ON THE ACC FOR LENGTHS OF EXTREMAL RAYS

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Abstract. We discuss the ascending chain condition for lengths of extremal rays. We prove that the lengths of extremal rays of n-dimensional $\mathbb{Q}$-factorial toric Fano varieties with Picard number one satisfy the ascending chain condition.

1. Introduction

We discuss the ascending chain condition (ACC, for short) for (minimal) lengths of extremal rays.

First, let us recall the definition of *$\mathbb{Q}$-factorial log canonical Fano varieties with Picard number one.*

Definition 1.1 ($\mathbb{Q}$-factorial log canonical Fano varieties with Picard number one)**.** Let X be a normal projective variety with only log canonical singularities. Assume that X is $\mathbb{Q}$-factorial, $-K_X$ is ample, and $\rho(X) = 1$. In this case, we call X a *$\mathbb{Q}$-factorial log canonical Fano variety with Picard number one.*

Definition 1.2 ((Minimal) lengths of extremal rays)**.** Let (X, Δ) be a log canonical pair and let $f : X \to Y$ be a projective surjective morphism. Let R be a $(K_X + \Delta)$-negative extremal ray of $\overline{NE}(X/Y)$. Then

$$\min_{[C]\in R} (-(K_X + \Delta) \cdot C)$$

is called the (*minimal*) *length of the* $(K_X + \Delta)$*-negative extremal ray* R.

From now on, we want to discuss the following conjecture. It seems to be the first time that the ascending chain condition for lengths of extremal rays is discussed in the literature.

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Conjecture 1.3 (ACC for lengths of extremal rays of $\mathbb{Q}$-factorial log canonical Fano varieties with Picard number one)**.** *We set*

$$\mathcal{L}_n := \left\{ l(X) \ ; \begin{array}{c} X \textit{ is an } n\textit{-dimensional } \mathbb{Q}\textit{-factorial log canonical} \\ \textit{Fano variety with Picard number one.} \end{array} \right\}.$$

Here

$$l(X) := \min_C (-K_X \cdot C)$$

where C runs over integral curves on X. For every n, the set $\mathcal{L}_n$ satisfies the ascending chain condition. This means that if X_k is an n-dimensional $\mathbb{Q}$-factorial log canonical Fano variety with Picard number one for every k such that

$$l(X_1) \leq l(X_2) \leq \cdots \leq l(X_k) \leq \cdots$$

then there is a positive integer l such that $l(X_m) = l(X_l)$ for every $m \geq l$.

We note that $l(X) \leq 2 \dim X$ when X is a $\mathbb{Q}$-factorial log canonical Fano variety with $\rho(X) = 1$ (see, for example, [Fj3, Theorem 18.2]).

Although, for inductive treatments, it may be better to consider the ascending chain condition for lengths of extremal rays of *log Fano* pairs (X, D) such that the coefficients of D are contained in a set satisfying the descending chain condition, we only discuss the case when $D = 0$ for simplicity. In this paper, we are mainly interested in $\mathbb{Q}$-factorial toric Fano varieties with Picard number one. Note that a $\mathbb{Q}$-factorial toric variety always has only log canonical singularities. So, we define

$$\mathcal{L}_n^{\text{toric}} := \left\{ l(X) \ ; \begin{array}{c} X \text{ is an } n\text{-dimensional } \mathbb{Q}\text{-factorial toric} \\ \text{Fano variety with Picard number one} \end{array} \right\}.$$

Let X be an n-dimensional $\mathbb{Q}$-factorial toric Fano variety with $\rho(X) = 1$. Then we have $l(X) \leq n+1$. Furthermore, $l(X) \leq n$ if $X \not\simeq \mathbb{P}^n$ (see [Fj1, Proposition 2.9]). We can easily see that $X \simeq \mathbb{P}(1, 1, 2, \ldots, 2)$ if and only if $l(X) = n$ (see [Fj1, Section 2], [Fj2, Proposition 2.1], and [Fj4]).

The following result is the main theorem of this paper, which supports Conjecture 1.3.

