal{P}_*(E_1 \otimes E_2)$ defines a filtered bundle over $E_1 \otimes E_2$, as desired. \square

Proposition 14.4. *The increasing family of \mathcal{O}_Δ -modules $\mathcal{P}_a \text{Hom}(E_1, E_2)$ ($a \in \mathbb{R}$) defines a filtered bundle structure over $\text{Hom}(E_1, E_2)$. This filtered bundle is denoted by $\text{Hom}(\mathcal{P}_* E_1, \mathcal{P}_* E_2)$.*

Proof of Proposition 14.4. Let $a \in \mathbb{R}$ and $f \in \mathcal{P}_a \text{Hom}(E_1, E_2)$. By definition,

$$f(\mathcal{P}_{b_i} E_1) \subset \mathcal{P}_{a+b_i} E_2$$

holds. Conversely, if an $\mathcal{O}_\Delta(*[0])$ -module morphism $f: E_1 \rightarrow E_2$ satisfies

$$f(\mathcal{P}_{b_i} E_1) \subset \mathcal{P}_{a+b_i} E_2$$

for all i , then $f \in \mathcal{P}_a \text{Hom}(E_1, E_2)$.

Hence,

$$\mathcal{P}_a \text{Hom}(E_1, E_2) = \{f \in \text{Hom}_{\mathcal{O}_\Delta(*[0])}(E_1, E_2) \mid f(\mathcal{P}_{b_i} E_1) \subset \mathcal{P}_{a+b_i} E_2 \text{ for all } i\}.$$

Define

$$m_{ij,a} := \max\{m \in \mathbb{Z} \mid m + c_j \leq b_i + a\}.$$

Then, by the above discussion,

$$\mathcal{P}_a \text{Hom}(E_1, E_2) = \sum_{i,j} \mathcal{O}_\Delta \cdot v_i^\vee \otimes z^{-m_{ij,a}} w_j.$$

It is clear that condition (1) in Definition 8.2 is satisfied. Condition (2) in Definition 8.2 follows from $m_{ij,a+n} = m_{ij,a} + n$. Choose $\epsilon_i > 0$ such that $\mathcal{P}_{a+b_i+\epsilon_i} E_2 = \mathcal{P}_{a+b_i} E_2$, and set $\epsilon := \min \epsilon_i$. Then,

$$\mathcal{P}_{a+\epsilon} \text{Hom}(E_1, E_2) = \mathcal{P}_a \text{Hom}(E_1, E_2).$$

Therefore, $\mathcal{P}_* \text{Hom}(E_1, E_2)$ defines a filtered bundle over $\text{Hom}(E_1, E_2)$. \square

15. DUAL BUNDLES REVISITED

In this section, we study prolongations of dual bundles within the framework of filtered bundles.

Proposition 15.1. *Let (E, h) be an acceptable vector bundle of rank r on Δ^* . Then*

$$\mathcal{P}_*^{h^\vee} E^\vee = \text{Hom}(\mathcal{P}_*^h E, \mathcal{P}_*^{(0)} \mathcal{O}_\Delta(*[0])).$$

Note that

$$\text{Hom}(\mathcal{P}_*^h E, \mathcal{P}_*^{(0)} \mathcal{O}_\Delta(*[0]))$$

in Proposition 15.1 is a filtered bundle, as described in Proposition 14.4, since both $\mathcal{P}_*^h E$ and $\mathcal{P}_*^{(0)} \mathcal{O}_\Delta(*[0])$ are filtered bundles (see Sections 4 and 8).

Proof of Proposition 15.1. Let $k \in \mathbb{R}$ be arbitrary. Take any $f \in \mathcal{P}_k \text{Hom}(E, \mathcal{O}_\Delta(*[0]))$. By definition, for any $a \in \mathbb{R}$, we have

$$f(\mathcal{P}_a^h E) \subset \mathcal{P}_{a+k}^{(0)} \mathcal{O}_\Delta(*[0]).$$

Take $a = -k + 1 - \epsilon$ with any $0 < \epsilon \ll 1$. Then

$$f(\mathcal{P}_{-k+1-\epsilon}^h E) \subset \mathcal{P}_{1-\epsilon}^{(0)} \mathcal{O}_\Delta(*[0]) = \mathcal{O}_\Delta.$$

