COMPACTIFICATION OF THE MODULI SPACE OF ABELIAN VARIETIES
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1. introduction

1. There are curves called stable curves of Deligne-Mumford. We know first smooth curves, then comes stable curves, relatively simple curves with mild singularities and finite automorphism.
2. The moduli $\overline{M}_g$ of stable curves of genus g, it is now known to be a projective scheme.
3. The moduli M_g of smooth curves of genus g is not compact, only quasi-projective. It is compactified as the moduli $\overline{M}_g$ of stable curves of genus g.
4. we wish to do the same for the moduli of abelian varieties AV.

2. Compactifications

1. There are many compactifications of the moduli of abelian varieties
2. Satake, Igusa
3. Mumford
4. FaltingsChai
5. However these are not compactification as the moduli of compact objects
6. I wish to construct a unique canonical compactification of the moduli of abelian varieties as the the moduli of compact objects, abelian varieties and their limits with extra structure, say, with non-commutative level structure
7. The compact objects are PSQAS, and actually we can construct a projective fine moduli scheme of the compact objects PSQASes.

3. Hesse cubics

3.1. **Definition.**

1. Let me give an example of the compactification
2. It is the moduli of Hesse cubics
3. Hesse cubic is thecurve defined by the equation
4. It is nonsingular elliptic curve for geenral μ, say, if $\mu \neq \infty$ or if $\mu^3 \neq 1$
5. if $\mu = \infty$ or if $\mu^3 \neq 1$ then it is a 3-gon,
6. if nonsingular, it contains 9 flexes K, a constant set independent of μ, hence any $C(\mu)$ contains K

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7. over $\mathbf{C}$, Let $E = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega$, a 1-dim torus,
$$\theta_k = \sum_{m \in \mathbf{Z}} e^{2\pi i(3m+k)^2\omega/6 + 2\pi i(3m+k)z)}$$
8. let $x_k = \theta_k$, and K is the image of 3-torsions. θ_k satisfy the equation of Hesse cubics.

3.2. **Moduli.**

1. Let us consider the moduli problem of the pair $(C(\mu), K)$
2. two pairs $(C(\mu), K)$ and $(C(\mu'), K)$ are isom. iff
there is isom $f : C(\mu) \to C(\mu')$ such that f is the identity of K.
3. Then $(C(\mu), K)$ and $(C(\mu'), K)$ are isom. iff $\mu = \mu'$.
4. because the isom f is 3×3 matrix, which fixes K, and sends 9 triple tangents to 9 triple tangents, it turns out to be the identity
5. Thus the set of isom of $(C(\mu), K)$ is just the set of μ, $\mathbf{P}^1$.

4. Noncommutative level structure

4.1. **A new difficulty.**

1. in higer dimension the similar moduli theory like $(C(\mu), K)$ leads us to a nonseparated moduli
2. because the degenerate abelian variety may be reducible in general, and the embeddings of K into them may not be equivalent to each other.
3. we should give up the embedded subscheme K
4. instead we consider the action which K induces on $H^0(C, O_C(1))$
5. This is the Heisenberg group and Schrödindger representatioin

4.2. **Non-commutative interpretation.**

1. any $x \in K$ is a 3-torsion,
2. translation by any $x \in K$ is lifted to $\gamma_x \in \mathrm{GL}(3)$
3. translation by $1/3$ is lifted to σ
(Recall that x_k is theta)
$\theta_k(z + 1/3) = \zeta_3^k \theta_k(z)$
4. translation by $1/3$ is lifted to τ
$[\theta_0, \theta_1, \theta_2](z + \omega/3) = [\theta_1, \theta_2, \theta_0](z)$
5. $\sigma(x_k) = \zeta_k x_k$
6. $\tau(x_k) = x_{k+1}$.
7. $G(K) = \langle \sigma, \tau \rangle$ is a subgroup of $\mathrm{GL}(3)$,
8. This is a particular case of the following general

4.3. **Heisenberg group.**

Definition 4.4. $G(K) = G_H$: Heisenberg group;
U_H : Schrödinger representation

$$K = H \oplus H^\vee, H \text{ finite abelian}, N = |H|$$
$$G_H = \{(a, z, \alpha); a \in \mu_N, z \in H, \alpha \in H^\vee\},$$
$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta),$$
$$V := V_H = \mathcal{O}[H^\vee],$$
$$(a, z, \alpha)v(\gamma) = a\gamma(z)v(\alpha + \gamma)$$

The action of $G(K)$ on V is denoted U_H, $\mathcal{O} = \mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$.

