§2.

## 2.1 instanton moduli spaces

$$\mathbb{P}^2 \supset \mathbb{A}^2 = \mathbb{P}^2 \setminus \ell_\infty$$

$\mathcal{M}(r,n)$ = framed moduli space of torsion free $E$ sheaves on $\mathbb{P}^2$ with $c_2(E) = n$, $rk = r$

$\cup$

$\mathcal{M}^{l.f.}(r,n)$ frame $\to$ $\varphi : E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus r}$

$\mathcal{M}^{l.f}(r,n) \cong$ framed instantons on $S^4$

Donaldson

<u>Th</u> $\mathcal{M}(r,n)$ is a nonsingular quasi-projective variety of $\dim = 2rn$

<u>1st proof</u> • develop the theory of stable pairs

· Thaddeus

· Le Potier

⋮

· Huybrechts-Lehn

deformation theory is controlled by $\mathrm{Ext}^i(E, E(-\ell_\infty))$

• existence of frame $\Rightarrow$ stability

• $\Rightarrow \mathrm{Ext}^0 = \mathrm{Ext}^2 = 0$

<u>2nd proof</u> quiver description of $\mathcal{M}(r,n)$

$$\begin{tikzcd} V \arrow[loop right, "B_1"] \arrow[loop above, "B_2"] \arrow[d, shift right, "b"'] \\ W \arrow[u, shift right, "a"'] \end{tikzcd}$$

$V$ : cpx vector sp of $\dim = n$

$W$ : 〃 $r$

$$(B_1, B_2, a, b) \in \underset{\dim\quad 2n^2}{\mathrm{End}(V)^{\oplus 2}} \oplus \underset{nr}{\mathrm{Hom}(W,V)} \oplus \underset{nr}{\mathrm{Hom}(V,W)} \curvearrowleft G = GL(V) \quad n^2$$

$$\mu(B_1, B_2, a, b) = [B_1, B_2] + ab \in \underset{n^2}{\mathrm{End}(V)}$$

<u>stable</u> $\overset{\text{def}}{\Longleftrightarrow}$ If $S \subset V$ subsp. $S \supset \mathrm{Im}\, a$ and inv under $B_1, B_2$ $\Rightarrow S = V$

("cyclic vector")

<u>Th</u> $\mathcal{M}(r,n) \cong \mu^{-1}(0)^{\text{stable}}/G$

$M(r,n)$ = framed moduli space of torsion free sheaves on $\mathbb{P}^2$ triv at $l_\infty$

$\mathbb{A}^2 = \mathbb{P}^2 \setminus l_\infty$ ; open K3

quiver description

$V \circlearrowleft B_1, B_2$, $\quad b \downarrow \uparrow a$, $\quad W$

- $\mu(B_1, B_2, a, b) = [B_1, B_2] + ab = 0$
- stable

$$M(r,n) \cong \mu^{-1}(0)^{\text{stable}} / G \qquad G = GL(V)$$

<u>proof</u> $\quad V \otimes \mathcal{O}(-1) \xrightarrow{\alpha} V^{\oplus 2} \otimes \mathcal{O} \oplus W \otimes \mathcal{O} \xrightarrow{\beta} V \otimes \mathcal{O}(1)$

$[x:y:z] \in \mathbb{P}^2 \qquad \alpha = \begin{bmatrix} zB_1 - x\,\mathrm{id} \\ zB_2 - y\,\mathrm{id} \\ zb \end{bmatrix} \qquad \beta = [-(zB_2 - y\,\mathrm{id}),\ (zB_1 - x\,\mathrm{id}),\ az]$

$\beta\alpha = 0 \Leftarrow \mu = 0$

$\beta$ : surjective $\Leftarrow$ stable

$\alpha$ : injective (as a sheaf hom)

$E = \operatorname{Ker}\beta / \operatorname{Im}\alpha \qquad$ torsion-free

$z = 0 \Rightarrow E|_{l_\infty} \cong W \otimes \mathcal{O} \qquad$ frame //

converse : $V = H^1(E(-l_\infty))$, $W = H^0(E|_{l_\infty})$

$2^{nd}$ proof of the smoothness

- $d\mu$ : surjective at a stable point
- $G \curvearrowright \{$stable pts$\}$ is free

