Minimal model theory for log surfaces in Fujiki's class $\mathcal{C}$

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§1 MMP for algebraic surfaces

X : smooth proj surface $/\mathbb{C}$

$$X =: X_0 \xrightarrow{\varphi_0} X_1 \to \cdots \to X_i \xrightarrow{\varphi_i} \cdots \to X_k$$

φ_i : contraction of a (-1)-curve s.t

$$\begin{cases} X_k : \text{good minimal model, i.e. } K_{X_k} : \text{semi-ample} \\ \quad \text{or} \\ X_k \simeq \mathbb{P}^2 \quad \text{Mori fiber space} \\ \quad \text{or} \\ X_k : \mathbb{P}^1\text{-bundle over a curve} \quad \text{"} \end{cases}$$

<u>Thm</u> 1 (O. Fujino) (X, Δ) : proj $\mathbb{Q}$-factorial log surface $/\mathbb{C}$

$\left(K_X + \Delta : \mathbb{Q}\text{-Cartier}, \ \Delta \in [0,1] \cap \mathbb{Q} \right)$

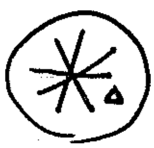
$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} \cdots \to (X_i, \Delta_i) \xrightarrow{\varphi_i} \cdots \to (X_k, \Delta_k)$$

φ_i : contraction of a $(K_{X_i} + \Delta_i)$-negative $\mathbb{P}^1$

s.t

$$\begin{cases} (X_k, \Delta_k) : \text{good minimal model, that is,} \\ \qquad\qquad K_{X_k} + \Delta_k : \text{semi-ample} \\ \text{or} \\ (X_k, \Delta_k) \to W : \text{Mori fiber space.} \end{cases}$$

<u>Rem 2</u> (X, Δ) : not necessarily log canonical.



X

X_i : $\mathbb{Q}$-factorial for $\forall i$ "

<u>Rem 3</u> X has only rational singularities

$\Longrightarrow X$: $\mathbb{Q}$-factorial "

<u>Thm 4</u> (MMP for log canonical surfaces)

(X, Δ) : projective log canonical surface $/\mathbb{C}$

$K_X + \Delta$: $\mathbb{Q}$-Cartier

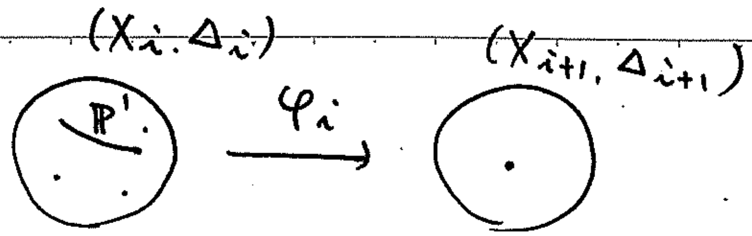
$$(X, \Delta) =: (X_0, \Delta_0) \to \cdots\cdots \to (X_k, \Delta_k)$$

s.t

$$\begin{cases} (X_k, \Delta_k) : \text{good minimal model} \\ \text{or} \\ (X_k, \Delta_k) \to W : \text{Mori fiber space} \end{cases}$$

<u>Rem 5</u> X in Thm 4 is not necessarily $\mathbb{Q}$-fac.

Rem 6



$C_i := \mathrm{Exc}(\varphi_i) \simeq \mathbb{P}^1$

$\mathrm{Exc}(\varphi_i)$ is disjoint from nonrational singularities of X_i

$\Rightarrow C_i$: $\mathbb{Q}$-Cartier (Key Point!) //

Thm 7 (H. Tanaka) Thm 1 and Thm 4 hold over any algebraically closed field $k = \overline{k}$ //

Kodaira-type vanishing theorems $\longrightarrow$ Thm 1, Thm 4

Artin-Keel contraction theorem $\longrightarrow$ Thm 7

$\uparrow$

Frobenius map

Surprisingly, the proof of Thm 7 is simpler than that of Theorems 1 and 4.

§2 MMP for compact complex surfaces.

