Looijenga

reductive Lie grp G with compact center

a sym space for G : mfd $\mathbb{X}$ on which G acts trans. s.t. isotropy grps are max comp subgrps of G

max comp subgrp single conj class $\longrightarrow$ $\mathbb{X}$ unique

sym. domain if $\mathbb{X}$ cpx mfd, $G \hookrightarrow \mathrm{Bihol}(\mathbb{X})$

É Cartan I–VI

III real v. sp V of even dim $2g$, $V \times V \xrightarrow{a} \mathbb{R}$ symplectic $G = Sp(V)$

$a_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \longrightarrow \mathbb{C}$

$h : (v, v') \longmapsto \sqrt{-1}\, a_{\mathbb{C}}(v, \overline{v'})$ sgn (g, g)

$\mathbb{X} := \mathbb{H}_V := \{ F \in Gr_g V_{\mathbb{C}} :$ isotropic for $a_{\mathbb{C}}$, h: positive $\}$

Ψ autom line bundle: Λ^g (taut bdl)

F stabilizer : $U(V)$

$I_{p,q}$ W cpx v. sp. of dim $p+q$, $h : W \times W \to \mathbb{C}$ herm of sgn (p, q)

$G = U(W)$ $\mathbb{X} := \mathbb{B}_W := \{ F \in Gr_p W : h|F > 0 \}$

$U(W)_F = U(F) \times U(F^{\perp})$ autom line bundle Λ^p(taut. bdl)

$p = 1$; $\mathbb{B}_W \subset \mathbb{P}W$ given by $h(v, v) > 0$ complex ball

III) $\sigma \in Sp(V)$ semisimple. suppose $\mathbb{H}_V^{\sigma} \neq \phi$; let $F \in \mathbb{H}_V^{\sigma}$

$\Rightarrow \sigma \in U(F) \times U(F^{\perp}) \cap Sp(V)$ $V_{\mathbb{C}} = \underset{\sigma}{\overset{\circlearrowleft}{F}} \oplus \underset{\sigma}{\overset{\circlearrowleft}{\overline{F}}}$

eigenv. on unit circle

inv under cpx conj

$V_{\mathbb{C}} = \oplus_{\lambda} V_{\mathbb{C}}^{\lambda}$ $\lambda = \pm 1 \Rightarrow V_{\mathbb{C}}^{\pm 1}$ defined over $\mathbb{R}$ a: symplectic on $V^{\pm 1}$

$F = \oplus_{\lambda} F^{\lambda}$ and $F^{\pm 1} \in \mathbb{H}_{V^{\pm}}$

$\mathrm{Im}(\lambda) > 0$ $V_{\mathbb{C}}^{\lambda}$ isotropic for $a_{\mathbb{C}}$, h has

$\mathbb{H}_V^{\sigma} \cong \mathbb{H}_{V^+} \times \mathbb{H}_{V^{-1}} \times \prod_{\mathrm{Im}(\lambda) > 0} \mathbb{B}_{V^{\lambda}}$ sgn $(p_{\lambda}, q_{\lambda})$, say (nondeg)

$F^{\lambda} \in \mathbb{B}_{V^{\lambda}}$

$Sp(V)_{\sigma}$ acts decompose accordingly On $V_{\mathbb{C}}^{\bar{\lambda}}$ h has sgn $(q_{\lambda}, p_{\lambda})$

and $F^{\bar{\lambda}}$ ann of F^{λ}

$F \in \mathbb{H}_V \quad V_{\mathbb{C}} = \underset{V_{\mathbb{C}}^{1,0}}{\underset{\shortparallel}{F}} \oplus \underset{V_{\mathbb{C}}^{0,1}}{\underset{\shortparallel}{\overline{F}}}$ HS pol by s

$\underline{\mathrm{IV}}$ V real v.sp of dim $2+n \geq 2$, endowed with a sym bil form $s : V \times V \longrightarrow \mathbb{R}$ of sgn $(2,n)$ $\quad O(V)$, $G = O(V)^\circ$

$\mathbb{D}_V := \left\{ F \in \mathbb{P}V : F \text{ is } s_{\mathbb{C}}\text{-isotropic and } h\text{-positive} \right\}$ $\quad s_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$, $h(v, v') = s_{\mathbb{C}}(v, \bar{v}')$

