NOTES ON ACCEPTABLE BUNDLES II

OSAMU FUJINO, TARO FUJISAWA, AND TAKASHI ONO

Abstract. The notion of acceptable bundles plays a fundamental role in the Simpson–Mochizuki theory. We study acceptable bundles on a partially punctured polydisk in detail. While this article is primarily expository, it also presents new arguments that differ from those of Mochizuki.

Contents

1. Introduction

This paper is a continuation of [FFO], where we gave a detailed study of acceptable bundles on a punctured disk. Here we extend the theory to higher-dimensional settings. More precisely, we investigate acceptable bundles on a partially punctured polydisk.

The notion of acceptable bundles plays a fundamental role in the Simpson–Mochizuki theory; see, for example, [S1], [S2], [M1], [M2], [M3], [M4], and [M5]. It has also found important applications in the study of higher-dimensional complex varieties. For related developments, we refer the reader to, for example, [Den], [DC], [DH], and [K].

Throughout this paper, we freely use the results established in [FFO]. In particular, the theory of acceptable bundles over a punctured disk developed there plays a decisive role in the present work. As in [FFO], one of the main purposes of this paper is to make

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Mochizuki's theory of acceptable bundles [M4, Chapter 21, Acceptable Bundles] more accessible to a broader audience.

Although we use an L^2 extension theorem of Ohsawa–Takegoshi type as a black box, we aim to present the theory of acceptable bundles in a form that is as self-contained as possible and accessible at the level of [Dem1].

Let E be a holomorphic vector bundle on the partially punctured polydisk $(\Delta^*)^l \times \Delta^{n-l}$, and let h be a smooth Hermitian metric on E. We denote by ω_P the Poincaré metric on $(\Delta^*)^l \times \Delta^{n-l}$, defined by

$$\omega_P := \sum_{j=1}^{l} \frac{\sqrt{-1}\, dz_j \wedge d\overline{z}_j}{|z_j|^2\bigl(-\log|z_j|^2\bigr)^2} + \sum_{k=l+1}^{n} \frac{\sqrt{-1}\, dz_k \wedge d\overline{z}_k}{(1-|z_k|^2)^2}.$$

We say that (E,h) is *acceptable* if the curvature $\sqrt{-1}\Theta_h(E)$, which is a smooth $\mathrm{Hom}(E,E)$-valued $(1,1)$-form, is bounded with respect to the metric induced by h and ω_P.

Let $\boldsymbol{a} = (a_1, \ldots, a_l) \in \mathbb{R}^l$. We define an $\mathcal{O}_{\Delta^n}$-module ${}_{\boldsymbol{a}}E$ as follows. For any open set $U \subset \Delta^n$, set

$$\Gamma(U, {}_{\boldsymbol{a}}E) := \left\{ s \in \Gamma\bigl(U \cap ((\Delta^*)^l \times \Delta^{n-l}), E\bigr) \;\middle|\; |s|_h = O\left(\frac{1}{\prod_{j=1}^{l} |z_j|^{a_j+\varepsilon}}\right) \text{ for every } \varepsilon > 0 \right\}.$$

We note that we sometimes write $\mathcal{P}_{\boldsymbol{a}}E$ instead of ${}_{\boldsymbol{a}}E$. Moreover, ${}_{\boldsymbol{0}}E$ is usually denoted by ${}^\diamond E$, where $\boldsymbol{0} = (0, \ldots, 0) \in \mathbb{R}^l$.

One of the main results of this paper is the following.

Theorem 1.1 (Prolongation by increasing orders, cf. [M4, Theorem 21.3.1])**.** *Let (E,h) be an acceptable vector bundle on a partially punctured polydisk $(\Delta^*)^l \times \Delta^{n-l}$. Then ${}_{\boldsymbol{a}}E$ is a locally free sheaf on Δ^n for any $\boldsymbol{a} \in \mathbb{R}^l$. Moreover, the family $\bigl({}_{\boldsymbol{a}}E \mid \boldsymbol{a} \in \mathbb{R}^l\bigr)$ naturally forms a filtered bundle.*

We make a brief remark on the assumptions in Theorem 1.1.

Remark 1.2. In [M4, Theorem 21.3.1], Mochizuki assumes for simplicity that $(\det E, \det h)$ is flat.

More precisely, we prove the following statement.

Theorem 1.3. *Let (E,h) be an acceptable vector bundle on a partially punctured polydisk $(\Delta^*)^l \times \Delta^{n-l}$ with* $\operatorname{rank} E = r$. *Fix $\boldsymbol{a} \in \mathbb{R}^l$. Then there exists a sufficiently small open neighborhood U of the origin $(0, \ldots, 0)$ in Δ^n such that there are a local frame $\boldsymbol{v} = \{v_1, \ldots, v_r\}$ of ${}_{\boldsymbol{a}}E$ and vectors $\boldsymbol{a}(v_j) \in \mathbb{R}^l$ $(j = 1, \ldots, r)$ with the following property: for any $\boldsymbol{b} \in \mathbb{R}^l$, we have*

$${}_{\boldsymbol{b}}E = \bigoplus_{j=1}^{r} \mathcal{O}_U\left(\sum_{i=1}^{l} \lfloor b_i - a_i(v_j) \rfloor D_i\right) \cdot v_j,$$

where $D_i := \{z_i = 0\} \subset \Delta^n$ for each i.

Theorem 1.1 implies that, for each $i = 1, \ldots, l$ and $b \in (a_i - 1, a_i]$, the image

$${}^{i}F_b\bigl({}_{\boldsymbol{a}}E|_{D_i}\bigr) \subset {}_{\boldsymbol{a}}E|_{D_i}$$

of the natural morphism ${}_{\boldsymbol{a}(i,b)}E|_{D_i} \to {}_{\boldsymbol{a}}E|_{D_i}$ is a subbundle. Here $\boldsymbol{a}(i,b)$ is defined by replacing the i-th component of $\boldsymbol{a}$ by b. The induced filtrations ${}^{i}F$ $(i = 1, \ldots, l)$ on ${}_{\boldsymbol{a}}E|_{D_i}$

are mutually compatible. These filtrations are referred to as the *parabolic filtrations associated with* ${}_{\boldsymbol{a}}E$.

Let $\mathbf{F} = ({}^iF \mid i = 1, \dots, l)$ denote the resulting tuple of filtrations. Let $\boldsymbol{v} = \{v_1, \dots, v_r\}$ be a local frame of ${}_{\boldsymbol{a}}E$ compatible with $\mathbf{F}$ near the origin, and let

$$a_i(v_k) = {}^i\deg^{\mathbf{F}}(v_k) := \deg^{{}^iF}(v_k)$$

be the corresponding weights. We define

$$v'_k := v_k \cdot \prod_{i=1}^{l} |z_i|^{a_i(v_k)}.$$

Let $H(h, \boldsymbol{v}')$ be the Hermitian matrix-valued function whose (p,q)-entry is given by $h(v'_p, v'_q)$. The following weak norm estimate is a fundamental tool in the study of acceptable bundles.

Theorem 1.4 (Weak norm estimate, cf. [M4, Theorem 21.3.2])**.** *There exist positive constants C and N such that, in a neighborhood of the origin,*

$$C^{-1}\left(-\sum_{i=1}^{l}\log|z_i|\right)^{-N} I_r \le H(h,\boldsymbol{v}') \le C\left(-\sum_{i=1}^{l}\log|z_i|\right)^{N} I_r.$$

Here I_r denotes the identity matrix of size r, and for Hermitian matrix-valued functions A and B, the notation $A \le B$ means that $B - A$ is positive semidefinite.

We can translate various results on acceptable bundles over Δ^* to the setting of partially punctured polydisks $(\Delta^*)^l \times \Delta^{n-l}$. Below we briefly explain some of these results for the reader's convenience.

Theorem 1.5 (Dual bundles, see [FFO, Theorem 1.12])**.** *Let (E,h) be an acceptable vector bundle on a partially punctured polydisk $(\Delta^*)^l \times \Delta^{n-l}$. Then, for any $\boldsymbol{a} \in \mathbb{R}^l$, we have*

$$\left({}_{\boldsymbol{a}}E\right)^{\vee} = {}_{-\boldsymbol{a}+\mathbf{1}-\boldsymbol{\varepsilon}}\left(E^{\vee}\right),$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^l$ and $\boldsymbol{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in \mathbb{R}^l$ with $0 < \varepsilon \ll 1$.

Theorem 1.6 (Tensor products, see [FFO, Theorem 1.14])**.** *Let (E_1,h_1) and (E_2,h_2) be acceptable vector bundles on a partially punctured polydisk $(\Delta^*)^l \times \Delta^{n-l}$. Then, for any $\boldsymbol{b} \in \mathbb{R}^l$, we have*

$${}_{\boldsymbol{b}}(E_1 \otimes E_2) = \sum_{\boldsymbol{a}_1+\boldsymbol{a}_2 \le \boldsymbol{b}} {}_{\boldsymbol{a}_1}E_1 \otimes {}_{\boldsymbol{a}_2}E_2.$$

Theorem 1.7 (Hom bundles, see [FFO, Proposition 17.1])**.** *Let (E_1,h_1) and (E_2,h_2) be acceptable vector bundles on a partially punctured polydisk $(\Delta^*)^l \times \Delta^{n-l}$. Then, for any $\boldsymbol{a} \in \mathbb{R}^l$, we have*

$${}_{\boldsymbol{a}}\mathrm{Hom}(E_1,E_2) = \left\{ f \in \mathrm{Hom}_{\mathcal{O}_{(\Delta^*)^l\times\Delta^{n-l}}}(E_1,E_2) \;\middle|\; f({}_{\boldsymbol{k}}E_1) \subset {}_{\boldsymbol{a}+\boldsymbol{k}}E_2 \text{ for all } \boldsymbol{k} \in \mathbb{R}^l \right\}.$$

As a special case of Theorem 1.7, we obtain the following statement.

Corollary 1.8 (see [M4, Proposition 21.3.3])**.** *Let (E,h) be an acceptable vector bundle on a partially punctured polydisk $(\Delta^*)^l \times \Delta^{n-l}$. Then ${}^{\diamond}\mathrm{End}(E)$ is canonically isomorphic to the sheaf of endomorphisms f of ${}_{\boldsymbol{a}}E$, for any $\boldsymbol{a} \in \mathbb{R}^l$, such that $f|_{D_i}$ preserves the filtration ${}^iF$ for each $i = 1, \dots, l$.*

As in [FFO], we adopt the following convention throughout this paper.

1.9 (Convention)**.** Let $\mathcal{F}$ be a sheaf on a topological space X. Unless explicitly stated otherwise, we write $f \in \mathcal{F}$ to indicate that f is a local section $f \in \mathcal{F}(U)$ over some open subset $U \subset X$.

In this paper, we do not distinguish between holomorphic vector bundles on a complex manifold X and the corresponding locally free $\mathcal{O}_X$-modules. These are treated as equivalent unless stated otherwise.

This paper is organized as follows. In Section 2, we introduce the notion of acceptable bundles on complex manifolds. In Section 3, we recall basic notions concerning increasing $\mathbb{R}$-indexed filtrations on vector spaces and vector bundles. Section 4 is devoted to a brief review of filtered bundles in the sense of Mochizuki. In Section 5, we recall basic definitions and properties of plurisubharmonic functions for the sake of completeness. Section 6 discusses fundamental properties of acceptable bundles on partially punctured polydisks, and Section 7 is devoted to several preliminary estimates in this setting. In Section 8, we explain a special case of the Ohsawa–Takegoshi L^2 extension theorem. Section 9 reviews results on acceptable bundles over a punctured disk, following [FFO]. In Section 10, we study the behavior of acceptable vector bundles over a punctured disk under pull-back by cyclic coverings.

Sections 11 and 12 form the technical core of this paper. In these sections, we develop the theory of prolongations of acceptable line bundles and vector bundles on $\Delta^* \times \Delta^{n-1}$, which provides the essential ingredients for the proofs of the main results given in the subsequent sections. Finally, in Section 13, we prove one of the main results of this paper, namely Theorem 1.1, and in Section 14 we establish the weak norm estimate stated in Theorem 1.4. In the final section, Section 15, we establish Theorems 1.5, 1.6, and 1.7, together with Corollary 1.8, completing the proofs of the remaining results via a systematic reduction to the curve case.

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2. Acceptable bundles on a complex manifold

Although our main interest lies in acceptable bundles on a partially punctured polydisk, we begin by recalling the general framework.

Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$, and let $D = \sum_{i \in I} D_i$ be a simple normal crossing divisor on X.

Definition 2.1 (Admissible coordinates, [M1, Definition 4.1])**.** Let $P \in X$, and let D_{i_j} $(j = 1, \dots, l)$ be the components of D passing through P. An *admissible coordinate system* around P is a pair $(\mathcal{U}, \varphi)$ satisfying:

- $\mathcal{U}$ is an open neighborhood of P in X;
- φ is a holomorphic isomorphism
$$\varphi : \mathcal{U} \xrightarrow{\sim} \Delta^n := \{(z_1, \dots, z_n) \mid |z_i| < 1\},$$
such that $\varphi(P) = (0, \dots, 0)$ and $\varphi(D_{i_j}) = \{z_j = 0\}$ for each $j = 1, \dots, l$.

Let (E, h) be a holomorphic vector bundle on $X \setminus D$ equipped with a smooth Hermitian metric h. Given a collection of real numbers $\boldsymbol{\alpha} = (\alpha_i)_{i \in I} \in \mathbb{R}^I$, we recall the notion of prolongation.

Definition 2.2 (Prolongation by increasing orders, [M2, Definition 4.2])**.** Let $U \subset X$ be open, and let $s \in \Gamma(U \setminus D, E)$ be a section. We say that the increasing order of s is at most $\boldsymbol{\alpha}$ if the following holds:

- For every $P \in U$, choose an admissible coordinate system $(\mathcal{U}, \varphi)$ around P. Then for every $\varepsilon > 0$ there exists a constant $C > 0$ such that on $\mathcal{U}$,
$$|s|_h \le \frac{C}{\prod_{j=1}^{l} |z_j|^{\alpha_{i_j}+\varepsilon}}.$$

In this case we write $-\operatorname{ord}(s) \le \boldsymbol{\alpha}$.

For $\boldsymbol{\alpha} \in \mathbb{R}^I$, we define an $\mathcal{O}_X$-module ${}_{\boldsymbol{\alpha}}E$ by setting

$$\Gamma(U, {}_{\boldsymbol{\alpha}}E) := \{\, s \in \Gamma(U \setminus (U \cap D), E) \mid -\operatorname{ord}(s) \le \boldsymbol{\alpha} \,\}$$

for any open subset $U \subset X$. The sheaf ${}_{\boldsymbol{\alpha}}E$ is called the *prolongation* of E of increasing order $\boldsymbol{\alpha}$.

Definition 2.3 (Poincaré metric)**.** On

$$(\Delta^*)^l \times \Delta^{n-l} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| < 1 \text{ for all } i,\ z_j \ne 0 \text{ for } j \le l\},$$

the *Poincaré metric* is defined by

$$\omega_P := \sum_{j=1}^{l} \frac{\sqrt{-1}\, dz_j \wedge d\bar{z}_j}{|z_j|^2(-\log|z_j|^2)^2} + \sum_{k=l+1}^{n} \frac{\sqrt{-1}\, dz_k \wedge d\bar{z}_k}{(1-|z_k|^2)^2}.$$

Equivalently,

$$\omega_P = -\sqrt{-1}\, \partial\bar{\partial} \log\left(\prod_{j=1}^{l} (-\log|z_j|^2) \prod_{k=l+1}^{n} (1-|z_k|^2) \right).$$

Let $P \in X$, and choose an admissible coordinate system $(\mathcal{U}, \varphi)$ around P. Via the isomorphism

$$\varphi \colon \mathcal{U} \setminus D \xrightarrow{\sim} (\Delta^*)^l \times \Delta^{n-l},$$

we pull back the Poincaré metric to obtain a Hermitian metric $g_{\mathbf{P}}$ on $\mathcal{U} \setminus D$.

Given the Hermitian metric h on E and the Poincaré metric $g_{\mathbf{P}}$ on $T_{\mathcal{U}\setminus D}$, we equip $\operatorname{Hom}(E,E) \otimes \Omega^{p,q}$ with the induced Hermitian metric $(\cdot,\cdot)_{h,g_{\mathbf{P}}}$ on $\mathcal{U} \setminus D$.

Definition 2.4 (Acceptable bundles, [M1, Definition 4.3])**.** Let (E, h) be a holomorphic vector bundle on $X \setminus D$ equipped with a smooth Hermitian metric h. Let $D_h = D'_h + \bar{\partial}$ denote its *Chern connection.* The curvature form of (E, h) is defined by

$$\sqrt{-1}\, \Theta_h(E) := \sqrt{-1}\, D_h^2,$$

which is a smooth $\operatorname{Hom}(E,E)$-valued $(1,1)$-form on $X \setminus D$.

We say that (E, h) is *acceptable at* P if, for an admissible coordinate system $(\mathcal{U}, \varphi)$ around P, the norm of the curvature $\sqrt{-1}\, \Theta_h(E)$ with respect to $(\cdot,\cdot)_{h,g_{\mathbf{P}}}$ is bounded on $\mathcal{U} \setminus D$.

If (E, h) is acceptable at every point of X, then we simply call it *acceptable.*

3. On Filtrations

In this section, we recall basic notions concerning increasing $\mathbb{R}$-indexed filtrations on vector spaces and vector bundles, following [M1] and [M3].

Definition 3.1 (cf. [M3, Definition 4.1])**.** Let V be a finite-dimensional vector space. An (*increasing*) *filtration* F of V indexed by $\mathbb{R}$ is a family of subspaces

$$\{F_\eta \mid \eta \in \mathbb{R}\}$$

satisfying the following conditions:

- $F_\eta \subset F_{\eta'}$ for $\eta \leq \eta'$;
- $F_\eta = V$ for any sufficiently large η.

When considering a tuple of filtrations, we write

$$\mathbf{F} = ({}^iF \mid i \in I).$$

For each $i \in I$, we denote by ${}^i\mathrm{Gr}^{\mathbf{F}}$ the graded space $\mathrm{Gr}^{{}^iF}$.