This implies

$$f \in (\mathcal{P}_{-k+1-\epsilon}^h E)^\vee = \mathcal{P}_{k+\epsilon-\delta}^{h^\vee} E^\vee$$

for any sufficiently small $\delta > 0$, and hence

$$f \in \mathcal{P}_{k+\epsilon'}^{h^\vee} E^\vee$$

holds for any $0 < \varepsilon' \ll 1$. By Lemma 7.9, we conclude

$$f \in \mathcal{P}_k^{h^\vee} E^\vee.$$

Thus,

$$\mathcal{P}_k \operatorname{Hom}(E, \mathcal{O}_\Delta(*[0])) \subset \mathcal{P}_k^{h^\vee} E^\vee.$$

We now prove the opposite inclusion. Fix $k \in \mathbb{R}$. It suffices to show that for any $f \in \mathcal{P}_k^{h^\vee} E^\vee$ and any $a \in \mathbb{R}$, we have

$$f(\mathcal{P}_a^h E) \subset \mathcal{P}_{a+k}^{(0)} \mathcal{O}_\Delta(*[0]).$$

Suppose $\operatorname{Par}(\mathcal{P}_0^h E) = \{b_1, \dots, b_r\}$, and let $\{v_i\}_{i=1}^r$ be a frame of $\mathcal{P}_0^h E$ compatible with the parabolic filtration. Let $\{v_i^\vee\}_{i=1}^r$ be the dual frame of $\{v_i\}_{i=1}^r$ as in Theorem 13.2. Define

$$n_i := \max\{n \in \mathbb{Z} \mid n - b_i \leq k\}.$$

Then $\{z^{-n_i} v_i^\vee\}_{i=1}^r$ is a frame of $\mathcal{P}_k^{h^\vee} E^\vee$ by Lemma 7.17 and Theorem 13.2.

Fix $a \in \mathbb{R}$, and define

$$m_i := \max\{m \in \mathbb{Z} \mid m + b_i \leq a\}.$$

By Lemma 7.17 again, the set $\{z^{-m_i} v_i\}_{i=1}^r$ forms a frame of $\mathcal{P}_a^h E$.

Therefore, in order to prove that $f(\mathcal{P}_a^h E) \subset \mathcal{P}_{a+k}^{(0)} \mathcal{O}_\Delta(*[0])$ for any $f \in \mathcal{P}_k^{h^\vee} E^\vee$, it suffices to check

$$z^{-n_i} v_i^\vee(z^{-m_i} v_i) \in \mathcal{P}_{a+k}^{(0)} \mathcal{O}_\Delta(*[0]) = \mathcal{O}_\Delta(\lfloor a + k \rfloor[0]).$$

This follows from the inequality

$$n_i + m_i = (n_i - b_i) + (m_i + b_i) \leq k + a,$$

which implies

$$n_i + m_i \leq \lfloor a + k \rfloor.$$

Hence, for all $f \in \mathcal{P}_k^{h^\vee} E^\vee$, we have

$$f(\mathcal{P}_a^h E) \subset \mathcal{P}_{a+k}^{(0)} \mathcal{O}_\Delta(*[0]).$$

Since $a \in \mathbb{R}$ is arbitrary, we obtain the inclusion

$$\mathcal{P}_k^{h^\vee} E^\vee \subset \mathcal{P}_k \operatorname{Hom}(E, \mathcal{O}_\Delta(*[0])).$$

Therefore,

$$\mathcal{P}_k^{h^\vee} E^\vee = \mathcal{P}_k \operatorname{Hom}(E, \mathcal{O}_\Delta(*[0])).$$

Since this equality holds for every $k \in \mathbb{R}$, the proof of Proposition 15.1 is complete. \square

16. ON TENSOR PRODUCTS

In this section, we discuss the prolongations of tensor products of acceptable bundles in detail. Throughout this section, we use the notation

$$\lceil a \rceil := \min\{n \in \mathbb{Z} \mid n \geq a\}$$

for $a \in \mathbb{R}$.

