ON NAKAYAMA'S THEOREM

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Dedicated to Professor Noboru Nakayama on the occasion of his sixtieth birthday

ABSTRACT. The main purpose of this paper is to make Nakayama's theorem more accessible. We give a proof of Nakayama's theorem based on the negative definiteness of intersection matrices of exceptional curves. In this paper, we treat Nakayama's theorem on algebraic varieties over any algebraically closed field of arbitrary characteristic although Nakayama's original statement is formulated for complex analytic spaces.

1. INTRODUCTION

In this paper, a *variety* means an integral separated scheme of finite type over an algebraically closed field k of any characteristic. The following theorem is very well known and plays a crucial role in the theory of higher-dimensional minimal models.

Theorem 1.1. *Let $f\colon X \to Y$ be a projective birational morphism from a smooth surface X to a normal surface Y. Then the intersection matrix of the f-exceptional curves is negative definite.*

The main purpose of this paper is to make the following theorem by Noboru Nakayama more accessible. Here we treat varieties over any algebraically closed field k of arbitrary characteristic although the original statement is formulated for complex analytic spaces.

Theorem 1.2 (Nakayama's theorem, see [N, Chapter III, 5.10. Lemma (3)])**.** *Let $f\colon X \to Y$ be a projective surjective morphism from a smooth variety X onto a normal variety Y. Let D be an $\mathbb{R}$-divisor on X. Then there exists an effective f-exceptional divisor E on X such that*

$$(f_*\mathcal{O}_X(\lfloor tD \rfloor))^{**} = f_*\mathcal{O}_X(\lfloor t(D+E) \rfloor)$$

holds for every positive real number t.

Theorem 1.2 has already played a fundamental role in [T], [PT], [CP], and so on. Nakayama's original proof in [N] uses his theory of *relative σ-decompositions* and *relative ν-decompositions* developed in [N, Chapter III, §1, §3, and §4]. Hence we had to study N_σ and N_ν to understand Theorem 1.2. Our argument in this paper clarifies that Theorem 1.2 is an easy consequence of Theorem 1.1. Roughly speaking, Theorem 1.2 is a variant of the negativity lemma (see, for example, [F, Lemma 2.3.26]). We do not need N_σ and N_ν.

2. PRELIMINARIES

Let us start with the definition of exceptional divisors.

Definition 2.1 (Exceptional divisors)**.** Let $f\colon X \to Y$ be a proper surjective morphism between normal varieties. Let E be a Weil divisor on X. We say that E is *f-exceptional* if $\operatorname{codim}_Y f(\operatorname{Supp} E) \geq 2$. We note that f is not always assumed to be birational.

In order to understand Theorem 1.2, we need the following definitions.

Definition 2.2. Let $D = \sum_i a_i D_i$ be an $\mathbb{R}$-divisor on a normal variety X such that D_i is a prime divisor on X for every i and that $D_i \neq D_j$ for $i \neq j$. We put

$$D^+ = \sum_{a_i > 0} a_i D_i \quad \text{and} \quad D^- = -\sum_{a_i < 0} a_i D_i \geq 0.$$

Note that

$$D = D^+ - D^-$$

obviously holds. For every real number x, $\lfloor x \rfloor$ is the integer defined by $x - 1 < \lfloor x \rfloor \leq x$. We put

$$\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$$

and call it the *round-down* of D.

Definition 2.3. Let $\mathcal{F}$ be a coherent sheaf on a normal variety X. We put

$$\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

and

$$\mathcal{F}^{**} = (\mathcal{F}^*)^*.$$

Then there exists a natural map $\mathcal{F} \to \mathcal{F}^{**}$. If this map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism, then $\mathcal{F}$ is called a *reflexive* sheaf.

For the basic properties of reflexive sheaves, see [H, Section 1]. We prepare an easy lemma for the reader's convenience.

