ON THE MINIMAL MODEL PROGRAM FOR PROJECTIVE MORPHISMS BETWEEN COMPACT COMPLEX ANALYTIC SPACES

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Abstract. We discuss the minimal model program for projective morphisms between compact complex analytic spaces. For simplicity, we consider only the minimal model program for divisorial log terminal pairs with ample scaling.

1. Introduction

We demonstrate that the minimal model program established in [F1] can be applied to the study of projective morphisms between compact complex analytic spaces. In this short note, we restrict our attention to the results of [F1] and [F2]. Naturally, by incorporating results from [EH1] and [H], one can obtain several more general versions of the minimal model program for projective morphisms of compact complex analytic spaces. We leave such generalizations to the interested reader as an exercise or further study.

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In this note, we freely use the notation and some results from [F1] and [F2].

2. Minimal model program for compact complex analytic spaces

Let $\pi\colon X \to Y$ be a projective morphism between complex analytic spaces. In [F1], the main focus is on the case where Y is Stein. More precisely, we fix a Stein compact subset W of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian and discuss the minimal model program over Y around W. In this short note, we note that the minimal model program can also work for projective morphisms between compact complex analytic spaces.

Theorem 2.1 (Minimal model program)**.** *Let $\pi\colon X \to Y$ be a projective morphism between compact complex analytic spaces. Let (X, Δ) be a $\mathbb{Q}$-factorial divisorial log terminal pair. Then we can run the $(K_X + \Delta)$-minimal model program over Y.*

Moreover, if $\mathcal{A}$ is a π-ample $\mathbb{R}$-line bundle on X such that $K_X + \Delta + \mathcal{A}$ is π-nef, then we can run the $(K_X + \Delta)$-minimal model program over Y with scaling of $\mathcal{A}$.

One of the main differences between the case where Y is Stein and the case where Y is compact is given in the following remark.

Remark 2.2. We note that in Theorem 2.1, the π-ample line bundle $\mathcal{A}$ may satisfy $H^0(X, \mathcal{A}^{\otimes m}) = 0$ for all $m > 0$.

Let us prove Theorem 2.1.

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Proof of Theorem 2.1. Since Y is compact, we have $\rho(X/Y;Y)<\infty$. Hence we can use the cone and contraction theorem established in [F2, Theorem 1.1.6]. Therefore, we can always run the $(K_X+\Delta)$-minimal model program over Y. We note that the existence of flips required for this minimal model program has already been established (see [F1, Theorem 1.14]). As usual, we put

$$\lambda := \inf\{t\geq 0 \mid K_X+\Delta+t\mathcal{A} \text{ is } \pi\text{-nef over } Y\}.$$

Then, by [F2, Theorem 1.1.6 (7)], there exists a $(K_X+\Delta)$-negative extremal ray R of $\overline{\mathrm{NE}}(X/Y;Y)$ such that $(K_X+\Delta+\lambda\mathcal{A})\cdot R=0$. Note that $K_X+\Delta+\lambda\mathcal{A}$ is π-nef over Y. Thus we can run the $(K_X+\Delta)$-minimal model program over Y with scaling of $\mathcal{A}$. Then we obtain a sequence of flips and divisorial contractions:

$$X=:X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} X_i \overset{\phi_i}{\dashrightarrow} \cdots$$

starting from $(X_0,\Delta_0):=(X,\Delta)$. We put $\mathcal{A}_0:=\mathcal{A}$, $\mathcal{A}_i:=\phi_{i-1*}\mathcal{A}_{i-1}$, and $\Delta_i:=\phi_{i-1*}\Delta_{i-1}$ for every $i\geq 1$. We note that (X_i,Δ_i) is a $\mathbb{Q}$-factorial divisorial log terminal pair for every i. We finish the proof of Theorem 2.1. □

In the proof of Theorem 2.1, we put

$$\lambda_i := \inf\{t\geq 0 \mid K_{X_i}+\Delta_i+t\mathcal{A}_i \text{ is } \pi\text{-nef over } Y\}.$$

Then can check that

$$\lambda=:\lambda_0\geq\lambda_1\geq\cdots\geq\lambda_i\geq\cdots\geq 0$$

holds as in the usual algebraic setting.

The minimal model program described in Theorem 2.1 is expected to terminate after finitely many steps. Unfortunately, the general termination is still a widely open problem. In what follows, we describe several important cases in which the minimal model program with scaling is known to terminate.