Proposition 16.1. *Let (E_1, h_1) and (E_2, h_2) be acceptable vector bundles of rank r_1 and r_2 , respectively. Suppose that*

$$\mathcal{P}ar(\mathcal{P}_0^{h_1} E_1) = \{b_1, \dots, b_{r_1}\}, \quad \mathcal{P}ar(\mathcal{P}_0^{h_2} E_2) = \{c_1, \dots, c_{r_2}\}.$$

Let $\{v_1, \dots, v_{r_1}\}$ and $\{w_1, \dots, w_{r_2}\}$ be frames of $\mathcal{P}_0^{h_1} E_1$ and $\mathcal{P}_0^{h_2} E_2$, respectively, such that

$$v_i \in \mathcal{P}_{b_i}^{h_1} E_1 \setminus \mathcal{P}_{< b_i}^{h_1} E_1, \quad w_j \in \mathcal{P}_{c_j}^{h_2} E_2 \setminus \mathcal{P}_{< c_j}^{h_2} E_2,$$

and are compatible with the parabolic filtrations. Then:

- (i) *The set $\{z^{\lceil b_i + c_j \rceil} v_i \otimes w_j\}_{1 \leq i \leq r_1, 1 \leq j \leq r_2}$ forms a frame of $\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)$.*
- (ii) *The set of parabolic weights of $\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)$ is given by*

$$\mathcal{P}ar(\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)) = \{b_i + c_j - \lceil b_i + c_j \rceil\}_{1 \leq i \leq r_1, 1 \leq j \leq r_2}.$$

In particular, the set $\{z^{\lceil b_i + c_j \rceil} v_i \otimes w_j\}_{1 \leq i \leq r_1, 1 \leq j \leq r_2}$ is compatible with the parabolic filtration.

Proof of Proposition 16.1. Since $|v_i \otimes w_j|_{h_1 \otimes h_2} = |v_i|_{h_1} \cdot |w_j|_{h_2}$, we have

$$v_i \otimes w_j \in \mathcal{P}_{b_i + c_j}^{h_1 \otimes h_2}(E_1 \otimes E_2),$$

and hence

$$z^{\lceil b_i + c_j \rceil} v_i \otimes w_j \in \mathcal{P}_{b_i + c_j - \lceil b_i + c_j \rceil}^{h_1 \otimes h_2}(E_1 \otimes E_2) \subset \mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2).$$

Step 1. In this step, we prove statement (i).

To prove (i), it suffices to show that for every $f \in \mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)$, there exist holomorphic functions $f_{ij} \in \mathcal{O}_\Delta$ such that

$$f = \sum_{i,j} f_{ij} z^{\lceil b_i + c_j \rceil} v_i \otimes w_j.$$

Since $\{v_i\}$ and $\{w_j\}$ are frames of E_1 and E_2 on Δ^* , respectively, the set $\{z^{\lceil b_i + c_j \rceil} v_i \otimes w_j\}_{i,j}$ forms a frame of $E_1 \otimes E_2$ on Δ^* . Hence, for any $f \in \mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)$, we can write

$$f = \sum_{i,j} f_{ij} z^{\lceil b_i + c_j \rceil} v_i \otimes w_j,$$

where each f_{ij} is holomorphic outside the origin. Therefore, it remains to show that f_{ij} is holomorphic at the origin.

Let $\{v_i^\vee\}$ and $\{w_j^\vee\}$ denote the dual frames of $\{v_i\}$ and $\{w_j\}$, respectively. Recall that

$$v_i^\vee \in \mathcal{P}_{-b_i}^{h_1^\vee} E_1^\vee, \quad w_j^\vee \in \mathcal{P}_{-c_j}^{h_2^\vee} E_2^\vee.$$

Therefore,

$$v_i^\vee \otimes w_j^\vee \in \mathcal{P}_{-b_i - c_j}^{h_1^\vee \otimes h_2^\vee}(E_1^\vee \otimes E_2^\vee).$$

By Proposition 15.1, we have

$$v_i^\vee \otimes w_j^\vee \in \mathcal{P}_{-b_i - c_j} \text{Hom}(E_1 \otimes E_2, \mathcal{O}_\Delta(*[0])).$$