For a nonzero vector $v \in V$, we define

$$\deg^F(v) := \min\{\eta \in \mathbb{R} \mid v \in F_\eta\}.$$

A basis $\boldsymbol{v} = \{v_1, \ldots, v_r\}$ of V is said to be *compatible with the filtration* F if there exists a decomposition

$$\boldsymbol{v} = \bigsqcup_{\lambda \in \mathbb{R}} \boldsymbol{v}_\lambda$$

such that, for each $\lambda \in \mathbb{R}$, the subset $\boldsymbol{v}_\lambda$ consists of vectors contained in F_λ and induces a basis of the graded piece $\mathrm{Gr}^F_\lambda V$.

Definition 3.2 (cf. [M3, Definition 4.2])**.** Let

$$\mathbf{F} = ({}^iF \mid i \in I)$$

be a tuple of $\mathbb{R}$-indexed filtrations of V. The tuple $\mathbf{F}$ is said to be *compatible* if there exists a direct sum decomposition

$$V = \bigoplus_{\boldsymbol{\eta} \in \mathbb{R}^I} U_{\boldsymbol{\eta}}$$

such that

$$ {}^IF_{\boldsymbol{\rho}} := \bigcap_{i \in I} {}^iF_{\rho_i} = \bigoplus_{\boldsymbol{\eta} \leq \boldsymbol{\rho}} U_{\boldsymbol{\eta}} \tag{3.1}$$

for all $\boldsymbol{\rho} = (\rho_i)_{i \in I} \in \mathbb{R}^I$. Here, $\boldsymbol{\eta} \leq \boldsymbol{\rho}$ means $\eta_i \leq \rho_i$ for all $i \in I$.

Any decomposition satisfying (3.1) is called a *splitting* of the compatible tuple $\mathbf{F}$.

From now on, we consider filtrations on vector bundles.

Definition 3.3 (cf. [M3, Definition 4.8])**.** Let X be a complex manifold and let V be a vector bundle on X. A filtration F of V indexed by $\mathbb{R}$ is a family of subbundles

$$\{F_\eta \subset V \mid \eta \in \mathbb{R}\}$$

such that $F_\eta \subset F_{\eta'}$ for $\eta \leq \eta'$ and $F_\eta = V$ for $\eta \gg 0$.

Let

$$\mathbf{F} = ({}^iF \mid i \in I)$$

be a tuple of filtrations of V. For a point $P \in X$, the induced tuple of filtrations on the fiber $V|_P$ is denoted by $\mathbf{F}|_P$.

To treat parabolic filtrations, we introduce the following notions.

Definition 3.4 (cf. [M2, Definition 3.12]). Let X be a complex manifold and let V be a vector bundle on X. Let

$$Y = \sum_{i \in I} Y_i$$

be a simple normal crossing divisor on X. For each $i \in I$, let ${}^{i}F$ be a filtration of $V|_{Y_i}$ in the sense of Definition 3.3. The tuple of filtrations

$$\mathbf{F} = \big({}^{i}F \mid i \in I\big)$$

is said to be *compatible* if, for any subset $J \subset I$, there exists, locally on

$$Y_J := \bigcap_{j \in J} Y_j,$$

a direct sum decomposition

$$V|_{Y_J} = \bigoplus_{\boldsymbol{\eta} \in \mathbb{R}^J} U_{\boldsymbol{\eta}}$$

such that

$${}^{J}F_{\boldsymbol{\rho}} := \bigcap_{j \in J} {}^{j}F_{\rho_j}\big|_{Y_J} = \bigoplus_{\boldsymbol{\eta} \leq \boldsymbol{\rho}} U_{\boldsymbol{\eta}}$$

holds for all $\boldsymbol{\rho} \in \mathbb{R}^J$.

Definition 3.5 (cf. [M1, Definition 2.16]). Let X be a complex manifold, V a vector bundle on X, and $Y \subset X$ a complex submanifold. Let F be a filtration of $V|_Y$ in the sense of Definition 3.3.

A smooth section f of V is said to be *compatible with the filtration* F if the value

$$\deg^{F|_P}(f(P))$$

is independent of the point $P \in Y$. In this case, we define

$$\deg^{F}(f) := \deg^{F|_P}(f(P))$$

for any $P \in Y$.

Definition 3.6 (cf. [M1, Definition 2.17]). Let $\boldsymbol{v} = \{v_1, \ldots, v_r\}$ be a smooth frame of V. It is said to be *compatible with the filtration F of $V|_Y$* if the following conditions are satisfied:

(1) Each v_i is compatible with F in the sense of Definition 3.5;
(2) For any point $P \in Y$, the frame $\boldsymbol{v}|_P$ is compatible with the filtration $F|_P$ in the sense of Definition 3.1.

Let

$$Y = \sum_{i \in I} Y_i$$

be a simple normal crossing divisor on X. For each $i \in I$, let ${}^{i}F$ be a filtration of $V|_{Y_i}$. The smooth frame $\boldsymbol{v} = \{v_1, \ldots, v_r\}$ of V is said to be *compatible with the tuple of filtrations*

$$\mathbf{F} = \big({}^{i}F \mid i \in I\big)$$

if $\boldsymbol{v}$ is compatible with ${}^{i}F$ for every $i \in I$.

4. Filtered bundles

In this section, we briefly review the notion of filtered bundles in the sense of Mochizuki, following [M5, 2.3 Filtered bundles].

Definition 4.1 (Filtered bundles in the local case, cf. [M5, 2.3.1])**.** Let U be an open neighborhood of $(0,\dots,0)$ in $\mathbb{C}^n$. We set

$$D_{U,i} := U \cap \{z_i = 0\}, \qquad D_U := \bigcup_{i=1}^{l} D_{U,i},$$

where $1 \leq l \leq n$. Let $\mathcal{V}$ be a locally free $\mathcal{O}_U(*D_U)$-module.

A *filtered bundle* $\mathcal{P}_*\mathcal{V}$ of $\mathcal{V}$ is a family of locally free $\mathcal{O}_U$-submodules $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ indexed by $\boldsymbol{a} \in \mathbb{R}^l$ satisfying the following conditions:

(1) If $\boldsymbol{a} \leq \boldsymbol{b}$ (i.e., $a_i \leq b_i$ for all $i = 1,\dots,l$), then

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{V} \subset \mathcal{P}_{\boldsymbol{b}}\mathcal{V}.$$

(2) There exists a frame $\boldsymbol{v} = \{v_1,\dots,v_r\}$ of $\mathcal{V}$ and vectors $\boldsymbol{a}(v_j) \in \mathbb{R}^l$ $(j = 1,\dots,r)$ such that, for any $\boldsymbol{b} \in \mathbb{R}^l$, we have

$$\mathcal{P}_{\boldsymbol{b}}\mathcal{V} = \bigoplus_{j=1}^{r} \mathcal{O}_U\left(\sum_{i=1}^{l} \lfloor b_i - a_i(v_j)\rfloor D_{U,i}\right) \cdot v_j. \tag{4.1}$$

Let X be a complex manifold with a simple normal crossing divisor D. Let

$$D = \bigcup_{i\in\Lambda} D_i$$

be the irreducible decomposition of D. For any point $P \in D$, a holomorphic coordinate neighborhood $(X_P, z_1,\dots,z_n)$ around P is called *admissible* (see Definition 2.1) if

$$D_P := D \cap X_P = \bigcup_{i=1}^{l(P)} \{z_i = 0\}.$$

For such an admissible coordinate neighborhood, there exists a uniquely determined map

$$\rho_P \colon \{1,\dots,l(P)\} \longrightarrow \Lambda$$

such that

$$D_{\rho_P(i)} \cap X_P = \{z_i = 0\}.$$

We define a map

$$\kappa_P \colon \mathbb{R}^\Lambda \longrightarrow \mathbb{R}^{l(P)}$$

by

$$\kappa_P(\boldsymbol{a}) := \big(a_{\rho_P(1)},\dots,a_{\rho_P(l(P))}\big).$$

Definition 4.2 (Filtered bundles, cf. [M5, 2.3.3])**.** Let $\mathcal{V}$ be a locally free $\mathcal{O}_X(*D)$-module. A *filtered bundle*

$$\mathcal{P}_*\mathcal{V} = \big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V} \mid \boldsymbol{a} \in \mathbb{R}^\Lambda\big)$$

of $\mathcal{V}$ is a family of locally free $\mathcal{O}_X$-submodules $\mathcal{P}_{\boldsymbol{a}}\mathcal{V} \subset \mathcal{V}$ satisfying the following conditions:

(1) For any $P \in D$, take an admissible coordinate neighborhood $(X_P, z_1,\dots,z_n)$ around P. Then, for any $\boldsymbol{a} \in \mathbb{R}^\Lambda$, the restriction $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{X_P}$ is determined only by $\kappa_P(\boldsymbol{a})$. We denote it by

$$\mathcal{P}^{(P)}_{\kappa_P(\boldsymbol{a})}\big(\mathcal{V}|_{X_P}\big).$$

(2) The family

$$\Big(\mathcal{P}^{(P)}_{\boldsymbol{b}}\big(\mathcal{V}|_{X_P}\big)\ \Big|\ \boldsymbol{b}\in\mathbb{R}^{l(P)}\Big)$$

is a filtered bundle over $\mathcal{V}|_{X_P}$ in the sense of Definition 4.1.

For any subset $I\subset\Lambda$, let $\boldsymbol{\delta}_I\in\mathbb{R}^\Lambda$ be the element whose j-th component is 1 for $j\in I$ and 0 for $j\in\Lambda\setminus I$. We set

$$D_I:=\bigcap_{i\in I}D_i,\qquad \partial D_I:=D_I\cap\Bigg(\bigcup_{j\in\Lambda\setminus I}D_j\Bigg).$$

Let $\mathcal{P}_*\mathcal{V}$ be a filtered bundle on (X,D). Fix $i\in\Lambda$ and $\boldsymbol{a}\in\mathbb{R}^\Lambda$. For any b satisfying $a_i-1\leq b\leq a_i$, we set

$$\boldsymbol{a}(b,i):=\boldsymbol{a}+(b-a_i)\boldsymbol{\delta}_i.$$

We define

$${}^iF_b\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_i}\big):=\mathcal{P}_{\boldsymbol{a}(b,i)}\mathcal{V}\big/\mathcal{P}_{\boldsymbol{a}(a_i-1,i)}\mathcal{V}.$$

It is naturally a locally free $\mathcal{O}_{D_i}$-module and can be regarded as a subbundle of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_i}$. In this way, we obtain a filtration ${}^iF$ of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_i}$ indexed by the interval $(a_i-1,a_i]$. If there is no risk of confusion, we simply write F.

For $I\subset\Lambda$ and $i\in I$, the filtrations ${}^iF$ induce a filtration on $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_I}$. Let $\boldsymbol{a}_I\in\mathbb{R}^I$ be the image of $\boldsymbol{a}$ under the natural projection $\mathbb{R}^\Lambda\to\mathbb{R}^I$, and set

$$(\boldsymbol{a}_I-\boldsymbol{\delta}_I,\boldsymbol{a}_I]:=\prod_{i\in I}(a_i-1,a_i].$$

For any $\boldsymbol{b}\in(\boldsymbol{a}_I-\boldsymbol{\delta}_I,\boldsymbol{a}_I]$, we define

$${}^IF_{\boldsymbol{b}}\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_I}\big):=\bigcap_{i\in I}{}^iF_{b_i}\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_I}\big).$$

By Definition 4.1, the following compatibility holds.

- Let P be a point of D_I. There exists an open neighborhood X_P of P in X and a (non-canonical) decomposition

$$\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_I\cap X_P}=\bigoplus_{\boldsymbol{b}\in(\boldsymbol{a}_I-\boldsymbol{\delta}_I,\boldsymbol{a}_I]}\mathcal{G}_{P,\boldsymbol{b}}$$

  such that, for any $\boldsymbol{c}\in(\boldsymbol{a}_I-\boldsymbol{\delta}_I,\boldsymbol{a}_I]$, we have

$$ {}^IF_{\boldsymbol{c}}\big(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_I\cap X_P}\big)=\bigoplus_{\boldsymbol{b}\leq\boldsymbol{c}}\mathcal{G}_{P,\boldsymbol{b}}. \tag{4.2}$$

  Indeed, there exists a frame $\boldsymbol{v}=\{v_1,\dots,v_r\}$ of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}$ around P with tuples $\boldsymbol{a}(v_j)\in\mathbb{R}^{l(P)}$ of real numbers satisfying (4.1), where $\boldsymbol{b}$ is replaced by $\boldsymbol{a}$. There exists a bijection

$$\kappa\colon I\simeq\{1,\dots,l(P)\}$$

  determined by $D_i\cap X_P=\{z_{\kappa(i)}=0\}$, by which we identify I with $\{1,\dots,l(P)\}$. Let $\mathcal{G}_{P,\boldsymbol{b}}$ be the subbundle of $\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_I\cap X_P}$ generated by $v_j|_{D_I\cap X_P}$ with $\boldsymbol{a}(v_j)=\boldsymbol{b}$. Then (4.2) follows.

For any $\boldsymbol{c} \in (\boldsymbol{a}_I - \boldsymbol{\delta}_I, \boldsymbol{a}_I]$, we define a locally free $\mathcal{O}_{D_I}$-module

$$ {}^I\mathrm{Gr}^F_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}) := \frac{{}^IF_{\boldsymbol{c}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_I})}{\sum_{\boldsymbol{b} \lneq \boldsymbol{c}} {}^IF_{\boldsymbol{b}}(\mathcal{P}_{\boldsymbol{a}}\mathcal{V}|_{D_I})}, $$

where $\boldsymbol{b} = (b_i) \lneq \boldsymbol{c} = (c_i)$ means that $b_i \leq c_i$ for all i and $\boldsymbol{b} \neq \boldsymbol{c}$.

5. Plurisubharmonic functions

For the sake of completeness, we recall the definition of plurisubharmonic functions, which play an important role throughout this paper.

Definition 5.1 (Plurisubharmonic functions)**.** Let Ω be an open subset of $\mathbb{C}^n$. A function $u\colon \Omega \to [-\infty, +\infty)$ is said to be *plurisubharmonic* (*psh*, for short) if

(1) u is upper semicontinuous, and
(2) for every complex line $L \subset \mathbb{C}^n$, the restriction $u|_{\Omega\cap L}$ is subharmonic on $\Omega \cap L$; that is, for all $a \in \Omega$ and $\xi \in \mathbb{C}^n$ with $|\xi| < d(a, \Omega^c)$,

$$ u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a + \xi e^{\sqrt{-1}\theta})\, d\theta. $$

For the basic properties of plurisubharmonic (psh, for short) functions, see, for example, [Dem1, 1.B. Plurisubharmonic Functions] and [NO, 3.3 Plurisubharmonic Functions]. In this paper, the notion of the Lelong number plays a crucial role, so we recall it here for the reader's convenience. For further details, see, for example, [Dem1, 2.B. Lelong Numbers].

Definition 5.2 (Lelong numbers)**.** Let u be a plurisubharmonic (psh) function on an open subset $\Omega \subset \mathbb{C}^n$. Then $\sqrt{-1}\,\partial\bar{\partial}u$ defines a closed positive $(1,1)$-current on Ω, hence determines a positive Radon measure. The *Lelong number* $\nu(u, x)$ of u at a point $x \in \Omega$ is defined by

$$ \nu(u, x) := \liminf_{z\to x} \frac{u(z)}{\log|z - x|} \in \mathbb{R}_{\geq 0}. $$

It is well known that

$$ \nu(u, x) = \lim_{r\to+0} \frac{1}{r^{2(n-1)}} \int_{B(x,r)} \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}u \wedge \left(\frac{\sqrt{-1}}{2\pi}\sum_{i=1}^n dz_i \wedge d\bar{z}_i\right)^{n-1} \tag{5.1} $$

holds, where $B(x, r) = \{z \in \mathbb{C}^n \mid |z - x| < r\}$.

By Siu's theorem (see, for example, [Dem1, (13.3) Corollary]), for every $c > 0$, the *upper level set of the Lelong number*

$$ E_c(u) := \{z \in \Omega \mid \nu(u, z) \geq c\} $$

is a closed analytic subset of Ω.

In Definition 5.2, the right-hand side of the equality (5.1) is the original definition of the Lelong number (see, for example, [Dem2, Chapter III, (5.7)]). Although the equality in (5.1) is not at all obvious, it is a well-known fact (see, for example, [Dem2, Chapter III, (6.9), Example]). A relatively accessible proof of Siu's theorem appearing in Definition 5.2 can be found in [Dem1, 13.A. Approximation of Plurisubharmonic Functions via Bergman kernels]. It is a particularly elegant application of the Ohsawa–Takegoshi L^2-extension theorem.

Lemma 5.3. *Let u be a plurisubharmonic function on an open subset $\Omega \subset \mathbb{C}^n$, and let $x \in \Omega$ be such that $\overline{B}(x, R_0) = \{z \in \mathbb{C}^n \mid |z - x| \leq R_0\} \subset \Omega$. Then the function*

$$\log r \longmapsto \sup_{|z-x|=r} u(z)$$

is convex and nondecreasing for $0 \leq r \leq R_0$.

Proof of Lemma 5.3. Since

$$\sup_{|z-x|=r} u(z) = \max_{z \in \overline{B}(x,r)} u(z),$$

it is clear that $\sup_{|z-x|=r} u(z)$ is a nondecreasing function of r (see, for example, [NO, (3.3.2) Theorem and (3.3.27) Remark]).

For each $\zeta \in \mathbb{C}^n$ with $|\zeta| = 1$, the function

$$\mathbb{C} \ni w \longmapsto u(x + \zeta w)$$

is subharmonic by definition. Hence

$$\mathbb{C} \ni w \longmapsto u(x + \zeta e^w)$$

is also subharmonic (see, for example, [Dem1, (1.8) Proposition] or [NO, (3.3.19) Theorem and (3.3.38) Remark]). It is easy to check that

$$\mathbb{C} \ni w \longmapsto \sup_{|\zeta|=1} u(x + \zeta e^w)$$

is upper semicontinuous and locally bounded from above. Therefore, by [NO, (3.3.3) Lemma (ii)] or [Dem2, Chapter I, (5.7) Theorem], it is also subharmonic. Since this function depends only on $\operatorname{Re} w$, it follows that $\sup_{|\zeta|=1} u(x + \zeta e^w)$ is convex as a function of $\operatorname{Re} w$. This proves the lemma. □

The following lemma is an elementary property of convex functions and is included here for completeness.