In the Hesse cubics case, $\mathcal{O} := \mathbf{Z}[\zeta_3, 1/3]$, $H = H^\vee = \mathbf{Z}/3\mathbf{Z}$, we identify $G(3)$ with $G(K)$:

$$\sigma = (1, 1, 0), \tau = (1, 0, 1) \in G(K), N = 3.$$
$$V_H = \mathcal{O}[H^\vee] = \mathcal{O} \cdot v(0) \oplus \mathcal{O} \cdot v(1) \oplus \mathcal{O} \cdot v(2)$$

4.5. **New formulation.**

1. classical level 3 str. = Fix the 3-division points K
2. new level 3 str.=Fix the matrix form of $G(K)$ on $V \simeq H^0(C, L)$
3. Let C: any smooth cubic, $L = O_C(1)$, Then the pair (C, L) always has a $G(K)$-action τ

Definition 4.6. For C any cubic with $L = O_C(1)$, (C, ψ, τ) is a level-$G(K)$ structure if

1. τ is a $G(K)$-action on the pair (C, L),
2. $\psi : C \to \mathbf{P}(V_H) = \mathbf{P}^2$ is the inclusion (it is a $G(K)$-equivariant closed immersion by τ)

Define : $(C, \psi, \tau) \simeq (C', \psi', \tau')$ isom. iff
$\exists\, (f, F) : (C, L) \to (C', L')$ $G(K)$-isom. with $\phi' \cdot f = \phi$
(This is equivalent to $f_{|K} = \mathrm{id}_K$ in the classical case.)

3. The second talk — Stable reduction theorem

Definition 3.1. $G(K) = G_H$: Heisenberg group;

U_H : Schrödinger representation

$$K = H \oplus H^\vee, H \text{ finite abelian}, N = |H|$$
$$H = H(e) = \oplus_{i=1}^{g}(\mathbf{Z}/e_i\mathbf{Z}), \text{ with } e_i|e_{i+1},$$
$$G_H = \{(a, z, \alpha); a \in \mu_N, z \in H, \alpha \in H^\vee\},$$
$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta),$$
$$V := V_H = \mathcal{O}[H^\vee],$$
$$(a, z, \alpha)v(\gamma) = a\gamma(z)v(\alpha + \gamma)$$

The action of $G(K)$ on V is denoted U_H, $\mathcal{O} = \mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$.

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$$\sigma = (1, 1, 0), \tau = (1, 0, 1) \in G(K), N = 3.$$
$$V_H = \mathcal{O}[H^\vee] = \mathcal{O} \cdot v(0) \oplus \mathcal{O} \cdot v(1) \oplus \mathcal{O} \cdot v(2)$$

3.2. Now we wish to construct limits of abelian schemes, PSQASes and TSQASes. We consider mainly $e_{\min}(K) := e_1 \geq 3$.

Let R be a CDVR, and $k(\eta)$ the fraction field of R. We start with

1. an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ and a pol. morphism
$$\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t := \text{Pic}\ ^0(G_\eta),$$
$$a \mapsto T_a^*(\mathcal{L}_\eta) \otimes \mathcal{L}_\eta^{-1}$$
2. let $K_\eta =: \ker(\mathcal{L}_\eta)$,
3. assume chara. $k(0) = R/m_R$ is prime to $|K_\eta|$,
4. over $\mathcal{O} := \mathcal{O}_N$, $N = \sqrt{|K_\eta|}$, we may assume
   $K_\eta =: \ker(\mathcal{L}_\eta) \simeq K = H(e) \oplus H(e)^\vee$, $(\exists e)$
5. $\Longrightarrow \mathcal{G}(K)$ acts on the pair $(G_\eta, \mathcal{L}_\eta)$,