Since $f \in \mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)$, it follows that

$$\begin{aligned} (v_i^\vee \otimes w_j^\vee)(f) &= (v_i^\vee \otimes w_j^\vee) \left(\sum_{i,j} f_{ij} z^{\lceil b_i + c_j \rceil} v_i \otimes w_j \right) \\ &= z^{\lceil b_i + c_j \rceil} f_{ij} \in \mathcal{P}_{-b_i - c_j}^{(0)} \mathcal{O}_\Delta(*[0]). \end{aligned}$$

Since $0 \leq [b_i + c_j] - (b_i + c_j) < 1$, we have

$$f_{ij} \in \mathcal{P}_{[b_i+c_j]-(b_i+c_j)}^{(0)} \mathcal{O}_\Delta(*[0]) = \mathcal{O}_\Delta(\lfloor [b_i + c_j] - b_i - c_j \rfloor [0]) = \mathcal{O}_\Delta.$$

This completes the proof of (i).

Step 2. In this step, we prove statement (ii).

As before, we define

$$\mathbf{v} := \{v_1, \dots, v_{r_1}\}, \quad \mathbf{w} := \{w_1, \dots, w_{r_2}\}, \quad \mathbf{v} \otimes \mathbf{w} := \{v_i \otimes w_j\}_{i,j}.$$

We further define

$$(\mathbf{v} \otimes \mathbf{w})^\# := \{z^{\lceil b_i+c_j \rceil} v_i \otimes w_j\}_{1 \leq i \leq r_1, 1 \leq j \leq r_2}.$$

As shown in Step 1, the set $(\mathbf{v} \otimes \mathbf{w})^\#$ forms a frame of $\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)$.

We consider the Hermitian matrix

$$H(h_1 \otimes h_2, \mathbf{v} \otimes \mathbf{w}) := (h_1(v_i, v_j) \cdot h_2(w_k, w_l)),$$

whose $((i-1)r_2 + k, (j-1)r_2 + l)$ -th entry is given by $h_1(v_i, v_j) \cdot h_2(w_k, w_l)$. Similarly, define

$$H(h_1 \otimes h_2, (\mathbf{v} \otimes \mathbf{w})^\#) := (z^{\lceil b_i+c_k \rceil} \cdot \bar{z}^{\lceil b_j+c_l \rceil} h_1(v_i, v_j) \cdot h_2(w_k, w_l)).$$

Then we have

$$\begin{aligned} \det H(h_1 \otimes h_2, (\mathbf{v} \otimes \mathbf{w})^\#) &= \left(\prod_{i,k} z^{\lceil b_i+c_k \rceil} \right) \left(\prod_{j,l} \bar{z}^{\lceil b_j+c_l \rceil} \right) \cdot (\det H(h_1, \mathbf{v}))^{r_2} \cdot (\det H(h_2, \mathbf{w}))^{r_1} \\ &= |z|^2 \sum_{i,j} \lceil b_i+c_j \rceil \cdot (\det H(h_1, \mathbf{v}))^{r_2} \cdot (\det H(h_2, \mathbf{w}))^{r_1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \gamma(\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)) &= -\frac{1}{2} \liminf_{z \rightarrow 0} \frac{\log \det H(h_1 \otimes h_2, (\mathbf{v} \otimes \mathbf{w})^\#)}{\log |z|} \\ (16.1) \quad &= -\frac{1}{2} \lim_{z \rightarrow 0} \frac{\log \det H(h_1 \otimes h_2, (\mathbf{v} \otimes \mathbf{w})^\#)}{\log |z|} \\ &= -\sum_{i,j} \lceil b_i+c_j \rceil + r_2 \cdot \gamma(\mathcal{P}_0^{h_1}(E_1)) + r_1 \cdot \gamma(\mathcal{P}_0^{h_2}(E_2)) \\ &= -\sum_{i,j} \lceil b_i+c_j \rceil + r_2 \cdot \sum_i b_i + r_1 \cdot \sum_j c_j \end{aligned}$$

by Corollary 7.6 and Theorem 12.3.