X : compact complex surface

$$X =: X_0 \xrightarrow{\varphi_0} \cdots\cdots \to X_i \xrightarrow{\varphi_i} \cdots \longrightarrow X_k$$

φ_i : contraction of a (-1)-curve.

What is X_k ?

Check the Enriques-Kodaira classification table!

Fortunately, if X is Kähler, then everything is OK!

<u>Assume</u> X : Kähler

$\Rightarrow$ " X_k : good minimal model, that is, ω_{X_k} : semi-ample

or

$X_k \to W$: Mori fiber space "

<u>Rem 8</u> X : Kähler $\Rightarrow$ X_i : Kähler for $\forall i$

<u>Rem 9</u> If X is Kähler with $\kappa(X) = -\infty$,

then X is <u>projective</u>.

If X is not Kähler with $\kappa(X) = -\infty$,

then X_k is a surface of class VII. //

The following result is very helpful.

Thm (..., Siu)

X : compact complex surface.

X : Kähler $\Longleftrightarrow$ $b_1(X)$: even. //

Goal: I would like to discuss MMP

for compact normal analytic surfaces.

§3 Classical results

X : compact normal analytic surface

<u>Def 11</u> (Mumford) D : $\mathbb{Q}$-divisor on X

$\pi : Y \to X$: resolution of singularities

$\mathrm{Exc}(\pi) = \sum_i E_i$

We can define

$$\pi^* D = \underset{\substack{\uparrow \\ \text{strict transform of } D \text{ on } Y}}{D^+} + \sum_i \alpha_i E_i$$

by $\quad (D^+ + \sum_i \alpha_i E_i) \cdot E_j = 0$ for ${}^\forall j$.

D, D' : $\mathbb{Q}$-divisors on X

We put $\quad D \cdot D' := (\pi^* D) \cdot (\pi^* D')$.

intersection pairing in the usual sense on Y

//

<u>Thm 12</u> (Grauert, Sakai)

C : irreducible curve on X s.t

$$C^2 = C \cdot C < 0$$

$$\Longrightarrow \quad {}^{\exists}\varphi : \begin{array}{ccc} X & \longrightarrow & Y \\ \cup & & \Psi \\ C & \longrightarrow & pt \end{array}$$ ← compact normal analytic surface

"

<u>Rem 13</u> Y is not necessarily projective even when X is projective. "

<u>naive MMP for compact normal analytic surfaces</u>

X : cpt normal analytic surface with

$$\kappa(X, \omega_X) \geqq 0.$$

$$\Rightarrow \quad X =: X_0 \xrightarrow{\varphi_0} \cdots \longrightarrow X_i \xrightarrow{\varphi_i} \cdots \longrightarrow X_k$$

φ_i : contraction of an ω_{X_i}-negative curve.

s.t

X_k : $\nexists\, \omega_{X_k}$-negative curves on X_k.

<u>Idea</u> : $\kappa(X, \omega_X) \geqq 0 \Longrightarrow \omega_X \equiv \sum_i a_i C_i$, $a_i \geqq 0$

$\omega_X \cdot C < 0 \Rightarrow {}^{\exists}i$ s.t $C_i^2 < 0$

We can contract C_i by Thm 12.

Unfortunately, in general, X_k is a surface with very bad singularities. Moreover, we have no informations on X when $\kappa(X, \omega_X) = -\infty$.

Goal: We have to find some ~~xxxxxx~~ reasonable conditions on X. "

§4 MMP for log surfaces in Fujiki's class $\mathcal{C}$

Thm 14 (X, Δ): log surface in Fujiki's class $\mathcal{C}$

X: $\mathbb{Q}$-factorial

$\Rightarrow$ $(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} \cdots \cdots \rightarrow (X_k, \Delta_k)$

s.t

$\begin{cases} (X_k, \Delta_k) : \text{good minimal model} \\ \quad \text{or} \\ (X_k, \Delta_k) \rightarrow W : \text{Mori fiber space.} \end{cases}$ //

Thm 15 (X, Δ): log canonical surface in Fujiki's class $\mathcal{C}$

$\Rightarrow$ $(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} \cdots \rightarrow (X_k, \Delta_k)$

s.t

$\begin{cases} (X_k, \Delta_k) : \text{good minimal model} \\ \quad \text{or} \\ (X_k, \Delta_k) \rightarrow W : \text{Mori fiber space.} \end{cases}$ //

Thm 14 is a generalization of Thm 1.

