ON THE MINIMAL MODEL PROGRAM FOR PROJECTIVE MORPHISMS BETWEEN COMPACT COMPLEX ANALYTIC SPACES

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Abstract. We study the minimal model program for projective morphisms between compact complex analytic spaces. For simplicity, we restrict ourselves to the minimal model program for divisorial log terminal pairs with ample scaling.

Contents

1. Introduction

We show that the minimal model program established in [F1] can be applied to projective morphisms between compact complex analytic spaces. In this note, we restrict our attention to results from [F1] and [F2]. By incorporating results from [EH1] and [H], one can obtain more general versions of the minimal model program for projective morphisms of compact complex analytic spaces. We do not pursue these generalizations here.

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In this note, we freely use the notation and some results from [F1] and [F2].

2. Minimal model program for compact complex analytic spaces

Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces. In [F1], the main focus is on the case where Y is Stein. More precisely, one fixes a Stein compact subset W of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian and discusses the minimal model program over Y around W. In this note, we observe that the minimal model program can also be carried out for projective morphisms between compact complex analytic spaces.

Theorem 2.1 (Minimal model program for projective morphisms between compact complex analytic spaces)**.** *Let $\pi\colon X \to Y$ be a projective morphism between compact complex analytic spaces. Let (X, Δ) be a $\mathbb{Q}$-factorial divisorial log terminal pair. Then we can run the $(K_X + \Delta)$-minimal model program over Y.*

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Moreover, if $\mathcal{A}$ is a π-ample $\mathbb{R}$-line bundle on X such that $K_X+\Delta+\mathcal{A}$ is π-nef, then we can run the $(K_X+\Delta)$-minimal model program over Y with scaling of $\mathcal{A}$.

One of the main differences between the case where Y is Stein and the case where Y is compact is highlighted in the following remark.

Remark 2.2. In Theorem 2.1, the π-ample line bundle $\mathcal{A}$ may satisfy $H^0(X, \mathcal{A}^{\otimes m})=0$ for all $m>0$.

Let us prove Theorem 2.1.

Proof of Theorem 2.1. Since Y is compact, we have $\rho(X/Y;Y)<\infty$. Hence, we can apply the cone and contraction theorem established in [F2, Theorem 1.1.6]. In particular, we can run the $(K_X+\Delta)$-minimal model program over Y. We note that the existence of flips required for this minimal model program has already been established (see [F1, Theorem 1.14]). As usual, we set

$$\lambda := \inf\{\, t\geq 0 \mid K_X+\Delta+t\mathcal{A} \text{ is } \pi\text{-nef over } Y\,\}.$$

Then, by [F2, Theorem 1.1.6 (7)], there exists a $(K_X+\Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y;Y)$ such that

$$(K_X+\Delta+\lambda\mathcal{A})\cdot R=0.$$

Note that $K_X+\Delta+\lambda\mathcal{A}$ is π-nef over Y. Thus, we can run the $(K_X+\Delta)$-minimal model program over Y with scaling of $\mathcal{A}$. Then we obtain a sequence of flips and divisorial contractions:

$$X=: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow \cdots$$

starting from $(X_0, \Delta_0):=(X,\Delta)$. We put $\mathcal{A}_0:=\mathcal{A}$, $\mathcal{A}_i:=\phi_{i-1*}\mathcal{A}_{i-1}$, and $\Delta_i:=\phi_{i-1*}\Delta_{i-1}$ for every $i\geq 1$. Note that (X_i,Δ_i) is a $\mathbb{Q}$-factorial divisorial log terminal pair for every i. ☐

In the proof of Theorem 2.1, we set

$$\lambda_i := \inf\{\, t\geq 0 \mid K_{X_i}+\Delta_i+t\mathcal{A}_i \text{ is } \pi\text{-nef over } Y\,\}.$$

Then one can check that

$$\lambda =: \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_i \geq \cdots \geq 0$$

holds as in the usual algebraic setting.

The minimal model program described in Theorem 2.1 is expected to terminate after finitely many steps. However, the general termination is still widely open (see, for example, [EH2]). In what follows, we describe several important cases in which the minimal model program with scaling is known to terminate.

Theorem 2.3. *In Theorem 2.1, if $K_X+\Delta$ is not π-pseudo-effective, then the minimal model program with ample scaling always terminates with a Mori fiber space over Y. If $K_X+\Delta$ is π-pseudo-effective, then $\lambda_\infty:=\lim_{i\to\infty}\lambda_i=0$.*

Proof of Theorem 2.3. We take a finite open cover $Y=\bigcup_{j\in J}U_j$ with $U_j\subset W_j\subset V_j$, where U_j and V_j are open subsets of Y, and W_j is a Stein compact subset of Y such that $\Gamma(W_j,\mathcal{O}_Y)$ is noetherian for every $j\in J$. From now on, we freely shrink V_j around W_j for every $j\in J$. In particular, we may assume that each V_j is Stein.

We put $X_j:=\pi^{-1}(V_j)$ for every $j\in J$. We can take a general effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A_j on X_j such that

$$A_j\sim_{\mathbb{R}}\mathcal{A}|_{X_j}$$

and $(X_j, \Delta_j + A_j)$ is divisorial log terminal for every $j \in J$, where $\Delta_j := \Delta|_{X_j}$.