There is a projective morphism

$$\pi : M(r,n) \longrightarrow M_0(r,n) = \mu^{-1}(0) /\!/ G = \operatorname{Spec}\left(\mathbb{C}[\mu^{-1}(0)]^G\right)$$

Uhlenbeck space

$= \{$closed $G$-orbits in $\mu^{-1}(0)\}$

$= \{\underbrace{\text{semisimple}}_{\text{direct sum of simple}}$ representations of the quiver with relation $\mu = 0\} / \text{Isom}$

**Prop** A simple representation is either

a) $(B_1, B_2, a, b)$ : stable & costable $\longleftarrow$ loc free sheaf

or

b) $W = 0$ & $V \cong \mathbb{C}$ $\quad B_1 = x,\ B_2 = y$ $\longleftarrow$ $(x, y) \in \mathbb{A}^2$

$$M_0(r,n) = \bigsqcup_{n' \leq n} M^{l.f}(r, n') \times S^{n-n'}\mathbb{A}^2$$

"$[(E,\varphi)]$"

$$\pi([E,\varphi]) = (E^W, \varphi) \times \underset{(\text{length})}{\mathrm{mult}(E^W/E)}$$

J. Li more algebro-geometric construction

**Ex** $r = 1$

$$M(1,n) = \mathrm{Hilb}^n \mathbb{A}^2$$
$$\downarrow$$
$$M_0(1,n) = S^n \mathbb{A}^2$$

3.2

$$\mathbb{T} = \underset{\substack{\uparrow \\ \text{frame}}}{T^r} \times \underset{\substack{\uparrow \\ \text{action on the base } \mathbb{A}^2}}{T^2} \curvearrowright M(r,n),\ M_0(r,n)$$

$$E|_{l_\infty} \xrightarrow{\varphi} \mathcal{O}_{l_\infty}^{\oplus r} \xrightarrow{t} \mathcal{O}_{l_\infty}^{\oplus r} \qquad (x,y) \mapsto (t_1 x, t_2 y)$$

$$H^*_{\mathbb{T}}(pt) = \mathbb{C}[\underbrace{\varepsilon_1, \varepsilon_2}_{\substack{\uparrow \\ T^2}}, \vec{a}\,]$$

$\vec{a} = (a_1, \cdots, a_r)$ $\leftarrow T^r$

**Claim** $M_0(r,n)^{\mathbb{T}} = \{0\}$

$$i_0 : \{0\} \hookrightarrow M_0(r,n)$$

Define the instanton partition function (Nekrasov) by

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda) = \sum_{n=0}^{\infty} \Lambda^{2nr} (i_{0*})^{-1} [M_0(r,n)]$$

$$\in \mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{a})[[\Lambda]]$$

$$
\begin{tikzcd}
M(r,n) \arrow[d, "\pi"', "\text{proper}"] & M(r,n)^{\mathbb{T}} \arrow[l, hook', "i"'] \arrow[d, "\pi^{\mathbb{T}}"] \arrow[dl, phantom, "\circlearrowright"] \\
M_0(r,n) \arrow[d, "a"] & \{0\} \arrow[l, hook', "i_0"'] \arrow[dl, equal] \\
pt &
\end{tikzcd}
$$

$M(r,n)^{\mathbb{T}} \cong \{\vec{Y} = (Y_1, \dots, Y_r) \mid \sum |Y_\alpha| = n\}$ finite set

$a \circ i_0 = id$

$i_{0*}^{-1} = $ " $a_* = \int_{M_0(r,n)}$ "

$a_* : H_*^{\mathbb{T}}(M_0(r,n)) \longrightarrow H_*^{\mathbb{T}}(pt)$

$i_{0*}^{-1}[M_0(r,n)] = $ " $\int_{M_0(r,n)} 1$ "