Theorem 1.4 (Main theorem)**.** *For every n, $\mathcal{L}_n^{\text{toric}}$ satisfies the ascending chain condition.*

In 2003, Professor Vyacheslav Shokurov explained his ideas on minimal log discrepancies, log canonical thresholds, and lengths of extremal rays to the first author at his office. He pointed out some analogies among them and asked the ascending chain condition for lengths of extremal rays. It is a starting point of this paper. For his ideas on minimal log discrepancies and log canonical thresholds, see, for example, [BS]. We note that Hacon–M$^{\mathrm{c}}$Kernan–Xu announced that they have established the ACC for log canonical thresholds (see [HMX]). We also note that the ACC for minimal log discrepancies is closely related to the termination of log flips (see [S]). We recommend the reader to see [K] and [T] for various aspects of log canonical thresholds.

We close this section with examples. Example 1.5 shows that the set $\mathcal{L}_n^{\mathrm{toric}}$ does not satisfy the descending chain condition. Example 1.6 implies that the ascending chain condition does not necessarily hold for (minimal) lengths of extremal rays of birational type.

Example 1.5. We consider $X_k = \mathbb{P}(1, k-1, k)$ with $k \geq 2$. Then

$$l(X_k) = \frac{2}{k-1}.$$

Therefore, $l(X_k) \to 0$ when $k \to \infty$.

Example 1.6. We fix $N = \mathbb{Z}^2$ and let $\{e_1, e_2\}$ be the standard basis of N. We consider the cone $\sigma = \langle e_1, e_2 \rangle$ in $N' = N + \mathbb{Z}e_3$, where $e_3 = \frac{1}{b}(1, a)$. Here, a and b are positive integers such that $\gcd(a, b) = 1$. Let $Y = X(\sigma)$ be the associated affine toric surface which has only one singular point P. We take a weighted blow-up of Y at P with the weight $\frac{1}{b}(1, a)$. This means that we divide σ by e_3 and obtain a fan Δ of $N'_{\mathbb{R}}$. We define $X = X(\Delta)$. It is obvious that X is $\mathbb{Q}$-factorial and $\rho(X/Y) = 1$. We can easily obtain

$$K_X = f^*K_Y + \left(\frac{1+a}{b} - 1\right)E,$$

where $E = V(e_3) \simeq \mathbb{P}^1$ is the exceptional curve of $f : X \to Y$, and

$$-K_X \cdot E = 1 - \frac{b-1}{a}.$$

We note that

$$-K_X \cdot E = \min_C(-K_X \cdot C)$$

where C runs over curves on X such that $f(C)$ is a point. We also note that $\overline{NE}(X/Y) = NE(X/Y)$ is spanned by E. In the above construction, we set $a = k^2$ and $b = mk + 1$ for arbitrary positive integers k, m. Then it is obvious that $\gcd(a, b) = 1$, and we obtain

$$-K_X \cdot E = 1 - \frac{m}{k}.$$

Therefore, the minimal lengths of K_X-negative extremal rays do not satisfy the ascending chain condition in this local setting. More precisely, the minimal lengths of K_X-negative extremal rays can take any values in $\mathbb{Q} \cap (0, 1)$ in this example.

We note that the minimal length of the K_X-negative extremal ray associated to a toric *birational* contraction morphism $f : X \to Y$ is bounded by $\dim X - 1$ (see [Fj4]).

For estimates of lengths of extremal rays of toric varieties and related topics, see [Fj1], [Fj2], and [Fj4].

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2. Preliminaries

In this section, we prepare various definitions and notation. We recommend the reader to see [Fj1, Section 2] for basic calculations.

2.1. Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n. A toric variety $X(\Delta)$ is associated to a *fan* Δ, a collection of convex cones $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ satisfying the following conditions:

(i) Each convex cone σ is a rational polyhedral cone in the sense there are finitely many $v_1, \ldots, v_s \in N \subset N_{\mathbb{R}}$ such that

$$\sigma = \{r_1 v_1 + \cdots + r_s v_s;\ r_i \geq 0\} =: \langle v_1, \ldots, v_s \rangle,$$

and it is strongly convex in the sense

$$\sigma \cap -\sigma = \{0\}.$$

(ii) Each face τ of a convex cone $\sigma \in \Delta$ is again an element in Δ.

(iii) The intersection of two cones in Δ is a face of each.