First, we consider the case where $K_X+\Delta$ is not π-pseudo-effective. We take a sufficiently small positive real number ε such that $K_X+\Delta+\varepsilon\mathcal{A}$ is not π-pseudo-effective. By applying [F1, Theorem E] to $(X_j, \Delta_j + \varepsilon A_j)$ for every $j \in J$, we obtain that the minimal model program over Y with scaling of $\mathcal{A}$ terminates with a Mori fiber space over Y.

Next, we consider the case where $K_X + \Delta$ is π-pseudo-effective. If the minimal model program terminates, then it is clear that $\lambda_\infty = 0$. Thus, we may assume that the minimal model program does not terminate and that $\lambda_\infty > 0$. In this case, the above minimal model program can be regarded as a $(K_X + \Delta + \frac{1}{2}\lambda_\infty\mathcal{A})$-minimal model program over Y with scaling of $\mathcal{A}$. By applying [F1, Theorem E] to $(X_j, \Delta_j + \frac{1}{2}\lambda_\infty A_j)$, we obtain a contradiction. Therefore, we always have $\lambda_\infty = 0$ when $K_X + \Delta$ is π-pseudo-effective. □

Theorem 2.4. *In Theorem 2.1, we further assume that* (X, Δ) *is kawamata log terminal and that one of the following conditions holds:*

(i) $K_X + \Delta$ *is* π*-big.*
(ii) $K_X + \Delta$ *is* π*-pseudo-effective and* Δ *is* π*-big.*
(iii) $K_X + \Delta$ *is not* π*-pseudo-effective.*

Then the $(K_X + \Delta)$*-minimal model program over* Y *with scaling of* $\mathcal{A}$ *in Theorem 2.1 always terminates. In* (i) *and* (ii)*, it terminates with a minimal model over* Y*. In* (iii)*, it terminates with a Mori fiber space over* Y*.*

Proof of Theorem 2.4. Note that (iii) has already been proved in Theorem 2.3. Thus, it suffices to treat (i) and (ii).

We use the same notation as in the proof of Theorem 2.3. We can take a general effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A_j on X_j such that

$$A_j \sim_{\mathbb{R}} \mathcal{A}|_{X_j}$$

and $(X_j, \Delta_j + A_j)$ is kawamata log terminal for every $j \in J$, where $\Delta_j := \Delta|_{X_j}$.

By the finiteness of weak log canonical models over V_j around W_j (see [F1, Theorem E]), the minimal model program in Theorem 2.1 terminates on X_j around W_j for every $j \in J$. These local terminations imply that the minimal model program terminates over Y. □

We close this note with the following useful theorem.

Theorem 2.5 (Dlt blow-ups for compact normal complex varieties)**.** *Let* X *be a compact normal complex variety and let* Δ *be an effective* $\mathbb{R}$*-divisor on* X *such that* $K_X + \Delta$ *is* $\mathbb{R}$*-Cartier. Then there exists a projective bimeromorphic morphism* $f\colon Z \to X$ *from a compact normal complex variety* Z *such that*

$$K_Z + \Delta_Z := f^*(K_X + \Delta)$$

and $(Z, \Delta_Z^{\leq 1} + \operatorname{Supp} \Delta_Z^{\geq 1})$ *is a* $\mathbb{Q}$*-factorial divisorial log terminal pair. Moreover, we may assume that* $a(E, X, \Delta) \leq -1$ *for every* f*-exceptional divisor* E *on* Z*. We note that if* (X, Δ) *is log canonical, then* $\Delta_Z = \Delta_Z^{\leq 1}$*.*

For the reader's convenience, we include a proof of Theorem 2.5.

Proof of Theorem 2.5. By resolution of singularities, we take a projective bimeromorphic morphism $g\colon V \to X$ from a smooth complex variety V such that

$$K_V + \Delta_V := g^*(K_X + \Delta).$$

We may assume that $\mathrm{Exc}(g)$ and $\mathrm{Supp}\, g_*^{-1}\Delta \cup \mathrm{Exc}(g)$ are simple normal crossing divisors on V.

We set

$$\Theta := g_*^{-1}\big(\Delta^{\leq 1} + \mathrm{Supp}\,\Delta^{>1}\big) + \sum_{E:\ g\text{-exceptional}} E.$$

Then we can write

$$K_V + \Theta = g^*(K_X + \Delta) + F,$$

where $-g_*F \geq 0$ by construction.

Since $g\colon V \to X$ is a projective morphism of compact normal complex varieties, we can take a g-ample line bundle $\mathcal{A}$ on V such that $K_V + \Theta + \mathcal{A}$ is g-nef. We run the $(K_V + \Theta)$-minimal model program over X with scaling of $\mathcal{A}$ as in Theorem 2.1.

By taking a suitable finite open cover of X as in the proof of Theorem 2.3, we can check that the positive part of F is contracted after finitely many steps of the minimal model program. For the details, see, for example, the proof of [F1, Theorem 1.21].

Thus we obtain the desired projective bimeromorphic morphism $f\colon Z \to X$. If (X, Δ) is log canonical, then it is clear that $\Delta_Z = \Delta_Z^{\leq 1}$. □

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