2 copies of a sym domain

stabilizer of F : $P_F = (F \oplus \overline{F})(\mathbb{R})$ pos def plane

$\cong SO(P_F) \times O(P_F^{\perp})$ $\quad P_F^{\perp}$ neg def

autom line bundle : taut bundle rel to $\mathbb{D}_V$ ~~HS polarised by s~~

Suppose $\sigma \in O(V)$ and $\mathbb{D}_V^{\sigma} \neq \emptyset$

$F \in \mathbb{D}_V^{\sigma}$ dec $V_{\mathbb{C}} = \underset{\mathcal{H}^{2,0}}{F} \oplus \underset{\mathcal{H}^{0,2}}{\overline{F}} \oplus \underset{\mathcal{H}^{1,1}}{(F \oplus \overline{F})^{\perp}}$ $\quad P_F \oplus P_F^{\perp}$ $\quad \sigma \in SO(P_F) \times O(P_F^{\perp})$

eigenv. on unit circle $\quad V_{\mathbb{C}} = \oplus_\lambda V_{\mathbb{C}}^{\lambda} \quad F \in V_{\mathbb{C}}^{\lambda}$ for some λ.

$\lambda = \pm 1 \implies V^{\pm 1}$ real and has sgn $(2, n')$

so $F \in \mathbb{D}_{V^{\pm 1}}$

$\lambda \notin \mathbb{R}$ $\quad V_{\mathbb{C}}^{\lambda}$ isotropic for $s_{\mathbb{C}}$, $\Rightarrow h$ has sgn $(1, n')$ for some n'

$\quad \cup$

$\quad F \qquad \overline{F} \subset V_{\mathbb{C}}^{\bar\lambda} \qquad (1, n')$

$$\mathbb{D}^{\sigma} \cong \begin{cases} \mathbb{D}_{V^{\pm}} \\ \mathbb{B}_{V^{\lambda}} \sqcup \mathbb{B}_{V^{\bar\lambda}} \end{cases}$$

Of the irreducible sym domains the following admit (complex) reflections

$I_{1,q}$, $\quad W^{q+1} \qquad \mathbb{B}_{W'} \subset \mathbb{B}_W$

$\quad \cup$ codim 1 $\qquad U(1) \hookrightarrow U(W)$ leaves W' pt wise fixed

$\quad W'$ sgn $(1, q-1)$ $\qquad$ has $\mathbb{B}_{W'}$ fixed pt set

IV $\quad V \quad (2,n) \qquad \sigma \in O(V)$ identity on V', -1 on $(V')^{\perp}$

$\quad \overset{\cup}{V'} \quad (2,n-1) \qquad D_{V'} \subset D_V$

$$D_{V'} = D_V^{\sigma}$$

$G \qquad X$

For arithmetic enhancement. G must be defined over a number field

over $\mathbb{Q}$ is already o.k.

III $\quad (V, a)/\mathbb{Q} \qquad$ IV $\quad (V, s)/\mathbb{Q}$

$\Gamma \subset G(\mathbb{Q})$ arithmetic subgrp when Γ stabilizes a lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$

and is of finite index in $G \cap GL(V_{\mathbb{Z}})$ ($\Rightarrow \Gamma$ discrete in G)

Γ will then acts properly on $\mathbb{H}_V$, $\mathbb{D}_V$ (and on $\mathbb{L}_V$)

autom line bundles : $\qquad$ Pass to Γ-orbit space

$$\begin{array}{ccc} \mathbb{L}_V & \mathbb{L}_V & \mathbb{L}_V^{\times} = \mathbb{L}_V - \text{zero section} \\ \downarrow & \downarrow & \\ \mathbb{H}_V & \mathbb{D}_V & \end{array}$$

$$\Gamma \backslash \begin{pmatrix} \mathbb{L}_V^{\times} \\ \downarrow \\ \mathbb{D}_V, \mathbb{H}_V \end{pmatrix} : \quad \begin{array}{c} (\mathbb{L}_V^{\times})_{\Gamma} \\ \downarrow \\ (\mathbb{D}_V)_{\Gamma}, (\mathbb{H}_V)_{\Gamma} \end{array} \qquad \begin{array}{c} \mathbb{L}_{\Gamma}^{\times} \\ \downarrow \\ \mathbb{X}_{\Gamma} \end{array} \quad \text{in general}$$