On the other hand, we have

$$(16.2) \quad z^{\lceil b_i+c_j \rceil} v_i \otimes w_j \in \mathcal{P}_{b_i+c_j-\lceil b_i+c_j \rceil}^{h_1 \otimes h_2}(E_1 \otimes E_2),$$

and

$$(16.3) \quad \sum_{i,j} (b_i + c_j - \lceil b_i + c_j \rceil) = -\sum_{i,j} \lceil b_i + c_j \rceil + r_2 \sum_i b_i + r_1 \sum_j c_j.$$

By Corollary 12.8, together with (16.1), (16.2), and (16.3), we conclude that $(\mathbf{v} \otimes \mathbf{w})^\#$ is a frame of $\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)$ compatible with the parabolic filtration,

$$\text{Par}(\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)) = \{b_i + c_j - \lceil b_i + c_j \rceil\}_{1 \leq i \leq r_1, 1 \leq j \leq r_2},$$

and

$$z^{\lceil b_i + c_j \rceil} v_i \otimes w_j \in \mathcal{P}_{b_i + c_j - \lceil b_i + c_j \rceil}^{h_1 \otimes h_2}(E_1 \otimes E_2) \setminus \mathcal{P}_{< b_i + c_j - \lceil b_i + c_j \rceil}^{h_1 \otimes h_2}(E_1 \otimes E_2).$$

Thus, statement (ii) is proved.

We now complete the proof of Theorem 16.1. \square

We are now ready to describe the behavior of the prolongation of the tensor product of acceptable bundles.

Theorem 16.2 (Tensor products, see Theorem 1.14). *Let (E_1, h_1) and (E_2, h_2) be acceptable vector bundles of rank r_1 and r_2 , respectively. Then the parabolic filtration on $E_1 \otimes E_2$ induced by $h_1 \otimes h_2$ coincides with the tensor product filtration:*

$$\mathcal{P}_*^{h_1 \otimes h_2}(E_1 \otimes E_2) = \mathcal{P}_*^{h_1}(E_1) \otimes \mathcal{P}_*^{h_2}(E_2).$$

Equivalently, for every $a \in \mathbb{R}$,

$$\mathcal{P}_a^{h_1 \otimes h_2}(E_1 \otimes E_2) = \sum_{a_1 + a_2 \leq a} \mathcal{P}_{a_1}^{h_1}(E_1) \otimes \mathcal{P}_{a_2}^{h_2}(E_2).$$

Proof of Theorem 16.2. Let $k \in \mathbb{R}$ be arbitrary. Then the following inclusion

$$\sum_{a+b \leq k} \mathcal{P}_a^{h_1}(E_1) \otimes \mathcal{P}_b^{h_2}(E_2) \subset \mathcal{P}_k^{h_1 \otimes h_2}(E_1 \otimes E_2)$$

holds obviously by definition. Hence, it suffices to prove the opposite inclusion.

Suppose that

$$\text{Par}(\mathcal{P}_0^{h_1} E_1) = \{b_1, \dots, b_{r_1}\}, \quad \text{Par}(\mathcal{P}_0^{h_2} E_2) = \{c_1, \dots, c_{r_2}\}.$$

Let $\{v_1, \dots, v_{r_1}\}$ and $\{w_1, \dots, w_{r_2}\}$ be frames of $\mathcal{P}_0^{h_1} E_1$ and $\mathcal{P}_0^{h_2} E_2$, respectively, compatible with the corresponding parabolic filtrations.

By Proposition 16.1, we have

$$\text{Par}(\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)) = \{b_i + c_j - \lceil b_i + c_j \rceil\}_{1 \leq i \leq r_1, 1 \leq j \leq r_2},$$

and the set

$$\{z^{\lceil b_i + c_j \rceil} v_i \otimes w_j\}_{1 \leq i \leq r_1, 1 \leq j \leq r_2}$$

forms a frame of $\mathcal{P}_0^{h_1 \otimes h_2}(E_1 \otimes E_2)$ compatible with the parabolic filtration.