Lemma 2.4. *Let V be a smooth surface and let $C_1, \ldots, C_m$ be effective Cartier divisors on V such that the intersection matrix $(C_i \cdot C_j)$ is negative definite and that $C_i \cdot C_j \geq 0$ for $i \neq j$. Let*

$$D = B + \sum_{i=1}^m a_i C_i$$

be an $\mathbb{R}$-divisor on V. Assume that

  (a) *$D \cdot C_i \leq 0$ (resp. $D \cdot C_i < 0$) for every i, and*
  (b) *$B \cdot C_i \geq 0$ for every i.*

Then $a_i \geq 0$ (resp. $a_i > 0$) for every i.

Remark 2.5. In Lemma 2.4, C_i may be reducible and disconnected. It may happen that C_i and C_j have some common irreducible components for $i \neq j$.

Proof of Lemma 2.4. By (b), $B \cdot C_j \geq 0$ for every j. Hence $(\sum_{i=1}^m a_i C_i) \cdot C_j \leq 0$ (resp. < 0) for every j by (a). Since $(C_i \cdot C_j)$ is negative definite and $C_i \cdot C_j \geq 0$ for $i \neq j$, $a_i \geq 0$ (resp. > 0) holds for every i. $\qquad\square$

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

Definition 3.1. Let $f \colon X \to Y$ be a projective surjective morphism from a smooth n-dimensional quasi-projective variety X onto a normal quasi-projective variety Y. Let H be a very ample Cartier divisor on Y and let A be a very ample Cartier divisor on X. Let E be an f-exceptional prime divisor on X with $\dim f(E) = e$. We put

$$C := E \cap f^* H_1 \cap \cdots \cap f^* H_e \cap A_1 \cap \cdots \cap A_{n-e-2},$$

where H_i is a general member of $|H|$ for every i and A_j is a general member of $|A|$ for every j, and call C a *general curve* associated to E, $f \colon X \to Y$, H, and A. We sometimes

simply say that C is a general curve associated to E. By construction, $E \cdot C < 0$ and $P \cdot C \geq 0$ for every prime divisor P on X with $P \neq E$. We note that C may be reducible and disconnected. We also note that

$$f^* H_1 \cap \cdots \cap f^* H_e \cap A_1 \cap \cdots \cap A_{n-e-2}$$

is a smooth surface when the characteristic of the base field k is zero by Bertini's theorem. Unfortunately, however, it may be singular in general.

For the proof of Theorem 1.2, we prepare several lemmas, which are easy applications of Theorem 1.1.

Lemma 3.2. *Let $f \colon X \to Y$, H, A be as in Definition 3.1. Let $E_1 \ldots, E_m$ be f-exceptional prime divisors on X such that $E_i \neq E_j$ for $i \neq j$ and $\dim f(E_i) = e$ for every i. Let C_i be a general curve associated to E_i, $f \colon X \to Y$, H, and A for every i. Then the matrix $(E_i \cdot C_j)$ is negative definite and that $E_i \cdot C_j \geq 0$ for $i \neq j$. Hence there exists an effective divisor E on X such that $\operatorname{Supp} E = \sum_{i=1}^m E_i$ and that $E \cdot C_i < 0$ for every i.*

Proof. We use the same notation as in Definition 3.1. By restricting everything to

$$f^* H_1 \cap \cdots \cap f^* H_e \cap A_1 \cap \cdots \cap A_{n-e-2},$$

we may assume that X is a pure two-dimensional quasi-projective scheme over k, $E_i = C_i$ for every i, and $E_i \cdot C_j \geq 0$ for $i \neq j$ (see Definition 3.1). We first treat the case where the characteristic of the base field k is zero. In this case, X is a smooth surface by Bertini's theorem. Then it is easy to see that $(E_i \cdot C_j)$ is negative definite (see Theorem 1.1). When the characteristic of the base field k is positive, X may be singular. In this case, by pulling everything back to a desingularization of

$$(f^* H_1 \cap \cdots \cap f^* H_e \cap A_1 \cap \cdots \cap A_{n-e-2})_{\mathrm{red}},$$

we can check the negative definiteness of $(E_i \cdot C_j)$ (see Theorem 1.1). Hence we can always take an effective divisor E with the desired properties (see Lemma 2.4). $\square$

