J. Math. Sci. Univ. Tokyo
7 (2000), 1–5.

Base Point Free Theorem of Reid-Fukuda Type

By Osamu Fujino*

Abstract. Let (X, Δ) be a proper dlt pair and L a nef Cartier divisor such that $aL-(K_X+\Delta)$ is nef and log big on (X, Δ) for some $a \in \mathbb{Z}_{>0}$. Then $|mL|$ is base point free for every $m \gg 0$. Furthermore, we give a partial answer to the four-dimensional log abundance conjecture in the appendix.

0. Introduction

The purpose of this paper is to prove the following theorem. This type of base point freeness was suggested by M. Reid in [Re, 10.4].

Theorem 0.1 (Base point free theorem of Reid-Fukuda type). *Let (X, Δ) be a proper dlt pair and L a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on (X, Δ) for some $a \in \mathbb{Z}_{>0}$. Then $|mL|$ is base point free for every $m \gg 0$, that is, there exists a positive integer m_0 such that $|mL|$ is base point free for every $m \geq m_0$.*

This theorem was proved by S. Fukuda in the case where X is smooth and Δ is a reduced simple normal crossing divisor in [Fk2]. In [Fk3], he proved it on the assumption that $\dim X \leq 3$ by using the log Minimal Model Program. Our proof is similar to [Fk3]. However, we do not use the log Minimal Model Program even in $\dim X \leq 3$. He also proved this theorem in $\dim X \geq 4$ under some extra conditions (see [Fk4]).

Acknowledgements. I would like to express my sincere gratitude to Dr. Daisuke Matsushita for giving me some comments. I am also grateful to Professor Shigefumi Mori, whose advice helped me to prove the appendix.

Notation. (1) We will make use of the standard notation and definitions as in [KoM].

1991 *Mathematics Subject Classification*. Primary 14C20; Secondary 14J10.
*Research Fellow of the Japan Society for the Promotion of Science.

(2) A pair (X, Δ) denotes that X is a normal variety over $\mathbb{C}$ and Δ is a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

(3) Diff denotes the different (see [Utah, Chapter 16]).

1. Preliminaries

In this section, we make some definitions and collect the necessary results.

DEFINITION 1.1 (cf. [Ka2, Definition 1.3]). A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, Δ), if there exists a proper birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y with the discrepancy $a(E, X, \Delta) \leq -1$ such that $\mu(E) = W$.

DEFINITION 1.2. Let (X, Δ) be lc and D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. The divisor D is called *nef and log big* on (X, Δ) if D is nef and big, and $(D^{\dim W} \cdot W) > 0$ for every center of log canonical singularities W for the pair (X, Δ).

REMARK 1.3. (1) Our definition of nef and log big is equivalent to that of Reid and Fukuda (see [Fk3, Definition]).

(2) The pair (X, Δ) is dlt if and only if it is wklt (see [Sz]).

(3) In [Fj], centers of log canonical singularities of dlt pairs were investigated (see [Fk, Definition 4.8, Lemma 4.9]).

The following proposition is a variant of Kawamata-Shokurov base point free theorem (cf. [Fk3, Proposition 2], for the proof, see [Ka1, Lemma 3] and [Fk2, Proof of Theorem 3]).

PROPOSITION 1.4. *Let (X, Δ) be a proper dlt pair and L a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{Z}_{>0}$. If* $\mathrm{Bs}|mL| \cap \llcorner \Delta \lrcorner = \emptyset$ *for every $m \gg 0$, then $|mL|$ is base point free for every $m \gg 0$, where* $\mathrm{Bs}|mL|$ *denotes the base locus of $|mL|$.*

The next lemma is a generalization of Kawamata-Viehweg vanishing theorem.

LEMMA 1.5 (cf. [Fk1, Lemma]). *Let X be a proper smooth variety and $\Delta = \sum_i d_i \Delta_i$ a sum of distinct prime divisors such that* $\mathrm{Supp}\Delta$ *is a simple normal crossing divisor and d_i is a rational number with $0 \le d_i \le 1$ for every i. Let D be a Cartier divisor on X. Assume that $D - (K_X + \Delta)$ is nef and log big on (X, Δ). Then $H^i(X, \mathcal{O}_X(D)) = 0$ for every $i > 0$.*

2. Proof of Theorem

PROOF OF THEOREM (0.1). By the definition of dlt pairs (see [Sh, 1.1]), there exists a log resolution (see [KoM, Notation 0.4 (10)]) $f : Y \to X$ of (X, Δ), which satisfies the following conditions:

(1) $K_Y + f_*^{-1}\Delta = f^*(K_X + \Delta) + \sum_i a_i E_i$ with $a_i > -1$ for every i, where E_i's are irreducible exceptional divisors,

(2) f induces an isomorphism at every generic point of center of log canonical singularities for the pair (X, Δ).