Lemma 5.4. *Let $f\colon (-\infty, b] \to \mathbb{R}$ be a convex function such that*

$$\lim_{x \to -\infty} \frac{f(x)}{x} = \nu \in \mathbb{R}.$$

Then

$$f(x) \leq \nu(x - b) + f(b)$$

for all $x \in (-\infty, b]$.

Proof of Lemma 5.4. Since f is convex, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2)$$

for all $x_1, x_2 \in (-\infty, b]$ and $\lambda \in [0, 1]$. This implies that the difference quotient

$$\frac{f(b) - f(x)}{b - x}$$

is nondecreasing in x on $(-\infty, b)$. In particular, for any $x_1, x_2 \in (-\infty, b)$ with $x_1 \leq x_2$, we have

$$\frac{f(b) - f(x_1)}{b - x_1} \leq \frac{f(b) - f(x_2)}{b - x_2}.$$

By the assumption

$$\lim_{x \to -\infty} \frac{f(x)}{x} = \nu,$$

we obtain

$$\lim_{x\to-\infty}\frac{f(b)-f(x)}{b-x}=\lim_{x\to-\infty}\frac{f(x)}{x}=\nu.$$

Hence, for every $x\in(-\infty,b)$,

$$\frac{f(b)-f(x)}{b-x}\geq\nu.$$

Multiplying both sides by $b-x>0$, we get

$$f(x)\leq\nu(x-b)+f(b),$$

which proves the lemma. □

By Lemma 5.3 and Lemma 5.4, we obtain the following corollary.

Corollary 5.5. *Let u be a plurisubharmonic function on an open subset $\Omega\subset\mathbb{C}^n$, and let $x\in\Omega$ satisfy $\overline{B}(x,R_0)\subset\Omega$. Then*

$$u(z)\leq\nu(u,x)\log\frac{|z-x|}{R_0}+\max_{w\in\overline{B}(x,R_0)}u(w)$$

for all $z\in\overline{B}(x,R_0)$.

Proof of Corollary 5.5. Note that

$$\nu(u,x)=\lim_{r\to+0}\frac{\sup_{|z-x|=r}u(z)}{\log r}.$$

By Lemma 5.3, the function $\sup_{|z-x|=r}u(z)$ is convex and nondecreasing in $\log r$. The desired inequality follows immediately from this fact by Lemma 5.4. □

We will need the following easy lemma in later sections.

Lemma 5.6. *On the polydisk Δ^n, consider*

$$\chi(0,N)=-N\left(\log(-\log|z_1|^2)+\sum_{k=2}^{n}\log(1-|z_k|^2)\right),$$

where $N>0$. Then both $\chi(0,N)$ and $\log|z_1|^2$ are plurisubharmonic functions.

For any $Q\in\Delta^{n-1}$, let $Q'=(0,Q)$. Then

$$\nu(\chi(0,N),Q')=0\quad\textit{and}\quad\nu(\log|z_1|^2,Q')=2.$$

Proof of Lemma 5.6. Note that $\chi(0,N)$ is smooth on $\Delta^*\times\Delta^{n-1}$. A direct computation shows that

$$\sqrt{-1}\,\partial\overline{\partial}\chi(0,N)=N\omega_P.$$

Thus $\chi(0,N)$ is plurisubharmonic on $\Delta^*\times\Delta^{n-1}$. We define $\chi(0,N)\equiv-\infty$ on $\{0\}\times\Delta^{n-1}$; then $\chi(0,N)$ extends to a plurisubharmonic function on Δ^n. For details, see [NO, (3.3.41) Theorem] and [Dem2, Chapter I, (5.24) Theorem].

It is easy to see that

$$0\leq\nu(\chi(0,N),Q')\leq\liminf_{z_1\to0}\frac{-N\log(-\log|z_1|^2)}{\log|z_1|}=0,$$

hence $\nu(\chi(0,N),Q')=0$.

Since z_1 is holomorphic on Δ^n, $\log|z_1|^2=2\log|z_1|$ is plurisubharmonic. It is well known that

$$\nu(\log|z_1|^2,Q')=2\,\mathrm{ord}_{Q'}(z_1)=2$$

(see, for example, [Dem1, (2.8) Theorem (b)]). This completes the proof. □

To make use of Siu's theorem in Definition 5.2, we prepare the following elementary lemma.

Lemma 5.7. *Let V be a connected complex manifold and let f be a real-valued function on V. Assume that for every $a, b \in \mathbb{R}$, the sets*

$$V_{\geq a} := \{x \in V \mid f(x) \geq a\}, \qquad V_{\leq b} := \{x \in V \mid f(x) \leq b\}$$

are closed analytic subsets of V. Then f is constant on V.

Proof of Lemma 5.7. Take $c \in f(V) \subset \mathbb{R}$. For every $\varepsilon > 0$, set

$$V_c^\varepsilon := V_{\geq c-\varepsilon} \cap V_{\leq c+\varepsilon}.$$

Then

$$V = V_{\geq c+\varepsilon} \cup V_c^\varepsilon \cup V_{\leq c-\varepsilon}.$$

Since $c \in f(V)$, both $V_{\geq c+\varepsilon}$ and $V_{\leq c-\varepsilon}$ are closed analytic subsets of V with $V_{\geq c+\varepsilon} \subsetneq V$ and $V_{\leq c-\varepsilon} \subsetneq V$, hence $V_c^\varepsilon = V$. Thus

$$V = \bigcap_{\varepsilon>0} V_c^\varepsilon = \{x \in V \mid f(x) = c\},$$

which means that f is constant on V. □

6. Basic properties of acceptable bundles on $(\Delta^*)^l \times \Delta^{n-l}$

In this section, we discuss basic properties of acceptable bundles on a polydisk punctured in the first l coordinates. A detailed description of acceptable bundles on a partially punctured polydisk is indispensable for the study of acceptable bundles on complex manifolds (see Definition 2.4). We employ the following definition of acceptable vector bundles on a partially punctured polydisk throughout the present paper.

Definition 6.1 (Acceptable bundles on a partially punctured polydisk)**.** Let E be a holomorphic vector bundle on $(\Delta^*)^l \times \Delta^{n-l}$, equipped with a smooth Hermitian metric h. We say that (E, h) is *acceptable* if its curvature $\sqrt{-1}\Theta_h(E)$, viewed as a smooth $\mathrm{Hom}(E, E)$-valued $(1,1)$-form on $(\Delta^*)^l \times \Delta^{n-l}$, is bounded with respect to the Hermitian metric $(\cdot,\cdot)_{h,\omega_P}$, which is the natural Hermitian metric on $\mathrm{Hom}(E, E) \otimes \Omega^{1,1}$ induced by the metric h on E and the Poincaré metric ω_P. In other words, there exists a constant $C > 0$ such that

$$|\sqrt{-1}\Theta_h(E)|_{h,\omega_P} \leq C \qquad \text{on } (\Delta^*)^l \times \Delta^{n-l}.$$

The following lemma is immediate.

Lemma 6.2. *Let (E, h) be an acceptable vector bundle on $(\Delta^*)^l \times \Delta^{n-l}$. Then the dual bundle $(E^\vee, h^\vee)$ and the determinant line bundle $(\det E, \det h)$ are also acceptable.*

Let (E_1, h_1) and (E_2, h_2) be acceptable vector bundles on $(\Delta^*)^l \times \Delta^{n-l}$. Then the tensor product $(E_1 \otimes E_2, h_1 \otimes h_2)$ and the Hom bundle $(\mathrm{Hom}(E_1, E_2), h_1^\vee \otimes h_2)$ are acceptable.

Proof of Lemma 6.2. The same argument as in [FFO, Lemma 2.2] applies verbatim in our setting. □

We now recall various notions of positivity for vector bundles on a complex manifold. For details, see for example [Dem1, Chapter 10] and [Dem2, Chapter VII, §6].

Definition 6.3. Let E be a holomorphic vector bundle on a complex manifold X, equipped with a smooth Hermitian metric h. Let D_h be the Chern connection of (E, h), and denote the curvature by $\Theta_h(E) = D_h^2$.

Fix $x \in X$, and choose a frame $e_1, \ldots, e_r$ of E at x with dual frame $e^1, \ldots, e^r$. Let $(z_1, \ldots, z_n)$ be local holomorphic coordinates centered at x. Then we may write

$$\sqrt{-1}\Theta_h(E) = \sum_{1\le j,k\le n} \sum_{1\le\alpha,\beta\le r} R^{\beta}_{j\bar{k}\alpha}\, dz_j \wedge d\bar{z}_k \otimes e^{\alpha} \otimes e_{\beta}.$$

We put

$$R_{j\bar{k}\alpha\overline{\beta}} := h_{\gamma\overline{\beta}}\, R^{\gamma}_{j\bar{k}\alpha}, \qquad h_{\gamma\overline{\beta}} := h(e_{\gamma}, e_{\beta}).$$

We say that (E, h) is *Nakano positive* (resp. *Nakano semipositive*) at x if

$$\sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\overline{\beta}}\, u^{j\alpha}\, \overline{u^{k\beta}} > 0 \quad (\text{resp.} \ge 0)$$

for any nonzero vector

$$u = \sum_{j,\alpha} u^{j\alpha} \left(\frac{\partial}{\partial z_j}\right) \otimes e_{\alpha} \in (T^{1,0}_X \otimes E)_x.$$

We say that (E, h) is *Griffiths positive* (resp. *Griffiths semipositive*) at x if

$$\sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\overline{\beta}}\, \xi^j \zeta^{\alpha}\, \overline{\xi^k \zeta^{\beta}} > 0 \quad (\text{resp.} \ge 0)$$

for all nonzero

$$\xi = \sum_j \xi^j \left(\frac{\partial}{\partial z_j}\right) \in T^{1,0}_{X,x}, \qquad \zeta = \sum_{\alpha} \zeta^{\alpha} e_{\alpha} \in E_x.$$

If (E, h) is Nakano positive (resp. Nakano semipositive, Griffiths positive, or Griffiths semipositive) at every point $x \in X$, then we simply say that (E, h) is Nakano positive (resp. Nakano semipositive, Griffiths positive, or Griffiths semipositive).

The notions of Nakano (semi)negativity and Griffiths (semi)negativity are defined similarly by reversing the inequalities.

Remark 6.4. By definition, Nakano (semi)positivity (resp. Nakano (semi)negativity) implies Griffiths (semi)positivity (resp. Griffiths (semi)negativity). The converse holds when $\dim X = 1$ or $\operatorname{rank} E = 1$.

Lemma 6.5 is a key estimate.

Lemma 6.5. *Let (E, h) be a holomorphic vector bundle on $X^* = (\Delta^*)^l \times \Delta^{n-l}$. If*

$$|\sqrt{-1}\Theta_h(E)|_{h,\omega_P} \le C,$$

that is, if (E, h) is acceptable on $(\Delta^*)^l \times \Delta^{n-l}$, then

$$-C\,\omega_P \otimes \mathrm{Id}_E \ \le_{\mathrm{Nak}}\ \sqrt{-1}\Theta_h(E)\ \le_{\mathrm{Nak}}\ C\,\omega_P \otimes \mathrm{Id}_E$$

on $(\Delta^*)^l \times \Delta^{n-l}$. Here $A \le_{\mathrm{Nak}} B$ means that $B - A$ defines a Nakano semipositive Hermitian form on $T^{1,0}_{X^*} \otimes E$ with respect to h.

Proof of Lemma 6.5. Fix $x \in (\Delta^*)^l \times \Delta^{n-l}$. Choose local coordinates $(w_1, \ldots, w_n)$ centered at x such that

$$\omega_P = \sqrt{-1} \sum_{i=1}^{n} dw_i \wedge d\overline{w}_i.$$

Choose a holomorphic frame $e_1, \dots, e_r$ of E which is orthonormal at x. Then

$$\sqrt{-1}\Theta_h(E) = \sum_{j,k,\alpha,\beta} R^{\beta}_{j\bar{k}\alpha}\, dw_j \wedge d\overline{w}_k \otimes e^{\alpha} \otimes e_{\beta},$$

and thus $R_{j\bar{k}\alpha\overline{\beta}}(x) = R^{\beta}_{j\bar{k}\alpha}(x)$.

Therefore,

$$\sum_{j,k,\alpha,\beta} |R_{j\bar{k}\alpha\overline{\beta}}(x)|^2 = |\sqrt{-1}\Theta_h(E)(x)|^2_{h,g_{\mathbf{P}}} \leq C^2.$$

For any

$$u = \sum_{j,\alpha} u^{j\alpha} \left(\frac{\partial}{\partial w_j}\right) \otimes e_{\alpha} \in (T^{1,0}_{X^*} \otimes E)_x,$$

we have

$$\begin{aligned}
\left|\sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\overline{\beta}}(x) u^{j\alpha}\overline{u^{k\beta}}\right|^2 &\leq \left(\sum_{k,\beta} \left|\sum_{j,\alpha} R_{j\bar{k}\alpha\overline{\beta}}(x) u^{j\alpha}\right|^2\right) \left(\sum_{k,\beta} |\overline{u^{k\beta}}|^2\right) \\
&\leq \left(\sum_{k,\beta}\left(\sum_{j,\alpha} \left|R_{j\bar{k}\alpha\overline{\beta}}(x)\right|^2\right)\left(\sum_{j,\alpha} |u^{j\alpha}|^2\right)\right)\left(\sum_{k,\beta} |\overline{u^{k\beta}}|^2\right) \\
&= |u|^4_{h,\omega_P} \cdot \sum_{j,k,\alpha,\beta} \left|R_{j\bar{k}\alpha\overline{\beta}}(x)\right|^2 \\
&\leq |u|^4_{h,\omega_P} \cdot C^2
\end{aligned}$$

by using the Cauchy–Schwarz inequality twice. This gives the desired inequality

$$-C|u|^2_{h,\omega_P} \leq \sum_{j,k,\alpha,\beta} R_{j\bar{k}\alpha\overline{\beta}}(x) u^{j\alpha}\overline{u^{k\beta}} \leq C|u|^2_{h,\omega_P}.$$

and completes the proof. □

Definition 6.6 (Twisted metric)**.** Let E be a holomorphic vector bundle on $(\Delta^*)^l \times \Delta^{n-l}$ with Hermitian metric h. For $\boldsymbol{a} = (a_1, \dots, a_l) \in \mathbb{R}^l$ and $N \in \mathbb{R}$, set

$$\chi(\boldsymbol{a}, N) := -\sum_{j=1}^{l} a_j \log|z_j|^2 - N\left(\sum_{j=1}^{l} \log(-\log|z_j|^2) + \sum_{k=l+1}^{n} \log(1-|z_k|^2)\right).$$

Define the twisted metric

$$h(\boldsymbol{a}, N) := h\, e^{-\chi(\boldsymbol{a},N)} = h \cdot \prod_{j=1}^{l} |z_j|^{2a_j}(-\log|z_j|^2)^N \prod_{k=l+1}^{n} (1-|z_k|^2)^N.$$

Then

$$\sqrt{-1}\Theta_{h(\boldsymbol{a},N)}(E) = \sqrt{-1}\Theta_h(E) + N\,\omega_P \otimes \mathrm{Id}_E\,.$$

By Lemma 6.5 and Definition 6.6, we have:

Corollary 6.7. *Let (E, h) be acceptable on $X^* = (\Delta^*)^l \times \Delta^{n-l}$. Then there exists N_0 such that for all $\boldsymbol{a} \in \mathbb{R}^l$ and all $N \geq N_0$,*

$$(E, h(\boldsymbol{a}, N)) \text{ is Nakano semipositive}, \qquad (E, h(\boldsymbol{a}, -N)) \text{ is Nakano seminegative.}$$

In particular, $h(\boldsymbol{a}, -N)$ is Griffiths seminegative for $N \geq N_0$. If $N > N_0$ then

$$(E, h(\boldsymbol{a}, N)) \text{ is Nakano positive}, \qquad (E, h(\boldsymbol{a}, -N)) \text{ is Nakano negative.}$$

Proof of Corollary 6.7. Since $h(\boldsymbol{a}, N) = h\, e^{-\chi(\boldsymbol{a},N)}$, we have

$$\sqrt{-1}\Theta_{h(\boldsymbol{a},N)}(E) = \sqrt{-1}\Theta_h(E) + N\,\omega_P \otimes \mathrm{Id}_E\,.$$

The claim follows immediately from Lemma 6.5. □

Lemma 6.8. *Let (E, h) be acceptable on $(\Delta^*)^l \times \Delta^{n-l}$. For any $\boldsymbol{b} \in \mathbb{R}^l$, set*

$$(E^\dagger, h^\dagger) := (E, h(\boldsymbol{b}, 0)).$$

Then

$$\sqrt{-1}\Theta_h(E) = \sqrt{-1}\Theta_{h^\dagger}(E^\dagger), \qquad {}_{\boldsymbol{a}}E = {}_{\boldsymbol{a}-\boldsymbol{b}}E^\dagger.$$

Proof of Lemma 6.8. Since $\partial\overline{\partial}\chi(\boldsymbol{b}, 0) = 0$ on $(\Delta^*)^l \times \Delta^{n-l}$, we have

$$\begin{aligned}\sqrt{-1}\Theta_{h^\dagger}(E^\dagger) &= \sqrt{-1}\Theta_h(E) + \sqrt{-1}\partial\overline{\partial}\chi(\boldsymbol{b}, 0) \otimes \mathrm{Id}_E \\ &= \sqrt{-1}\Theta_h(E)\end{aligned}$$

The identity of the prolongations is immediate from the definition. □

The following lemma is very well known and it plays a crucial role in the theory of acceptable bundles through Corollary 6.7.

Lemma 6.9. *Let (E, h) be a vector bundle on a complex manifold X with $\sqrt{-1}\Theta_h(E)$ Griffiths seminegative. Then for every holomorphic section s of E, the function $\log|s|_h^2$ is plurisubharmonic on X.*

Proof of Lemma 6.9. Let $\{\bullet, \bullet\}_h$ denote the sesquilinear pairing

$$C^\infty(X, \wedge^p T_X^\vee \otimes E) \times C^\infty(X, \wedge^q T_X^\vee \otimes E) \to C^\infty(X, \wedge^{p+q} T_X^\vee \otimes \mathbb{C})$$

induced by the Hermitian metric h.