Thm 15 is a generalization of Thm 4.

§5 Definitions

Def 16 X : compact analytic surface.

$$a(X) := \text{trans.deg}_{\mathbb{C}}\, \mathcal{M}(X)$$

$$0 \leqq a(X) \leqq \dim X = 2.$$

$\uparrow$ algebraic dimension of X

$$\dim X = a(X) \underset{\text{def}}{\Longleftrightarrow} X : \text{Moishezon}.$$ //

Def 17 X : compact analytic surface

X : Fujiki's class $\mathcal{C}$

$\underset{\text{def}}{\Longleftrightarrow}$ $X \underset{\text{bim}}{\sim}$ a compact Kähler surface

$\Longleftrightarrow \begin{cases} {}^{\forall} f : Y \to X \text{ resolution} \\ \quad \text{then } Y : \text{Kähler} \end{cases}$ //

Def 18 X : compact analytic surface

$$\iota : \underset{\substack{\parallel \\ X \setminus \operatorname{Sing} X}}{U} \hookrightarrow X$$

$\omega_X := \iota_* \omega_U$: canonical sheaf of X.

(Abuse of notation : $\omega_X \simeq \mathcal{O}_X(K_X)$) //

<u>Def 19</u> (log surfaces) X: compact analytic $\underbrace{\text{surface}}_{\text{normal}}$

$\Delta \in [0.1] \cap \mathbb{Q}$

If $K_X + \Delta$: $\mathbb{Q}$-Cartier, then $(X.\Delta)$ is called a <u>log surface</u>.

<u>Rem 20</u> $K_X + \Delta$: $\mathbb{Q}$-Cartier

$$\underset{\text{def}}{\iff} \begin{cases} \exists m \in \mathbb{Z}_{>0} \text{ s.t } m\Delta : \mathbb{Z}\text{-divisor} \\ (\omega_X^{\otimes m} \otimes \mathcal{O}_X(m\Delta))^{**} : \text{loc. free} \end{cases}$$
//

<u>Def 21</u> X: normal, compact analytic surface.

X: $\mathbb{Q}$-factorial

$\underset{\text{def}}{\iff}$ ${}^\forall$prime div D on X is $\mathbb{Q}$-Cartier.

$({}^\exists m \in \mathbb{Z}_{>0}$ s.t mD: Cartier$)$

//

<u>Def 22</u> $(X.\Delta)$: log surface.

$(X.\Delta)$: Fujiki's class $\mathcal{C}$ $\underset{\text{def}}{\iff}$ X: Fujiki's class $\mathcal{C}$

$(X.\Delta)$: $\mathbb{Q}$-factorial $\underset{\text{def}}{\iff}$ X: $\mathbb{Q}$-factorial.

$(X.\Delta)$: log canonical $\underset{\text{def}}{\iff}$ $a(E.X.\Delta) > -1$ for ${}^\forall E$ //

§6 Projectivity criteria

Lem 23 X : $\mathbb{Q}$-fac compact analytic surface.
X : Moishezon
$\Rightarrow X$: projective.

($\because$) X : algebraic space by Artin's GAGA
Nakai-Moishezon's ampleness criterion
for algebraic spaces $\longrightarrow X$: projective.

$$\left(\begin{array}{l} f : Y \longrightarrow X \quad \text{resol.} \\ \quad \uparrow \\ \text{projective} \quad H : \text{ample on } Y. \\ \quad f_* H : \text{ample on } X \end{array} \right)$$

//

Lem 24 (X, Δ) : $\mathbb{Q}$-fac. log surface in
Fujiki's class $\mathcal{C}$ with $\kappa(X, K_X+\Delta) = -\infty$
$\Rightarrow X$: projective.