For each $(i, j) \in \{1, \dots, r_1\} \times \{1, \dots, r_2\}$, define

$$n_{ij} := \max \{n \in \mathbb{Z} \mid n + b_i + c_j - \lceil b_i + c_j \rceil \leq k\}.$$

Then the set

$$\{z^{-n_{ij} + \lceil b_i + c_j \rceil} v_i \otimes w_j\}_{i,j}$$

is a frame of $\mathcal{P}_k^{h_1 \otimes h_2}(E_1 \otimes E_2)$. Since

$$z^{-n_{ij} + \lceil b_i + c_j \rceil} v_i \in \mathcal{P}_{n_{ij} - \lceil b_i + c_j \rceil + b_i}^{h_1}(E_1), \quad w_j \in \mathcal{P}_{c_j}^{h_2}(E_2),$$

and $n_{ij} + b_i + c_j - \lceil b_i + c_j \rceil \leq k$, it follows that

$$z^{-n_{ij} + \lceil b_i + c_j \rceil} v_i \otimes w_j \in \mathcal{P}_{n_{ij} - \lceil b_i + c_j \rceil + b_i}^{h_1}(E_1) \otimes \mathcal{P}_{c_j}^{h_2}(E_2) \subset \sum_{a+b \leq k} \mathcal{P}_a^{h_1}(E_1) \otimes \mathcal{P}_b^{h_2}(E_2).$$

Therefore, we obtain the inclusion

$$\mathcal{P}_k^{h_1 \otimes h_2}(E_1 \otimes E_2) \subset \sum_{a+b \leq k} \mathcal{P}_a^{h_1}(E_1) \otimes \mathcal{P}_b^{h_2}(E_2),$$

and hence the desired equality

$$\mathcal{P}_k^{h_1 \otimes h_2}(E_1 \otimes E_2) = \sum_{a+b \leq k} \mathcal{P}_a^{h_1}(E_1) \otimes \mathcal{P}_b^{h_2}(E_2)$$

holds for every $k \in \mathbb{R}$. This completes the proof of Theorem 16.2. \square

17. ON HOM BUNDLES

In this final section, we prove that the parabolic filtration on $\text{Hom}(E_1, E_2)$ induced by $h_1^\vee \otimes h_2$ coincides with the filtration on the filtered bundle $\text{Hom}(\mathcal{P}_*^{h_1} E_1, \mathcal{P}_*^{h_2} E_2)$.

Proposition 17.1. *Let (E_1, h_1) and (E_2, h_2) be acceptable vector bundles of rank r_1 and r_2 , respectively, defined on Δ^* . Then the parabolic filtration on the Hom bundle $\text{Hom}(E_1, E_2)$ induced by the metric $h_1^\vee \otimes h_2$ coincides with the filtration on the filtered bundle $\text{Hom}(\mathcal{P}_*^{h_1} E_1, \mathcal{P}_*^{h_2} E_2)$:*

$$\mathcal{P}_*^{h_1^\vee \otimes h_2} \text{Hom}(E_1, E_2) = \text{Hom}(\mathcal{P}_*^{h_1} E_1, \mathcal{P}_*^{h_2} E_2).$$

Proof of Proposition 17.1. As usual, we denote the filtered bundle $\text{Hom}(\mathcal{P}_*^{h_1} E_1, \mathcal{P}_*^{h_2} E_2)$ by $\mathcal{P}_* \text{Hom}(E_1, E_2)$ (see Proposition 14.3). By Theorem 16.2, for any $k \in \mathbb{R}$, we have

$$\mathcal{P}_k^{h_1^\vee \otimes h_2} \text{Hom}(E_1, E_2) = \mathcal{P}_k^{h_1^\vee \otimes h_2}(E_1^\vee \otimes E_2) = \sum_{b+c \leq k} \mathcal{P}_b^{h_1^\vee} E_1^\vee \otimes \mathcal{P}_c^{h_2} E_2.$$

Let $f \otimes u \in \mathcal{P}_b^{h_1^\vee} E_1^\vee \otimes \mathcal{P}_c^{h_2} E_2$ with $b+c \leq k$, and let $x \in \mathcal{P}_a^{h_1} E_1$. Then, by Proposition 15.1, we have

$$f(x) \in \mathcal{P}_{b+a}^{(0)} \mathcal{O}_\Delta(*[0]).$$

Therefore,

$$(f \otimes u)(x) = f(x) \cdot u \in \mathcal{P}_{b+a+c}^{h_2} E_2 \subset \mathcal{P}_{a+k}^{h_2} E_2.$$

