Investigations of moduli of stab. bundles on surf.

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\( X \) smooth proj. surf

\[
\begin{align*}
\text{fix } r, c_1, c_2 & \quad C_1 \in NS(X) \\
M = M(r, c_1, c_2) & \quad \text{moduli sp. of stab. vect. bundles} \\
& \quad \text{on } X \text{ with rank } \neq r \text{ given } c_1, c_2.
\end{align*}
\]

General properties

\( r, c_1 \text{ fix. } c_2 \gg 0 \Rightarrow M \text{ is good i.e. generically smooth of expected dim. and irreducible.} \)

Bagaudov-Gieseker inequality

\[
\begin{align*}
c_2 - 2 \frac{r-1}{r} c_1^2 & < 0 \\
\Rightarrow & \quad M = \emptyset.
\end{align*}
\]

Quest. What happens intermediate value \( c_2 \)?

Can we understand in some case?
Mestrano 1997.

3 examples $X \subset \mathbb{P}^3$ deg 7\#7. where

$M_x(2,1,2) \text{ is not irreducible }$

(at least one good and

one not good camp)

Our subject:

Recall if $X$ is Calabi-Yau (K3 or Abelian)

$\Rightarrow \quad \mathcal{M} \text{ is a smooth symplectic variety.}$

(Mukai, ...) irreducible (Yoshioka).

This applies to deg 4 hypersurft in $\mathbb{P}^3$.

Our situation, $X \subset \mathbb{P}^3$, deg $X = 5$. smooth, very general.

$\Rightarrow \quad K_X = O_X(1)$

Pic($X$) = $\mathbb{Z} \cdot O_X(1) \Rightarrow "\text{Stability is well-det}"

\text{rank } r = 2 \cdot C = O_X(1) \text{ i.e. } \text{det}(E) = O_X(1)

\text{slope \ Semi-stability implies stability.}

$\Rightarrow \quad \text{all points of } M_x(2,1,2) \text{ are stable.}$

$\Rightarrow \quad H^0(\text{End}(E)) = C.$
1st quest. When is the moduli good?

2nd quest. What about irreducibility?

   Discussion of $c_2 < 9$.

2. Deformation to the boundary, irreducibility.

3. The case $c_2 = 0$.

$x \subset \mathbb{P}^3$, $\deg x = 5$.

$k_x = \mathcal{O}_x(1)$

$h^1(\mathcal{O}_x(n)) = 0$ $(\forall n)$

$\rho_{ic}(x) = 2.\mathcal{O}_x(1)$

$x(\mathcal{O}_x(n)) = \frac{5}{2}(n^2 - n) + 5$.

Fix $c_2 \in \mathbb{N}$.

$M =$ the moduli space of stable bundles $E$

with rank = 2, $\text{det} E = \mathcal{O}_x(1)$, $c_2(E) = c_2$.

$\Rightarrow E = E(n)$

$x(E(n)) = 5n^2 + 10 - c_2$ $(\text{if } c_2 = 0 \Rightarrow x = 5n^2)$

$\deg E(n) = 2n + 1$.

$H^0(\mathbb{P}^3, \mathcal{O}(n)) \twoheadrightarrow H^0(x, \mathcal{O}(n)) \xrightarrow{\cdot n} (n \leq 4)$.
Expected dimensions:

\[ E \otimes E = \text{End}(E) = \text{End}^0(E) \oplus \mathcal{O} \times 1_E \]

\( \mathcal{O} \) trace-free end.

deflection & obstruction are given by:

\[ H^1(X, \text{End}^0(E)) \quad \& \quad H^2(X, \text{End}^0(E)) \]

Kuranishi theory: the moduli space can be locally viewed as the set of the map:

\[ H^1(\text{End}^0(E)) \rightarrow H^2(\text{End}^0(E)) \]

\[ \dim_E(M) \geq h^1(\text{End}^0(E)) - h^2(\text{End}^0(E)) = -\chi(\text{End}^0(E)) \quad \text{expected dim.} \]

Def: \( M \) is good at \( E \) if \( \dim_E(M) = -\chi(\text{End}^0(E)) \)

Notice if \( h^2(\text{End}^0(E)) = 0 \Rightarrow M \) is good and smooth at \( E \).