Definition 2.2. The *dimension* $\dim \sigma$ of σ is the dimension of the linear space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$ spanned by σ.

We denote by N_σ the sublattice of N generated (as a subgroup) by $\sigma \cap N$, i.e.,

$$N_\sigma := \sigma \cap N + (-\sigma \cap N).$$

If σ is a k-dimensional simplicial cone, and $v_1, \ldots, v_k$ are the first lattice points along the edges of σ, the *multiplicity* of σ is defined to be the *index* of the lattice generated by the $\{v_i\}$ in the lattice N_σ;

$$\operatorname{mult}(\sigma) := |N_\sigma : \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k|.$$

We note that $X(\sigma)$, which is the affine toric variety associated to σ, is non-singular if and only if $\mathrm{mult}(\sigma) = 1$.

Let us recall a well-known fact. See, for example, [M, Lemma 14-1-1].

Lemma 2.3. *A toric variety $X(\Delta)$ is $\mathbb{Q}$-factorial if and only if each cone $\sigma \in \Delta$ is simplicial.*

2.4. The *star* of a cone τ can be defined abstractly as the set of cones σ in Δ that contain τ as a face. Such cones σ are determined by their images in $N(\tau) := N/N_\tau$, that is, by

$$\overline{\sigma} = (\sigma + (N_\tau)_{\mathbb{R}})/(N_\tau)_{\mathbb{R}} \subset N(\tau)_{\mathbb{R}}.$$

These cones $\{\overline{\sigma}; \tau \prec \sigma\}$ form a fan of $N(\tau)$, and we denote this fan by $\mathrm{Star}(\tau)$. We set $V(\tau) = X(\mathrm{Star}(\tau))$. It is well known that $V(\tau)$ is an $(n-k)$-dimensional torus invariant closed subvariety of $X(\Delta)$, where $k = \dim \tau$. If $\dim V(\tau) = 1$ (resp. $n-1$), then we call $V(\tau)$ a *torus invariant curve* (resp. *torus invariant divisor*). For the details about the correspondence between τ and $V(\tau)$, see [Fl, 3.1 Orbits].

2.5 (Intersection Theory)**.** Assume that Δ is simplicial. If $\sigma, \tau \in \Delta$ span γ with $\dim \gamma = \dim \sigma + \dim \tau$, then

$$V(\sigma) \cdot V(\tau) = \frac{\mathrm{mult}(\sigma) \cdot \mathrm{mult}(\tau)}{\mathrm{mult}(\gamma)} V(\gamma)$$

in the *Chow group* $A^*(X)_{\mathbb{Q}}$. For the details, see [Fl, 5.1 Chow groups]. If σ and τ are contained in no cone of Δ, then $V(\sigma) \cdot V(\tau) = 0$.

2.6 ($\mathbb{Q}$-factorial toric Fano varieties with Picard number one)**.** Now we fix $N \simeq \mathbb{Z}^n$. Let $\{v_1, \ldots, v_{n+1}\}$ be a set of primitive vectors such that $N_{\mathbb{R}} = \sum_i \mathbb{R}_{\geq 0} v_i$. We define n-dimensional cones

$$\sigma_i := \langle v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1} \rangle$$

for $1 \leq i \leq n+1$. Let Δ be the complete fan generated by n-dimensional cones σ_i and their faces for all i. Then we obtain a complete toric variety $X = X(\Delta)$ with Picard number $\rho(X) = 1$. It is well known that X has only log canonical singularities (see, for example, [M, Proposition 14-3-2]) and that $-K_X$ is ample. We call it a *$\mathbb{Q}$-factorial toric Fano variety with Picard number one* (see also Lemma 2.7 below). We define $(n-1)$-dimensional cones $\mu_{i,j} = \sigma_i \cap \sigma_j$ for $i \neq j$. We can write $\sum_i a_i v_i = 0$, where $a_i \in \mathbb{Z}_{>0}$ for every i and $\gcd(a_1, \ldots, a_{n+1}) = 1$. From now on, we simply write $V(v_i)$ to denote $V(\langle v_i \rangle)$ for every i. Note that $\mathrm{mult}(\langle v_i \rangle) = 1$ for every i. Then we obtain