This implies that, for every $a \in \mathbb{R}$, we have

$$(f \otimes u)(\mathcal{P}_a^{h_1} E_1) \subset \mathcal{P}_{a+k}^{h_2} E_2,$$

and hence $f \otimes u \in \mathcal{P}_k \text{Hom}(E_1, E_2)$. Thus, we obtain the inclusion

$$\mathcal{P}_k^{h_1^\vee \otimes h_2} \text{Hom}(E_1, E_2) \subset \mathcal{P}_k \text{Hom}(E_1, E_2).$$

We now prove the opposite inclusion. Let

$$\text{Par}(\mathcal{P}_0^{h_1} E_1) = \{b_1, \dots, b_{r_1}\}, \quad \text{Par}(\mathcal{P}_0^{h_2} E_2) = \{c_1, \dots, c_{r_2}\}.$$

Let $\{v_1, \dots, v_{r_1}\}$ and $\{w_1, \dots, w_{r_2}\}$ be frames of $\mathcal{P}_0^{h_1} E_1$ and $\mathcal{P}_0^{h_2} E_2$, respectively, compatible with the parabolic filtrations, such that

$$v_i \in \mathcal{P}_{b_i}^{h_1} E_1 \setminus \mathcal{P}_{< b_i}^{h_1} E_1, \quad w_j \in \mathcal{P}_{c_j}^{h_2} E_2 \setminus \mathcal{P}_{< c_j}^{h_2} E_2.$$

By Theorem 13.2, we have

$$\mathcal{P}_{1-\epsilon}^{h_1^\vee} E_1^\vee \simeq (\mathcal{P}_0^{h_1} E_1)^\vee$$

for sufficiently small $\epsilon > 0$. Moreover, by Theorem 13.2,

$$\text{Par}(\mathcal{P}_{1-\epsilon}^{h_1^\vee} E_1^\vee) = \{-b_1, \dots, -b_{r_1}\},$$

and the dual frame $\{v_1^\vee, \dots, v_{r_1}^\vee\}$ is compatible with this parabolic filtration. Fix an arbitrary $k \in \mathbb{R}$, and let $f \in \mathcal{P}_k \text{Hom}(E_1, E_2)$. Then, for any $a \in \mathbb{R}$, we have

$$f(\mathcal{P}_a^{h_1} E_1) \subset \mathcal{P}_{a+k}^{h_2} E_2.$$

In particular,

$$f(\mathcal{P}_{b_i}^{h_1} E_1) \subset \mathcal{P}_{b_i+k}^{h_2} E_2$$

for all i . Define

$$n_{ij} := \max\{n \in \mathbb{Z} \mid n + c_j \leq b_i + k\}.$$

Then, by Lemma 7.17, the set $\{z^{-n_{ij}} w_j\}_{j=1}^{r_2}$ forms a frame of $\mathcal{P}_{b_i+k}^{h_2} E_2$. Since $f(\mathcal{P}_{b_i}^{h_1} E_1) \subset \mathcal{P}_{b_i+k}^{h_2} E_2$, there exist holomorphic functions $f_{ij} \in \mathcal{O}_\Delta$ such that

$$f = \sum_{i,j} f_{ij} \cdot v_i^\vee \otimes z^{-n_{ij}} w_j.$$

Since $v_i^\vee \in \mathcal{P}_{-b_i}^{h_1^\vee} E_1^\vee$ and $z^{-n_{ij}} w_j \in \mathcal{P}_{b_i+k}^{h_2} E_2$, it follows that

$$v_i^\vee \otimes z^{-n_{ij}} w_j \in \mathcal{P}_{-b_i}^{h_1^\vee} E_1^\vee \otimes \mathcal{P}_{b_i+k}^{h_2} E_2 \subset \sum_{b+c \leq k} \mathcal{P}_b^{h_1^\vee} E_1^\vee \otimes \mathcal{P}_c^{h_2} E_2 = \mathcal{P}_k^{h_1^\vee \otimes h_2} \text{Hom}(E_1, E_2).$$