More precisely, let Ω be an open subset of X, and assume that $E|_\Omega$ is trivialized as $\Omega \times \mathbb{C}^r$ by a C^∞ frame $\{e_\lambda\}$. Then for any sections

$$u = \sum_\lambda u_\lambda \otimes e_\lambda, \quad v = \sum_\mu v_\mu \otimes e_\mu,$$

we have

$$\{u, v\}_h = \sum_{\lambda,\mu} u_\lambda \wedge \overline{v}_\mu \cdot h(e_\lambda, e_\mu).$$

Let $D_h = D'_h + \overline{\partial}$ denote the Chern connection associated with (E, h). We may assume that $s \not\equiv 0$. Outside the zero set of s, we have

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\log|s|_h^2 &= \sqrt{-1}\frac{\{D'_h s, D'_h s\}_h}{|s|_h^2} - \sqrt{-1}\frac{\{D'_h s, s\}_h \wedge \{s, D'_h s\}_h}{|s|_h^4} - \frac{\{\sqrt{-1}\Theta_h(E)s, s\}_h}{|s|_h^2} \\ &\geq -\frac{\{\sqrt{-1}\Theta_h(E)s, s\}_h}{|s|_h^2} \geq 0.\end{aligned}$$

We note that the first inequality is due to Cauchy–Schwarz inequality and the second one holds since $\sqrt{-1}\Theta_h(E)$ is Griffiths seminegative. Thus we have

$$\sqrt{-1}\partial\overline{\partial}\log|s|_h^2 \geq 0$$

outside the zero set of s. That is, $\log|s|_h^2$ is subharmonic on $X \setminus \{s = 0\}$.

Moreover, since $\log|s|_h^2$ is locally bounded from above, it extends to a subharmonic function on all of X (see, for example, [NO, (3.3.41) Theorem] or [Dem2, Chapter I, (5.24) Theorem]).

This completes the proof of Lemma 6.9. □

We end this section with a very important remark.

Remark 6.10. In [M4, 21.2. Twist of the metric of an acceptable bundle], Mochizuki sets $\tau(\boldsymbol{a}, N) := \chi(\boldsymbol{a}, -N)$ and defines $h_{\boldsymbol{a},N} := h\, e^{-\tau(\boldsymbol{a},N)}$. Thus $h(\boldsymbol{a}, N) = h_{\boldsymbol{a},-N}$ in our notation. If N is sufficiently large, Corollary 6.7 shows that $(E, h(\boldsymbol{a}, N))$ is Nakano positive and $(E, h(\boldsymbol{a}, -N))$ is Griffiths negative. In Mochizuki's notation, the roles of N and $-N$ are reversed. We find our convention more natural, and therefore adopt $\chi(\boldsymbol{a}, N)$ in this paper.

7. Some preliminary estimates

In this section, we collect several preliminary estimates for acceptable bundles on a partially punctured polydisk. Although all the results in this section can be found in [M4, 21.2], we present them here in detail, since we adopt a different convention (see Remark 6.10).

We set

$$X := \Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| < 1 \text{ for all } i\}$$

and

$$D := \sum_{i=1}^{l} D_i,$$

where $D_i := \{z_i = 0\}$ for each i. We put $X^* := X \setminus D$. Then clearly

$$X^* = (\Delta^*)^l \times \Delta^{n-l}.$$

For $i = 1, \dots, n$, let

$$\pi_i \colon X^* \to D_i$$

denote the natural projection. We set

$$D_i^\circ := D_i \setminus \bigcup_{\substack{j \neq i \\ j \leq l}} D_j.$$

For any point $P \in D_i^\circ$, we see that

$$\pi_i^{-1}(P) \simeq \begin{cases} \Delta^*, & 1 \leq i \leq l, \\ \Delta, & l+1 \leq i \leq n. \end{cases}$$

For $0 < R \leq 1$, we define

$$X(R) := \{(z_1, \dots, z_n) \in X \mid |z_i| < R \text{ for all } i\},$$

and set $X^*(R) := X(R) \cap X^*$.

As a direct consequence of Lemma 6.9, we obtain the following corollary.

Corollary 7.1 ([M4, Corollary 21.2.5])**.** *Let (E, h) be an acceptable vector bundle on X^*. Assume that $(E, h(0, -N_0))$ is Griffiths seminegative. Let F be a holomorphic section of E on $X^*(R)$ such that*

$$\|F|_{X^*(R)}\|_{h(\boldsymbol{a},-N)} < \infty$$

for some $0 < R \leq 1$. Here

$$\|F|_{X^*(R)}\|^2_{h(\boldsymbol{a},-N)} := \int_{X^*(R)} |F|^2_{h(\boldsymbol{a},-N)} \, \mathrm{dvol}_{X-D},$$

where dvol_{X-D} *is the volume form on* $X^* = (\Delta^*)^l \times \Delta^{n-l}$ *with respect to the Poincaré metric* ω_P*, that is,*

$$\mathrm{dvol}_{X-D} = \frac{\omega_P^n}{n!}.$$

Then for every $1 \leq j \leq l$ *and every* $P \in D_j^\circ$*, we have*

$$\int_{\pi_j^{-1}(P)\cap X^*(R')} \left|F|_{\pi_j^{-1}(P)\cap X^*(R')}\right|^2_{h(\boldsymbol{a},-M)} \mathrm{dvol}_{\pi_j^{-1}(P)} < \infty,$$

for any $0 < R' < R$ *and any* $M \geq \max\{N_0, N\}$*, where* $\mathrm{dvol}_{\pi_j^{-1}(P)}$ *is the volume form induced by the restriction* $\omega_P|_{\pi_j^{-1}(P)}$*.*

More precisely, there exists a constant $C > 0$ *such that*

$$\int_{\pi_j^{-1}(P)\cap X^*(R')} \left|F|_{\pi_j^{-1}(P)\cap X^*(R')}\right|^2_{h(\boldsymbol{a},-M)} \mathrm{dvol}_{\pi_j^{-1}(P)} < C\|F|_{X^*(R)}\|^2_{h(\boldsymbol{a},-N)} < \infty.$$

Proof of Corollary 7.1. Since $M \geq N_0$, Lemma 6.9 implies that

$$|F|^2_{h(\boldsymbol{a},-M)} = \exp(\log |F|^2_{h(\boldsymbol{a},-M)})$$

is plurisubharmonic. Hence for any complex submanifold V of $X^*(R)$, the restriction $|F|^2_{h(\boldsymbol{a},-M)}|_V$ is subharmonic.

Let U be a small ball centered at P in D_j°. Then $\pi_j^{-1}(U) \simeq \pi_j^{-1}(P) \times U$. Let dvol_U be the Euclidean volume form on U. There exists a constant $C_1 > 0$ such that

$$\mathrm{dvol}_U \cdot \mathrm{dvol}_{\pi_j^{-1}(P)} \leq C_1 \,\mathrm{dvol}_{X-D} \quad \text{on } \pi_j^{-1}(U) \cap X^*(R'). \tag{7.1}$$

For $Q \in \pi_j^{-1}(P) \cap X^*(R')$, the function $\left|F|_{\{Q\}\times U}\right|^2_{h(\boldsymbol{a},-M)}$ is plurisubharmonic, hence

$$|F|^2_{h(\boldsymbol{a},-M)}(Q,P) \leq \frac{1}{\mathrm{Vol}(U)} \int_{\{Q\}\times U} \left|F|_{\{Q\}\times U}\right|^2_{h(\boldsymbol{a},-M)} \mathrm{dvol}_U \tag{7.2}$$

by the mean value inequality, where

$$\mathrm{Vol}(U) := \int_U 1 \,\mathrm{dvol}_U < \infty.$$

Since $M \geq N$, there exists a constant $C_2 > 0$ such that

$$|F|^2_{h(\boldsymbol{a},-M)} \leq C_2 |F|^2_{h(\boldsymbol{a},-N)} \quad \text{on } X^*(R'). \tag{7.3}$$

Combining (7.1), (7.2), and (7.3), we obtain

$$\begin{aligned}
&\int_{\pi_j^{-1}(P)\cap X^*(R')} \left|F|_{\pi_j^{-1}(P)\cap X^*(R')}\right|^2_{h(\boldsymbol{a},-M)} \mathrm{dvol}_{\pi_j^{-1}(P)} \\
&\leq \frac{1}{\mathrm{Vol}(U)} \int_{(\pi_j^{-1}(P)\times U)\cap X^*(R')} |F|^2_{h(\boldsymbol{a},-M)} \,\mathrm{dvol}_U \,\mathrm{dvol}_{\pi_j^{-1}(P)} \\
&\leq \frac{C_1}{\mathrm{Vol}(U)} \int_{(\pi_j^{-1}(P)\times U)\cap X^*(R')} |F|^2_{h(\boldsymbol{a},-M)} \,\mathrm{dvol}_{X-D} \\
&\leq \frac{C_1 C_2}{\mathrm{Vol}(U)} \|F|_{X^*(R')}\|^2_{h(\boldsymbol{a},-N)} < \infty.
\end{aligned}$$

More precisely, the first inequality follows from the mean value inequality (7.2), the second one follows from (7.1), the third one is due to (7.3), and the final one follows from the assumption. This completes the proof. ∎

The following lemma is also a direct consequence of the mean value inequality for subharmonic functions.

Lemma 7.2 ([M4, Lemma 21.2.6])**.** *Let (E, h) be an acceptable vector bundle on $X^* = \Delta^*$. Let f be a holomorphic section of E on*

$$\Delta^*(R) := \{z \in \mathbb{C} \mid 0 < |z| < R\}$$

for some $0 < R \leq 1$, and assume that

$$\|f|_{\Delta^*(R)}\|_{h(b,-M_0)} < \infty$$

for some $b, M_0 \in \mathbb{R}$. Suppose that $(E, h(0, -N_0))$ is Griffiths seminegative. Let $M \geq \max\{N_0, M_0 + 2\}$. Then

$$|f(z)|_h^2 \leq B \cdot \|f|_{\Delta^*(R)}\|^2_{h(b,-M_0)} \, |z|^{-2b}\big(-\log |z|\big)^M$$

holds on $X^*(R/5) = \Delta^*(R/5)$, where $B > 0$ is independent of f.

Proof of Lemma 7.2. Since $M \geq N_0$, Lemma 6.9 implies that $\log |f(z)|_{h(b,-M)}$ is subharmonic on $\Delta^*(R)$. Let dvol denote the Euclidean volume form, and let dvol_{ω_P} be the volume form associated to the Poincaré metric. For $0 < |z| \leq R/5 \leq 1/5$, we have

$$\begin{aligned}
\log |f(z)|_{h(b,-M)} &\leq \frac{4}{\pi|z|^2} \int_{|w-z|\leq|z|/2} \log |f(w)|^2_{h(b,-M)} \, \mathrm{dvol} \\
&\leq \log \left(\frac{4}{\pi|z|^2} \int_{|w-z|\leq|z|/2} |f(w)|^2_{h(b,-M)} \, \mathrm{dvol} \right) \\
&\leq \log \left(\frac{9}{\pi} \int_{|w-z|\leq|z|/2} \frac{|f(w)|^2_{h(b,-M)}}{|w|^2} \, \mathrm{dvol} \right) \\
&\leq \log \left(\frac{9}{\pi} \int_{|w-z|\leq|z|/2} |f(w)|^2_{h(b,-M_0)} \, \mathrm{dvol}_{\omega_P} \right) \\
&\leq \log \left(\frac{9}{\pi} \|f|_{\Delta^*(R)}\|^2_{h(b,-M_0)} \right).
\end{aligned}$$

The first inequality is the mean value inequality; the second follows from Jensen's inequality; the third uses $|w| \leq \frac{3}{2}|z|$; the fourth follows from $M \geq M_0 + 2$ and $\log |w| < -1$ for $|w| \leq \frac{3}{2}|z| \leq \frac{3}{10} < e^{-1}$. This proves the desired estimate. □

Although the following lemma is elementary, it plays a crucial role in this paper.

Lemma 7.3 ([M4, Lemma 21.2.7])**.** *Let (E, h) be an acceptable vector bundle on $X^* = \Delta^*$. Let f be a holomorphic section of E such that*

$$|f|_h = O\left(\frac{1}{|z|^{a+\varepsilon}}\right)$$

for every $\varepsilon > 0$ on $\Delta^*(R)$ for some $0 < R \leq 1$. Let N_0 be such that $(E, h(0, -N_0))$ is Griffiths seminegative, and let $M \geq N_0$. Define

$$H(z) := |f(z)|_h^2 \, |z|^{2a} \left(-\log |z|\right)^{-M}.$$

Then $H(z)$ is bounded near the origin. More precisely,

$$\max_{|z|\leq R'} H(z) = \max_{|z|=R'} H(z)$$

for every $0 < R' < R$.

Proof of Lemma 7.3. For $\varepsilon > 0$, set

$$H_\varepsilon(z) := H(z)\,|z|^{2\varepsilon}.$$

By Lemma 6.9, $\log H_\varepsilon(z)$ is subharmonic on $\Delta^*(R)$. The assumption on f implies

$$\lim_{z\to 0} \log H_\varepsilon(z) = -\infty.$$

Hence $\log H_\varepsilon$ extends as a subharmonic function to $\Delta(R)$ (see [NO, (3.3.25) Theorem]). Therefore,

$$\max_{|z|\le R'} H_\varepsilon(z) = \max_{|z|=R'} H_\varepsilon(z). \tag{7.4}$$

Since $H(z)$ is continuous on $\{|z| = R'\}$ and $H_{\varepsilon_1}(z) \le H_{\varepsilon_2}(z)$ holds for $0 \le \varepsilon_2 \le \varepsilon_1 \le 1$, letting $\varepsilon \to 0$ in (7.4) yields the boundedness of $H(z)$ on $\{|z| \le R'\}$. This completes the proof. □

Proposition 7.4 is a direct consequence of Lemma 7.3, and it will play a crucial role in the following sections.

Proposition 7.4 ([M4, Proposition 21.2.8])**.** *Let (E,h) be an acceptable vector bundle on $X^* = X \setminus D = (\Delta^*)^l \times \Delta^{n-l}$. Let F be a holomorphic section of E on $X^*(R)$ for some $0 < R \le 1$. Assume that there exist real numbers a_i $(1 \le i \le l)$ such that:*

- *For every $\varepsilon > 0$, every $1 \le i \le l$, and every $P \in D_i^\circ$, we have*

$$\left| F|_{\pi_i^{-1}(P)} \right|_h = O\left(\frac{1}{|z_i|^{a_i+\varepsilon}} \right).$$

Let N_0 be such that $(E, h(0,-N_0))$ is Griffiths seminegative, and let $M \ge N_0$. Fix any real number $0 < R' < R$. Then there exists a constant $B > 0$, independent of F, such that

$$|F|_h^2 \le B \cdot \prod_{j=1}^{l} \left(|z_j|^{-2a_j} (-\log|z_j|)^M \right) \cdot \max_{\substack{|z_j|=R' \\ 1\le j\le l}} |F|_h^2 \quad \text{on } X^*(R').$$

Proof of Proposition 7.4. Set

$$H(z_1,\dots,z_l) := |F|_h^2 \cdot \prod_{j=1}^{l} \left(|z_j|^{2a_j} (-\log|z_j|)^{-M} \right).$$

Applying Lemma 7.3 to each coordinate z_i successively, we obtain the desired inequality. This completes the proof. □

Similarly, we obtain the following:

Corollary 7.5 ([M4, Corollary 21.2.9])**.** *Let (E,h) be an acceptable vector bundle on $X^* = (\Delta^*)^l \times \Delta^{n-l}$. Suppose that F is a holomorphic section of E on $X^*(R)$ for some $0 < R \le 1$, and that*

$$\|F|_{X^*(R)}\|_{h(\boldsymbol{a},-M_0)} < \infty.$$

Let N_0 be such that $(E, h(0,-N_0))$ is Griffiths seminegative, and let $M \ge \max\{N_0, M_0 + 2\}$. Then, for any $0 < R' < R$, we have on $X^*(R')$:

$$|F|_h^2 \le B \cdot \prod_{j=1}^{l} \left(|z_j|^{-2a_j} (-\log|z_j|)^M \right) \cdot \max_{\substack{|z_j|=R' \\ 1\le j\le l}} |F|_h^2,$$

where $B > 0$ is independent of F. In particular, $F \in {}_{\boldsymbol{a}}E$.

Proof of Corollary 7.5. The desired estimate follows directly from Corollary 7.1, Lemma 7.2, and Proposition 7.4. This completes the proof. □

In the following sections, we will repeatedly use Proposition 7.4 and Corollary 7.5.

8. L^2 extension theorem of Ohsawa–Takegoshi type

In this paper, we use an L^2 extension theorem of Ohsawa–Takegoshi type as a black box. We remark that Mochizuki does not rely on the Ohsawa–Takegoshi L^2 extension theorem; instead, he develops the theory within the framework of Andreotti–Visentini (see [M4, 21.1. Some general results on vector bundles on Kähler manifolds], as well as [AV] and [CG]). The theorem stated below is a very special case of [O1, Theorem] and [GZ, Corollary 3.13]. Since optimal constants are not needed for our purposes, we restrict ourselves to this weaker formulation.

Theorem 8.1 (see [O1, Theorem] and [GZ, Corollary 3.13])**.** *Let V be a bounded Stein open subset of $\mathbb{C}^n$ and let $(\mathcal{E}, h)$ be a Nakano semipositive vector bundle over V. Let φ be any smooth plurisubharmonic function on V and let $s_1, \dots, s_m$ be linear functions such that*

$$W := \{x \in V \mid s_1(x) = \cdots = s_m(x) = 0\}$$

is a closed complex submanifold of codimension m. We put

$$c_k = (\sqrt{-1})^{k^2}$$

for any positive integer k. Then, given a holomorphic $\mathcal{E}$-valued $(n-m)$-form g on W with

$$\int_W e^{-\varphi} c_{n-m} \{g, g\}_h < \infty,$$

for any $\varepsilon > 0$, there exists a holomorphic $\mathcal{E}$-valued n-form G_ε on V which coincides with

$$g \wedge ds_1 \wedge \cdots \wedge ds_m$$

on W and satisfies

$$\int_V e^{-\varphi} \left(1 + \sum_{i=1}^m |s_i|^2\right)^{-m-\varepsilon} c_n \{G_\varepsilon, G_\varepsilon\}_h \leq \frac{C}{\varepsilon} \int_W e^{-\varphi} c_{n-m} \{g, g\}_h < \infty,$$

where C is a positive constant independent of g.

