ON RATIONAL CHAIN CONNECTEDNESS

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The results in this short note seems to be more or less well known to the experts although they are formulated and stated only for algebraic varieties in [HM]. Here, we do not repeat the proof of [HM] in the complex analytic setting. We will use a different approach.

Definition 1 (Rational chain connectedness)**.** Let X be a projective complex analytic space. We say that X is *rationally chain connected* if for arbitrary points $P_1, P_2 \in X$ there exists a connected curve C on X such that $P_1, P_2 \in C$ and that every irreducible component of C is rational.

In Definition 1, X may be reducible and highly singular.

Theorem 2. *Let $\pi\colon X \to S$ be a projective morphism of complex analytic spaces with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$ such that (X, Δ) is log canonical. Assume that $-(K_X + \Delta)$ is π-ample over some open neighborhood of $P \in S$. Then $\pi^{-1}(P)$ is rationally chain connected.*

Proof. After shrinking S around P, we can naturally see $[X, K_X + \Delta]$ as a quasi-log complex analytic space with $\mathrm{Nqlc}(X, K_X + \Delta) = \emptyset$ (see [F2]). Therefore, by [F2, Theorem 9.8], we obtain that $\pi^{-1}(P)$ is rationally chain connected. □

Note that the proof of [F2, Theorem 9.8] essentially depends on [HM]. Hence, this note does not give an alternative proof of the results in [HM] established for algebraic varieties.

Corollary 3. *Let $\pi\colon X \to S$ be a projective bimeromorphic morphism of normal complex varieties with $\pi_*\mathcal{O}_X \simeq \mathcal{O}_S$ such that (X, Δ) is kawamata log terminal. Let $P \in S$ be an arbitrary point. Assume that $K_X + \Delta$ is π-numerically trivial over P, that is, $(K_X + \Delta) \cdot C = 0$ holds for every curve C with $\pi(C) = P$. Then $\pi^{-1}(P)$ is rationally chain connected.*

Proof. After shrinking S around P, we can find an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on X such that $(X, \Delta + D)$ is still kawamata log terminal and that $-D$ is π-ample. Hence $-(K_X + \Delta + D)$ is π-ample over some open neighborhood of P. Therefore, by Theorem 2, $\pi^{-1}(P)$ is rationally chain connected. □

The following lemma is obvious. We will use it in the proof of Theorem 5.

Lemma 4. *Let $g\colon Z \to Y$ and $f\colon Y \to X$ be proper surjective morphisms of complex analytic spaces. If $(f \circ g)^{-1}(P)$ is rationally chain connected, then $f^{-1}(P)$ is also rationally chain connected.*

Proof. This is because $(f \circ g)^{-1}(P) \to f^{-1}(P)$ is surjective. □

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This note will be contained in [F1] or [F2].

By using the minimal model program for projective morphisms between complex analytic spaces (see [F1]), we can prove the following result as an easy application of Corollary 3.

Theorem 5 (Kawamata log terminal singularities)**.** *Let X be a normal complex variety such that (X, Δ) is kawamata log terminal for some effective $\mathbb{R}$-divisor Δ on X. Let $f\colon Y \to X$ be any proper bimeromorphic morphism from a complex variety Y. Then $f^{-1}(P)$ is rationally chain connected for every $P \in X$.*

The author learned the following proof in [DH].

Proof. Throughout this proof, we will freely shrink X around P without mentioning it explicitly. By applying Chow's lemma for proper bimeromorphic morphisms (see [H, Corollary 2]) and taking a suitable resolution of singularities (see [BM]), we may assume that Y is smooth and f is projective by Lemma 4. We may further assume that the exceptional locus $\mathrm{Exc}(f)$ of f is a divisor and that the union of the support of $f^{-1}_*\Delta$ and $\mathrm{Exc}(f)$ is a simple normal crossing divisor on Y. Then we can write

$$K_Y + \Delta_Y = f^*(K_X + \Delta) + E$$

such that (Y, Δ_Y) is kawamata log terminal and that E is effective with $\mathrm{Supp}\, E = \mathrm{Exc}(f)$. By [F1], we can run a minimal model program with respect to $K_Y + \Delta_Y$ over X around P starting from (Y, Δ_Y):

$$(Y, \Delta_Y) =: (Y_0, \Delta_{Y_0}) \overset{\varphi_0}{\dashrightarrow} (Y_1, \Delta_{Y_1}) \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{i-1}}{\dashrightarrow} (Y_i, \Delta_{Y_i}) \overset{\varphi_i}{\dashrightarrow} \cdots \overset{\varphi_{m-1}}{\dashrightarrow} (Y_m, \Delta_{Y_m}),$$

where $\Delta_{Y_{i+1}} := (\varphi_i)_*\Delta_{Y_i}$ for every $i \geq 0$, such that $K_{Y_m} + \Delta_{Y_m}$ is semi-ample over some open neighborhood of P. We note that each step φ_i exists only after shrinking X around P suitably. We also note that f is a projective bimeromorphic morphism. Hence $K_Y + \Delta_Y$ is automatically big over X. Therefore, we finally get a small projective bimeromorphic morphism $f_m\colon Y_m \to X$ such that $K_{Y_m} + \Delta_{Y_m} = f_m^*(K_X + \Delta)$ holds. By Corollary 3, $f_m^{-1}(P)$ is rationally chain connected. By applying Theorem 2 to each flipping contraction and each divisorial contraction in the above minimal model program, we can check that $\pi^{-1}(P)$ is rationally chain connected with the aid of Lemma 4. □

The following corollary of Theorem 5 is well known for proper rational maps of algebraic varieties.

Corollary 6. *Let $f\colon X \dashrightarrow Y$ be a meromorphic map of complex analytic spaces such that (X, Δ) is kawamata log terminal. Assume that there are no rational curves on Y. Then f is a morphism.*

Proof. Let Γ be the graph of $f\colon X \dashrightarrow Y$. By Theorem 5, any positive-dimensional fiber of $p\colon \Gamma \to X$ is rationally chain connected, where $p\colon \Gamma \to X$ is the projection. Since Y has no rational curves, $p\colon \Gamma \to X$ is an isomorphism. This means that f is a morphism. □

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