Then we have a stable reduction theorem

Theorem 3.3. (A refined version of Alexeev-Nakamura's stable reduction theorem) ([AN99], [N99]) *Assume* $e_{\min}(K) \geq 3$. $\exists$ *proper flat projective schemes* $(Q, \mathcal{L}_Q)$ *(PSQAS) and* $(P, \mathcal{L}_P)$ *(TSQAS) over* R*, by a finite base change if necessary, such that*

(0) $(Q_\eta, \mathcal{L}_\eta) \simeq (P_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta)$,
(1) $(P, \mathcal{L}_P)$ *is the normalization of* $(Q, \mathcal{L}_Q)$,
(2) P_0 *is reduced,*
(3) $\mathcal{L}_Q$ *is very ample,*
(4) $G(K)$ *acts on* $(Q, \mathcal{L}_Q)$ *and* $(P, \mathcal{L}_P)$ *extending the action of it on* $(G_\eta, \mathcal{L}_\eta)$,

Definition 3.4. The triple (X, ϕ, τ) or (X, L, ϕ, τ) is a PSQAS with level-$G(K)$ str. if

1. τ is a $G(K)$-action on the pair (X, L),
2. $\phi : X \to \mathbf{P}(V)$ a $G(K)$-equiv. closed immersion such that $\phi^* : V \simeq H^0(X, L)$, $L = \phi^* O_{\mathbf{P}(V)}(1)$.

Define : $(X, \phi, \tau) \simeq (X', \phi', \tau')$ isom. iff

$\exists$ $(f, F) : (X, L) \to (X', L')$ $G(K)$-isom. which makes the diagram commutative

$$\phi' \cdot (f, F) = \phi : (X, L) \to (\mathbf{P}(V_H), O_{\mathbf{P}}(1))$$

- Theorem proves that *the moduli is proper*,
- $(Q_0, \mathcal{L}_0)$: PSQAS — projectively stable quasi-abelian scheme,
- $(P_0, \mathcal{L}_0)$: TSQAS — torically stable quasi-abelian scheme (= variety),
- In dim. ≤ 4, any PSQAS=TSQAS, in dim. one it is a smooth elliptic or an N-gon,
- In dim. 8, PSQAS $\neq$ TSQAS for E8,
- The next theorem proves that *the moduli is separated.*

Theorem 3.5. ([N99],[N10],[N13]) *Suppose* $e_{\min}(K) \geq 3$*. Then* $(Q, \mathcal{L})$ *and* $(P, \mathcal{L})$ *are uniquely determined by* $(G_\eta, \mathcal{L}_\eta)$.

Suppose given $(Q, \mathcal{L})$, $(Q', \mathcal{L}')$ over R, suppose

$$(Q_\eta, \mathcal{L}_\eta) \simeq_{k(\eta)} (Q'_\eta, \mathcal{L}'_\eta) \Longrightarrow (Q, \mathcal{L}) \simeq_R (Q', \mathcal{L}').$$

How can we construct $(Q, \mathcal{L})$?

4. PSQAS in dimension one

4.1. **Hesse cubics and thetas.** R be a CDVR, q uniformizer, $I = qR$. (Say, $y^2 = x^3 - x^2 - q$)

$$\begin{aligned}\theta_0(q,w) &= \sum_{m\in\mathbf{Z}} q^{9m^2} w^{3m}\\ &= 1 + q^9w^3 + q^9w^{-3} + q^{36}w^6 + \cdots,\\ \theta_1(q,w) &= \sum_{m\in\mathbf{Z}} q^{(3m+1)^2} w^{3m+1}\\ &= qw + q^4w^{-2} + q^{16}w^4 + \cdots,\\ \theta_2(q,w) &= \sum_{m\in\mathbf{Z}} q^{(3m+2)^2} w^{3m+2}\\ &= qw^{-1} + q^4w^2 + q^{16}w^{-4} + q^{25}w^5 + \cdots.\end{aligned}$$