An \( \text{irr} \) comp of \( M \) is good \( \iff h^2(\text{End}^0(E)) = 0 \)

at general pt.

In K-theory:

\[ E \sim \mathcal{O} \oplus \mathcal{O}(t) - c_2(pt) \]

\[ E^* \sim \mathcal{O} \oplus \mathcal{O}(-t) - c_2(pt) \]

\[ E \otimes E^* \sim \mathcal{O} \oplus \mathcal{O}(-t) \oplus \mathcal{O} \oplus \mathcal{O}(t) - 4c abd(pt) \]

\[ \text{End}^0(E) \sim \mathcal{O} \oplus \mathcal{O}(t) \oplus \mathcal{O}(t) - 4c abd(pt) \]

\[ \chi(\text{End}^0(E)) = \frac{t}{2} (\chi^2 - \chi^t) + 5 + 5 + \frac{5}{2} (t^2 - 1) + 5 - 4c \]

\[ = 20 - 4c_2. \]
Expected dim = 4g - 20.

Serre duality: use $K_X \otimes O_X(1)$

$H^i(E(1)) \cong H^{2-g-i}(E(-1))$.

For $\text{End}^0(E) \sim \text{End}^0(E)^*$

$H^i(\text{End}^0(E)) \cong H^{2-g-i}(\text{End}^0(E) \otimes K_X)^*$

in any case $H^2(\text{End}^0(E))$ is dual to $\text{Hom}(E \otimes \mathcal{K}_X)$.

So an element of the dual of the space of obstructions may be seen as a Higgs field $\phi: E \to E \otimes K_X$. $\text{Tr} \phi = 0$

$(E, \phi)$ is called a "Hitchin pair".

In particular,

$\text{Tr}(\phi) = 0 \Rightarrow \phi$ has a spectral cover.

In our case $K_X = O_X(1)$

$Z \in \text{Tot}(\mathcal{K}_X)$

$Z$ is given by $\Gamma - \text{det}(\phi)$,

$\text{det}(\phi) \in H^0(K_X^{\otimes 2}) = H^0(O_X(2))$. 
Strategy to understand if \( M \) is good.

To try to bound the dimensions of the space of co-constructed bundles \( \{(E, g)\} \).

If we can show \( \dim \{ (E, g) \} < 4c^2 - 20 \).

\[ \Rightarrow \] all irr comp are good.

What can \( Y \) look like? \( \varphi = 0 \).

(1) \( \text{dec}(Y) = 0 \), \( Y \) is nilpotent. Will see then \( Y \) comes from an exact seq.

\[ \begin{array}{c}
0 \rightarrow \mathcal{O}_\mathbb{P} \rightarrow E \rightarrow \mathcal{O}_\mathbb{P}(1) \rightarrow 0 \quad \text{lpl = c}\nu. \\
\end{array} \]

\[ \text{Via} \quad E \rightarrow \mathcal{O}_\mathbb{P}(1) \frac{\varphi}{\alpha} \mathcal{O}_\mathbb{P}(1) \rightarrow E(1) \]

(2) \( \text{det} Y = \alpha^2 \in H^0(\mathcal{O}_\mathbb{P}(2)) \) for \( \alpha \in H^0(\mathcal{O}_\mathbb{P}(1)) \).

The spectral cover \( \mathcal{X} \) is reducible a union. In this case one can see \( \exists \) exact seq of \( (T-\alpha)(T+\alpha) \).
(3)\[\text{clearly it is not a square, it defines } R^X,\]
\[\text{reduced div in } \{0x(2)\}.
\]
\[Z^1 \text{ is ramified cover}.
\]
\[\text{ramified along } R.
\]
Thus \(M\) is good for \(c > 10\). not good for \(c < 9\).

proof by knowing \(t\) the dimensions of the case (i), (ii), and (iii).
Yesterday

\[ E \ni \exists \psi, E \rightarrow E(1) \]