Therefore, $f \in \mathcal{P}_k^{h_1^\vee \otimes h_2} \text{Hom}(E_1, E_2)$, and hence

$$\mathcal{P}_k \text{Hom}(E_1, E_2) \subset \mathcal{P}_k^{h_1^\vee \otimes h_2} \text{Hom}(E_1, E_2).$$

Combining both inclusions, we conclude that

$$\mathcal{P}_k \text{Hom}(E_1, E_2) = \mathcal{P}_k^{h_1^\vee \otimes h_2} \text{Hom}(E_1, E_2) \quad \text{for all } k \in \mathbb{R}.$$

This completes the proof of Proposition 17.1. □

REFERENCES

- [B1] O. Biquard, Fibrés paraboliques stables et connexions singulières plates, *Bull. Soc. Math. France* **119** (1991), no. 2, 231–257.
- [B2] O. Biquard, Prolongement d’un fibre holomorphe hermitien à courbure L^p sur une courbe ouverte, *Internat. J. Math.* **3** (1992), no. 4, 441–453.
- [BB] O. Biquard, P. Boalch, Wild non-abelian Hodge theory on curves, *Compos. Math.* **140** (2004), no. 1, 179–204.
- [C] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. **45**. Cambridge University Press, New York, 1957.
- [CG] M. Cornalba, P. Griffiths, Analytic cycles and vector bundles on non-compact algebraic varieties, *Invent. Math.* **28** (1975), 1–106.
- [Dem1] J.-P. Demailly, *Analytic methods in algebraic geometry*, *Surveys of Modern Mathematics*, **1**. International Press, Somerville, MA; Higher Education Press, Beijing, 2012.
- [Dem2] J.-P. Demailly, *Complex Analytic and Differential Geometry*, available at <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>
- [DH] Y. Deng, F. Hao, Vanishing theorem for tame harmonic bundles via L^2 -cohomology, *Compos. Math.* **160** (2024), no. 12, 2828–2855.
- [F] O. Forster, *Lectures on Riemann surfaces*, Translated from the German by Bruce Gilligan. Graduate Texts in Mathematics, **81**. Springer-Verlag, New York–Berlin, 1981.
- [FFO] O. Fujino, T. Fujisawa, T. Ono, Notes on acceptable bundles II, preprint (2026). arXiv:2604.05233 [math.AG]
- [M1] T. Mochizuki, Asymptotic behaviour of tame nilpotent harmonic bundles with trivial parabolic structure, *J. Differential Geom.* **62** (2002), no. 3, 351–559.
- [M2] T. Mochizuki, Kobayashi–Hitchin correspondence for tame harmonic bundles and an application, *Astérisque* No. **309** (2006), viii+117 pp.
- [M3] T. Mochizuki, Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D -modules. I, *Mem. Amer. Math. Soc.* **185** (2007), no. 869, xii+324 pp.
- [M4] T. Mochizuki, Wild harmonic bundles and wild pure twistor D -modules, *Astérisque* No. **340** (2011), x+607 pp.

- [M5] T. Mochizuki, Good wild harmonic bundles and good filtered Higgs bundles, SIGMA Symmetry Integrability Geom. Methods Appl. **17** (2021), Paper No. 068, 66 pp.
- [M6] T. Mochizuki, *Periodic monopoles and difference modules*, Lecture Notes in Mathematics, **2300**. Springer, Cham, [2022], ©2022.
- [M7] T. Mochizuki, *Private notes*, April 11, 2025.
- [NO] J. Noguchi, T. Ochiai, *Geometric function theory in several complex variables*, Translated from the Japanese by Noguchi. Translations of Mathematical Monographs, **80**. American Mathematical Society, Providence, RI, 1990.
- [SS1] C. Sabbah, C. Schnell, *The MHM Project* (Version 2), available at <https://perso.pages.math.cnrs.fr/users/claude.sabbah/MHMProject/mhm.html>
- [SS2] C. Sabbah, C. Schnell, Degenerating complex variations of Hodge structure in dimension one, preprint (2022). arXiv:2206.08166 [math.AG]
- [S1] C. T. Simpson, Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918.
- [S2] C. T. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc. **3** (1990), no. 3, 713–770.
- [S3] C. T. Simpson, *Private notes*, June 11, 2025.

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