$$0 < V(v_l) \cdot V(\mu_{k,l}) = \frac{\mathrm{mult}(\mu_{k,l})}{\mathrm{mult}(\sigma_k)},$$

$$V(v_i)\cdot V(\mu_{k,l}) = \frac{a_i}{a_l}\cdot\frac{\mathrm{mult}(\mu_{k,l})}{\mathrm{mult}(\sigma_k)},$$

and

$$\begin{aligned}-K_X\cdot V(\mu_{k,l}) &= \sum_{i=1}^{n+1} V(v_i)\cdot V(\mu_{k,l})\\ &= \frac{1}{a_l}(\sum_{i=1}^{n+1} a_i)\frac{\mathrm{mult}(\mu_{k,l})}{\mathrm{mult}(\sigma_k)},\end{aligned}$$

where $K_X = -\sum_{i=1}^{n+1} V(v_i)$ is a *canonical divisor* of X. For the procedure to compute intersection numbers, see 2.5 or [Fl, p.100].

We note the following well-known fact.

Lemma 2.7. *Let X be an n-dimensional $\mathbb{Q}$-factorial complete normal variety with Picard number one. Assume that X is toric. Then X is an n-dimensional $\mathbb{Q}$-factorial toric Fano variety with Picard number one.*

Let us recall the following easy lemma, which will play crucial roles in the proof of our main theorem: Theorem 1.4. The proof of Lemma 2.8 is obvious by the description in 2.6.

Lemma 2.8. *We use the notations in* 2.6. *We consider the sublattice N' of N spanned by $\{v_1,\ldots,v_{n+1}\}$. Then the natural inclusion $N'\to N$ induces a finite toric morphism $f: X'\to X$ from a weighted projective space X' such that f is étale in codimension one. In particular, $X(\Delta)$ is a weighted projective space if and only if $\{v_1,\ldots,v_{n+1}\}$ generates N.*

For a toric description of *weighted projective spaces*, see [Fj1, Section 2].

2.9. In Lemma 2.8, we consider $C = V(\mu_{k,l})\simeq\mathbb{P}^1\subset X$ and the unique torus invariant curve $C'\subset X'$ such that $f(C') = C$. We set

$$m_{k,l} := \deg(f|_{C'}: C'\to C)\in\mathbb{Z}_{>0}$$

for every (k,l). Then we can check that

$$m_{k,l} = |N(\mu_{k,l})/N'(\mu_{k,l})|$$

by definitions, where $N'(\mu_{k,l}) = N'/N'_{\mu_{k,l}}$ and $N(\mu_{k,l}) = N/N_{\mu_{k,l}}$. Let D be a Cartier divisor on X. Then we obtain

$$C\cdot D = \frac{1}{m_{k,l}}(C'\cdot f^*D)$$

by the projection formula. Therefore, we have

$$\begin{aligned} C \cdot V(v_k) &= V(\mu_{k,l}) \cdot V(v_k) \\ &= \frac{\mathrm{mult}(\mu_{k,l})}{\mathrm{mult}(\sigma_l)} = \frac{\gcd(a_k, a_l)}{m_{k,l} a_l}. \end{aligned}$$

This is because

$$(C' \cdot f^* V(v_k)) = \frac{\gcd(a_k, a_l)}{a_l}$$

since X' is a weighted projective space.

2.10 (Lemma on the ACC)**.** We close this section with an easy lemma for the ascending chain condition.

Lemma 2.11. *We have the following elementary properties on ACC.*

(1) *If A satisfies the ascending chain condition, then any subset B of A satisfies the ascending chain condition.*

(2) *If A and B satisfy the ascending chain condition, then so does*

$$A + B = \{a + b\,;\, a \in A, b \in B\}.$$

(3) *If there exists a real number t_0 such that*

$$A \subset \{x \in \mathbb{R}\,;\, x \geq t_0\}$$

and $A \cap \{x \in \mathbb{R}\,;\, x > t\}$ is a finite set for any $t > t_0$, then A satisfies the ascending chain condition.

All the statements in Lemma 2.11 directly follow from definitions.

3. Proof of the main theorem

In this section, we prove the main theorem of this paper: Theorem 1.4. We will freely use the notation in Section 2.