Theorem 2.3. *In Theorem 2.1, if $K_X+\Delta$ is not π-pseudo-effective, then the minimal model program with ample scaling always terminates at a Mori fiber space over Y. If $K_X+\Delta$ is π-pseudo-effective, then $\lambda_\infty:=\lim_{i\to\infty}\lambda_i=0$ holds.*

Proof of Theorem 2.3. We take a finite open cover $Y=\bigcup_{j\in J}U_j$ with $U_j\subset W_j\subset V_j$, where U_j and V_j are open subsets of Y and W_j is a Stein compact subset of Y such that $\Gamma(W_j,\mathcal{O}_Y)$ is noetherian, for every $j\in J$. From now on, we may freely shrink V_j around W_j for every $j\in J$. In particular, we may assume that V_j is Stein. We put $X_j:=\pi^{-1}(V_j)$ for every $j\in J$. We can take an effective general $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A_j on X_j with $A_j\sim_{\mathbb{R}}\mathcal{A}|_{X_j}$ such that (X_j,Δ_j+A_j) is divisorial log terminal for every $j\in J$, where $\Delta_j:=\Delta|_{X_j}$. We first treat the case where $K_X+\Delta$ is not π-pseudo-effective. We take a small positive real number ε such that $K_X+\Delta+\varepsilon\mathcal{A}$ is not π-pseudo-effective. By applying [F1, Theorem E] to $(X_j,\Delta_j+\varepsilon A_j)$ for every $j\in J$, we obtain that the minimal model program over Y with scaling of $\mathcal{A}$ terminates at a Mori fiber space over Y. Next, we treat the case where $K_X+\Delta$ is π-pseudo-effective. If the minimal model program terminates, then it is obvious that $\lambda_\infty=0$. So we assume that the minimal model program does not terminate with $\lambda_\infty>0$. In this case, the above minimal model program can be seen as a $(K_X+\Delta+\frac{1}{2}\lambda_\infty\mathcal{A})$-minimal model program over Y with scaling of $\mathcal{A}$. By applying [F1, Theorem E] to $(X_j,\Delta_j+\frac{1}{2}\lambda_\infty A_j)$, we get a contradiction. Anyway, we always have $\lambda_\infty=0$ when $K_X+\Delta$ is π-pseudo-effective. We finish the proof of Theorem 2.3. □

Theorem 2.4. *In Theorem 2.1, we further assume that (X,Δ) is kawamata log terminal and that one of the following conditions hold.*

(i) *$K_X + \Delta$ is π-big.*
(ii) *$K_X + \Delta$ is π-pseudo-effective and Δ is π-big.*
(iii) *$K_X + \Delta$ is not π-pseudo-effective.*

Then the $(K_X + \Delta)$-minimal model program over Y with scaling of $\mathcal{A}$ in Theorem 2.1 always terminates. In (i) *and* (ii)*, it terminates at a minimal model over Y. In* (iii)*, it terminates at a Mori fiber space over Y.*

Proof of Theorem 2.4. Note that (iii) has already been proved in Theorem 2.3. Hence it is sufficient to treat (i) and (ii). We use the same notation as in the proof of Theorem 2.3. We can take an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor A_j on X_j with $A_j \sim_{\mathbb{R}} \mathcal{A}|_{X_j}$ such that $(X_j, \Delta_j + A_j)$ is kawamata log terminal for every $j \in J$, where $\Delta_j := \Delta|_{X_j}$. By using the finiteness of weak log canonical models over V_j around W_j (see [F1, Theorem E]), we can check that the minimal model program in Theorem 2.1 terminates on X_j around W_j for every $j \in J$. Hence the minimal model program terminates over Y. We finish the proof of Theorem 2.4. □

We close this short note with a useful theorem.

Theorem 2.5 (Dlt blow-ups)**.** *Let X be a compact normal complex variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then there exists a projective bimeromorphic morphism $f \colon Z \to X$ from a compact normal complex variety Z with $K_Z + \Delta_Z := f^*(K_X + \Delta)$ such that $(Z, \Delta_Z^{\leq 1} + \operatorname{Supp} \Delta^{>1})$ is a $\mathbb{Q}$-factorial divisorial log terminal pair. Moreover, we may assume that $a(E, X, \Delta) \leq -1$ holds for every f-exceptional divisor E on Z. We note that if (X, Δ) is log canonical then $\Delta_Z = \Delta_Z^{\leq 1}$ holds.*

For the reader's convenience, we give a proof of Theorem 2.5 here.

Proof of Theorem 2.5. By the resolution of singularities, we take a projective bimeromorphic morphism $g \colon V \to X$ from a smooth complex variety V with $K_V + \Delta_V := g^*(K_X + \Delta)$. We may assume that $\operatorname{Supp} \Delta_V$ is a simple normal crossing divisor on V. We put

$$\Theta := g_*^{-1}\left(\Delta^{\leq 1} + \operatorname{Supp} \Delta^{>1}\right) + \sum_{E\colon g\text{-exceptional}} E.$$

Then we can write

$$K_V + \Theta = g^*(K_X + \Delta) + F$$

with $-g_*F \geq 0$ by definition. Since $g \colon V \to X$ is a projective morphism of compact normal complex varieties, we can always take a g-ample line bundle $\mathcal{A}$ on V such that $K_V + \Theta + \mathcal{A}$ is g-nef. Then we run the $(K_V + \Theta)$-minimal model program over X with scaling of $\mathcal{A}$ explained in Theorem 2.1. By suitably taking a finite open cover of X as in the proof of Theorem 2.3, we can check that the positive part of F is contracted after finitely many steps of the minimal model program. For the details, see, for example, the proof of [F1, Theorem 1.21]. Hence we obtain a desired projective bimeromorphic morphism $f \colon Z \to X$. If (X, Δ) is log canonical, then it is obvious that $\Delta_Z = \Delta_Z^{\leq 1}$ holds. We finish the proof of Theorem 2.5. □

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