($\because$) $f : Y \longrightarrow X$: minimal resol.
$\uparrow$
Kähler surface with $\kappa(Y) = -\infty$

$$\Downarrow \text{ Enriques-Kodaira}$$
Y : projective
$$\Downarrow$$
X : Moishezon
$$\Downarrow \text{ Lem 23}$$
X : projective //

<u>Idea</u> of the proof of Thms 14 and 15.

(X, Δ): $\mathbb{Q}$-fac log surface or log canonical surface.

X : Fujiki's class $\mathcal{C}$.

- X: projective $\Longrightarrow$ we can apply MMP for algebraic log surfaces.
- X : not projective $\Longrightarrow$ we use Sakai's contraction thm to obtain a minimal model. //

§7 Abundance theorem

(X, Δ): $\mathbb{Q}$-fac log surface in Fujiki's class $\mathcal{C}$

Thm 25 (Non-vanishing thm)

$(K_X+\Delta)\cdot C \geqq 0$ for $\forall$curve C on X

$\Rightarrow \kappa(X, K_X+\Delta) \geqq 0$. "

(∵) $f: Y \to X$: minimal resol.

$\uparrow$ Kähler

$$K_Y + \underset{\substack{\rotatebox{90}{$\leqq$} \\ 0}}{\Delta_Y} = f^*(K_X+\Delta)$$

If $\kappa(Y, K_Y) \geqq 0 \Rightarrow \kappa(X, K_X+\Delta)$
$= \kappa(Y, K_Y+\Delta_Y)$
$\geqq \kappa(Y, K_Y) \geqq 0$.

If $\kappa(Y, K_Y) = -\infty$, then Y is projective.

By Lem 23, X : projective.

$\underset{\uparrow}{\Longrightarrow} \kappa(X, K_X+\Delta) \geqq 0$.

MMP is ok for projective varieties.

Thm 26 (Abundance theorem)

$(K_X+\Delta)\cdot C \geqq 0$ for $\forall$curve C on X

$\Longrightarrow K_X + \Delta$: semi-ample.

$(\because)$ By Thm 25, $\kappa(X, K_X+\Delta) = 0, 1,$ or 2.

- $\kappa(X, K_X+\Delta) \geqq 2 \Longrightarrow X$: projective $\Longrightarrow$ o.k.
- $\kappa(X, K_X+\Delta) = 1 \Longrightarrow$ elementary exercise.
- $\kappa(X, K_X+\Delta) = 0 \Longrightarrow X$: proj $\Longrightarrow$ o.k

$\Downarrow$

X : non-proj $\rightsquigarrow X \underset{\mathrm{bim}}{\sim} K3$ or torus

$\rightsquigarrow$ o.k //

<u>Rem 27</u> (X, Δ) : $\mathbb{Q}$-fac. proj log surface,

the proof of abundance for (X, Δ) with

$\kappa(X, K_X+\Delta) = 2$ is very difficult!

$\kappa(X, K_X+\Delta) = 0$ case is also nontrivial.

The proof is based on techniques by

T. Fujita, Sakai, ⋯ .

4

§8 MMP for $\mathbb{Q}$-fac log surfaces in Fujiki's class $\mathcal{C}$ (Theorem 14).

(X, Δ): $\mathbb{Q}$-fac. log surfaces in Fujiki's class $\mathcal{C}$.

- $\kappa(X, K_X+\Delta) = -\infty \Rightarrow X$: projective (Lem 24)

  $\Rightarrow$ we can run the MMP for algebraic surfaces.

  $\Rightarrow$ we get a Mori fiber space.

- $\kappa(X, K_X+\Delta) \geqq 0 \Rightarrow$ we get

  $$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\varphi_0} \cdots\cdots \to (X_k, \Delta_k)$$

  by Sakai's contraction theorem s.t

  $(K_{X_k} + \Delta_k) \cdot C \geqq 0$ for $^\forall C$: curve on X_k.

Rem 28. we can check that

X_i: $\mathbb{Q}$-factorial for $^\forall i$.

(This is nontrivial.).

By Theorem 26, $K_{X_k} + \Delta_k$: semi-ample.

§9 Contraction for log canonical surfaces.