Proof of Theorem 1.4. Let X be an n-dimensional $\mathbb{Q}$-factorial toric Fano variety with Picard number one as in 2.6. It is sufficient to consider $\{v_1, \ldots, v_{n+1}\}$ with the condition

$$\frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \leq \frac{\mathrm{mult}(\mu_{k,l})}{a_k \mathrm{mult}(\sigma_l)}$$

for every (k, l). We note that

$$\frac{\mathrm{mult}(\mu_{k,l})}{a_k \mathrm{mult}(\sigma_l)} = \frac{\mathrm{mult}(\mu_{k,l})}{a_l \mathrm{mult}(\sigma_k)}$$

for every $k \neq l$. We also note that we can easily check that

$$l(X) = \min_{1 \leq i \leq n+1} (-K_X \cdot V(v_i))$$

(cf. [M, Proposition 14-1-2]). In our notation, we have

$$l(X) = \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \sum_{i=1}^{n+1} a_i$$

for this $\{v_1, \ldots, v_{n+1}\}$ by the formula in 2.6. Therefore, we can write

$$\mathcal{L}_n^{\mathrm{toric}} = \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \sum_{i=1}^{n+1} a_i \, ; \, \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \leq \frac{\mathrm{mult}(\mu_{k,l})}{a_k \mathrm{mult}(\sigma_l)} \text{ for every } (k,l) \right\}.$$

It is sufficient to prove that

$$\mathcal{M}_i = \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i \, ; \, \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \leq \frac{\mathrm{mult}(\mu_{k,l})}{a_k \mathrm{mult}(\sigma_l)} \text{ for every } (k,l) \right\}$$

satisfies the ascending chain condition. This is because $\mathcal{L}_n^{\mathrm{toric}}$ is contained in

$$\left\{ \frac{\mathrm{mult}(\mu_{1,2})}{\mathrm{mult}(\sigma_2)} \right\} + \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{\mathrm{mult}(\sigma_1)} \right\} + \mathcal{M}_3 + \cdots + \mathcal{M}_{n+1}.$$

We note that

$$\left\{ \frac{\mathrm{mult}(\mu_{1,2})}{\mathrm{mult}(\sigma_2)} \right\}, \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{\mathrm{mult}(\sigma_1)} \right\} \subset \left\{ \frac{1}{m} \, ; \, m \in \mathbb{Z}_{>0} \right\}.$$

Therefore, it is sufficient to prove the following proposition by Lemma 2.11.

Proposition 3.1. *For* $3 \leq i \leq n+1$, $\mathcal{M}_i \cap \{x \in \mathbb{R} \, ; \, x > \varepsilon\}$ *is a finite set for every* $\varepsilon > 0$.

From now on, we fix i with $3 \leq i \leq n+1$. Since

$$\mathcal{M}_i = \left\{ \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i \, ; \, \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \leq \frac{\mathrm{mult}(\mu_{k,l})}{a_k \mathrm{mult}(\sigma_l)} \text{ for every } (k,l) \right\},$$

we have

$$\begin{aligned} \varepsilon &< \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i \\ &= \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \cdot \frac{a_i \mathrm{mult}(\sigma_j)}{\mathrm{mult}(\mu_{i,j})} \cdot \frac{\mathrm{mult}(\mu_{i,j})}{\mathrm{mult}(\sigma_j)} \\ &\leq \frac{\mathrm{mult}(\mu_{i,j})}{\mathrm{mult}(\sigma_j)} \end{aligned}$$

for every $1 \leq j \leq n+1$ with $j \neq i$. Therefore, we obtain

$$\frac{\mathrm{mult}(\sigma_j)}{\mathrm{mult}(\mu_{i,j})} \leq \llcorner \varepsilon^{-1} \lrcorner$$

for every $1 \leq j \leq n+1$ with $j \neq i$, where $\llcorner \varepsilon^{-1} \lrcorner$ is the integer satisfying $\varepsilon^{-1} - 1 < \llcorner \varepsilon^{-1} \lrcorner \leq \varepsilon^{-1}$. We set

$$\mathbb{Z}(i,j) = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{i-1} + \mathbb{Z}v_{i+1} + \cdots + \mathbb{Z}v_{j-1} + \mathbb{Z}v_{j+1} + \cdots + \mathbb{Z}v_{n+1}$$

for $j \neq i$ and

$$\mathbb{Z}(j) = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_{j-1} + \mathbb{Z}v_{j+1} + \cdots + \mathbb{Z}v_{n+1}.$$