(See also [Sz, Divisorial Log Terminal Theorem].) We define $E := \sum_i \ulcorner a_i \urcorner E_i \ge 0$ and $F := f_*^{-1}\Delta + E - \sum_i a_i E_i$. Then $K_Y + F = f^*(K_X + \Delta) + E$. If $\llcorner \Delta \lrcorner = 0$, then (X, Δ) is klt. So we can assume that $\llcorner \Delta \lrcorner \ne 0$. We take an irreducible component S of $\llcorner \Delta \lrcorner$. By [KoM, Corollary 5.52], S is normal. Therefore, $(S, \mathrm{Diff}(\Delta - S))$ is dlt by [Sh, 3.2.3] (see also [KoM, Definition 2.37] and [Utah, 17.2 Theorem]). We put $T := f_*^{-1}S$ and $M := f^*L$. We consider the following exact sequence:

$$0 \to \mathcal{O}_Y(-T) \to \mathcal{O}_Y \to \mathcal{O}_T \to 0.$$

Tensoring with $\mathcal{O}_Y(mM + E)$ for $m \ge a$, we have the exact sequence:

$$0 \to \mathcal{O}_Y(mM + E - T) \to \mathcal{O}_Y(mM + E) \to \mathcal{O}_T(mM + E) \to 0.$$

By Lemma (1.5), $H^1(Y, \mathcal{O}_Y(mM + E - T)) = 0$. We note that M is nef and $mM + E - T - (K_Y + F - T) = f^*(mL - (K_X + \Delta))$ is nef and log big on $(Y, F - T)$. Then we have that

$$H^0(Y, \mathcal{O}_Y(mM + E)) \to H^0(T, \mathcal{O}_T(mM + E))$$

is surjective. By the projection formula, we have that

$$H^0(Y, \mathcal{O}_Y(mM + E)) \simeq H^0(X, f_*\mathcal{O}_Y(mM + E)) \simeq H^0(X, \mathcal{O}_X(mL))$$

and

$$H^0(T, \mathcal{O}_T(mM+E)) \supset H^0(T, \mathcal{O}_T(mM)) \simeq H^0(S, \mathcal{O}_S(mL)).$$

Note that E is effective and f-exceptional and that $E|_T$ is effective but not necessarily $f|_T$-exceptional, where $f|_T : T \to S$. We consider the following commutative diagram:

$$\begin{array}{ccccc}
H^0(Y, \mathcal{O}_Y(mM+E)) & \longrightarrow & H^0(T, \mathcal{O}_T(mM+E)) & \longrightarrow & 0 \\
\uparrow \cong & & \uparrow \iota & & \\
H^0(X, \mathcal{O}_X(mL)) & \longrightarrow & H^0(S, \mathcal{O}_S(mL)). & &
\end{array}$$

Since the left vertical arrow is an isomorphism and ι is injective by the above argument, the map ι is an isomorphism and

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(S, \mathcal{O}_S(mL))$$

is surjective. By induction on dimension, $|mL|_S|$ is base point free for every $m \gg 0$ since $(aL - (K_X+\Delta))|_S = aL|_S - (K_S + \mathrm{Diff}(\Delta - S))$ is nef and log big on $(S, \mathrm{Diff}(\Delta - S))$. So we have that $\mathrm{Bs}|mL| \cap \llcorner \Delta \lrcorner = \emptyset$. By Proposition (1.4), we get the result. $\square$

3. Appendix

The following theorem is a partial answer to the four-dimensional log abundance conjecture.

THEOREM 3.1. *Let (X, Δ) be a proper dlt fourfold and $K_X + \Delta$ nef and big. Then $K_X + \Delta$ is semi-ample.*

PROOF. Let a be a positive integer such that $a(K_X + \Delta)$ is Cartier. We define $L := a(K_X + \Delta)$, $S := \llcorner \Delta \lrcorner$, and $T := f_*^{-1}S = \llcorner f_*^{-1}\Delta \lrcorner$, where f is the log resolution in the proof of Theorem (0.1). Apply the same proof as that of Theorem (0.1) and the abundance theorem for the semi divisorial log terminal threefold $(S, \mathrm{Diff}(\Delta - S))$ (see [Fj]). Note that S is seminormal and $f|_T : T \to S$ has connected fibers by the connectedness lemma ([Utah, 17.4 Theorem]). $\square$

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(Received April 1, 1999)

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
E-mail: fujino@kurims.kyoto-u.ac.jp