Baily-Borel package

For $d \in \mathbb{Z}$, $A_d(\mathbb{L}^{\times}) = \{ f : \mathbb{L}^{\times} \to \mathbb{C}$ hom of degree $-d$: $f(cv) = c^{-d} f($
$\quad A_{\cdot}(\mathbb{L}^{\times})$ closed under mult : graded alg $\qquad$ + growth condition (often empty)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ↑

finiteness. $A_{\cdot}(\mathbb{L}^{\times})^{\Gamma}$ is a f.g. graded subalg with $\qquad$ inv under $G(\mathbb{Q})$

pos degree gen.

$\mathbb{L}_{\Gamma}^{\times\, bb} := \mathrm{Spec}\,(A_{\cdot}(\mathbb{L}^{\times})^{\Gamma}) \qquad \mathrm{Proj}\,(A_{\cdot}(\mathbb{L}^{\times})^{\Gamma}) =: \mathbb{X}_{\Gamma}^{bb}$

quasi-hom cone with base

<u>separation</u> $A_\bullet(\mathbb{L}^\times)^\Gamma$ separates the Γ-orbits, $\mathbb{L}^\times_\Gamma \hookrightarrow (\mathbb{L}^\times_\Gamma)^{bb}$ and

<u>topology</u> $\mathbb{L}^{\times bb}_\Gamma$ arises as a Γ-orbit space of a Γ extends $\mathbb{L}^{\times bb} \supset \mathbb{L}^\times$ which comes with a natural part of complex manifolds (of same type as $\mathbb{L}^\times$) with a topology invariant under $G(\mathbb{Q}) \times \mathbb{C}^\times$ such that if $\mathcal{O}_{\mathbb{L}^{\times bb}}$ sheaf of cont piecewise hol fns, then $\mathbb{L}^{\times bb}_\Gamma$ with its anal structure is the Γ-orbit space ($\mathbb{L}^\times$ appears in this partition)

This partition gives rise to a stratification of $\mathbb{L}^{\times bb}_\Gamma$ inv under $\mathbb{C}^\times$ and hence also one for X^{bb}_Γ

$\Rightarrow$ X_Γ and $\mathbb{L}^\times_\Gamma$ have a quasi proj scheme

$I_{p,q}$ $\mathbb{C}$ v.sp W, with Herm form h of sign (p,q)

Here $G = U(W)$ $\mathbb{B}_W$

(V, a) symplectic $\sigma \in Sp(V)$

$(*)$ $\mathbb{H}_V^{\sigma} = \mathbb{H}_{V^{+1}} \times \mathbb{H}_{V^{-1}} \times \prod_{\mathrm{Im}(\lambda)>0} \mathbb{B}_{V^\lambda}$

If (V, a) defined over $\mathbb{Q}$ and $\sigma \in Sp(V_{\mathbb{Q}})$

and $(*)$ reduces to one factor $\cong \mathbb{B}_W$

Then $Sp(V)_\sigma$ defined over $\mathbb{Q}$ and $\mathbb{B}_W$ is its sym dom

More intrinsically: we assume (W, h) defined over a CM field K

require: $\iota : K \hookrightarrow \mathbb{C}$

$(W \otimes_\iota \mathbb{C}, h \otimes_\iota \mathbb{C})$ has for some pair $(\iota_0, \bar{\iota}_0)$ sign (p,q)

and is definite for all other emb

situation nice if σ has finite order $m \in \{3, 4, 6\}$

for then $e^{2\pi i/m}$, $e^{-2\pi i/m}$ are the only prim m-th roots of 1

Then focus on the sum of corr eigen sp

This is defined $/\mathbb{Q}$ and has desired property

BB-extension in the other case

$(V, s : V \times V \to \mathbb{R})$ sign $(2, n)$

$/\mathbb{Q}$ and suppose $\Gamma \subset O(V_{\mathbb{Q}})$ arithm

Let $\mathcal{I} = \{$isotropic subsp of $V / \mathbb{Q}\}$ including $\{0\}$

Fact: Γ has only finitely many orbits in $\mathcal{I}$

For $I \in \mathcal{I}$

$$\pi_{I^\perp} : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} / I_{\mathbb{C}}^\perp = (V/I^\perp)_{\mathbb{C}} \cong I_{\mathbb{C}}^*$$