<u>Th 29</u> $(X . \Delta)$: compact log canonical surface.

not nec. in Fujiki's class $\mathcal{C}$

C : irr curve on X s.t

$-(K_X + \Delta) \cdot C > 0$ and $C^2 < 0$

Mumford's sense

$\Rightarrow$ $\exists\, \varphi : X \longrightarrow Y$ projective bim. morphism.

s.t $\mathrm{Exc}(\varphi) = C \simeq \mathbb{P}^1$

C passes through no nonrational singular points of X.

In particular, C : $\mathbb{Q}$-Cartier.

Moreover, $(Y . \Delta_Y)$: log canonical,

where $\Delta_Y = \varphi_* \Delta$ //

$(\because)$ $\exists\, \varphi :$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \cup & & \Psi \\ C & \longrightarrow & P \end{array}$$

Sakai's contraction.

$(X . \Delta)$: lc. $-(K_X + \Delta) \cdot C$

$\Longrightarrow (Y . \Delta_Y)$: lc

By the negativity lemma, Y is <u>log terminal</u> at P.

$\Rightarrow$ Y : rational singularities around P.

$\Rightarrow$ X : rational singularities in a nbd of C.

$R^i\varphi_* \mathcal{O}_X = 0$

$\Rightarrow$ C : $\mathbb{Q}$-Cartier.

§10 Projectivity for lc surfaces.

Thm 30 (X, Δ): log canonical surface in Fujiki's class $\mathcal{C}$.

$\kappa(X, K_X + \Delta) = -\infty$

$\Rightarrow X$: projective.

Rem 31 X: not nec. $\mathbb{Q}$-factorial.

"Thm 30 is a key result for MMP of lc surfaces in Fujiki's class $\mathcal{C}$."

Lem 31 (X, Δ): lc surface.

$P \in X$: not rational sing.

$\Rightarrow P \notin \operatorname{Supp} \Delta$, X: Gorenstein at P.

($\because$) This follows from the classification table //

Proof of Thm 30

Step 1 $f: Y \to X$ minimal resol

$$K_Y + \Delta_Y = f^*(K_X + \Delta)$$

$\quad\quad\quad\;\; \mathrm{VII}$
$\quad\quad\quad\;\; 0$

Y : smooth proj surface $\leftarrow \left| \begin{array}{l} \kappa(Y) = -\infty \\ Y : \text{Kähler} \end{array} \right.$

$\Rightarrow X$: Moishezon.

<u>Step 2</u> $X : \mathbb{Q}$-fac $\underset{\substack{\uparrow \\ \text{Lem 23}}}{\Longrightarrow}$ X : projective.

"We may assume that X : <u>not</u> $\mathbb{Q}$-fac"

<u>Step 3</u> By applying Th 29 finitely many times,

"we may assume that

$\left[\begin{array}{l} \text{if } C : \text{curve on } X \text{ with } -(K_X+\Delta)\cdot C > 0 \\ \text{then } C^2 \geqq 0. \end{array} \right.$"

<u>Step 4</u>

$$\begin{tikzcd} Y \arrow[d, "h"'] \arrow[dr, "f"] & \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

$g : Z \to X$
minimal resol of nonrational singularities

$$K_Z + \Delta_Z = g^*(K_X + \Delta)$$

$$\mathrm{Exc}(g) = \sum_i E_i$$

$$\underset{\substack{\uparrow \\ \text{Lem 31}}}{\Longrightarrow} \quad \Delta_Z = \sum_i E_i + g_*^{-1}\Delta$$

Z: $\mathbb{Q}$-factorial, projective

$\cdot$ $\kappa(Z, K_Z + \Delta_Z) = -\infty$.