Proof of Theorem 8.1. This theorem is a direct consequence of [O1, Theorem]. Readers interested in optimal constants are referred to [GZ, Corollaries 3.13 and 3.14]. The above non-optimal version suffices for our purposes. □

Since Theorem 8.1 is not a standard formulation of the Ohsawa–Takegoshi L^2 extension theorem, we include below a more familiar version for the reader's convenience. Of course, Theorem 8.2 is a special case of Theorem 8.1.

Theorem 8.2 (Ohsawa–Takegoshi L^2 extension theorem)**.** *Let $V \subset \mathbb{C}^n$ be a bounded Stein open set, and let $\mathcal{E}$ be a holomorphic vector bundle over V equipped with a smooth Hermitian metric h that is Nakano semipositive. Let φ be a smooth plurisubharmonic function on V. Let s be a nonzero linear function on $\mathbb{C}^n$, and set*

$$H := V \cap \{s = 0\}.$$

Let f be a holomorphic section of $\mathcal{E}|_H$ such that

$$\int_H |f|_h^2 e^{-\varphi}\, d\lambda_{n-1} < \infty,$$

where $d\lambda_{n-1}$ denotes the Lebesgue measure on $\mathbb{C}^{n-1} = \{s = 0\}$. Then there exists a holomorphic section F of $\mathcal{E}$ on V satisfying $F|_H = f$ and

$$\int_V |F|_h^2 e^{-\varphi}\, d\lambda_n \leq C'' \int_H |f|_h^2 e^{-\varphi}\, d\lambda_{n-1},$$

where $d\lambda_n$ denotes the Lebesgue measure on $\mathbb{C}^n$ and $C'' > 0$ is a constant independent of f

Proof of Theorem 8.2. The original Ohsawa–Takegoshi L^2 extension theorem is formulated for holomorphic functions. However, the same argument applies to holomorphic sections of Nakano semipositive vector bundles. Indeed, Theorem 8.2 follows from the standard proof given in [OT] and [O2, 2 Proof of Theorem 0.2], combined with a variant of Kodaira–Nakano's vanishing theorem (see [O2, Theorem 1.7] and [O3, Theorem 5]). We omit the details. □

Corollary 8.3. *Let V be a bounded Stein open subset of $\mathbb{C}^n$ and let $(\mathcal{E}, h)$ be a Nakano semipositive vector bundle over V. Let φ be any smooth plurisubharmonic function on V. Let $(z, w_2, \dots, w_n)$ be a coordinate system of $\mathbb{C}^n$. We put*

$$W := \{x \in V \mid w_2(x) = \cdots = w_n(x) = 0\}.$$

Let $f(z)$ be a holomorphic section of $\mathcal{E}|_W$ on W such that

$$\int_W |f|_h^2 e^{-\varphi} \frac{\sqrt{-1}}{2} dz \wedge d\overline{z} < \infty.$$

Then there exists a holomorphic section F of $\mathcal{E}$ on V such that

$$\int_V |F|_h^2 e^{-\varphi} d\lambda_n \leq C' \int_W |f|_h^2 e^{-\varphi} \frac{\sqrt{-1}}{2} dz \wedge d\overline{z} < \infty,$$

where $d\lambda_n$ denotes the Lebesgue measure of $\mathbb{C}^n$ and C' is a positive number which does not depend on f.

Proof of Corollary 8.3. The corollary is an immediate consequence of Theorem 8.1. For the reader's convenience, we briefly indicate the argument.

Set $g := f\, dz$. Then g is a holomorphic $\mathcal{E}|_W$-valued 1-form on W satisfying

$$\int_W e^{-\varphi} c_1 \{g, g\}_h < \infty.$$

Applying Theorem 8.1, we obtain a holomorphic $\mathcal{E}$-valued n-form G on V of the form

$$G = F\, dz \wedge dw_2 \wedge \cdots \wedge dw_n,$$

such that $F|_W = f$ and

$$\int_V e^{-\varphi} c_n \{G, G\}_h \leq C^\sharp \int_W e^{-\varphi} c_1 \{g, g\}_h < \infty,$$

where $C^\sharp > 0$ is independent of f. Since $c_n\{G, G\}_h$ is a constant multiple of $|F|_h^2 d\lambda_n$, the desired estimate follows. □

Remark 8.4. Corollary 8.3 can alternatively be obtained by applying Theorem 8.2 inductively along the flag

$$W = W_2 \subset W_3 \subset \cdots \subset W_n \subset V,$$

where

$$W_i := \{x \in V \mid w_j(x) = 0 \text{ for } i \leq j \leq n\}.$$

In the subsequent sections, we will frequently use the following form of the Ohsawa–Takegoshi L^2 extension theorem, which is a direct consequence of Corollary 8.3.

Proposition 8.5. *Let $0 < R < 1$. Define*

$$X^*(R) := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \,\middle|\, \begin{array}{l} 0 < |z_i| < R \quad \textit{for } 1 \leq i \leq l, \\ |z_i| < R \quad \textit{for } l+1 \leq i \leq n \end{array} \right\}.$$

Then $X^*(R)$ is a bounded Stein open subset of $\mathbb{C}^n$. Let $(\mathcal{E}, h)$ be a Nakano semipositive vector bundle over $X^*(R)$. We define the new coordinates as follows:

$$z := z_1, \qquad w_i := \begin{cases} z_i - z_1 & \textit{for } 2 \leq i \leq l, \\ z_i & \textit{for } l+1 \leq i \leq n. \end{cases}$$

Define the submanifold

$$Y^*(R) := \{(z_1, \dots, z_n) \in X^*(R) \mid w_2 = \cdots = w_n = 0\}.$$

Set the weight functions

$$\psi := \frac{1}{l} \sum_{i=1}^{l} \log |z_i|^2, \qquad \phi := -\left(1 - \frac{1}{l}\right) \sum_{i=1}^{l} \log |z_i|^2, \quad \textit{and} \quad \phi_{\boldsymbol{a}} := -\sum_{i=1}^{l} a_i \log |z_i|^2.$$

Let f be a holomorphic section of $\mathcal{E}|_{Y^*(R)}$ satisfying

$$\int_{Y^*(R)} |f|_h^2 e^{-\psi - \phi_{\boldsymbol{a}}} \cdot \frac{\sqrt{-1}}{2} \, dz \wedge d\overline{z} < \infty.$$

Then there exists a holomorphic section F of $\mathcal{E}$ on $X^*(R)$ such that

$$F|_{Y^*(R)} = f, \qquad \int_{X^*(R)} |F|_h^2 e^{-\psi - \phi_{\boldsymbol{a}}} \, d\lambda_n < \infty. \tag{8.1}$$

Therefore, we also have

$$\int_{X^*(R)} |F|_h^2 e^{-\phi - \phi_{\boldsymbol{a}}} \frac{\omega_P^n}{n!} < \infty. \tag{8.2}$$

We note that $\phi \equiv 0$ when $l = 1$.

Proof of Proposition 8.5. Note that $\psi + \phi_{\boldsymbol{a}}$ is a smooth plurisubharmonic function on $X^*(R)$ for any $\boldsymbol{a} \in \mathbb{R}^l$. Hence, by Corollary 8.3, there exists a holomorphic section F of $\mathcal{E}$ on $X^*(R)$ satisfying (8.1).

Moreover, since $0 < R < 1$, there exists a positive constant $C^\dagger$ such that

$$e^{-\phi - \phi_{\boldsymbol{a}}} \frac{\omega_P^n}{n!} \leq C^\dagger e^{-\psi - \phi_{\boldsymbol{a}}} \, d\lambda_n \quad \text{on } X^*(R).$$

Therefore, the integrability condition (8.2) follows immediately. □

9. Acceptable bundles on Δ^*

In this section, we briefly recall acceptable bundles on Δ^* following [FFO]. We strongly recommend the interested reader to see [FFO].

Theorem 9.1 (see [FFO, Theorem 1.9])**.** *Let (E, h) be an acceptable vector bundle on Δ^* with $\operatorname{rank} E = r$. Then ${}_aE$ is a holomorphic vector bundle for every $a \in \mathbb{R}$. Let $\{v_1, \ldots, v_r\}$ be a local frame of ${}_aE$ near the origin. Define*

$$\gamma({}_aE) := -\frac{1}{2} \liminf_{z \to 0} \frac{\log \det H(h, \boldsymbol{v})}{\log |z|},$$

where $H(h, \boldsymbol{v})$ is the $r \times r$ matrix $(h(v_i, v_j))$. Then $\gamma({}_aE)$ is a well-defined real-valued invariant of ${}_aE$.

Furthermore, if we let

$$\mathcal{P}ar_a(E, h) =: \{b_1, \ldots, b_r\},$$

then we have

$$\gamma({}_aE) = -\frac{1}{2} \lim_{z \to 0} \frac{\log \det H(h, \boldsymbol{v})}{\log |z|} = \sum_{i=1}^{r} b_i.$$

Note that if we define

$$\{\lambda_1, \ldots, \lambda_k\} := \{\lambda \in (a-1, a] \mid {}_\lambda E / {}_{<\lambda} E \neq 0\}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$, then

$$\sum_{i=1}^{r} b_i = \sum_{i=1}^{k} \lambda_i \dim_{\mathbb{C}} \left({}_{\lambda_i} E / {}_{<\lambda_i} E \right).$$

The following easy lemma will be used in Section 11.

Lemma 9.2. *Let (E, h) be an acceptable vector bundle on Δ^*.*

(i) *Let $\mathcal{D}$ be a dense subset of $\mathbb{R}$. Then the family $\{\gamma({}_aE)\}_{a \in \mathcal{D}}$ uniquely determines $\{\gamma({}_aE)\}_{a \in \mathbb{R}}$.*

(ii) *For any $\alpha \in \mathbb{R}$, $\mathcal{P}ar_\alpha(E, h)$ is uniquely determined by $\{\gamma({}_aE)\}_{a \in \mathbb{R}}$.*

In particular, for any dense subset $\mathcal{D} \subset \mathbb{R}$, the family $\{\gamma({}_aE)\}_{a \in \mathcal{D}}$ uniquely determines $\mathcal{P}ar_\alpha(E, h)$ for all $\alpha \in \mathbb{R}$.

Proof of Lemma 9.2. Since $\gamma({}_aE)$ is right-hand continuous by [FFO, Lemma 7.10], we can recover $\{\gamma({}_aE)\}_{a \in \mathbb{R}}$ by $\{\gamma({}_aE)\}_{a \in \mathcal{D}}$. Thus we have (i). We write

$$\mathcal{P}ar_\alpha(E, h) := \{\underbrace{\lambda_1, \ldots, \lambda_1}_{l_1 \text{ times}}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{l_k \text{ times}}\}.$$

We note that $\gamma({}_\lambda E) - \gamma({}_{\lambda-\varepsilon} E) \neq 0$ for $0 < \varepsilon \ll 1$ if and only if $\lambda \in \mathcal{P}ar_\alpha(E, h)$. Moreover, we have

$$\gamma({}_{\lambda_i} E) - \gamma({}_{\lambda_i - \varepsilon} E) = l_i$$

for $0 < \varepsilon \ll 1$ by [FFO, Lemma 7.12]. Hence we obtain (ii). □

For the details of acceptable bundles on a punctured disk, see [FFO].

10. Pull-back and descent revisited

The behavior of acceptable vector bundles on a punctured disk under pull-back by cyclic coverings has already been discussed in [FFO, Section 11]. In this section, we revisit this topic from a slightly different perspective. Because the literature employs various notational conventions, one of our aims here is to clarify the notation that will be used in the following sections. Throughout this section, we closely follow [M4, 21.4.2. Pull-back and descent].

Definition 10.1. For any $a, b \in \mathbb{R}$, we define

$$\nu(a,b) := \lfloor b-a \rfloor \in \mathbb{Z},$$

that is, $\nu(a,b)$ is the unique integer satisfying

$$b-1 < \nu(a,b) + a \leq b.$$

We now examine the behavior of acceptable vector bundles on a punctured disk under pull-back via cyclic coverings.

Let $X := \Delta$ and $X^* := \Delta^*$. Fix a positive integer c, and let

$$\psi_c \colon X \to X, \qquad \psi_c(z) = z^c,$$

be the cyclic covering of degree c. Let (E,h) be an acceptable vector bundle on the target space X^*. Then its pull-back

$$(\widetilde{E}, \widetilde{h}) := \psi_c^*(E,h)$$

is again an acceptable vector bundle on the source space X^*. We sometimes simply write $\psi_c^{-1}E$ to denote $(\widetilde{E}, \widetilde{h})$.

Let $\boldsymbol{v} = \{v_1, \dots, v_r\}$ be a frame of ${}^\diamond E = {}_0E$ compatible with the parabolic filtration, so that $v_i \in {}_{a_i}E \setminus {}_{<a_i}E$ for each i, where $a_i \in (-1, 0]$. Define

$$\widetilde{v}_i := z^{-\nu(ca_i, b)}\, \psi_c^*(v_i).$$

Lemma 10.2. *Let $\widetilde{\boldsymbol{v}} = \{\widetilde{v}_1, \dots, \widetilde{v}_r\}$. Then $\widetilde{\boldsymbol{v}}$ is a frame of ${}_b\widetilde{E}$ compatible with the parabolic filtration. In particular,*

$$\mathcal{P}ar({}_b\widetilde{E}) = \{\nu(ca,b) + ca \mid a \in \mathcal{P}ar({}^\diamond E)\}.$$

Proof of Lemma 10.2. This is proved in [FFO, Lemma 11.2]. We refer the reader to the proof there for details, although the notation used here is slightly different. □

Let $\mu_c := \mathbb{Z}/c\mathbb{Z}$ denote the Galois group of $\psi_c \colon X \to X$, and let g be a generator of μ_c. The group μ_c acts on X by multiplication, and this action lifts to ${}_b\widetilde{E}$. For each i, we have

$$g^*(\widetilde{v}_i) = \zeta^{-\nu(ca_i,b)}\widetilde{v}_i,$$

where ζ is a primitive c-th root of unity.

From now on, assume that $0 \leq b < 1/2$. If c is sufficiently large, then:

- $0 \leq \nu(ca,b) \leq c-1$ for every $a \in \mathcal{P}ar({}^\diamond E)$, and
- the map $\mathcal{P}ar({}^\diamond E) \to \mathbb{Z}$, $a \mapsto \nu(ca,b)$, is injective.

Let 0 denote the origin of $X = \Delta$. We obtain the following vector space decomposition:

$$ {}_b\widetilde{E}|_0 = \bigoplus_{0 \leq p \leq c-1} V_p, \tag{10.1}$$

where

$$V_p = \langle \widetilde{v}_j|_0 \mid \nu(ca_j, b) = p \rangle.$$

Then g acts on V_p by multiplication by ζ^{-p}.

By definition, for each p with $V_p \neq 0$, there exists a unique

$$\chi(p) \in \mathcal{P}ar({}^\diamond E) \quad \text{such that} \quad p = \nu(c\chi(p), b).$$

Thus we obtain an injection

$$\chi\colon \{0 \leq p \leq c-1 \mid V_p \neq 0\} \longrightarrow \mathcal{P}ar({}^\diamond E). \tag{10.2}$$

Set

$$\varphi(p) := \nu(c\chi(p), b) + c\chi(p) \in \mathcal{P}ar({}_b\widetilde{E}),$$

giving a map

$$\varphi : \{0 \leq p \leq c-1 \mid V_p \neq 0\} \longrightarrow \mathcal{P}ar({}_b\widetilde{E}). \tag{10.3}$$

The decomposition (10.1) induces a splitting of the parabolic filtration F of ${}_b\widetilde{E}$:

$$F_d({}_b\widetilde{E}|_0) = \bigoplus_{\varphi(p)\leq d} V_p.$$

Conversely, let

$$\widetilde{\boldsymbol{u}} = \{\widetilde{u}_1, \ldots, \widetilde{u}_r\}$$

be a μ_c-equivariant frame of ${}_b\widetilde{E}$, so that

$$g^*\widetilde{u}_j = \zeta^{-p_j}\widetilde{u}_j \quad (0 \leq p_j \leq c-1).$$

Then $\widetilde{u}_j|_0 \in V_{p_j}$, and in particular, $\widetilde{\boldsymbol{u}}$ is compatible with the filtration F. Set

$$u_j := z^{p_j}\widetilde{u}_j.$$

Then each u_j is μ_c-invariant and hence descends to a section of E, which we also denote by u_j.

Lemma 10.3. *The set $\boldsymbol{u} = \{u_1, \ldots, u_r\}$ is a frame of ${}^\diamond E$ compatible with the parabolic filtration.*

Proof of Lemma 10.3. Let w be the coordinate on the target space X, so $w = \psi_c(z) = z^c$. By definition, for every $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$|\widetilde{u}_j|_{\widetilde{h}} \leq \frac{C}{|z|^{\varphi(p_j)+\varepsilon}}.$$

Since

$$\varphi(p_j) = \nu(c\chi(p_j), b) + c\chi(p_j) = p_j + c\chi(p_j),$$

we obtain

$$|u_j|_h = |z^{p_j}\widetilde{u}_j|_{\widetilde{h}} \leq \frac{C}{|w|^{\chi(p_j)+\varepsilon/c}}.$$

Thus $u_j \in {}_{\chi(p_j)}E \subset {}^\diamond E$. Because $\widetilde{u}_j \in {}_{\varphi(p_j)}\widetilde{E} \setminus {}_{<\varphi(p_j)}\widetilde{E}$, we also have $u_j \in {}_{\chi(p_j)}E \setminus {}_{<\chi(p_j)}E$. As in the proof of [FFO, Lemma 11.3], this implies that $\boldsymbol{u}$ is a frame of ${}^\diamond E$ compatible with the parabolic filtration. □

We end this section with an important remark.

Remark 10.4. In [M4, 21.4.2], Mochizuki uses the weak norm estimate (see [M4, Theorem 21.3.2]). In contrast, in [FFO, Section 11], we do not make use of the weak norm estimate (see [FFO, Theorem 1.13]). This is because, in [FFO], the weak norm estimate is proved in [FFO, Section 13], where the argument depends on the results of [FFO, Section 11].