Lemma 3.3. *Let $f \colon X \to Y$, H, and A be as in Definition 3.1. Let $E_1, \ldots, E_m$ be f-exceptional prime divisors on X such that $E_i \neq E_j$ for $i \neq j$. Let C_i be a general curve associated to E_i, $f \colon X \to Y$, H, and A for every i. Then there is an effective divisor E on X such that $\operatorname{Supp} E = \sum_{i=1}^m E_i$ and that $E \cdot C_i < 0$ for every i.*

Proof. By Lemma 3.2, we can find an effective divisor E^j on X such that

$$\operatorname{Supp} E^j = \sum_{\dim f(E_i) = j} E_i$$

and that $E^j \cdot C_i < 0$ where C_i is a general curve associated to E_i with $\dim f(E_i) = j$. We note that $E^j \cdot C_i = 0$ (resp. ≥ 0) when C_i is a general curve associated to E_i with $\dim f(E_i) > j$ (resp. $< j$) by construction. We put

$$E = \sum_{j=0}^{n-2} m_j E^j$$

with

$$m_0 \gg m_1 \gg \cdots \gg m_{n-2} > 0.$$

Then E is an effective divisor on X with the desired properties. $\square$

Lemma 3.4. *Let $f \colon X \to Y$, H, and A be as in Definition 3.1. Let D be an $\mathbb{R}$-divisor on X such that $\operatorname{Supp} D^-$ is f-exceptional and that $D \cdot C \leq 0$ for any general curve C associated to any f-exceptional divisor on X. Then D is effective.*

Proof. Let $\mathcal{E} = \{E_1, \ldots, E_m\}$ be the set of all f-exceptional divisors on X. By pulling everything back to a desingularization V of

$$(f^*H_1 \cap \cdots \cap f^*H_{n-2})_{\mathrm{red}}$$

and using Lemma 2.4 on V, we obtain that the pull-back of D to V is effective. This means that the coefficient of E_i in D is nonnegative when $\dim f(E_i) = n - 2$. Assume that the coefficient of E_i in D is nonnegative when $\dim f(E_i) \geq e + 1$. Then we pull everything back to a desingularization of

$$(f^*H_1 \cap \cdots \cap f^*H_e \cap A_1 \cap \cdots \cap A_{n-e-2})_{\mathrm{red}}$$

and use Lemma 2.4 again. Then we obtain that the coefficient of E_i in D is nonnegative when $\dim f(E_i) \geq e$. We repeat this process finitely many times. Then we finally obtain that D is effective. $\qquad\square$

Lemma 3.5. *Let $f\colon X \to Y$ be a projective surjective morphism from a smooth variety X onto a normal variety Y. Let D be an $\mathbb{R}$-divisor on X. Then there exists an effective f-exceptional divisor E on X such that if G is any $\mathbb{R}$-divisor on X and U is any Zariski open subset of Y with the following properties:*

(a) *$G|_{f^{-1}(U)} \equiv_U D|_{f^{-1}(U)}$, that is, $G|_{f^{-1}(U)}$ is relatively numerically equivalent to $D|_{f^{-1}(U)}$ over U, and*

(b) *the support of $G^-|_{f^{-1}(U)}$ is f-exceptional,*

then $(G + E)|_{f^{-1}(U)}$ is effective, equivalently, $G^-|_{f^{-1}(U)} \leq E|_{f^{-1}(U)}$ holds.

Proof. In Step 1, we will treat the case where Y is affine. In Step 2, we will treat the general case.

Step 1. In this step, we will construct a desired divisor E under the extra assumption that Y is affine.