**Lemma** (1) $\pi_* i_* = i_{0*} \circ \pi_*^{\mathbb{T}}$

(2) $\pi_*[M(r,n)] = [M_0(r,n)]$

**Th** $i_{0*}^{-1}[M_0(r,n)] = \pi_*^{\mathbb{T}} \underline{i_*^{-1}[M(r,n)]}$

$$\int_{M_0(r,n)} 1 \qquad\qquad \frac{1}{e(T_{\vec{Y}} M(r,n))}[\vec{Y}]$$

$$= \sum_{\vec{Y}} \frac{1}{e(T_{\vec{Y}} M(r,n))}$$

**Ex** $r = 1$

$$i_{0*}^{-1}[S^n \mathbb{A}^2] = \int_{S^n \mathbb{A}^2} 1 = \frac{1}{n!} \int_{\mathbb{A}^{2n}} 1$$

$$\mathbb{A}^{2n}/\mathfrak{S}_n \qquad = \frac{1}{n!}\left(\int_{\mathbb{A}^2} 1\right)^n$$

$$\int_{\mathbb{A}^2} 1 = \frac{1}{e(T_0 \mathbb{A}^2)} = \frac{1}{\varepsilon_1 \varepsilon_2}$$

$$\sum_Y \frac{1}{e(T_Y \mathrm{Hilb}^n \mathbb{A}^2)} = \frac{1}{n!} \frac{1}{(\varepsilon_1 \varepsilon_2)^n}$$

↑ nontrivial combinatorial identity

(Cauchy formula for Jack polynomial)

$r \geq 2$

<u>Nekrasov</u> conj

$$\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1, \varepsilon_2, \vec{a})\Big|_{\varepsilon_1=\varepsilon_2=0}$$ is regular at $\varepsilon_1=\varepsilon_2=0$

and is computable by period integral of <u>hyperelliptic curves</u> (genus $= r-1$)

<u>proved</u> N + Yoshioka

Nekrasov - Okounkov

Braverman + Etingof

<u>Fixed pt</u> $M(r,n)^{\mathbb{T}}$ $\quad \mathbb{T} = T^2 \times T^r$

$$M(r,n)^{T^r} = \coprod_{n_1+\cdots+n_r=n} M(1,n_1)\times\cdots\times M(1,n_r)$$

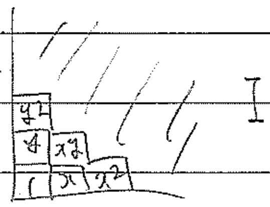
$$\begin{array}{ccc} (E,\varphi) & \mathcal{O}_{l_\infty}^{\oplus r} \xrightarrow{\ t\ } \mathcal{O}_{l_\infty}^{\oplus r} & \\ \rotatebox{90}{$\in$} & \uparrow \quad \circlearrowleft \quad \uparrow & \text{rk } 1 \\ (E, t\varphi) & E \xrightarrow{\exists! \varphi_t} E & \Rightarrow E = E_1 \oplus \cdots \oplus E_r \end{array}$$

$$M(1,n)^{T^2} = \{ \text{monomial ideals} \}$$

$\|$

ideal sheaf

$\longleftrightarrow I \subset \mathbb{C}[x,y]$




3.3. Sketch of a proof of Nekrasov conjecture for $r=2$

I assume that we already know $\varepsilon_1\varepsilon_2 \log Z$ is regular at $\varepsilon_1=\varepsilon_2=0$

$\|$

$\vec{a} \to a$ $\quad \underline{F_0(a,\Lambda)}$ + higher

$(a_1, a_2)$ $\quad (a_2 = -a_1)$

may assume $a_1 + a_2 = 0$.