(i) \( \psi \) is nilpotent
(ii) \( \det(\psi) = \alpha^2 \quad \alpha \in H^0(X, \mathcal{O}_X(1)) \)
(iii) \( \det(\psi) \) is not square \( \Leftrightarrow \mathcal{O}_X(2) \) reduced div.

\[ \text{Tot}(\mathcal{K}_X) \supset Z = 2:1 \text{ covering of } X \text{ branched over } R \]

\[ ^{\wedge} \text{spectral covering of } \psi \]

\( Z \) is irreducible.

\[ \hat{Z} \rightarrow Z \text{ resolution} \quad (\mathbb{Q}/\mathbb{Z} \text{ equiv}) \]

outside of codim 2 subset \( (E, \psi) \leftrightarrow q \).

line bundle \( L \) on \( Z \) and \( E = \mathcal{O}_X \cdot L \).

\( \hat{L} : \hat{Z} \rightarrow X \cdot \text{ extend } L \text{ to } \hat{Z} \).

\[ \hat{E} = \hat{L} \cdot (\hat{Z})^* \]

\( (E, \psi) \) is determined by \( \hat{E} \in \text{Pic}(\hat{Z}) \)

\( \dim \text{ count} \quad \dim \{ E \in \text{case } (iii) \} \)

\[ \leq \left\{ \overline{\text{Space of } R \in \mathcal{O}_X(2)} \right\} + \max \dim \text{Pic}^0(\hat{Z}) \]

for general \( R, \text{Pic}^0(\hat{Z}) = 0 \)
Take a curve \( C = \cup_0 X \) \( U \) = general plane \( \text{in } \mathbb{P}^3 \).

\[ \mathbb{P}^2 \downarrow \mathbb{P}^1 \xrightarrow{\text{Pic}(\mathbb{P}^2)} \text{Prym}(Z_0/X) \]

\( \mathfrak{m}_C \xrightarrow{\mathfrak{m}_C} \mathfrak{m}_C \xrightarrow{\mathfrak{m}_C} \mathfrak{m}_C \)

\( C \) has genus \( 16 \).

\( \text{Prym}(Z_0/X) \text{ dim } 10 \).

\( \text{total dim (case (ii)) } \leq 18 \).

\( \mathfrak{m}_C \) \( \exp \text{ dim } \geq 20 \) this case doesn't contribute to a general \( \mathfrak{m}_C \).

\( \text{Case (i) } \mathfrak{m}_C \text{ has factor as } \)

\( E \xrightarrow{-} \mathfrak{f}_p(1) \xrightarrow{-} \mathfrak{f}_p(1) \xrightarrow{-} E(1) \)

\( (\star) \) \( 0 \xrightarrow{-} \mathfrak{f}_p \xrightarrow{-} E \xrightarrow{-} \mathfrak{f}_p(1) \xrightarrow{-} 0 \).

\( \text{Case (ii) } 2^{\text{nd}} (\star) \text{ an exact seq } \text{ no } \phi \).

\( M \supset \mathcal{V} = \{ E \mid \mathfrak{t}^0(E) \geq 1 \} \).

For \( V \supset E \), we have at least one exact seq (\( \star \)).
This figure in O'Grady's approach:
Nisse for bundles on quintic (1998)

Understanding bundles in $V$

Thus, fundamental fact point about the Serre construction is that $P$ is a.R.C. subscheme which satisfies the Cayley--B-- conditions $CB(p)$ for $CB(2)$

$$\xi \in \text{Ext}^1(I_{p/k}(1), O_k) = H^1(I_p(2))^*$$

exact class of $(*)$.

$$0 \to I_p(2) \to O_k(2) \to O_p(2) \to 0.$$  

$$0 \to H^0(I_p(2)) \to H^0(O_k(2)) \to H^0(O_p(2)) \to H^1(I_p(2)) \to 0.$$  

$H^1(I_p(2))^*$ may be seen as the space of torsion:

$$H^0(O_p(2)) \xrightarrow{\text{natural}} \xi \text{ s.t. } \xi |_{H^0(O_k(2))} = 0.$$  

In K--theory $E \sim O_k \oplus O_k(1) - |P| \cdot \text{pt}$.

$\Rightarrow C(E) = P$.