11. Acceptable line bundles on $\Delta^* \times \Delta^{n-1}$

In this section, we study an acceptable line bundle (L, h) on a partially punctured polydisk

$$X^* := \Delta^* \times \Delta^{n-1}.$$

Our approach is a natural extension of the method developed in [FFO], and appears to be new and different from that of Mochizuki.

We consider the projection

$$\pi\colon \Delta^* \times \Delta^{n-1} \to \Delta^{n-1}.$$

Lemma 11.1. *Let (L, h) be an acceptable line bundle on a partially punctured polydisk $X^* = \Delta^* \times \Delta^{n-1}$. Set $P := (0, \dots, 0) \in \Delta^{n-1}$. Assume that*

$$\alpha \notin \mathcal{P}ar_\alpha\big(L|_{\pi^{-1}(P)},\, h|_{\pi^{-1}(P)}\big)\,. \tag{11.1}$$

Then ${}_\alpha L$ is a line bundle on $X(R) = \Delta(R)^n$ for some $0 < R < 1$.

A more detailed description of a local generator of ${}_\alpha L$ can be found in the proof of Lemma 11.1.

Proof of Lemma 11.1. Choose a sufficiently large positive integer N such that

$$h(\alpha, N) := h \cdot e^{-\chi(\alpha,N)}$$

is Nakano semipositive. Let f be a generator of ${}_\alpha\big(L|_{\pi^{-1}(P)}\big)$. By assumption (11.1), we have

$$\int_{\pi^{-1}(P)\cap X(R)} |f|^2_{h(\alpha,N)} e^{-\psi} \frac{\sqrt{-1}}{2} dz_1 \wedge d\overline{z}_1 < \infty$$

for any $0 < R < 1$, where $\psi = \log |z_1|^2$.

By Proposition 8.5, there exists a holomorphic section F of L on $X^*(R)$ such that $F|_{\pi^{-1}(P)} = f$ and

$$\|F|_{X^*(R)}\|^2_{h(\alpha,N)} = \int_{X^*(R)} |F|^2_{h(\alpha,N)} \frac{\omega_P^n}{n!} < \infty.$$

Hence, by Corollary 7.5, we have $F \in {}_\alpha L$.

Set $g := f^{-1}$. Applying the same argument to $L^\vee$, we obtain a holomorphic section G of $L^\vee$ on $X^*(R)$ such that $G|_{\pi^{-1}(P)} = g$ and $G \in {}_{1-\alpha-\varepsilon}(L^\vee)$ for some $0 < \varepsilon \ll 1$. Since $(F \cdot G)|_{\pi^{-1}(P)} \equiv 1$, after replacing G by $\frac{G}{F \cdot G}$ we may assume that $F \cdot G \equiv 1$ on $X(R)$.

Claim 11.2. *The section F is a generator of ${}_\alpha L$, and G is a dual generator of ${}_{1-\alpha-\varepsilon}(L^\vee)$ on $X(R)$. In particular, both ${}_\alpha L$ and ${}_{1-\alpha-\varepsilon}(L^\vee)$ are line bundles on $X(R)$, and*

$$({}_\alpha L)^\vee = {}_{1-\alpha-\varepsilon}(L^\vee).$$

Proof of Claim 11.2. Let $\Phi \in {}_\alpha L$. Then $\Phi \cdot G \in {}_{1-\varepsilon}(\mathcal{O}_{X^*}) = \mathcal{O}_X$, and hence $\Phi = (\Phi \cdot G)F$. This shows that ${}_\alpha L = \mathcal{O}_X \cdot F$ on $X(R)$. The statement for $L^\vee$ follows similarly. □

Using the same argument, for any $Q \in \Delta(R)^{n-1}$, the restrictions $F|_{\pi^{-1}(Q)}$ and $G|_{\pi^{-1}(Q)}$ generate ${}_\alpha(L|_{\pi^{-1}(Q)})$ and ${}_{1-\alpha-\varepsilon}(L^\vee|_{\pi^{-1}(Q)})$, respectively. In particular,

$${}_\alpha(L|_{\pi^{-1}(Q)}) = ({}_\alpha L)|_{\pi^{-1}(Q)}.$$

This completes the proof of Lemma 11.1. □

Lemma 11.3 below is one of the key points of our approach.

Lemma 11.3. *In the setting of Lemma 11.1,* $\gamma\big({}_{\alpha}(L|_{\pi^{-1}(Q)})\big)$ *is independent of* $Q \in \Delta(R)^{n-1}$.

Proof of Lemma 11.3. By trivializing ${}_{\alpha}L$ using F, we may assume $F \equiv G \equiv 1$ on $X(R)$. We take a sufficiently large positive real number N. Then

$$\log |F|_{h\cdot e^{-\chi(\alpha,-N)}}$$

is plurisubharmonic on $X^*(R)$ by Lemma 6.9. Thus, for any $\alpha' > \alpha$, we see that

$$\log |F|_{h\cdot e^{-\chi(\alpha',-N)}}$$

is plurisubharmonic on $X(R)$ (see, for example, [NO, (3.3.41) Theorem] or [Dem2, Chapter I, (5.24) Theorem]). Similarly, we may assume that

$$\log |G|_{h^{\vee}\cdot e^{-\chi(\beta',-N)}}$$

is also plurisubharmonic on $X(R)$ for any $\beta' > 1-\alpha-\varepsilon$.

We can write $h = |\cdot|^2 e^{-2\varphi_\alpha}$. Then we obtain that

$$-2\varphi_\alpha - \chi(\alpha', -N) \tag{11.2}$$

and

$$2\varphi_\alpha - \chi(\beta', -N) \tag{11.3}$$

are plurisubharmonic. By considering the Lelong number at $Q' = (0, Q)$, we obtain

$$\begin{aligned} &\liminf_{z\to Q'} \frac{-2\varphi_\alpha - \chi(\alpha', -N)}{\log|z-Q'|} \\ &= \lim_{r\to+0} \frac{1}{r^{2(n-1)}} \int_{B(Q',r)} \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}(-2\varphi_\alpha) \wedge \left(\frac{\sqrt{-1}}{2\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i\right)^{n-1} + 2\alpha' \end{aligned} \tag{11.4}$$

and

$$\begin{aligned} &\liminf_{z\to Q'} \frac{2\varphi_\alpha - \chi(\beta', -N)}{\log|z-Q'|} \\ &= \lim_{r\to+0} \frac{1}{r^{2(n-1)}} \int_{B(Q',r)} \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}(2\varphi_\alpha) \wedge \left(\frac{\sqrt{-1}}{2\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i\right)^{n-1} + 2\beta' \end{aligned} \tag{11.5}$$

by Lemma 5.6. We put

$$\Psi(Q') := \lim_{r\to+0} \frac{1}{r^{2(n-1)}} \int_{B(Q',r)} \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}(2\varphi_\alpha) \wedge \left(\frac{\sqrt{-1}}{2\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i\right)^{n-1}$$

Then Ψ is an $\mathbb{R}$-valued function on $V := \{0\} \times \Delta(R)^{n-1}$. By Siu's theorem in Definition 5.2 (see, for example, [Dem1, (13.3) Corollary]),

$$\{x \in V \mid \Psi(x) \geq a\} \quad \text{and} \quad \{x \in V \mid \Psi(x) \leq b\}$$

are closed analytic subsets of V. By Lemma 5.7, we obtain that Ψ is constant on V. We put $\Psi := 2A \in \mathbb{R}$. Then, by (11.4) and the convexity properties of plurisubharmonic functions, that is, Lemma 5.3 and Corollary 5.5, we have

$$-2\varphi_\alpha - \chi(\alpha', -N) \leq (-2A + 2\alpha') \log \frac{|z-Q'|}{R_0} + M_1, \tag{11.6}$$

where M_1 is the maximum of $-2\varphi_\alpha - \chi(\alpha', -N)$ on $\overline{B}(Q', R_0) \subset X(R)$. Similarly, by (11.5) and Corollary 5.5, we have

$$2\varphi_\alpha - \chi(\beta', -N) \leq (2A + 2\beta') \log \frac{|z - Q'|}{R_0} + M_2, \tag{11.7}$$

where M_2 is the maximum of $2\varphi_\alpha - \chi(\beta', -N)$ on $\overline{B}(Q', R_0) \subset X(R)$. By (11.6), we obtain

$$\liminf_{z_1 \to 0} \frac{-\varphi_\alpha|_{\pi^{-1}(Q)}}{\log |z_1|} \geq -A. \tag{11.8}$$

By (11.7), we have

$$\liminf_{z_1 \to 0} \frac{\varphi_\alpha|_{\pi^{-1}(Q)}}{\log |z_1|} \geq A. \tag{11.9}$$

Therefore, by (11.8) and (11.9), we have

$$A \leq \liminf_{z_1 \to 0} \frac{\varphi_\alpha|_{\pi^{-1}(Q)}}{\log |z_1|} \leq \limsup_{z_1 \to 0} \frac{\varphi_\alpha|_{\pi^{-1}(Q)}}{\log |z_1|} \leq A. \tag{11.10}$$

Thus, we obtain

$$\gamma\left({}_\alpha(L|_{\pi^{-1}(Q)})\right) = \lim_{z_1 \to 0} \frac{\varphi_\alpha|_{\pi^{-1}(Q)}}{\log |z_1|} = A. \tag{11.11}$$

This means that

$$\gamma\left({}_\alpha(L|_{\pi^{-1}(Q)})\right)$$

is constant with respect to $Q \in (\Delta(R))^{n-1}$. This is what we wanted. □

By the above results, we have the following statement.

Proposition 11.4. *For any $\alpha \in \mathbb{R}$, ${}_\alpha L$ is a line bundle on $X(R)$ for some $0 < R < 1$. Moreover, $\gamma\left({}_\alpha(L|_{\pi^{-1}(Q)})\right)$ is independent of $Q \in \Delta(R)^{n-1}$.*

Proof of Proposition 11.4. By Lemma 11.1, we may assume that

$$\alpha \in \mathcal{P}ar_\alpha\left(L|_{\pi^{-1}(P)}, h|_{\pi^{-1}(P)}\right).$$

Let f be a generator of ${}_\alpha(L|_{\pi^{-1}(P)})$ on $\pi^{-1}(P)$. We take a sufficiently small positive real number δ. Then

$$\int_{\pi^{-1}(P) \cap X(R)} |f|^2_{h \cdot e^{-\chi(\alpha+\delta, N)}} e^{-\psi} \cdot \frac{\sqrt{-1}}{2} dz_1 \wedge d\overline{z}_1 < \infty,$$

where $\psi = \log |z_1|^2$. By Proposition 8.5 and Corollary 7.5, we can take a holomorphic section F of L on $X^*(R)$ such that $F|_{\pi^{-1}(P)} = f$ and $F \in {}_{\alpha+\delta}L$. We may assume that $\alpha + \delta \notin \mathcal{P}ar_{\alpha+\delta}\left(L|_{\pi^{-1}(P)}, h|_{\pi^{-1}(P)}\right)$. By Lemma 11.3 and [FFO, Lemma 13.1], we have

$$\alpha = \gamma\left({}_{\alpha+\delta}(L|_{\pi^{-1}(P)})\right) = \gamma\left({}_{\alpha+\delta}(L|_{\pi^{-1}(Q)})\right)$$

for any $Q \in (\Delta(R))^{n-1}$. Thus, by Proposition 7.4, we have $F \in {}_\alpha L$. Similarly, by Proposition 8.5 and Corollary 7.5, we can extend $g := f^{-1}$ and obtain $G \in {}_{1-\alpha-\delta-\varepsilon}(L^\vee)$ on $X^*(R)$ such that $G|_{\pi^{-1}(P)} = g$, where ε is a sufficiently small positive real number. Hence, by the same argument as in the proof of Lemma 11.1, ${}_\alpha L$ and ${}_{1-\alpha-\delta-\varepsilon}(L^\vee)$ are line bundles on $X(R)$ for some $0 < R < 1$. Moreover, F is a generator of ${}_\alpha L$ and G is a generator of ${}_{1-\alpha-\delta-\varepsilon}(L^\vee)$ on $X(R)$ for some $0 < R < 1$. We can easily check that

$${}_\alpha(L|_{\pi^{-1}(Q)}) = ({}_\alpha L)|_{\pi^{-1}(Q)}$$

for every $Q \in \Delta(R)^{n-1}$. By the same proof of Lemma 11.3, we see that $\gamma\left({}_{\alpha}(L|_{\pi^{-1}(Q)})\right)$ is independent of $Q \in \Delta(R)^{n-1}$. We finish the proof of Proposition 11.4. □

The following theorem is the main result of this section.

Theorem 11.5. *Let (L,h) be an acceptable line bundle on a partially punctured polydisk $\Delta^* \times \Delta^{n-1}$. Then ${}_{\alpha}L$ is a line bundle on Δ^n for any $\alpha \in \mathbb{R}$. Moreover, $({}_{\alpha}L)|_{\pi^{-1}(Q)} = {}_{\alpha}\left(L|_{\pi^{-1}(Q)}\right)$ holds for every $Q \in \Delta^{n-1}$. We also have that $\gamma\left({}_{\alpha}\left(L|_{\pi^{-1}(Q)}\right)\right)$ is independent of $Q \in \Delta^{n-1}$.*

Proof of Theorem 11.5. We take an arbitrary point $P \in \Delta^{n-1}$. After shifting and rescaling the coordinate system around P, we apply Proposition 11.4. Then we have the desired properties. □

12. Acceptable vector bundles on $\Delta^* \times \Delta^{n-1}$

In this section, we study an acceptable vector bundle (E,h) with $\operatorname{rank} E \geq 2$ on a partially punctured polydisk

$$X^* = \Delta^* \times \Delta^{n-1}.$$

Our approach heavily relies on the results established in Section 11.

Theorem 12.1. *Let (E,h) be an acceptable vector bundle on a partially punctured polydisk $\Delta^* \times \Delta^{n-1}$. Then, for any $\alpha \in \mathbb{R}$, ${}_{\alpha}E$ is locally free on Δ^n. Moreover,*

$${}_{\alpha}(E|_{\pi^{-1}(Q)}) = ({}_{\alpha}E)|_{\pi^{-1}(Q)}$$

for every $Q \in \Delta^{n-1}$. We also note that

$$\mathcal{P}ar_{\alpha}(E|_{\pi^{-1}(Q)}, h|_{\pi^{-1}(Q)})$$

is independent of $Q \in \Delta^{n-1}$.

A more detailed description of the sheaf ${}_{\alpha}E$ and its local frames can be found in the proof of Theorem 12.1.

Proof of Theorem 12.1. We divide the proof into several steps.

Step 1. Set $P := (0,\ldots,0) \in \Delta^{n-1}$. In this step, we prove that ${}_{\alpha}E$ is locally free on $X(R)$ for some $0 < R < 1$, under the assumption that

$$\alpha \notin \mathcal{P}ar_{\alpha}(E|_{\pi^{-1}(P)}, h|_{\pi^{-1}(P)}).$$

Let $\{v_1,\ldots,v_r\}$ be a frame of ${}_{\alpha}(E|_{\pi^{-1}(P)})$. We note that ${}_{\alpha}(E|_{\pi^{-1}(P)})$ is locally free by Theorem 9.1. Fix $0 < R < 1$. As in the line bundle case (cf. the proof of Lemma 11.1), by Proposition 8.5 and Corollary 7.5, there exist holomorphic sections $\{V_1,\ldots,V_r\}$ of ${}_{\alpha}E$ on $X^*(R)$ such that

$$V_i|_{\pi^{-1}(P)} = v_i \qquad (1 \leq i \leq r).$$

Consider the dual frame

$$\{w_1,\ldots,w_r\} := \{v_1^{\vee},\ldots,v_r^{\vee}\}$$

of ${}_{1-\alpha-\varepsilon}(E^{\vee}|_{\pi^{-1}(P)})$ for some $0 < \varepsilon \ll 1$ (see [FFO, Theorem 1.12]). By the same argument, we obtain holomorphic sections $\{W_1,\ldots,W_r\}$ of ${}_{1-\alpha-\varepsilon}(E^{\vee})$ on $X^*(R)$ such that

$$W_i|_{\pi^{-1}(P)} = w_i = v_i^{\vee}.$$

Claim 12.2. *For some $0 < R < 1$, the families $\{V_1,\ldots,V_r\}$ and $\{W_1,\ldots,W_r\}$ form frames of ${}_{\alpha}E$ and ${}_{1-\alpha-\varepsilon}(E^{\vee})$ on $X(R)$, respectively.*

Proof of Claim 12.2. Note that, by construction,

$$V_i \cdot W_j \in {}_{1-\varepsilon}(\mathcal{O}_{X^*}) = \mathcal{O}_X, \qquad (V_i \cdot W_j)\big|_{\pi^{-1}(P)} = \delta_{ij}.$$

Let

$$A := (V_i \cdot W_j)_{i,j}$$

be the associated $r \times r$ matrix-valued holomorphic function. Shrinking the radius if necessary, we may assume that

$$\det A \neq 0 \quad \text{on } X(R)$$

for some $0 < R < 1$. Define

$$\begin{pmatrix} W_1' \\ \vdots \\ W_r' \end{pmatrix} = A^{-1} \begin{pmatrix} W_1 \\ \vdots \\ W_r \end{pmatrix}.$$

Replacing W_i by W_i', we may further assume that

$$V_i \cdot W_j = \delta_{ij} \quad \text{on } X(R).$$

We also assume that

$$V_1 \wedge \cdots \wedge V_r \neq 0 \quad \text{on } X^*(R).$$

Let $\Phi \in {}_\alpha E$. Then Φ admits the expansion

$$\Phi = \sum_{i=1}^{r} (\Phi \cdot W_i)\, V_i,$$

where

$$\Phi \cdot W_i \in {}_{1-\varepsilon}(\mathcal{O}_{X^*}) = \mathcal{O}_X.$$

It follows that $\{V_1, \dots, V_r\}$ forms a local frame of ${}_\alpha E$ on $X(R)$.