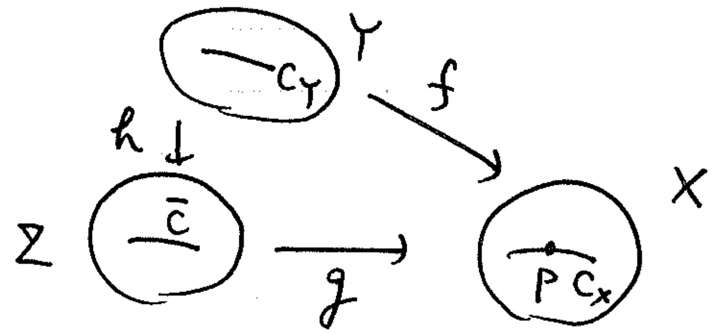
$\exists$ $(K_Z + \Delta_Z)$-negative extremal ray

$R = \mathbb{R}_{\geqq 0} [\bar{C}]$ //

<u>Step 5</u> <u>Claim</u> $\bar{C}^2 \geqq 0$.

($\because$) Assume $\bar{C}^2 < 0$.

$-(K_Z + \Delta_Z) \cdot \bar{C} > 0 \Rightarrow \bar{C}$ not g-excep.



$C_X := g_* \bar{C}$ C_X passes through at least one nonrational singular point $P \in X$.

$C_X \not\subset \mathrm{Supp}\,\Delta$.

$\cdot$ C_Y: strict transform of $\bar{C}$ on Y

$$\Rightarrow (C_Y)^2 < 0, \quad C_Y \not\subset \operatorname{Supp} \Delta_Y .$$

$$-K_Y \cdot C_Y \geqq -(K_Y + \Delta_Y) \cdot C_Y$$

$$= -(K_Z + \Delta_Z) \cdot \overline{C} > 0 .$$

$\therefore$ C_Y : (-1)-curve.

$$\Rightarrow -K_Y \cdot C_Y = 1 .$$

On the other hand, $\Delta_Y \cdot C_Y \geqq 1$

since $P \in C_X$.

$$\Rightarrow -(K_Y + \Delta_Y) \cdot C_Y = 1 - \Delta_Y \cdot C_Y \leqq 0 .$$

$\lightning$

$$\therefore \overline{C}^2 \geqq 0$$

//

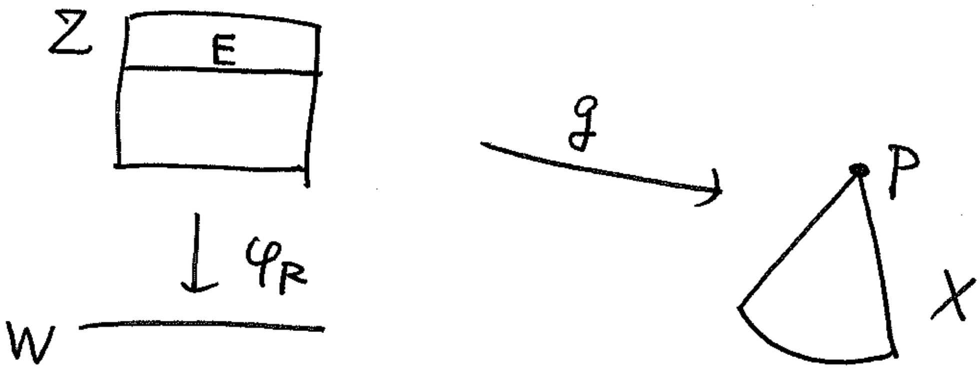
<u>Step 6</u> ($\overline{C}^2 > 0$ case)

$\Rightarrow \rho(Z) = 1$ we get a contradiction!

<u>Step 7</u> ($\overline{C}^2 = 0$ case)

$$\underset{\substack{\| \\ \mathbb{R}_{\geq 0}[\overline{C}]}}{\varphi_R} : Z \longrightarrow \underset{\substack{\uparrow \\ \text{smooth curve}}}{W} \quad : \text{extremal contraction},$$

we can get the following picture



$$E = \mathrm{Exc}(g)$$

$\varphi_R : E \xrightarrow{\sim} W$: elliptic curve

We can directly check that $-K_X$: ample.

$\Rightarrow X$: projective. //

§11 MMP for lc surfaces in Fujiki's class $\mathcal{C}$

(X, Δ): lc surface in Fujiki's class $\mathcal{C}$.

not nec. $\mathbb{Q}$-fac.

- $\kappa(X, K_X+\Delta) = -\infty \Longrightarrow X$: proj by Thm 30.