Hence

$$\lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q,w) = [1,0,0]$$

This looks strange. But

$$\begin{aligned}\theta_0(q, q^{-1}u) &= 1 + q^6u^3 + \cdots,\\ \theta_1(q, q^{-1}u) &= u + q^6u^{-2} + \cdots,\\ \theta_2(q, q^{-1}u) &= q^2u^2 + \cdots.\end{aligned}$$

Hence

$$\lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{-1}u) = [1, \overline{u}, 0]$$

$$\lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{-2}u) = \lim_{q\to 0} [1, q^{-1}u, u^2] = [0,1,0] \quad \text{in } \mathbf{P}^2.$$

In fact,

Let $w = q^{-2\lambda}u$ (a section over a finite extension of $k(\eta)$) and $u \in R \setminus I$.

$$(1) \qquad \lim_{q\to 0} [\theta_0, \theta_1, \theta_2](q, q^{-2\lambda}u) = \begin{cases} [1,0,0] & (\text{if } -1/2 < \lambda < 1/2),\\ [1,\overline{u},0] & (\text{if } \lambda = 1/2),\\ [0,1,0] & (\text{if } 1/2 < \lambda < 3/2),\\ [0,1,\overline{u}] & (\text{if } \lambda = 3/2),\\ [0,0,1] & (\text{if } 3/2 < \lambda < 5/2).\\ [\overline{u},0,1] & (\text{if } \lambda = 5/2),\end{cases}$$

Thus $\lim_{\tau\to\infty}$ of the image of $E(\tau)$ is the 3-gon $x_0x_1x_2 = 0$.

Summary 4.2.

1. This is a set-theoretic computation.
2. The limit is computed from the distribution of minima of

   $$(3m+k)^2 - (3m+k)\lambda, \quad (m \in \mathbf{Z}, k = 0,1,2)$$

   for fixed λ,
3. The distri. of minima is described by Delaunay decomposition:
4. Picture of Delaunay decom. (Tomorrow)

Definition 4.3. For $\lambda \in X \otimes_{\mathbf{Z}} \mathbf{R}$ fixed, let

$$F_\lambda := a^2 - 2\lambda a \quad (a \in X = \mathbf{Z}).$$

We define a Delaunay cell

$$D(\lambda) := \begin{matrix}\text{the convex closure of all } a \in X \\ \text{that attain the minimum of } F_\lambda\end{matrix}$$

For example, $D(j + (1/2)) = [j, j+1]$ and $\lambda = j + (1/2)$, then (by forgetting any 0)

$$[\bar\theta_k]_{k=0,1,2} := \lim_{q \to 0} [\theta_k(q, q^{-2(j+(1/2))} u))]_{k=0,1,2} = [\bar u^j, \bar u^{j+1}],$$

Hence we have the limit

$$\{[\bar u^j, \bar u^{j+1}] \in \mathbf{P}^1; \bar u \in \mathbf{G}_m\} \simeq \mathbf{G}_m.$$

4.4. **The complex case.** Come back to Hesse cubics, θ_k.

1. θ_k is Y-inv. where $Y = 3\mathbf{Z}$,
2. we wish to think
$$\begin{aligned} E(\omega) &\simeq \text{Proj } \mathbf{C}[\theta_k \vartheta, k = 0,1,2] \\ &=^* \text{Proj}\,(\mathbf{C}[[a(x) w^x \vartheta, x \in X]])^{Y-\text{inv}} \\ &\simeq^* \text{Proj } \mathbf{C}[a(x) w^x \vartheta, x \in X])/Y \end{aligned}$$
3. because $U = \text{Spec } A$ is affine, G a finite group acting on U, then
$$U/G = \text{Spec } A^{G\text{-inv}}.$$
4. Over $\mathbf{C}$, $a(x) \in \mathbf{C}^\times$, and
$$\mathbf{G}_m = \text{Proj } \mathbf{C}[a(x) w^x \vartheta, x \in X],$$
because
$$U_k = \text{Spec } \mathbf{C}[a(x) w^x \vartheta / a(k) w^k \vartheta; x \in X] = \text{Spec } \mathbf{C}[w, w^{-1}] = \mathbf{G}_m,$$
5. Hence over $\mathbf{C}$ we may think so:
$$\begin{aligned} E(\omega) &\simeq \mathbf{G}_m / w \mapsto q^6 w \\ &\simeq \mathbf{G}_m / \{w \mapsto q^{2y} w; y \in 3\mathbf{Z}\} \\ &\simeq (\text{Proj } \mathbf{C}[a(x) w^x \vartheta, x \in X])/Y. \end{aligned}$$