Cayley--B-- the conditions to insure that $E$ is defined by $\xi$, is loc--free:

such that $\xi$ proper, $H^0(O_k(2)) \nrightarrow 0$,

$$H^0(O_p(2)) \times \text{pt}.$$
Def \( P \) satisfies CB(1) if

\[ p' \subset p \text{ colength 1 subsch} \]

\[ \text{Ker} \\
\text{Ker} \rightarrow H^0(O_X(n)) \rightarrow H^0(O_p(n)) \rightarrow H^0(O_{p'}(n)) \]

\( P \) imposes the same cond. on \( P \) on \( H^0(O_X(n)) \)

(\( \exists \exists \exists \) s.t. CB(1) \( \Rightarrow \) CB(2))

1st example: \( P \) consists of \( n \) points in general position on \( X \).

\[ h^0(O_X(1)) = 10 \text{ any } p' \subset p \text{ imposes vanishing on } P \]

So \( P \) satisfies CB(2)

For \( c > 11 \), \( V \subset \{ \text{all } P \} \)

For a given choice of subscheme \( P \), the space of ext.

is \( \text{Ext} (P, H^0(f_p(1)^*))) \)
$$\dim V(\alpha) = 3c_2 - 1.$$ 

\[\begin{align*}
\{ E \} & \rightarrow \{ (E, s) \} = \{ (p, \beta) \} \rightarrow \{ p \}_{2c_2} \\
h^0(E) = 1
\end{align*}\]

$$\dim c_2 = 10 - 1$$

$$c_2 \geq 10 \Rightarrow 3c_2 - 11 \leq \exp(\dim).$$

$$c_2 = 10 \Rightarrow X(E_{11}) = 5n^2 \leq 3^{10} = 1200.$$ 

For \(c_2 \geq 11\) the sing pes form a strict subset of the moduli sp \(\Rightarrow\) good.

\(c_2 = 10 \Rightarrow 10\) general pts don't satisfy \(\mathcal{CB}(2)\).

\(\text{vanish at 9 pts leave } \frac{1}{2}\text{ of } H^0(\mathcal{O}(21)).\)

which is non zero at 10 point.

To construct a \(\mathcal{CB}(2)\) collection of 10 pts:

Choose a sect. of \(H^0(\mathcal{O}(21))\) defining \(Y \subseteq X \cap 1_{\mathcal{O}(21)}\).

Then choose \(P = 10\) general pt on \(Y\).

If we start with 9 general pts on \(Y\), \(Y\) is the only sect allowed, it vanishes at 10 pts.
\{ P \} = \{( Y, p) \}, \quad Y \text{ dim } 9, \quad h^0(\mathcal{O}_Y(1)) = 10.
\{ P \} \text{ has dim } 10. \quad V(1) \text{ has dim } 19 \geq 3g - 11.
\leq \text{ dim } = 20.
\Rightarrow M(C_2) \text{ is good}.

**Theorem** For \( C_2 \geq 10 \), \( M(C_2) \) is good. (Nisse 1995, \( C_2 \geq 13 \))

Next question: What do the \( M(C_2) \) look like \( C_2 \leq 9 \)?

\[ X(E) = h^0(E) - h^1(E) + h^1(E) = 10 - C_2. \]

\[ \Rightarrow 2 - h^0(E) - h^1(E) = 10 - C_2. \]

\[ C_2 \leq 9 \Rightarrow h^0(E) > 0 \quad \Rightarrow \exists \text{ exact seq. (x)} \]
\[ C_2 \leq 7 \Rightarrow h^0(E) \geq 2 \quad \Rightarrow h^0(\mathcal{O}_E(1)) > 1 \quad P \subset \text{plane} \]
\[ C_2 \leq 6 \Rightarrow h^0(E) \geq 3 \quad \Rightarrow \mathcal{O}_P \text{ (plane) } n(\text{plane}) = \text{line} \]

\[ M(C_2) \subset V(C_2) \text{ for } C_2 \leq 9 \]

\[ 0 \to \mathcal{O}_X \to E \to \mathcal{O}_P(1) \to 0 \]
\[ C_2 \leq 5 \quad P \leq l \text{ in } \mathbb{P}^3 \]