$\mathbb{L}^\times := \{ v \in V_{\mathbb{C}} \;\; s_{\mathbb{C}}(v, v) = 0, \;\; s_{\mathbb{C}}(v, \bar{v}) > 0 \}$

$\downarrow /\mathbb{C}^\times$ put here Satake topology

$\mathbb{P}(\mathbb{L}^\times) =: \mathbb{D}$

pt if $I = \{0\}$

$I_{\mathbb{C}}^* \setminus \{0\}$ if $\dim I = 1$

$$(\mathbb{L}^\times)^{bb} := \mathbb{L}^\times \sqcup \coprod_{I \in \mathcal{I}} \pi_{I^\perp}(\mathbb{L}^\times)$$

$\cong \mathrm{Isom}_{\mathbb{R}}(I, \mathbb{C})$ if $\dim I = 2$

$\mathbb{C}^\times$ $O(V_{\mathbb{Q}})$ - acts here

Γ acts

$(\dim I = 2)$

Γ_I acts in the last case through an arith grp (like $SL(2,\mathbb{Z})$)

orbit space is a $\mathbb{C}^\times$-bundle over 1 or 2 modular curves

$\Gamma \backslash (\mathbb{L}^\times)^{bb} = (\mathbb{L}^\times)^{bb}_\Gamma = \mathbb{L}^\times_\Gamma \sqcup$ modular curves (fin. number) $\mathbb{C}^\times$-bundles $\sqcup$ finitely many copies of $\mathbb{C}^\times \sqcup \{*\}$

$I_{1,n}$ (W, h) Hermitian v.sp. / CM field K

$\Gamma \subset U(W)$ "arithm"

$\mathcal{J} = \{$isotropic subspaces defined over $K\}$

Γ acts on this with fin many orbits

$I \in \mathcal{J} \Rightarrow \dim I \leq 1$

$\pi_{I^\perp} : W \longrightarrow W/I^\perp \cong I^*$

$\mathbb{L}^\times = \{ w \in W : h(w,w) > 0 \}$

$\downarrow /\mathbb{C}^\times$

$\mathbb{P}(\mathbb{L}^\times) = \mathbb{B}_W \qquad \pi_{I^\perp}(\mathbb{L}^\times) = \begin{cases} \{*\} & \text{if } I = \{0\} \\ I^* - \{0\} & \text{if } \dim I = 1 \end{cases}$

$$(\mathbb{L}^\times)^{bb} = \mathbb{L}^\times \sqcup \bigsqcup_{I \in \mathcal{J}} \pi_{I^\perp}(\mathbb{L}^\times)$$

$(\mathbb{L}^\times)^{bb}_\Gamma = (\mathbb{L}^\times)_\Gamma \sqcup$ fin. many copies of $\mathbb{C}^\times \sqcup \{*\}$

$\mathbb{D}^{bb}_\Gamma$ and $\mathbb{B}^{bb}_\Gamma$ are obtained in the obvious way

add a finite set

In these two cases we have (complex) reflections, (totally geodesic hyperplane sections

$\mathbb{X} = \mathbb{D}$ or $\mathbb{B}$, Γ as above

<u>Γ-arrangement</u> : collection $\{H \cap \mathbb{X}\}_{H \in \mathcal{H}}$ hyperplane sections (tot. geod) given as a finite union of Γ-orbits

Such a set is loc finite

$\mathbb{X}_{\mathcal{H}} := \bigcup_{H \in \mathcal{H}} \mathbb{X} \cap H$ closed Γ-inv (Cartier div)

its image $(\mathbb{X}_{\mathcal{H}})_\Gamma \subset \mathbb{X}_\Gamma$ $\mathbb{Q}$-Cartier divisor

(Its closure in X_P^{bb} is in general <u>not</u> $\mathbb{Q}$-Cartier)

<u>Example</u> (Deligne - Mostow)

<u>GIT basics</u> :

G . complex red alg grp

H : f.d. $\mathbb{C}$rep of G

- $\mathbb{C}[H]^G$ is f.g. graded alg with pos. deg. gen.
- $\mathrm{Spec}\, \mathbb{C}[H]^G$ base $\mathrm{Proj}\, \mathbb{C}[H]^G$

$\quad\ \|$ closed pts

$G /\!/ H$ $\updownarrow$

closed G-orbits

$H^{ss} := \{ v \in H : \overline{Gv} \not\ni 0 \}$

$G /\!/ \mathbb{P}(H^{ss}) \rightleftarrows \mathrm{Proj}\, \mathbb{C}[H]^G$

closed G-orbits in $\mathbb{P}(H^{ss}) \leftrightarrow$ closed pts here

$H^{st} = \{ v \in H^{ss} ; G_v : \text{finite} \}$

open in H^{ss}

$G \backslash \mathbb{P}(H^{st}) \hookrightarrow G /\!/ \mathbb{P}(H^{ss})$.