4.5. **The scheme-theoretic limit.** What happes over a CDVR R ?
Let $a(x) = q^{x^2}$ for $x \in X$, $X = \mathbf{Z}$, $Y = 3\mathbf{Z}$.

1. let
$$\widetilde{R} := R[a(x) w^x \vartheta, x \in X],$$
$$Z = \text{Proj } \widetilde{R}/Y.$$
2. define S_y action of Y on $\widetilde{R}$
$$S_y(a(x) w^x \vartheta) = a(x+y) w^{x+y} \vartheta$$
by imitating θ_k.

3.

$$\begin{aligned}\mathcal{X} &= \text{Proj } R[a(x)w^x\vartheta, x \in X],\\ U_n &= \text{Spec } R[a(x)w^x/a(n)w^n, x \in X]\\ &= \text{Spec } R[(a(n+1)/a(n))w, (a(n-1)/a(n))w^{-1}]\\ &= \text{Spec } R[q^{2n+1}w, q^{-2n+1}w^{-1}]\\ &\simeq \text{Spec } R[x_n, y_n]/(x_ny_n - q^2).\end{aligned}$$

4. Let $\mathcal{X}_0 := \mathcal{X} \otimes_R (R/qR)$ and $V_n = \mathcal{X}_0 \cap U_n$.
   Then $\mathcal{X}_0$ is an infinite chain of $\mathbf{P}^1$:
   $V_n = \text{Spec } k[x_n, y_n]/(x_ny_n)$,
5. $\mathcal{X}_0/Y$: 3-gon

4.6. **PSQASes in the general case.** Let a CDVR R, $k(\eta) = \text{Frac}(R)$. We can const. similar degenerations of AV over R if $\exists$ a lattice X, $Y \subset X$ $[X : Y] < \infty$, and

$$a(x) \in k(\eta)^\times, \quad (x \in X)$$

such that

(i) $a(0) = 1$,
(ii) $b(x, y) := a(x + y)a(x)^{-1}a(y)^{-1}$ is a symm. bilin. form on $X \times X$,
(iii) $B(x, y) := \text{val}_q\, b(x, y)$ is pos. def.,
(iv)* B is even and $\text{val}_q\, a(x) = B(x, x)/2$.

We assume here a stronger condition (4)* for simplicity.

1. These data do exist in general, (Faltings-Chai)
2. Suppose an abelian scheme $(G_\eta, \mathcal{L}_\eta)$, $\lambda(\mathcal{L}_\eta) : G_\eta \to G^t_\eta$
3. $(G, \mathcal{L})$:Neron model of G_η
4. totally degenerate case: Suppose G_0 is a split torus over R/qR,
5. let G^{for} : formal completion of G along G_0,
6. Thm(SGA): G^{for} is a formal torus over R

$$G_{\text{for}} \simeq \mathbf{G}^g_{m,R,\text{for}} = \text{Spf } R[[w^x; x \in X]]^{I\text{-adic}}$$

7. any $\theta \in \Gamma(G, \mathcal{L})$ is a conv. Fourier series, $\theta \in\in R[[w^x; x \in X]]^{I\text{-adic}}$,

Theorem 4.7. *If G is totally deg., $\exists$ $\{a(x); x \in X\}$ subj. to (i)-(iv)* and*
(v)

$$\Gamma(G_\eta, \mathcal{L}_\eta) = \bigoplus_{\bar{x} \in X/Y} k(\eta)\ \theta_{\bar{x}}$$

where

$$\theta_{\bar{x}} := \sum_{y \in Y} a(x + y)w^{x+y}.$$

The condition (v) proves $(Q_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta)$ in Theorem 3.3.