Similarly, one checks that $\{W_1, \dots, W_r\}$ is a local frame of ${}_{1-\alpha-\varepsilon}(E^\vee)$. In particular, we obtain the identification

$${}_{1-\alpha-\varepsilon}(E^\vee) = ({}_\alpha E)^\vee.$$

We complete the proof. □

By the same argument, for any $Q \in \Delta(R)^{n-1}$, the restrictions $\{V_i|_{\pi^{-1}(Q)}\}$ and $\{W_i|_{\pi^{-1}(Q)}\}$ form frames of ${}_\alpha(E|_{\pi^{-1}(Q)})$ and ${}_{1-\alpha-\varepsilon}(E^\vee|_{\pi^{-1}(Q)})$, respectively.

The wedge product $V_1 \wedge \cdots \wedge V_r$ defines a frame of $\det({}_\alpha E)$, and similarly $W_1 \wedge \cdots \wedge W_r$ defines the dual frame of $\det({}_{1-\alpha-\varepsilon}(E^\vee))$ on $X(R)$. Moreover, for any $Q \in \Delta(R)^{n-1}$, the restrictions

$$(V_1 \wedge \cdots \wedge V_r)|_{\pi^{-1}(Q)} \quad \text{and} \quad (W_1 \wedge \cdots \wedge W_r)|_{\pi^{-1}(Q)}$$

provide a frame of $\det({}_\alpha(E|_{\pi^{-1}(Q)}))$ and the dual frame of $\det({}_{1-\alpha-\varepsilon}(E^\vee|_{\pi^{-1}(Q)}))$, respectively. By the same argument as in the proof of Lemma 11.3, it follows that

$$\gamma\big({}_\alpha(E|_{\pi^{-1}(Q)})\big)$$

is independent of $Q \in \Delta(R)^{n-1}$.

Step 2. In this step, we prove that $\mathcal{P}ar_\alpha(E|_{\pi^{-1}(Q)}, h|_{\pi^{-1}(Q)})$ is independent of $Q \in \Delta^{n-1}$.

Set

$$\Lambda := \{(z_2, \dots, z_n) \in \Delta^{n-1} \mid \operatorname{Re} z_i, \operatorname{Im} z_i \in \mathbb{Q} \text{ for } 2 \le i \le n\},$$

and define

$$\mathcal{P} := \{\alpha \in \mathbb{R} \mid \alpha \in \mathcal{P}ar_\alpha(E|_{\pi^{-1}(Q)}, h|_{\pi^{-1}(Q)}) \text{ for some } Q \in \Lambda\}.$$

Let $P \in \Lambda$ be any point. After shifting and rescaling coordinates around P, we may apply Step 1 to any $\alpha \in \mathbb{R} \setminus \mathcal{P}$. It follows that, for such α, $\gamma({}_{\alpha}(E|_{\pi^{-1}(Q)}))$ is independent of $Q \in \Delta^{n-1}$. Since $\mathcal{P}$ is countable, Lemma 9.2 (i) implies that this holds for all $\alpha \in \mathbb{R}$. By Lemma 9.2 (ii), we conclude that $\mathcal{P}ar_{\alpha}(E|_{\pi^{-1}(Q)}, h|_{\pi^{-1}(Q)})$ is independent of $Q \in \Delta^{n-1}$.

Step 3. In this step, we prove that for any $\alpha \in \mathbb{R}$, ${}_{\alpha}E$ is locally free on $X(R)$ for some $0 < R < 1$.

Set $P := (0, \dots, 0) \in \Delta^{n-1}$. By Step 1, we may assume that

$$\alpha \in \mathcal{P}ar_{\alpha}(E|_{\pi^{-1}(P)}, h|_{\pi^{-1}(P)}).$$

Let $\{v_1, \dots, v_r\}$ be a frame of ${}_{\alpha}(E|_{\pi^{-1}(P)})$. As in Step 1, there exist holomorphic sections $\{V_1, \dots, V_r\}$ of ${}_{\alpha+\delta}E$ on $X^*(R)$ for some $0 < \delta \ll 1$ such that

$$V_i|_{\pi^{-1}(P)} = v_i \qquad (1 \leq i \leq r).$$

Since $0 < \delta \ll 1$, by Step 2,

$$\alpha + \delta \notin \mathcal{P}ar_{\alpha+\delta}\left(E|_{\pi^{-1}(P)}, h|_{\pi^{-1}(P)}\right) = \mathcal{P}ar_{\alpha+\delta}\left(E|_{\pi^{-1}(Q)}, h|_{\pi^{-1}(Q)}\right)$$

holds for any $Q \in \Delta^{n-1}$. Thus, by Proposition 7.4, we see that $V_i \in {}_{\alpha}E$ for every i since $0 < \delta \ll 1$.

Let

$$\{w_1, \dots, w_r\} := \{v_1^{\vee}, \dots, v_r^{\vee}\}$$

be the dual frame of ${}_{1-\alpha-\delta-\varepsilon}(E^{\vee}|_{\pi^{-1}(P)})$ for some $0 < \varepsilon \ll 1$ (see [FFO, Theorem 1.12]). Arguing as in Step 1, we obtain holomorphic sections $\{W_1, \dots, W_r\}$ of ${}_{1-\alpha-\delta-\varepsilon}(E^{\vee})$ on $X^*(R)$ such that

$$W_i|_{\pi^{-1}(P)} = w_i = v_i^{\vee} \qquad (1 \leq i \leq r).$$

By the argument in the proof of Claim 12.2, $\{V_1, \dots, V_r\}$ forms a frame of ${}_{\alpha}E$ on $X(R)$ for some $0 < R < 1$. In particular, ${}_{\alpha}E$ is locally free on $X(R)$.

Step 4. In this final step, we prove that ${}_{\alpha}E$ is locally free on Δ^n for any $\alpha \in \mathbb{R}$.

Let $P \in \Delta^{n-1}$ be an arbitrary point. After shifting and rescaling coordinates around P, we apply Step 3. It follows that ${}_{\alpha}E$ is locally free on Δ^n for every $\alpha \in \mathbb{R}$.

We complete the proof of Theorem 12.1. □

13. Acceptable bundles on $(\Delta^*)^l \times \Delta^{n-l}$

In this section, we prove Theorem 1.1, which is one of the main results of this paper, in full generality.

Let (E, h) be an acceptable vector bundle on a partially punctured polydisk

$$X^* := (\Delta^*)^l \times \Delta^{n-l},$$

where $l \geq 2$. As before, we set

$$X := \Delta^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| < 1 \text{ for all } i\},$$

and

$$X^* = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{ll} 0 < |z_i| < 1 & \text{for } 1 \leq i \leq l, \\ |z_i| < 1 & \text{for } l+1 \leq i \leq n \end{array} \right\}.$$

For each $1 \leq i \leq n$, let

$$\pi_i \colon X^* \to D_i := \{z_i = 0\}$$

denote the natural projection. For $1 \leq i \leq l$, we also set

$$D_i^\circ := D_i \setminus \bigcup_{\substack{j \neq i \\ j \leq l}} D_j.$$

We put $I := \{1, \dots, l\}$ and define

$$D_I := \bigcap_{i \in I} D_i.$$

We begin with the following basic lemma.

Lemma 13.1. *Let (E, h) be an acceptable vector bundle on a partially punctured polydisk $X^* = (\Delta^*)^l \times \Delta^{n-l}$. For any $\boldsymbol{a} = (a_1, \dots, a_l) \in \mathbb{R}^l$, define*

$$\mathcal{P}ar({}_{\boldsymbol{a}}E, i) := \mathcal{P}ar\left({}_{a_i}\left(E|_{\pi_i^{-1}(P)}\right)\right)$$

for a point $P \in D_i^\circ$. Then $\mathcal{P}ar({}_{\boldsymbol{a}}E, i)$ is independent of the choice of $P \in D_i^\circ$, and hence is well defined.

Proof of Lemma 13.1. This follows immediately from Theorem 12.1 and its proof. □

We first consider a special case.

Proposition 13.2. *Let (E, h) be an acceptable vector bundle on a partially punctured polydisk $X^* = (\Delta^*)^l \times \Delta^{n-l}$. Assume that*

$$\mathcal{P}ar({}^\diamond E, i) \subset \left(-\frac{1}{l}, 0\right] \tag{13.1}$$

for every $1 \leq i \leq l$. Then ${}^\diamond E$ is locally free on $X(R)$ for some $0 < R < 1$.

The proof of Proposition 13.2 closely follows the arguments in Sections 11 and 12.

Proof. By Theorem 12.1, we may assume that $l \geq 2$. Set

$$Y^* := \{(z_1, \dots, z_n) \in X^* \mid z_1 = \cdots = z_l, \ z_{l+1} = \cdots = z_n = 0\}.$$

Let $\{v_1, \dots, v_r\}$ be a frame of ${}^\diamond(E|_{Y^*})$. Choose a sufficiently large positive real number N such that $h(0, N)$ is Nakano semipositive. As in Proposition 8.5, define

$$\psi := \frac{1}{l} \sum_{i=1}^{l} \log |z_i|^2, \qquad \phi := -\left(1 - \frac{1}{l}\right) \sum_{i=1}^{l} \log |z_i|^2,$$

and

$$\phi_{\varepsilon_1/l} := -\frac{\varepsilon_1}{l} \sum_{i=1}^{l} \log |z_i|^2,$$

where $\varepsilon_1 > 0$ is sufficiently small. Let $0 < R < 1$ and put $z := z_1 = \cdots = z_l$ on Y^*. Then

$$\int_{Y^*(R)} |v_j|_{h(0,N)} e^{-\psi - \phi_{\varepsilon_1/l}} \frac{\sqrt{-1}}{2} \, dz \wedge d\overline{z} < \infty$$

for every j. Hence, by Proposition 8.5, there exist holomorphic sections $\{V_1, \dots, V_r\}$ of E on $X^*(R)$ such that $V_j|_{Y^*} = v_j$ and

$$\int_{X^*(R)} |V_j|_{h(0,N)} e^{-\phi - \phi_{\varepsilon_1/l}} \frac{\omega_P^n}{n!} < \infty$$

for all j. By Corollary 7.5, we obtain

$$V_j \in {}_{\left(1-\frac{1}{l}+\frac{\varepsilon_1}{l},\ldots,1-\frac{1}{l}+\frac{\varepsilon_1}{l}\right)}E$$

for every j.

Let $\{w_1, \ldots, w_r\} := \{v_1^\vee, \ldots, v_r^\vee\}$ be the dual frame of ${}_{1-\varepsilon}(E^\vee|_{Y^*})$ for some $0 < \varepsilon \ll 1$. We may assume that $h^\vee(0, N) := h^\vee \cdot e^{-\chi(0,N)}$ is Nakano semipositive since N is sufficiently large. We put

$$\phi_{(\mathbf{1}-\boldsymbol{\varepsilon}_2)/l} := -\frac{1-\varepsilon_2}{l}\sum_{i=1}^{l}\log|z_i|^2,$$

where $0 < \varepsilon_2 < \varepsilon$. Then

$$\int_{Y^*(R)} |w_j|_{h^\vee(0,N)}\, e^{-\psi-\phi_{(\mathbf{1}-\boldsymbol{\varepsilon}_2)/l}} \frac{\sqrt{-1}}{2}\, dz \wedge d\overline{z} < \infty.$$

Applying Proposition 8.5 and Corollary 7.5 once again, we obtain holomorphic sections $\{W_1, \ldots, W_r\}$ of $E^\vee$ on $X^*(R)$ such that $W_j|_{Y^*} = w_j = v_j^\vee$ and

$$W_j \in {}_{\left(1-\frac{1}{l}+\frac{1-\varepsilon_2}{l},\ldots,1-\frac{1}{l}+\frac{1-\varepsilon_2}{l}\right)}(E^\vee)$$

for every j.

By (13.1) and Proposition 7.4, we have $V_j \in {}^\diamond E$ for all j, since $0 < \varepsilon_1 \ll 1$. Moreover,

$$W_j \in {}_{\left(1-\frac{\varepsilon_2}{l},\ldots,1-\frac{\varepsilon_2}{l}\right)}(E^\vee)$$

for every j. Arguing as in the proof of Claim 12.2, we conclude that $\{V_1, \ldots, V_r\}$ forms a frame of ${}^\diamond E$ on $X(R)$ and that $\{W_1, \ldots, W_r\}$ is the dual frame, for some $0 < R < 1$. □

As an immediate consequence of Proposition 13.2, we obtain the following corollary.

Corollary 13.3. *Let (E, h) be an acceptable vector bundle on a partially punctured polydisk $X^* = (\Delta^*)^l \times \Delta^{n-l}$. Let $\boldsymbol{a} = (a_1, \ldots, a_l) \in \mathbb{R}^l$. Assume that*

$$\mathcal{P}ar({}_{\boldsymbol{a}}E, i) \subset \left(-\frac{1}{l} + a_i,\, a_i\right]$$

for every $1 \le i \le l$. Then ${}_{\boldsymbol{a}}E$ is locally free on X.

Proof. By Lemma 6.8, we may assume that $a_i = 0$ for all i. Let $P \in X \setminus X^*$ be an arbitrary point. After shrinking and rescaling coordinates around P, we apply Proposition 13.2. It follows that ${}_{\boldsymbol{a}}E$ is locally free on Δ^n for every $\boldsymbol{a} \in \mathbb{R}^l$. □

From now on, we study acceptable vector bundles on $X^* = (\Delta^*)^l \times \Delta^{n-l}$ in general. Lemma 13.4 follows easily from Corollary 13.3.

Lemma 13.4. *Let (E, h) be an acceptable vector bundle on $X^* = (\Delta^*)^l \times \Delta^{n-l}$, and let $\eta > 0$ be sufficiently small. Then there exists a positive integer c such that*

$$\mathcal{P}ar\left({}_{\boldsymbol{\eta}}(\psi_c^{-1}E), i\right) \subset (-\eta, \eta)$$

for each $i = 1, \ldots, l$, where $\boldsymbol{\eta} = (\eta, \ldots, \eta) \in \mathbb{R}^l$ and $\psi_c \colon X = \Delta^n \to X = \Delta^n$ is the finite cover defined by

$$\psi_c(z_1, \ldots, z_l, z_{l+1}, \ldots, z_n) = (z_1^c, \ldots, z_l^c, z_{l+1}, \ldots, z_n).$$

More precisely, for any positive integer m, we can choose c divisible by m. Moreover, if $0 < \eta < \frac{1}{2l}$, then ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$ is locally free on X.

Proof. Note first that $\psi_c^*\omega_P = \omega_P$. Hence, $\psi_c^*(E,h)$ is an acceptable vector bundle on X^*. From now on, we simply write $\psi_c^{-1}E$ to denote $\psi_c^*(E,h)$.

To analyze the parabolic weights of $\psi_c^{-1}E$ along $\{z_i = 0\}$, it suffices to consider the case $l = 1$. In this case, the behavior of parabolic weights under ψ_c follows from the curve case (see Lemma 10.2 and [FFO, Section 11]).

Applying Diophantine approximation (cf. [FFO, Lemma 12.2] and [C, Chapter I, Theorem VI]), we may choose a sufficiently large and divisible positive integer c such that

$$\mathcal{P}ar\big({}_{\boldsymbol{\eta}}(\psi_c^{-1}E), i\big) \subset (-\eta, \eta)$$

for all $i = 1, \ldots, l$. Moreover, [FFO, Lemma 12.2] ensures that c can be taken to be divisible by any given positive integer m.

Finally, if $0 < \eta < \frac{1}{2l}$, Corollary 13.3 implies that ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$ is locally free on X. □

The following theorem is the main result of this section. Although we use Lemma 13.4, which differs slightly from [M4, Lemma 21.7.2], the proof of Theorem 13.5 is essentially the same as that given in [M4, 21.7.2. Proof of Theorem 21.3.1].

Theorem 13.5 (Prolongation by increasing orders)**.** *Let (E,h) be an acceptable vector bundle on a partially punctured polydisk $X^* = (\Delta^*)^l \times \Delta^{n-l}$. Then, for any $\boldsymbol{a} \in \mathbb{R}^l$, ${}_{\boldsymbol{a}}E$ is a locally free sheaf on $X = \Delta^n$.*

Proof of Theorem 13.5. We first note that, by Lemma 6.8, we may assume without loss of generality that $\boldsymbol{a} = \mathbf{0}$.

We divide the proof into two steps. In Step 1, we prove that ${}_{\boldsymbol{a}}E = {}^\diamond E$ is locally free on X. In Step 2, we give a supplementary remark on the parabolic filtrations; the description obtained there will be used in the proof of Theorem 1.3.

Step 1. The case $l = 1$ has already been treated in Section 11. Hence, we assume $l \geq 2$ throughout this step.

Let $0 < \eta < \frac{1}{2l}$ and set $\boldsymbol{\eta} = (\eta, \ldots, \eta) \in \mathbb{R}^l$. We consider $X = \Delta^n$ and $X^* = (\Delta^*)^l \times \Delta^{n-l}$. For a positive integer c, define

$$\psi_c \colon X \to X, \quad \psi_c(z_1, \ldots, z_n) = (z_1^c, \ldots, z_l^c, z_{l+1}, \ldots, z_n).$$

We choose c so that

$$\mathcal{P}ar\big({}_{\boldsymbol{\eta}}(\psi_c^{-1}E), i\big) \subset (-\eta, \eta), \quad i = 1, \ldots, l.$$

By Lemma 13.4, the sheaf ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$ is locally free.

Let $\mu_c = \mathbb{Z}/c\mathbb{Z} = \langle g \rangle$. There is a natural μ_c^l-action on X given by

$$(g_1, \ldots, g_l)^*(z_1, \ldots, z_n) = (\zeta_1 z_1, \ldots, \zeta_l z_l, z_{l+1}, \ldots, z_n),$$

where g_i is a generator of the i-th factor $\mu_c^{(i)}$ of μ_c^l and ζ_i is a primitive c-th root of unity. This action lifts to ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$, and each $\mu_c^{(i)}$ acts on ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)|_{D_i}$.