<u>Example</u> (Deligne - Mostow)

U 2-dim $\mathbb{C}$-v.sp. $\quad \mathrm{Sym}^{12} U^* =: H$

group $SL(U)$ $\quad SL(U)$-rep

$\mathbb{P}H$: complete linear system

$H^{ss} = \{$ assoc. div has no pt of mult $> 6 \}$

$\cup$

$H^{st} = \{$ — — $\geq 6 \}$

$\cup$

$H^{\circ} = \{$ — — $\geq 2 \}$

$F \in H^\circ \longrightarrow \Delta_F \subset \mathbb{P}U$ 12-elt subset

$$\Delta_F \subset \begin{array}{c} C_F \\ \downarrow \mu_6 \\ \mathbb{P}U \end{array} \quad \text{given by } w^6 = F(u)$$

where (w, u) and $(\lambda^2 w, \lambda u)$ define same pt

Consider the differential ω_F on C_F

"defined" by $\dfrac{\alpha}{w}$ α transl 2-form on U

degree zero

μ_6 acts on C_F

Let χ be such that $\frac{1}{w}$ is in χ-eigenvector

$\Rightarrow$ ω_F a χ-eigenvector

$g(C_F) = 25$ $H^1(C_F, \mathbb{Z})$ sympl. unim rk 50

$H^1(C_F, \mathbb{C})^{\chi^i}$ of dim 10 for $i = 1, \dots, 5$

$\cup$

~~max pos~~ def subsp $H^{1,0}(C_F, \mathbb{C})^{\chi^i}$ dim is $2i - 1$ (for $i = 1$, spanned by ω_F)

$H^1(C_F, \mathbb{C})^\chi$ has sgn $(1, 9)$

Suggests: fix $(V_\mathbb{Z}, a_\mathbb{Z})$ sympl lattice unim genus 25, endowed with a char of μ_6 such that

$\exists$: $H^1(C_F, \mathbb{Z}) \xrightarrow[\varphi]{\cong} V_\mathbb{Z}$ ambiguity is in $Sp(V_\mathbb{Z})_{\mu_6}$

$\circlearrowleft \mu_6$ $\circlearrowleft \mu_6$

$W := V_\mathbb{C}^\chi$ Γ image of $Sp(V_\mathbb{Z})_{\mu_6}$

$\cup$

$\mathbb{L}^\times \ni \varphi(\omega_F)$

$H^\circ \ni F \longmapsto \mathbb{L}^\times_\Gamma$ Γ-orbit of $\varphi(\omega_F)$

Get: $SL(U) \backslash H^\circ \longrightarrow \mathbb{L}^\times_\Gamma$ open emb.

$\cap$

$SL(U)\backslash H^{st} \overset{\cong}{\longrightarrow} (\mathbb{L}^\times_\Gamma)^{bb}$ $A(\mathbb{L}^\times)^\Gamma$

$\subset SL(U)\backslash\backslash H^{ss} \overset{\cong}{\longrightarrow}$ $\mathbb{C}[H]^{SL(U)} \cong$

U cx vsp of dim 3

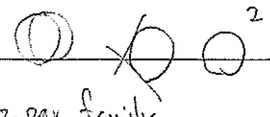
$H = \mathrm{Sym}^4 U^*$ $\quad SL(U)$-rep

$H \supset H^{ss} \supset H^{st} \supset H^{0}$

A_3-sing $\quad$ nodes A_1 & cusps A_2 $\quad$ smooth quartic curves

$\uparrow$

closed orbits



2-par. family

Let $F \in H^0 \to C_F \subset \mathbb{P}U$ smooth quartic

$$\begin{array}{ccc} & S_F \subset \mathbb{P}(\mathbb{C}\oplus U) & w^4 = F(u) \\ C_F \nearrow & \downarrow \mu_4 & \\ \searrow & \mathbb{P}U & \end{array}$$