We consider the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}(i,j) & \longrightarrow & \mathbb{Z}(j) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{\mu_{i,j}} & \longrightarrow & N & \longrightarrow & N/N_{\mu_{i,j}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_j} & & \downarrow & & \\
0 & \longrightarrow & N_{\mu_{i,j}}/\mathbb{Z}(i,j) & \longrightarrow & N/\mathbb{Z}(j) & \xrightarrow{p_j} & \mathcal{A}^j_{(i,j)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We note that

$$\left| \mathcal{A}^j_{(i,j)} \right| = \frac{\mathrm{mult}(\sigma_j)}{\mathrm{mult}(\mu_{i,j})} \leq \llcorner \varepsilon^{-1} \lrcorner.$$

Therefore, for any $v \in N$, we have

$$p_j \circ \pi_j \left((\llcorner \varepsilon^{-1} \lrcorner)! v \right) = 0$$

in $\mathcal{A}^j_{(i,j)}$. Thus,

$$\pi_j \left((\llcorner \varepsilon^{-1} \lrcorner)! v \right) \in N_{\mu_{i,j}}/\mathbb{Z}(i,j).$$

This holds for every $1 \leq j \leq n+1$ with $j \neq i$. Let us consider the natural projection $\pi : N \to N/N'$ where $N' = \sum_{k=1}^{n+1} \mathbb{Z}v_k$. Then, by the above argument, we obtain that

$$\pi \left((\llcorner \varepsilon^{-1} \lrcorner)! v \right) \in \bigcap_{j \neq i} N_{\mu_{i,j}}/(N' \cap N_{\mu_{i,j}}) \subset N/N'.$$

Claim. $\pi\left((\llcorner \varepsilon^{-1} \lrcorner)! v\right) = 0$ *in* N/N', *equivalently,* $(\llcorner \varepsilon^{-1} \lrcorner)! v \in N'$.

Proof of Claim. By replacing v_i with v_{n+1}, we may assume that $i = n+1$. We embed N and N' into $\mathbb{Q}^n$ by setting $v_1 = (1, 0, \ldots, 0)$,

$v_2 = (0, 1, 0, \dots, 0)$, $\dots$, and $v_n = (0, \dots, 0, 1)$. Then it is easy to see that

$$\bigcap_{1 \le j \le n} (N' + N_{\mu_{n+1,j}}) = N'.$$

On the other hand, we have

$$N_{\mu_{n+1,j}}/(N' \cap N_{\mu_{n+1,j}}) \simeq (N' + N_{\mu_{n+1,j}})/N'$$

for $1 \le j \le n$. Therefore,

$$\pi\left((\llcorner \varepsilon^{-1} \lrcorner)!v\right) \in \bigcap_{1 \le j \le n} N_{\mu_{n+1,j}}/(N' \cap N_{\mu_{n+1,j}}) \subset N/N'$$

implies that

$$\pi\left((\llcorner \varepsilon^{-1} \lrcorner)!v\right) = 0$$

in N/N', equivalently,

$$(\llcorner \varepsilon^{-1} \lrcorner)!v \in N'.$$

This completes the proof of Claim. □

Thus, we obtain

$$1 \le m_{1,2} \le (\llcorner \varepsilon^{-1} \lrcorner)!.$$

Moreover,

$$\varepsilon < \frac{\mathrm{mult}(\mu_{1,i})}{\mathrm{mult}(\sigma_1)} = \frac{\gcd(a_1, a_i)}{m_{1,i} a_1} \le \frac{\gcd(a_1, a_i)}{a_1}.$$

By the same way, we obtain

$$\varepsilon < \frac{\gcd(a_2, a_i)}{a_2}.$$

We note the following obvious inequality:

$$\frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i \le \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} \cdot \frac{a_i \mathrm{mult}(\sigma_2)}{\mathrm{mult}(\mu_{2,i})} \le 1.$$