\[ H^0(\Omega_{\mathbb{P}^3}(2)) \xrightarrow{\sim} H^0(\mathcal{O}_l(2)) \xrightarrow{\text{not true in case } |P'| = 3} H^0(\mathcal{O}_l(2)) \xrightarrow{\text{vanishing of } S < H^0(\Omega_{\mathbb{P}^3}(2))} \]

\[ \text{vanishing on } P' \Rightarrow \text{vanishing on } l \]
\[ \Rightarrow \text{vanishing on } P \]

\[ P \subset l \text{ satisfies } CB(2) \iff |P'| > 4 \]

\[ |ln x| = 5 \Rightarrow 3 \text{ cases here} \]
\[ C_2 = 4, \quad P = 4 \text{ points out of the 5 points in } x \cap l \]
\[ C_2 = 5, \quad P = x \cap l \]

Today's conditions to understand CB(2) or CB(4)...

we need to understand curves in $\mathbb{P}^3$. 
It turns out in fact:

\[ C_2 = 4 \]

\[ 0 \rightarrow R_{\phi} \rightarrow \mathcal{O}(U) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{(1)} \rightarrow 0 \]

\[ \text{rank} = 2 \]

\[ \dim M(4) = 2, \quad M(4) \cong X = \{ q \in X \} \]

\[ R_{\psi} \leftarrow q \]

\[ C_2 = 5, \quad M(5) \leq \dim = 3, \quad R_{\phi} \]

\[ \{ q \in \mathbb{P}^3 \setminus X \} \rightarrow q \]
There moduli sp are non cpo.
(except the 1st one $M(41)$)

We can compactify consider the compactification by torsion free sheaves
(Shobuich compact.)

$\tilde{M} = \tilde{M}(2,1,c_2)$ be the moduli sp of tor tree
sheaves $E$ on $X$.
$t = 2 \quad \det = \mathcal{O}(1), \quad G(E) = C_2$

This has what O'Grady the double dual
stratification: if $E \in M(2)$

$$0 \to E \to E^w \to \mathcal{A} \to 0$$

$E$ is torsion, length $d$

If $d = 0 \Rightarrow E = E^w$

If $E \in \mathcal{M}_0 \triangleq \left\{ E: \text{non locally free} \right\}$

$C_2(E^w) = C_2(E) - d$

$E^w$ is stable, $E^w \in M(21-d)$
The invariant $d$ defines stratification of $\tilde{M}(c)$ with strata defined by $M(c, c_d)$, we have:

\[ \tilde{M}(c, c_d) \xrightarrow{\phi} M(c) \]

the fibers of this map have $\dim = 3$.

and are irreducible with dense open subset consisting of points $E$ such:

\[ A = E^w / E = \bigoplus_{i=1}^{d} E_i, \quad E_i = c_{y_i} \]

\[ E \xleftarrow{\psi} \text{quot} \xrightarrow{\pi} \mathcal{S} \xrightarrow{\pi} 0 \]

\[ \bigoplus_{i=1}^{d} \text{points } y_1 \ldots y_d \in X \]

at each pt $E_i$ quotient:

\[ (E^w)_y \rightarrow C_{y_i} \rightarrow 0 \]

Thin $(L_i, \text{Ellin} \ldots) \sim \text{Lehn}$

The closure of this open set $= \text{Quot}(E^w)$.

$\overline{M(c)}$ is projective.
<table>
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<th>$d$</th>
<th>$M(\mathbb{C}(\mathbb{Q}_d))$</th>
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</tr>
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$\dim_{\mathbb{C}} \text{H}^{\text{full}}_{\text{generic}} = 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0$

**Note**

$\overline{M}(\mathbb{C}) = \text{closure of } M(\mathbb{C}) \subseteq \overline{M}(\mathbb{C})$

$\overline{M}(\mathbb{C}) = \overline{M}(\mathbb{C}) \cup M(\mathbb{C}, 4)$

**Theorem 1**

$M(\mathbb{C})$ is irreducible for $c_2 \leq 9$
Theorem 2
Define $M^{sm}(10) \subseteq M(10)$ components
generically seminatural

$H^i(E) = H^i(E(1)) = \ldots = H^i(E(n)) = 0$.