α transl invar 3-form on U

$\omega_F := \mathrm{Res}\,\dfrac{\alpha}{w^3}$ 2-form on S_F

S_F K3 surface

Let $\chi : \mu_4 \hookrightarrow \mathbb{C}^\times$ s.t. w equiv for χ^{+1}

Then ω_F also eigenv for χ

$H^2(S_F, \mathbb{Z})$ even unim of sign $(3,19)$

$\circlearrowleft \mu_4$

$H^2(S_F, \mathbb{Z})^{\mu_4}$ image of $H^2(\mathbb{P}U, \mathbb{Z}) \cong \mathbb{Z}$

χ^i occurs in $H^2(S_F, \mathbb{C})$ with mult 7 for $i = 1,2,3$

$H^2(S_F, \mathbb{C})^{\chi}$ dim 7

$\cup$

$\mathbb{C}\omega_F = H^{2,0}(S_F)$ with complement of type $(1,1)$ prim. hence neg def $\Big\}$ sign $(1,6)$

Fix an even unim lattice Λ of sign $(3,19)$ with μ_4-action so that

$\exists\ H^2(S_F, \mathbb{Z}) \xrightarrow[\varphi]{\cong} \Lambda$ unique up an ele of $O(\Lambda)_{\mu_4} \supseteq \Gamma$ acts here via Γ

$\circlearrowleft \mu_4 \qquad \circlearrowleft \mu_4 \qquad W := (\Lambda \otimes \mathbb{C})^{\chi}$ sign $(1,6)$

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$\varphi_{\mathbb{C}}(\omega_F) \in \mathbb{L}^\times = \{ w \in W : h(w,w) > 0 \}$

$F \mapsto \Gamma . \varphi_{\mathbb{C}}(\omega_F)$

$$\begin{array}{ccc}
H^0 & \longrightarrow & \mathbb{L}^\times_\Gamma \\
\downarrow & & \\
SL(U)\backslash H^0 & & \\
\cap & & \\
SL(U)\backslash H^{st} & \longrightarrow & (\mathbb{L}^*_{\mathcal{K}})_\Gamma \\
\dim 6 & \cong &
\end{array}$$

not onto : image arrangement complement

$\mathbb{C}[SL(U)\backslash H^{st}] = \mathbb{C}[SL(U)\backslash H^{ss}]$

$\|$

this isom realizes $\mathbb{C}[H]^{SL(U)}$ $\quad \mathbb{C}[H]^{SL(U)}$

as the algebra of anal fun in $\mathbb{L}^\times_{\mathcal{K}}$, invar under Γ

that are finite sum of homogeneous one. (surprising)

type IV example sextic curves

$H = \mathrm{Sym}^6 U^* \supset H^{ss} \supset H^{st} \supset H^0$

additional closed orbit 3-dim family (each 20 dim sp)

allow simp sing

$F \in H^0$ $\quad S_F \quad w^2 = F(u)$, 6

$C_F \quad \downarrow \mu_2 \quad \deg(w) = 3$

$\mathbb{P}U \quad (w, u) \sim (\lambda^3 w, \lambda u)$

$\lambda \in \mathbb{C}^\times$

$\omega_F = \mathrm{Res} \frac{\alpha}{w}$

2-form on S_F $\cup$ K3 surface

involn ι $H^2(S_F, \mathbb{Z})$ $\circlearrowleft$

$H^2(S_F, \mathbb{Z})^{\mu_2} \cong H^2(\mathbb{P}U, \mathbb{Z}) \cong \mathbb{Z}$

Λ : K3 lattice $\quad H^2(S_F, \mathbb{Z})$

$$\iota \circlearrowright \Lambda \xleftarrow[\varphi]{\cong} H^2(S_F,\mathbb{Z}) \circlearrowleft \iota$$

$V \subset (\Lambda \otimes \mathbb{R})^{-1}$ sign $(2, 19)$

$\mathbb{L}^\times \subset V_\mathbb{C} \quad \{ v \in V_\mathbb{C} \; ; \; s_\mathbb{C}(v,v) = 0, \; s_\mathbb{C}(v,\bar{v}) > 0 \}$