We have a vector bundle decomposition

$${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)|_{D_i} = \bigoplus_{0 \leq p \leq c-1} {}^iV_p,$$

where g_i acts on ${}^iV_p$ by multiplication by ζ_i^{-p}. As in the curve case (see (10.3)), we define a map

$$\varphi_i \colon \{\, 0 \leq p \leq c-1 \mid {}^iV_p \neq 0 \,\} \longrightarrow \mathcal{P}ar({}_{\boldsymbol{\eta}}(\psi_c^{-1}E), i).$$

For $\eta - 1 < b \leq \eta$, we define a filtration ${}^iF'$ of ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)|_{D_i}$ in the category of vector bundles on D_i by

$$ {}^iF'_b := \bigoplus_{\varphi_i(p) \leq b} {}^iV_p. \tag{13.2} $$

The collection of filtrations $\big({}^iF' \mid i = 1, \ldots, l\big)$ is compatible in the sense of Definition 3.4, since μ_c^l is abelian. In particular, we obtain a vector bundle decomposition

$$ {}_{\boldsymbol{\eta}}\big(\psi_c^{-1}E\big)\,\big|_{D_I} = \bigoplus_{\boldsymbol{p}} {}^IV_{\boldsymbol{p}}, \tag{13.3} $$

where $\boldsymbol{p} = (p_1, \ldots, p_l) \in \{0, 1, \ldots, c-1\}^l$, and g_i acts on ${}^IV_{\boldsymbol{p}}$ by multiplication by $\zeta_i^{-p_i}$ for each $1 \leq i \leq l$.

We set

$$ \boldsymbol{\delta}_i := (\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0) \in \mathbb{R}^l. $$

For $-1 < b < 0$, we define a subsheaf ${}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E)'$ of ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$ by

$$ {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E)' := \operatorname{Ker}\left(\pi\colon {}_{\boldsymbol{\eta}}(\psi_c^{-1}E) \longrightarrow \frac{{}_{\boldsymbol{\eta}}(\psi_c^{-1}E)|_{D_i}}{{}^iF'_{\eta+b}}\right), $$

where π is the natural morphism of $\mathcal{O}_X$-modules.

Claim 13.6. *For any $-1 < b < 0$, we have*

$$ {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E)' = {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E). $$

In particular, the parabolic filtration ${}^iF$ coincides with ${}^iF'$.

Proof of Claim 13.6. Let $f \in {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E)$. Viewing f as a section of ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$, we set $\overline{f} := \pi(f)$. For any point $P \in D_i^{\circ}$, we have $f|_{\pi_i^{-1}(P)} \in {}_{\boldsymbol{\eta}}(\psi_c^{-1}E|_{\pi_i^{-1}(P)})$. By the curve case,

$$ f(P) = f|_{\pi_i^{-1}(P)}(P) \in {}^iF'_{\eta+b}|_P, $$

and hence $\overline{f}(P) = 0$. Since this holds for all $P \in D_i^{\circ}$, we obtain $\overline{f} = 0$ on D_i, which shows $f \in {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E)'$.

Conversely, let $f \in {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E)'$. For any $P \in D_i^{\circ}$, the curve case implies

$$ \Big| f|_{\pi_i^{-1}(P)} \Big|_h = O\left(\frac{1}{|z_i|^{\eta+b+\varepsilon}}\right) $$

for all $\varepsilon > 0$. By Proposition 7.4, this shows that $f \in {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E)$.

Hence, we have the desired equality

$$ {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E)' = {}_{\boldsymbol{\eta}+b\boldsymbol{\delta}_i}(\psi_c^{-1}E), $$

which completes the proof of Claim 13.6. □

We record the following elementary observation.

Claim 13.7. *Let $v \in {}^IV_{\boldsymbol{p}}$, and let $v^{\sharp}$ be a holomorphic section of ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$ on $X(R)$ for some $0 < R < 1$ such that $v^{\sharp}|_{D_I} = v$. Define*

$$ v^{\flat} := \frac{1}{c^l} \sum_{k_1=0}^{c-1} \cdots \sum_{k_l=0}^{c-1} \zeta_1^{k_1p_1} \cdots \zeta_l^{k_lp_l} (g_1^{k_1}, \ldots, g_l^{k_l})^* v^{\sharp}. $$

Then $v^{\flat}|_{D_I} = v$, and $v^{\flat}$ is a μ_c^l-equivariant holomorphic section of ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$ on $X(R)$.

We now return to the proof of Theorem 13.5. By (13.3) and Claim 13.7, we can choose a μ_c^l-equivariant frame $\boldsymbol{v} = \{v_1, \dots, v_r\}$ of ${}_{\boldsymbol{\eta}}(\psi_c^{-1}E)$ on $X(R)$ for some $0 < R < 1$ such that

$$(g_1, \dots, g_l)^* v_i = \prod_{j=1}^{l} \zeta_j^{-p_j(v_i)} \, v_i$$

for integers $0 \le p_j(v_i) \le c-1$. By construction, the frame $\boldsymbol{v}$ is compatible with the parabolic filtrations ${}^{j}F$ for all $1 \le j \le l$ (see Definition 3.5).

For each i, define

$$\overline{v}_i := \prod_{j=1}^{l} z_j^{p_j(v_i)} \cdot v_i.$$

Since $\overline{v}_i$ is μ_c^l-invariant, it descends to a section of E. By the curve case (Lemma 10.3), each $\overline{v}_i$ is a section of ${}^{\diamond}E$. Moreover, for any $P \in D_i^{\circ}$, the restrictions $\overline{\boldsymbol{v}}|_{\pi_i^{-1}(P)}$ form a frame of ${}^{\diamond}(E|_{\pi_i^{-1}(P)})$. It follows that $\overline{\boldsymbol{v}}$ is a frame of ${}^{\diamond}E$ on a neighborhood of the origin. Hence, ${}^{\diamond}E$ is locally free on $X(R)$ for some $0 < R < 1$. As in Step 4 of the proof of Theorem 12.1, we conclude that ${}^{\diamond}E$ is locally free on X.

Step 2. In this step, we give a more direct description of the parabolic filtrations of ${}^{\diamond}E$ in a neighborhood of the origin. This description will be used in the proof of Theorem 1.3.

As in the curve case (see (10.2) in Section 10), we have a map

$$\chi_i \colon \{\, 0 \le p \le c-1 \mid {}^{i}V_p \neq 0 \,\} \longrightarrow \mathcal{P}ar({}^{\diamond}E, i).$$

We set

$$a_i(v_j) := \chi_i\big(p_i(v_j)\big). \tag{13.4}$$

We define a filtration ${}^{i}F'_b$ of ${}^{\diamond}E|_{D_i}$ by vector subbundles by

$${}^{i}F'_b := \big\langle \overline{v}_j|_{D_i} \,\big|\, a_i(v_j) \le b \big\rangle,$$

that is, ${}^{i}F'_b$ is the vector subbundle of ${}^{\diamond}E|_{D_i}$ generated by those $\overline{v}_j|_{D_i}$ with $a_i(v_j) \le b$. Here $\overline{\boldsymbol{v}} = \{\overline{v}_1, \dots, \overline{v}_r\}$ denotes the local frame of ${}^{\diamond}E$ constructed in Step 1.

For $-1 < b \le 0$, we define a subsheaf

$${}_{b\cdot\boldsymbol{\delta}_i}(E)' := \operatorname{Ker}\left(\pi \colon {}^{\diamond}E \longrightarrow \frac{{}^{\diamond}E|_{D_i}}{{}^{i}F'_b}\right),$$

where π denotes the natural morphism of $\mathcal{O}_X$-modules.

Claim 13.8. *We have*

$${}_{b\cdot\boldsymbol{\delta}_i}E = {}_{b\cdot\boldsymbol{\delta}_i}(E)',$$

and consequently ${}^{i}F_b = {}^{i}F'_b$.

Proof of Claim 13.8. Let $f \in {}_{b\cdot\boldsymbol{\delta}_i}E$. We regard f as a section of ${}^{\diamond}E$. For any $P \in D_i^{\circ}$, applying the curve case to

$$f|_{\pi_i^{-1}(P)} \in {}^{\diamond}(E|_{\pi_i^{-1}(P)}),$$

we obtain $f(P) \in {}^{i}F'_b|_P$. Hence, $f \in {}_{b\cdot\boldsymbol{\delta}_i}(E)'$.

Conversely, let $f \in {}_{b\cdot\boldsymbol{\delta}_i}(E)'$. By the curve case, we have

$$f|_{\pi_i^{-1}(P)} \in {}_{b}(E|_{\pi_i^{-1}(P)}) \quad \text{for all } P \in D_i^{\circ}.$$

Therefore, by Proposition 7.4, we conclude that $f \in {}_{b\cdot\boldsymbol{\delta}_i}E$.

Thus, we obtain

$$ {}_{b\cdot\boldsymbol{\delta}_i}E = {}_{b\cdot\boldsymbol{\delta}_i}(E)', $$

and consequently ${}^iF_b = {}^iF'_b$. □

By construction, ${}^iF'$ defines a filtration in the category of vector bundles on D_i, and the tuple $\left({}^iF' \mid i = 1, \dots, l\right)$ is compatible in the sense of Definition 3.4. Hence, the same holds for ${}^iF$: it defines a filtration by vector subbundles on D_i, and the tuple $\left({}^iF \mid i = 1, \dots, l\right)$ is compatible.

We conclude the proof of Theorem 13.5. □

We prove Theorems 1.1 and 1.3.

Proof of Theorem 1.3. As in the proof of Theorem 13.5, we may assume that $\boldsymbol{a} = \mathbf{0}$, i.e., ${}_{\boldsymbol{a}}E = {}^\diamond E$. Let $\overline{\boldsymbol{v}} = \{\overline{v}_1, \dots, \overline{v}_r\}$ be a local frame of ${}^\diamond E$ on a sufficiently small open neighborhood U of the origin in Δ^n, constructed in Step 1 of the proof of Theorem 13.5. Then we have

$$ {}^\diamond E|_U = \bigoplus_{j=1}^r \mathcal{O}_U \cdot \overline{v}_j. $$

For $1 \le j \le r$ and $1 \le i \le l$, we set

$$ a_i(\overline{v}_j) := a_i(v_j) \in (-1, 0], $$

as in (13.4) of Step 2 in the proof of Theorem 13.5. By the curve case result and Proposition 7.4, it follows that for any $\boldsymbol{b} \in \mathbb{R}^l$,

$$ {}_{\boldsymbol{b}}E|_U = \bigoplus_{j=1}^r \mathcal{O}_U \left(\sum_{i=1}^l \lfloor b_i - a_i(\overline{v}_j) \rfloor D_i \right) \cdot \overline{v}_j. $$

This completes the proof. □

Proof of Theorem 1.1. In Theorem 13.5, we have already shown that ${}_{\boldsymbol{a}}E$ is a locally free sheaf on Δ^n for any $\boldsymbol{a} \in \mathbb{R}^l$. By Theorem 1.3, the family

$$ \left({}_{\boldsymbol{a}}E \mid \boldsymbol{a} \in \mathbb{R}^l\right) $$

naturally forms a filtered bundle in the sense of Mochizuki (see Definitions 4.1 and 4.2). This completes the proof. □

14. Weak norm estimates

In this section, we prove the weak norm estimate stated in Theorem 1.4.

Proof of Theorem 1.4. Let $\boldsymbol{v} = \{v_1, \dots, v_r\}$ be a frame of ${}_{\boldsymbol{a}}E$ defined in a neighborhood of the origin $0 \in \Delta^n$, which is compatible with the parabolic filtrations

$$ \mathbf{F} := \left({}^iF \mid i = 1, \dots, l\right). $$

See Definition 3.6 for details.

For $1 \le i \le l$ and $1 \le j \le r$, we set

$$ a_i(v_j) := {}^i\deg^{\mathbf{F}}(v_j) = \deg^{{}^iF}(v_j). $$

Define

$$ v'_j := v_j \cdot \prod_{i=1}^l |z_i|^{a_i(v_j)}, \qquad \boldsymbol{v}' := \{v'_1, \dots, v'_r\}. $$

By the construction of $\boldsymbol{v}'$ and Proposition 7.4, there exist constants $C_1 > 0$ and $M_1 > 0$ such that

$$H(h, \boldsymbol{v}') \leq C_1 \Big(-\sum_{i=1}^{l} \log |z_i| \Big)^{M_1} I_r.$$

Let $\boldsymbol{v}^\vee = \{v_1^\vee, \dots, v_r^\vee\}$ be the dual frame of $\boldsymbol{v}$. For any point $P \in D_i^\circ$, the restriction $\boldsymbol{v}^\vee|_{\pi_i^{-1}(P)}$ is a frame of

$${}_{-a_i+(1-\varepsilon)}E^\vee \Big|_{\pi_i^{-1}(P)}$$

for $0 < \varepsilon \ll 1$, compatible with the induced parabolic filtration. Hence, for $0 < \varepsilon \ll 1$, $\boldsymbol{v}^\vee$ defines a local frame of

$${}_{-\boldsymbol{a}+(1-\varepsilon)\boldsymbol{\delta}}E^\vee$$

around the origin $0 \in \Delta^n$, compatible with the parabolic filtrations, where

$$\boldsymbol{\delta} = (1, \dots, 1) \in \mathbb{R}^l.$$

By the curve case, we have

$${}^i\deg^{\mathbf{F}}(v_j^\vee) = \deg^{{}^iF}(v_j^\vee) = -a_i(v_j)$$

for all i and j. We define

$$(v_j^\vee)' := v_j^\vee \cdot \prod_{i=1}^{l} |z_i|^{-a_i(v_j)}, \qquad (\boldsymbol{v}^\vee)' := \{(v_1^\vee)', \dots, (v_r^\vee)'\}.$$

Applying Proposition 7.4 again, there exist constants $C_2 > 0$ and $M_2 > 0$ such that

$$H\big(h^\vee, (\boldsymbol{v}^\vee)'\big) \leq C_2 \Big(-\sum_{i=1}^{l} \log |z_i| \Big)^{M_2} I_r.$$

This implies that there exist constants $C_3 > 0$ and $M_3 > 0$ such that

$$C_3 \Big(-\sum_{i=1}^{l} \log |z_i| \Big)^{-M_3} I_r \leq H(h, \boldsymbol{v}').$$

Combining the above estimates, we obtain the desired weak norm estimate. □

15. Basic properties via reduction to curves

In this final section, we establish Theorems 1.5, 1.6, and 1.7, together with Corollary 1.8, by systematically reducing the statements to the curve case.

Proof of Theorem 1.5. We first note the inclusion

$${}_{-\boldsymbol{a}+\mathbf{1}-\boldsymbol{\varepsilon}}(E^\vee) \subset ({}_{\boldsymbol{a}}E)^\vee.$$

This follows directly from the definition.

To prove the reverse inclusion, let $\boldsymbol{v} = \{v_1, \dots, v_r\}$ be a local frame of ${}_{\boldsymbol{a}}E$ compatible with the parabolic filtration, and let $\boldsymbol{v}^\vee = \{v_1^\vee, \dots, v_r^\vee\}$ denote the dual frame. For any $P \in D_i^\circ$, we consider the restriction $v_j^\vee|_{\pi_i^{-1}(P)}$.

By the curve case results [FFO, Theorems 1.12 and 13.2] together with Proposition 7.4, we conclude that

$$v_j^\vee \in {}_{-\boldsymbol{a}+\mathbf{1}-\boldsymbol{\varepsilon}}(E^\vee) \quad \text{for all } j.$$

This implies

$$({}_{\boldsymbol{a}}E)^\vee \subset {}_{-\boldsymbol{a}+\mathbf{1}-\boldsymbol{\varepsilon}}(E^\vee).$$

Combining the two inclusions, we obtain the desired equality

$$({}_{\boldsymbol{a}}E)^{\vee} = {}_{-\boldsymbol{a}+\boldsymbol{1}-\boldsymbol{\varepsilon}}\left(E^{\vee}\right).$$

We complete the proof of Theorem 1.5. □

Proof of Theorem 1.6. By definition, we have the inclusion

$$\sum_{\boldsymbol{a}_1+\boldsymbol{a}_2\leq \boldsymbol{b}} {}_{\boldsymbol{a}_1}E_1 \otimes {}_{\boldsymbol{a}_2}E_2 \subset {}_{\boldsymbol{b}}(E_1\otimes E_2). \tag{15.1}$$

Thus, it suffices to show that this inclusion is in fact an equality.

Set

$$Y := \{z_1 = \cdots = z_l,\ z_{l+1} = \cdots = z_n = 0\} \subset \Delta^n.$$

We consider the restriction ${}_{\boldsymbol{b}}(E_1 \otimes E_2)|_Y$. Applying the curve case result [FFO, Theorem 1.14] to this restriction, we obtain that the inclusion (15.1) is an equality in a neighborhood of the origin.

The same argument applies after translating the center to any point of $\Delta^n \setminus (\Delta^*)^l \times \Delta^{n-l}$. Hence, the inclusion (15.1) is an equality everywhere, which completes the proof of Theorem 1.6. □

Proof of Theorem 1.7. Recall that

$$\mathrm{Hom}(E_1, E_2) = E_1^{\vee} \otimes E_2$$

is an acceptable vector bundle (see Lemma 6.2). By definition, a section $f \in {}_{\boldsymbol{a}}\mathrm{Hom}(E_1, E_2)$ satisfies the condition that

$$f({}_{\boldsymbol{k}}E_1) \subset {}_{\boldsymbol{a}+\boldsymbol{k}}E_2 \quad \text{for all } \boldsymbol{k} \in \mathbb{R}^l.$$

Conversely, let $f \in \mathrm{Hom}(E_1, E_2)$ be a morphism satisfying

$$f({}_{\boldsymbol{k}}E_1) \subset {}_{\boldsymbol{a}+\boldsymbol{k}}E_2 \quad \text{for all } \boldsymbol{k} \in \mathbb{R}^l.$$

For any $P \in D_i^{\circ}$, we consider the restriction $f|_{\pi_i^{-1}(P)}$. By the curve case result [FFO, Proposition 17.1] together with Proposition 7.4, we conclude that

$$f \in {}_{\boldsymbol{a}}\mathrm{Hom}(E_1, E_2).$$

This completes the proof. □

Corollary 1.8 follows directly from Theorem 1.7.

Proof of Corollary 1.8. The assertion follows immedizately from Theorem 1.7, since

$${}^{\diamond}\mathrm{End}(E) = {}_{\boldsymbol{0}}\mathrm{Hom}(E, E).$$

This completes the proof. □

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Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: fujino@math.kyoto-u.ac.jp

Department of Mathematics and Data Science, Center for Liberal Arts and Sciences, Tokyo Denki University, Tokyo, Japan

Email address: fujisawa@mail.dendai.ac.jp

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

Email address: takashio@kurims.kyoto-u.ac.jp