When Y is affine, we can take an effective f-exceptional divisor E on X such that $E \cdot C < 0$ for any general curve associated to any f-exceptional divisor on X by Lemma 3.3. By replacing E with mE for some positive integer m, we may further assume that $(D + E) \cdot C \leq 0$ for any general curve C associated to any f-exceptional divisor on X. Let U be any Zariski open subset of Y and let G be any $\mathbb{R}$-divisor on X satisfying (a) and (b). Then,

$$(G + E)|_{f^{-1}(U)} \cdot C = (D + E)|_{f^{-1}(U)} \cdot C \leq 0$$

holds for any general curve C associated to any f-exceptional divisor on X by (a) and the construction of E, and the support of $(G + E)^-|_{f^{-1}(U)}$ is f-exceptional by (b). Hence, by Lemma 3.4, $(G + E)|_{f^{-1}(U)}$ is effective.

Step 2. In this step, we will treat the general case.

We take a finite affine Zariski open cover

$$Y = \bigcup_\alpha U_\alpha$$

of Y. We consider $f\colon f^{-1}(U_\alpha) \to U_\alpha$ for every α. By Step 1, we have an effective f-exceptional divisor E_α on $f^{-1}(U_\alpha)$ with the desired property for every α. Let E be an effective f-exceptional divisor on X such that $E_\alpha \leq E|_{f^{-1}(U_\alpha)}$ holds for every α. Then E obviously satisfies the desired property.

We complete the proof. $\qquad\square$

Let us prove Theorem 1.2.

Proof of Theorem 1.2. We prove this theorem in the following two steps.

Step 1. Let E be any f-exceptional divisor on X. Then

$$(f_*\mathcal{O}_X(\lfloor tD \rfloor))^{**} = (f_*\mathcal{O}_X(\lfloor t(D+E) \rfloor))^{**}$$

holds. Therefore, the inclusion

$$f_*\mathcal{O}_X(\lfloor t(D+E) \rfloor) \subset (f_*\mathcal{O}_X(\lfloor tD \rfloor))^{**}$$

always holds.

Step 2. Let E be an effective f-exceptional divisor on X satisfying the property in Lemma 3.5. Let Σ denote the smallest Zariski closed subset of Y such that f is equidimensional over $Y \setminus \Sigma$. Note that $\operatorname{codim}_Y \Sigma \geq 2$. By [H, Corollary 1.7], $f_*\mathcal{O}_X(\lfloor tD \rfloor)|_{Y \setminus \Sigma}$ is a reflexive sheaf on $Y \setminus \Sigma$. Let $\mathcal{E} = \{E_1, \dots, E_m\}$ be the set of all f-exceptional divisors on X. Hence $\operatorname{codim}_X \left(f^{-1}(\Sigma) \setminus \sum_{i=1}^m E_i \right) \geq 2$ holds by definition. We take any affine Zariski open subset U of Y. Then we have the following natural inclusion and equalities

$$
\begin{aligned}
\Gamma\left(U, (f_*\mathcal{O}_X(\lfloor tD \rfloor))^{**}\right) &\subset \Gamma\left(U \setminus \Sigma, (f_*\mathcal{O}_X(\lfloor tD \rfloor))^{**}\right) \\
&= \Gamma\left(U \setminus \Sigma, f_*\mathcal{O}_X(\lfloor tD \rfloor)\right) \\
&= \Gamma\left(f^{-1}(U) \setminus f^{-1}(\Sigma), \mathcal{O}_X(\lfloor tD \rfloor)\right) \\
&= \Gamma\left(f^{-1}(U) \setminus \sum_{i=1}^m E_i, \mathcal{O}_X(\lfloor tD \rfloor)\right),
\end{aligned}
$$

(3.1)

where the inclusion follows from the torsion-freeness of $(f_*\mathcal{O}_X(\lfloor tD \rfloor))^{**}$, the first equality is due to the fact that $f_*\mathcal{O}_X(\lfloor tD \rfloor)|_{Y \setminus \Sigma}$ is a reflexive sheaf on $Y \setminus \Sigma$, the second equality is obvious, and the final equality follows from $\operatorname{codim}_X \left(f^{-1}(\Sigma) \setminus \sum_{i=1}^m E_i \right) \geq 2$. We note that