  We can apply MMP for algebraic surfaces. We get a Mori fiber space.

- $\kappa(X, K_X+\Delta) \geqq 0 \Longrightarrow$ By Th 29, we get

  $$(X, \Delta) =: (X_0, \Delta_0) \to \cdots\cdots \to (X_k, \Delta_k)$$

  s.t $(K_{X_k}+\Delta_k)\cdot C \geqq 0$ for $^\forall C$.

  $\Longrightarrow$ $K_{X_k}+\Delta_k$: semi-ample.

  Thm 26
  (Abundance)

"

Thm 15 holds true.

§12 Appendix: Complete nonprojective algebraic surfaces (mainly due to Kento Fujita).

Question 32 $\exists$? (X, Δ) : log canonical complete nonprojective algebraic surface ??

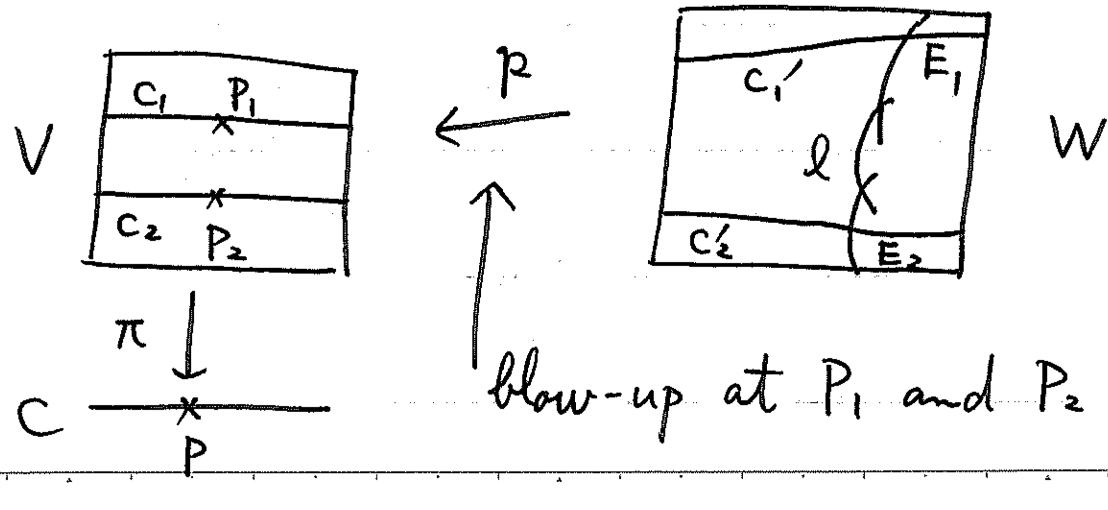
Answer 33 (Kento Fujita) Yes!!

Example 34 C : smooth elliptic curve.

$$\mathcal{L} = \mathcal{O}_C(L) \in \mathrm{Pic}^0(C)$$

$\uparrow$ nontorsion element

$$\pi : V = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$$



l: strict transform of $\pi^{-1}(P)$

E_1, E_2: exceptional curve

$g : W \longrightarrow S$: contraction of

C_1', C_2' and l.

strict transform of C_1

Then $\mathrm{Pic}(S) = \{0\}$.

In particular, S is nonprojective.

We can directly check that

$K_S \sim 0$, Calabi-Yau,

S : log canonical,

$\pi_1(S) = \{1\}$

"

Rem 35 k: algebraically closed field

s.t $k \neq \overline{\mathbb{F}_p}$

The above construction

works over k.

Therefore, we have complete nonprojective algebraic surfaces / k

$k = \overline{k}$
$k \neq \overline{\mathbb{F}_p}$ //

Rem 36 $k = \overline{\mathbb{F}_p}$

X: complete algebraic surface / k

$\underset{\substack{\uparrow \\ \text{Artin}}}{\Longrightarrow}$ X: projective.

Problem 37 $\exists$? complete nonprojective algebraic log canonical surface (X, Δ) with $\kappa(X, K_X + \Delta) = 2$??