5. The third talk — The moduli space $SQ_{g,K}$

By Theorem 3.3, any level $G(K)$ PSQAS $(Q_0, \mathcal{L}_0)$ is $G(K)$-*equivariantly embedded into* $\mathbf{P}(V)$ if $e_{\min}(K) \geq 3$ where $V = V_H := \mathcal{O}_N[v(\mu); \mu \in H^\vee]$.

5.1. **The $G(K)$-action and the $G(K)$-linearization.** The $G(K)$-action τ on (Z, L) is ess. the same as $G(K)$-linearization

$$\{\phi_g; g \in G(K)\}$$

(i) $\phi_g : \mathcal{L} \to T_g^*(\mathcal{L})$ is a bundle isomorphism,
(ii) $\phi_{gh} = T_h^* \phi_g \circ \phi_h$ for any $g, h \in G(K)(T)$.

the action τ on (Z, L) is recovered from it as follows : By the isomorphisms

$$L \xrightarrow{\phi_h} T_h^*(L) \xrightarrow{T_h^*\phi_g} T_h^*(T_g^*(L)) = T_{gh}^*(L),$$

for $x \in Z$, $\xi \in L_x$,

$$\tau(h) \cdot (z, \xi) = (T_h(z), \phi_h(z) \cdot \xi).$$

Then $\tau(gh) = \tau(g)\tau(h)$ iff $\phi_{gh} = T_h^* \phi_g \circ \phi_h$.

Now we wish to define the action of $G(K)$ on $H^0(Z, L)$:

$$\rho_L(g)(\theta) := T_{g^{-1}}^*(\phi_g(\theta)), \quad \rho(gh) = \rho(g)\rho(h).$$

For a level-$G(K)$-PSQAS

1. $H^0(Q_0, \mathcal{L}_0)$ is an irreducible $G(K)$-module [NS06]
2. V_H is a unique irred. repres. of wt one of $G(K)$ over $\mathbf{Z}[\zeta_N, 1/N]$, hence
3. $H^0(Q_0, \mathcal{L}_0) \simeq V_H$ over $k = R/qR$,

Lemma 5.2. *Assume* $e_{\min}(K) \geq 3$. *Then*
for a level-$G(K)$ *PSQAS* (Q_0, ϕ_0, τ_0),
$\exists$ *a unique level-*$G(K)$ *PSQAS* (Q_0', i, U_H) *isom. to* (Q_0, ϕ_0, τ_0)
where $i : Q_0' = \phi_0(Q_0) \subset \mathbf{P}(V_H)$ *: inclusion.*

Proof.
- Let $(Q_0, \mathcal{L}_0, \phi_0, \tau_0) = (Z, L, \phi, \tau)$.
- Since $\phi^* : V_H \simeq H^0(Z, L)$ isom, let

$$\rho(\phi, \tau)(g)(\theta) := (\phi^*)^{-1} \rho_L(g)(\theta) \phi^* \quad \theta \in H^0(Z, L).$$

- $\rho(\phi, \tau) \in \mathrm{End}\,(V_H)$.
- By Schur's lemma, $\exists\ A \in \mathrm{GL}(V_H)$ s.t.

$$U_H = A^{-1} \rho(\phi, \tau) A = (\phi^* A)^{-1} \rho_L(g)(\theta)(\phi^* A).$$

- Hence choose ψ (closed imm.) by $\psi^* = \phi^* A$. Then $U_H = \rho(\psi, \tau)$,
- let $Z' = \psi(Z)$, $i : Z' \subset \mathbf{P}(V_H)$. (Z', i, U_H) equiv. to (Z, ϕ, τ)