Define

$$\tau := -\frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial^2 F_0}{\partial a^2} + 8\log\frac{-2\sqrt{-1}\,a}{\Lambda}\right) \quad \in \mathbb{C}(\vec{a})[[\Lambda]]$$

$$u := -\frac{1}{4}\frac{\partial F_0}{\partial \log\Lambda} + a^2$$

$$\omega := -2\pi\sqrt{-1}\left(\frac{\partial u}{\partial a}\right)^{-1} \qquad \omega' = \omega\tau$$

Consider the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z}\omega + \mathbb{Z}\omega'$
and the corresponding $\wp$-function

$\rightsquigarrow$ Weierstrauss form of $E_\tau$

$$y^2 = 4x^3 - g_2 x - g_3$$

Th $E_\tau$ is

$$y^2 = 4x^3 - \left(\frac{4}{3}u^2 - 4\Lambda^4\right)x - \left(\frac{4}{3}u\Lambda^4 - \frac{8}{27}u^3\right)$$

(Seiberg-Witten curve) elliptic curve

$\mathbb{C}[u, \Lambda]$

$F_0$ can be recovered from this.

$$\underset{\underset{-2\pi\sqrt{-1}\frac{\partial a}{\partial u}}{\shortparallel}}{\omega} = \omega(u, \Lambda) = \int_A \frac{dx}{y}$$

$$\Rightarrow \quad a = a(u, \Lambda) \quad \underset{\text{inverse function}}{\Longrightarrow} \quad \underset{\underset{\frac{\partial F_0}{\partial \log\Lambda}}{\shortparallel}}{u} = u(a, \Lambda)$$

$Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$

$$\log Z = \frac{1}{\varepsilon_1\varepsilon_2}\Big(F_0 + \varepsilon_1\varepsilon_2 A + \frac{\varepsilon_1^2+\varepsilon_2^2}{3} B + \cdots \Big)$$

$(\varepsilon_1+\varepsilon_2)H$ vanish

$$Z = \sum \Lambda^{2nr} \int_{M_0(n,r)} 1 \qquad \text{cf. Hilbert series}$$

limit of Hilbert series of the coord ring of $M_0(r,n)$

$\mathbb{A}^2 \curvearrowleft T^2 \ni (t_1, t_2)$ $\quad x^i y^j$ weight $t_1^i t_2^j$

$$\operatorname{ch} \mathcal{O}[\mathbb{A}^2] = \frac{1}{(1-t_1)(1-t_2)}$$

$t_1 = e^{\beta\varepsilon_1}$

$t_2 = e^{\beta\varepsilon_2}$

$$\xrightarrow[\beta \to 0]{} \frac{1}{\varepsilon_1\varepsilon_2} = \int_{\mathbb{A}^2} 1$$

$$\tau := -\frac{1}{2\pi\sqrt{-1}}\left(\frac{\partial^2 F_a}{\partial a^2} + 8 \log \frac{-2\sqrt{-1}a}{\Lambda}\right) \qquad q = e^{2\pi\tau}$$

$$u := -\frac{1}{4}\frac{\partial F_0}{\partial \log \Lambda} + a^2$$

$$\omega := -2\pi\sqrt{-1}\left(\frac{\partial u}{\partial a}\right)^{-1} \qquad \omega' = \tau\omega$$

$E_\tau = \mathbb{C}/\mathbb{Z}\omega + \mathbb{Z}\omega'$ . $\wp$ function

$$u + \cdots = \frac{\partial}{\partial \log \Lambda} \log Z = \frac{1}{Z} \sum \Lambda^{4n}\, 4n \int_{M_0(2,n)} 1$$

$$4n = \dim M_0(2,n) = c_2(\mathcal{E})/[\mathbb{P}^2]$$

$\mathcal{E}$: universal sheaf

Th $E_\tau$ is $y^2 = 4x^3 - (\frac{4}{3}u^2 - 4\Lambda^4)x - (\frac{4}{3}u\Lambda^4 - \frac{8}{27}u^3)$.