$\Rightarrow M^{sm}(10)$ is irr.

Theorem 3
$M(2)$ is irr for $c > 0$.

Discuss Theorem 1
$c_0 < q \Rightarrow 3 \geq \deg_1 - c_0 \rightarrow E \rightarrow \psi(1) \rightarrow 0$

$c_2 = 4.5$ we saw yesterday $p$~line

$c_3 = 6.9$, $p$~plane (exercise)

$c_4 = 8.9$,

First expected you think of let $N \subset P^3$

rat normal curve.

$N \neq P^3$, $\psi(1)$ has deg 3

Recall we are looking for $P \subset X \subseteq P^3$
satisfying $\text{CB}(2)$ i.e. C.B. for $H^0(\psi(1))$

$= H^0(\psi(2))$

$P = \text{any 8 pts in } N \subseteq P^3$

$\psi_3(2)|N$ = a line bundle of deg 6 on $P^1$

$P \cap CP$ contains 7 pts $f \in H^0(CP(1))$, $f(P) = 0$

$\Rightarrow f|N = 0 \Rightarrow f(P) = 0$
$P \leq X \cap N$ choose 8 or 9 pts.

The full families

$Q = 8 \quad \text{"Cayley acted"}$

Choose $W^3 \subseteq \mathcal{C}^0 = H^0(\mathbb{P}^3(2,1))$

Let $Z = \bigcap_{w \in W} \text{Zero}(w) = \text{not of dim 3 good}$.

for $W$ general it has 8 pts call that $P$

$H^0(\mathbb{P}^3(2)(-P')) = W \subseteq \mathcal{C}^3 \subseteq \mathcal{C}^0$

all $w \in W$ vanish on $P$.

If $P \leq X \Rightarrow \text{set } (P, 3) = (E, \lambda)$

$\{P, 3\} = \{E, \lambda\} \quad \text{if } P \neq \text{Plane } \Rightarrow h^0(E) = 1.$

$\begin{align*}
\{P\} & \subseteq \{P, f(\omega)\} = 1. \\
\{P\} & \neq \{P, f(\omega)\} = 1. \\
\end{align*}$

When is $\{P\} \leq X$ irr of dim 13.
\{ X \in \mathbb{P} \} \dim = 49 + 21 = 68

\{ \mathbb{P} \subset \mathbb{P}^3 \} \quad \{ x \} \quad \dim = 21

\text{Argue that for irreducibility of the plane}

\{ X = \mathbb{P} \} \text{ irr } \Rightarrow \text{ Galois act transitively.}

\text{It suffices to find a canonical comp for } X \text{ for ex using } P(N) \text{ or better using }

\begin{array}{c}
\text{P} = \\
\text{Is a (2,2) complete Intersect.}
\end{array}

O'Grady's method allows one to show that 
\{ \mathbb{Z} \neq \phi \} \text{ for an irr comp } \mathbb{Z} \in \mathcal{M}(s)

\text{Lemma (O'Grady) if } \mathbb{Z} \neq \phi \Rightarrow \text{codim } \mathbb{Z} = 1
Theorem (Nijssen)
for $c > c_0$, any irreducible $\overline{Z} \subset \overline{M(g)}$
then $\overline{Z} = \emptyset$.

Thus we have seen the families

part 1: dimension count to show there are
no strange components, which might
come from $P \leq 2$.

(2.2) complete but in $P_1$
but $\overline{Z}$ reducible
$2 = 2' \cup \overline{Z}' \cup \cdots$

Thus similar quest for (3.3) complete fut.

Thus 1 + 2 + Theorem (Nijssen) $\Rightarrow$ Theorem 3.

$C_2 = 0$ we can check that all points
$\overline{M}(10)$ corresponds to $H^1(\overline{E(11)}) = 0 \Rightarrow \overline{m}$.

$C_2 = 11$ check that general pt on
$\overline{M}(11,10) \cap \overline{M}(11,4)$ is smooth.

Thus $\overline{M}(11)$ is smooth.

$C_2 > 12$ only one boundary comp.