$\varphi(\omega_F) \in \mathbb{L}^\times \qquad O(\Lambda)_{/\Lambda_2} =: \Gamma \subset O(V_\mathbb{Z})$

$\hookrightarrow \mathbb{L}^\times_\Gamma$

$$\begin{tikzcd}
H^0 \arrow[r] \arrow[d] & \mathbb{L}^\times_\Gamma \\
SL(U)\backslash H^0 \arrow[ur] \arrow[d, phantom, "\cap"] & \\
SL(U)\backslash H^{st} \arrow[r, "\cong"'] \arrow[uur] & (\mathbb{L}^\times_{\mathcal{H}})_\Gamma \arrow[uu, "\text{emb}"', "\text{J. Shah}"]
\end{tikzcd}$$

$\mathbb{C}[H]^{SL(U)} \equiv \Gamma$-inv fcns on $\mathbb{L}^\times_{\mathcal{H}}$

finite sum of hom elts

$SL(U)\backslash\backslash H^{ss}$

g-hom cone

BB package for ball case

(W, h) herm v.sp of sign $(1, n)$ $n \geq 1$

$\Gamma \subset U(W)$ arithm. $\quad$ ((W, h) defined over CM field $K \subset \mathbb{C}$)

Let be given a Γ-arrangement $\mathcal{H}$. $\quad \Gamma \in U(W_K)$

every $H \in \mathcal{H}$ defined over K

$\mathbb{L}^\times = \{ w \in W \; ; \; h(w, w) > 0 \}$

$\Gamma \circlearrowright \; \cup$

$\mathbb{L}^\times_{\mathcal{H}}$

Usual BB $\quad I \subset W$ isotropic $/K$

$$\mathbb{L}^\times \subset W \xrightarrow{\pi_{I^\perp}} W/I^\perp$$

$$\bigsqcup_I \pi_{I^\perp}(\mathbb{L}^\times) \sqcup \mathbb{L}^\times = (\mathbb{L}^\times)^{bb}$$

Let $\mathcal{J}_{\mathcal{H}}$ be the collection of nonpos subsp of W that are intersections label. from $\mathcal{H} \cup \{I^\perp\}_{I \subset J/K}$. This includes W (empty inters) but not $\{0\}$ (positive)

$J \in \mathcal{J}_{\mathcal{H}} \quad \mathbb{L}^\times \subset W \xrightarrow{\pi_J} W/J$

$\mathbb{L}^\times \supset \mathbb{L}^\times_{\mathcal{H}}$ $\quad$ If J is Lorentz sign, then $\pi_J(\mathbb{L}^\times) = W/J$

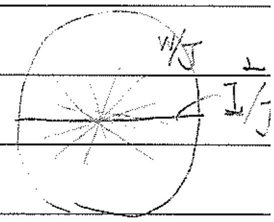
$\pi_J(\mathbb{L}^\times_{\mathcal{H}})$ arrangement complement

Γ_J acts through a finite grp.

If J degenerate

$J \cap J^\perp =: I$ isotropic line then $\pi_J(\mathbb{L}^\times) = W/J - I^\perp/J$

$\pi_J(\mathbb{L}^\times_{\mathcal{H}})$



Γ_J acts here (after passage to a subgrp finite index)

via a lattice, orbit space $\mathbb{C}^\times$-bundle over an abelian torsor

$$(\mathbb{L}^\times_{\mathcal{H}})^{bb} := \mathbb{L}^\times_{\mathcal{H}} \sqcup \bigsqcup_{J \in \mathcal{J}_{\mathcal{H}}} \pi_J(\mathbb{L}^\times_{\mathcal{H}})$$

Satake type topology $\Gamma \times \mathbb{C}^\times$-invariant

$$(\mathbb{L}^\times_{\mathcal{H}})^{bb}_{\Gamma} = \mathrm{Spec}\,(A^{\cdot}(\mathbb{L}^\times_{\mathcal{H}})^{\Gamma})$$

$$\mathbb{C}[H]^{SL(U)} \nearrow \quad A^d(\mathbb{L}^\times_{\mathcal{H}}) = \{ f : \mathbb{L}^\times_{\mathcal{H}} \to \mathbb{C} \text{ hol hom of degree } -d \text{ + growth cond} \}$$