(Seiberg-Witten curve)

Rem. analogous to mirror symmetry

(sketch of a proof)

$\hat{\mathbb{P}}^2$ = blowup of $\mathbb{P}^2$ at $0 \in \mathbb{A}^2$ $\supset C$ : exceptional divisor $\quad l_\infty$

$\hat{M}(2,k,n)$ = moduli space of framed sheaves $(E, \varphi)$

rk 2 . $c_1(E) = kC$ , $n = c_2(E) - \frac{1}{4}c_1(E)^2$

$\mathcal{E}$ : universal sheaf on $\hat{\mathbb{P}}^2 \times \hat{M}(2,k,n)$

$$\mu(C) = \left(c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2\right)/[C] \in H^2_{\mathbb{T}}(\hat{M})$$

$k=0,1$, $\quad \hat{Z}_k(\varepsilon_1, \varepsilon_2, a; t, \Lambda) = \sum_n \Lambda^{4n r} \int_{\hat{M}(2,k,n)} e^{t\mu(C)}$ $\leftarrow$ compute $\mu(C)|_{(\vec{k}, \vec{Y}^P, \vec{Y}^Q)}$ explicit

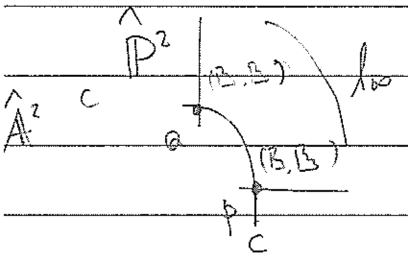
<u>Idea</u> Compute $\uparrow$ in two ways & made ① = ②

① localization $\mathbb{T}^2 \times T^2 = \mathbb{T}$ $\qquad M(2,n)^{\mathbb{T}} = \{\vec{Y} = (Y_1, Y_2)\}$

$$\hat{M}(2,k,n)^{\mathbb{T}} = \{(\vec{k}, \vec{Y}^P, \vec{Y}^Q) \mid \text{some constraint}\}$$

$\vec{k} \in \mathbb{Z}^2$

$(E, \varphi) = (I_1(k_1 C), \varphi_1) \oplus (I_2(k_2 C), \varphi_2)$ $\qquad (k_1, k_2) \in \mathbb{Z}^2$, $k_1 + k_2 = k$




$I_1 = I_{Z_1}$ $\qquad Z_1, Z_2 = \begin{matrix} Z_1^P \cup Z_1^Q \\ Z_2^P \cup Z_2^Q \end{matrix}$

$I_2 = I_{Z_2}$

in toric coord around $P, Q$

$Z_1^P, Z_1^Q$ ... are monomial ideals $\Rightarrow$ Young diagrams

$e(T_{(\vec{k}, \vec{Y}^P, \vec{Y}^Q)}\hat{M})$ = 3 factors

1) $\leftarrow$ line b'dles $\mathrm{Ext}^1(\mathcal{O}(k_1 C), \mathcal{O}(k_2 C - l_\infty))$ $\nwarrow$ explicitly written

2) $\leftarrow P$ $\nwarrow$

3) $\leftarrow Q$ $\leftarrow$ the same as $T^2$-equiv the case for $\mathbb{A}^2$

$$(\hat{\mathbb{A}}^2, P) \underset{\text{loc. isom}}{\cong} (\mathbb{A}^2, 0)$$

$\Rightarrow \hat{Z}_{k=0 \text{ or } 1} = \sum_{l \in \mathbb{Z} \text{ or } \mathbb{Z} + \frac{1}{2}} Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, a + \varepsilon_1 l; \Lambda e^{t\varepsilon_1})$ $\leftarrow$ 2)

$\times Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, a + \varepsilon_2 l; \Lambda e^{t\varepsilon_2})$ $\leftarrow$ 3)

$\times \dfrac{\Lambda^{2l^2}}{l(\varepsilon_1, \varepsilon_2, a)}$ $\leftarrow$ explicit from 1)

$u = \dfrac{\partial F_0}{\partial \log \Lambda}$

$\varepsilon_1, \varepsilon_2 \longrightarrow 0$

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\hat{Z}_{k=1}}{Z} = \exp\left(A - B + \frac{t^2}{8}\frac{\partial u}{\partial \log \Lambda}\right) \underset{\sum_{\ell\in\mathbb{Z}+\frac{1}{2}}}{\theta_{11}} \Big(\frac{\sqrt{-1}}{2\pi}\underbrace{\frac{\partial u}{\partial a}}_{\frac{1}{\omega}}\Big|\tau\Big)$$

$\varepsilon_1\varepsilon_2 \log Z$

$\quad \shortparallel$

$F_0 + \cdots \ \ \varepsilon_1\varepsilon_2 A$.