$$\Gamma\left(f^{-1}(U) \setminus \sum_{i=1}^m E_i, \mathcal{O}_X(\lfloor tD \rfloor)\right) = \{\phi \in k(X) \mid ((\phi) + \lfloor tD \rfloor)|_{f^{-1}(U) \setminus \sum E_i} \geq 0\} \cup \{0\},$$

where $k(X)$ stands for the rational function field of X and (ϕ) is the divisor associated to $\phi \in k(X)$. By the definition of E, if

$$((\phi) + \lfloor tD \rfloor)|_{f^{-1}(U) \setminus \sum E_i} \geq 0$$

holds, then

$$((\phi) + t(D+E))|_{f^{-1}(U)} \geq 0$$

holds. Therefore, by taking the round-down, we obtain that

$$((\phi) + \lfloor t(D+E) \rfloor)|_{f^{-1}(U)} \geq 0.$$

This implies that

$$
\begin{aligned}
\Gamma\left(f^{-1}(U) \setminus \sum_{i=1}^m E_i, \mathcal{O}_X(\lfloor tD \rfloor)\right) & \\
\subset \{\phi \in k(X) \mid ((\phi) + \lfloor t(D+E) \rfloor)|_{f^{-1}(U)} \geq 0\} &\cup \{0\} \\
= \Gamma\left(f^{-1}(U), \mathcal{O}_X(\lfloor t(D+E) \rfloor)\right)& \\
= \Gamma\left(U, f_*\mathcal{O}_X(\lfloor t(D+E) \rfloor)\right).&
\end{aligned}
$$

(3.2)

Hence we get the following inclusion

$$\Gamma\left(U, (f_*\mathcal{O}_X(\lfloor tD \rfloor))^{**}\right) \subset \Gamma\left(U, f_*\mathcal{O}_X(\lfloor t(D+E) \rfloor)\right)$$

by (3.1) and (3.2). This means that the opposite inclusion

$$(f_*\mathcal{O}_X(\lfloor tD \rfloor))^{**} \subset f_*\mathcal{O}_X(\lfloor t(D+E) \rfloor)$$

holds.

By combining Step 1 with Step 2, we see that the effective f-exceptional divisor E on X with the property in Lemma 3.5 is a desired one. $\qquad\square$

Finally, we note the following statement, which is similar to [N, Chapter III, 5.10. Lemma (4)].

Proposition 3.6. *Let $f\colon X \to Y$ be a projective surjective morphism from a smooth quasi-projective variety X onto a normal quasi-projective variety Y. Let D be an $\mathbb{R}$-divisor on X. Assume that $D \cdot C \leq 0$ for any general curve C associated to any f-exceptional divisor on X. Then $f_*\mathcal{O}_X(\lfloor D \rfloor)$ is reflexive.*

Proof. If we further assume that Y is quasi-projective and $D \cdot C \leq 0$ for any general curve C associated to any f-exceptional divisor on X in Lemma 3.5, then we see that $G|_{f^{-1}(U)}$ is effective by Lemma 3.4 and the proof of Lemma 3.5. Hence, the argument in Step 2 in the proof of Theorem 1.2 works with $E = 0$ and $t = 1$. Therefore, we obtain

$$f_*\mathcal{O}_X(\lfloor D \rfloor) = (f_*\mathcal{O}_X(\lfloor D \rfloor))^{**}.$$

This means that $f_*\mathcal{O}_X(\lfloor D \rfloor)$ is reflexive. $\qquad\square$

References

[CP] F. Campana, M. Păun, Foliations with positive slopes and birational stability of orbifold cotangent bundles, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 1–49.

[F]   O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.

[H]   R. Hartshorne, Stable reflexive sheaves, Math. Ann. **254** (1980), no. 2, 121–176.

[N]   N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, **14**. Mathematical Society of Japan, Tokyo, 2004.

[PT] M. Păun, S. Takayama, Positivity of twisted relative pluricanonical bundles and their direct images, J. Algebraic Geom. **27** (2018), no. 2, 211–272.

[T]   S. Takayama, A local ampleness criterion of torsion free sheaves, Bull. Sci. Math. **137** (2013), no. 5, 659–670.

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