Since a_1, a_2, and a_i are positive integers, we have

$$\gcd(l, a_i) = \frac{\gcd(a_1, a_i) \cdot \gcd(a_2, a_i)}{\gcd(d, a_i)}$$

where $d := \gcd(a_1, a_2)$ and $l := \mathrm{lcm}(a_1, a_2) = a_1 a_2 / d$. Therefore, we obtain

$$\frac{\gcd(l, a_i)}{l} = \frac{\gcd(a_1, a_i)}{a_1} \cdot \frac{\gcd(a_2, a_i)}{a_2} \cdot \frac{d}{\gcd(d, a_i)} > \varepsilon^2 \frac{d}{\gcd(d, a_i)} \ge \varepsilon^2.$$

This means that

$$\frac{l}{\gcd(l, a_i)} \le \varepsilon^{-2}.$$

Thus, we have

$$1 \geq \frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i = \frac{a_i}{m_{1,2} l} = \frac{\gcd(l, a_i)}{l} \cdot \frac{a_i}{m_{1,2} \gcd(l, a_i)}$$
$$\geq \varepsilon^2 \frac{a_i}{m_{1,2} \gcd(l, a_i)}.$$

So, we obtain

$$\frac{a_i}{\gcd(l, a_i)} \leq \varepsilon^{-2} m_{1,2} \leq \varepsilon^{-2} (\llcorner \varepsilon^{-1} \lrcorner)!.$$

On the other hand,

$$\frac{\mathrm{mult}(\mu_{1,2})}{a_1 \mathrm{mult}(\sigma_2)} a_i = \frac{a_i}{m_{1,2} l}.$$

We note that

$$\frac{a_i}{m_{1,2} l} = \frac{\dfrac{a_i}{\gcd(l, a_i)}}{m_{1,2} \dfrac{l}{\gcd(l, a_i)}}.$$

This implies that $\mathcal{M}_i \cap \{x \in \mathbb{R}\,;\, x > \varepsilon\}$ is a finite set. This is because

$$\frac{a_i}{\gcd(l, a_i)}, \ \frac{l}{\gcd(l, a_i)}, \ m_{1,2}$$

are positive integers and

$$\frac{a_i}{\gcd(l, a_i)} \leq \varepsilon^{-2} (\llcorner \varepsilon^{-1} \lrcorner)!, \ \frac{l}{\gcd(l, a_i)} \leq \varepsilon^{-2}, \ m_{1,2} \leq (\llcorner \varepsilon^{-1} \lrcorner)!.$$

Thus we proved the proposition and $\mathcal{L}_n^{\mathrm{toric}}$ satisfies the ascending chain condition. □

References

[BS] C. Birkar, and V. V. Shokurov, Mld's vs thresholds and flips, J. Reine Angew. Math. **638** (2010), 209–234.

[Fj1] O. Fujino, Notes on toric varieties from Mori theoretic viewpoint, Tohoku Math. J. (2) **55** (2003), 551–564.

[Fj2] O. Fujino, Toric varieties whose canonical divisors are divisible by their dimensions, Osaka J. Math. **43** (2006), 275–281.

[Fj3] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), 727–789.

[Fj4] O. Fujino, Notes on toric varieties from Mori theoretic viewpoint, II, in preparation.

[Fl] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, **131**. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[HMX] C. D. Hacon, J. M$^{\mathrm{c}}$Kernan, C. Xu, Boundedness of volumes and Shokurov's ACC conjecture, in preparation.

[K] J. Kollár, Singularities of pairs, Algebraic geometry—Santa Cruz 1995, 221–287, Proc. Sympos. Pure Math., **62**, Part 1, Amer. Math. Soc., Providence, RI, 1997.

[M] K. Matsuki, *Introduction to the Mori program*, Universitext. Springer-Verlag, New York, 2002.

[S] V. V. Shokurov, Letters of a bi-rationalist. V. Minimal log discrepancies and termination of log flips, Proc. Steklov Inst. Math. 2004 (246), 315–336.

[T] B. Totaro, The ACC conjecture for log canonical thresholds (after de Fernex, Ein, Mustaţă, Kollár), Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026. Astérisque No. **339** (2011), Exp. No. 1025, ix, 371–385.

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