② Structure result

Higher rank case

Th [NY] fix $r, k$

$\exists!$ class $\Omega_j(\varepsilon, t)$ (independent of $n$) $\in \mathbb{C}[c_i(\mathcal{E})/[0], \varepsilon_1, \varepsilon_2][[t]]$ (over $\mathbb{P}^2 \times M(r,n)$)

s.t. $$\int_{\hat{M}(r,k,n)} e^{t\mu(C)} = \sum_j \int_{M(r,\hat{n}-j)} \Omega_j(\varepsilon, t)$$

$\hat{n} = n - \dfrac{k(r-k)}{2r}$

$$\begin{array}{ccc} \hat{M}_0(r,k,\hat{n}) & & \hat{M}(r,k,n) \\ & & \vdots \\ \big\downarrow & & M(r,\hat{n}) \\ M_0(r,\hat{n}) & \swarrow & \\ \shortparallel & & \\ M_0^{l.f.}(r,n') \times S^{\hat{n}-n'}\mathbb{A}^2 & & \end{array}$$

Rem We do not know explicit forms of $\Omega_j$ in general, but $\exists$ algorithm to compute them

e.g. for small $j$ $\Longrightarrow$ enough to compute small $n$

Using $\sigma(t) = t - \dfrac{g_2}{2}\dfrac{t^5}{5!} - 6g_3\dfrac{t^7}{7!} + \cdots$

we can determine $g_2, g_3$ from $\Omega_j$ with small $j$.

Fixed point

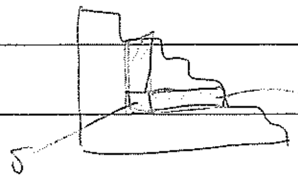
For simplicity $r=1$, $\mathbb{C}^* \subset T^2$ $\qquad \mathbb{T} = T^1 \times T^2$

$\{(t, t^{-1})\}$ $\qquad \mathbb{C}^* \curvearrowright \mathbb{A}^2$, $\omega = dx \wedge dy$ $\quad (t_1 x, t_2 y)$

$M(1,n)^{\mathbb{C}^*} = \{ Y \mid |Y| = n \}$

$\leftarrow$ 2n dim'l

$$\mathrm{ch}\, T_Y M(1,n) = \sum_{s \in Y} \left( t^{h_Y(s)} + t^{-h_Y(s)} \right)$$



s

$h_Y(s)$ hook length

$$e(T_Y M(1,n)) = (-1)^n \varepsilon^{2n} \underbrace{h(Y)}_{\prod_{s \in Y} h_Y(s)}{}^2 \qquad \varepsilon_1 = \varepsilon = -\varepsilon_2$$

$\bigoplus_n H^*_{\mathbb{T}}(M(r,n))$ $\leftarrow$ operators acting on

e.g. $\cdot\ c_i(\mathcal{E})/[0]$

$\cdot\ M(r, n, n+1) = \{ (E_1, E_2, \varphi) \mid E_1 \supset E_2 \}$ $\qquad \uparrow$ $r=1$, $2(n+1)$ dim nonsingular

$$\begin{array}{ccc} & M(r,n,n+1) & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2 \text{ proper}} \\ M(r,n) & & M(r,n+1) \end{array}$$

$$H^*_{\mathbb{T}}(M(r,n)) \xrightarrow{[M(r,n,n+1)]} H^*_{\mathbb{T}}(M(r,n+1))$$

$$c \longmapsto p_{2*}(p_1^*(c))$$

$\longleftarrow$ similar

$\underline{r=1}$ $\Rightarrow$ Heisenberg algebra $\longleftrightarrow$ symmetric functions

Virasoro-alg

r: general $\cdot$ W-alg $\qquad$ Maulik-Okounkov, Schiffmann-Vasserot