□

- $\mathrm{Hilb}^{\chi(n)}$: the Hilbert scheme parametrizing subschemes (Z, L) of $\mathbf{P}(V_H)$ with $\chi(Z, L^n) = n^g \sqrt{|K|} =: \chi(n)$
- $(\mathrm{Hilb}^{\chi(n)})^{G(K)\text{-inv}}$: the $G(K)$-inv. part of it,
- $A_{g,K}$: moduli of level $G(K)$-AV
- By Lemma 5.2, $A_{g,K} = \{(A_0', i, U_H); A_0' : \mathrm{AV}\}$,
- $A_{g,K} \subset (\mathrm{Hilb}^{\chi(n)})^{G(K)\text{-inv}}$,

We define

$$SQ_{g,K} := \overline{A_{g,K}} \subset (\mathrm{Hilb}^{\chi(n)})^{G(K)\text{-inv}}.$$

Theorem 5.3. *Suppose $H = \oplus_{i=1}^{g}(\mathbf{Z}/e_i\mathbf{Z})$, $e_{\min} \geq 3$. For any closed field k of characteristic prime to $|H| = \prod_{i=1}^{g} e_i$,*

$$SQ_{g,K}(k) = \{(Q_0, i, U_H); PSQAS, i : Q_0 \subset \mathbf{P}(V_H)\}$$

Proof. Choose $x_0 \in SQ_{g,K}$. Then $x_0 = (Z_0, \mathcal{L}_0) \in SQ_{g,K}$, $(Z_0, \mathcal{L}_0) \in$ Hilb.

- $\exists$ a proper flat $\pi : (Z, \mathcal{L}) \to \mathrm{Spec}\ R$ s.t. $(Z_\eta, \mathcal{L}_\eta)$ is an $U_H(G(K))$-inv. level $G(K)$-AV,
- so is $(Z, \mathcal{L})$.
- by Theorem 3.3, by a finite base change if necessary $\exists$ a level-$G(K)$ PSQAS $(Q, \mathcal{L}_Q)$ s.t.
$$(Q_\eta, \mathcal{L}_{Q,\eta}) \simeq (Z_\eta, \mathcal{L}_\eta)$$
- By Lemma 5.2 and Theorem 3.3, we may assume $(Q, \mathcal{L}) : U_H(G(K))$-invariant R-subsch. of $\mathbf{P}(V_H)_R$.
- hence $(Z_\eta, \mathcal{L}_\eta) = (Q_\eta, \mathcal{L}_{Q,\eta})$ by the uniqueness of Lemma 5.2.
- $(Z, \mathcal{L}) = (Q, \mathcal{L})$, hence $x_0 = (Z_0, \mathcal{L}_0) = (Q_0, \mathcal{L}_0)$ is a PSQAS.

□

Theorem 5.4. *Suppose $e_{\min}(K) \geq 3$. Let $N := \sqrt{|K|}$. The functor $\mathcal{SQ}_{g,K}$ of level-$G(K)$ PSQASes (Q, ϕ, τ) over reduced base schemes is represented by the projective $\mathbf{Z}[\zeta_N, 1/N]$-scheme $SQ_{g,K}$.*

$$SQ_{g,K}(T) = \{(Q, \phi, \tau); \text{PSQAS with level-}G(K)\text{ str. over }T\}$$

(I will not explain in detail.)

6. The space of closed orbits

6.1. **Example.** Define the action of $G = \mathbf{C}^*$ on $\mathbf{C}^2$:

$$(\alpha, x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in \mathbf{C}^*)$$

The quotient space of $\mathbf{C}^2$ by $\mathbf{C}^*$ is

$$\mathbf{C}^2 /\!/ \mathbf{C}^* = \mathrm{Spec}\ \mathbf{C}[x, y]^{G\text{-inv}} = \mathrm{Spec}\ \mathbf{C}[t]$$

Is this the space of orbits ?
$\exists$ four kinds of orbits:

$$O(a, 1) = \{(x, y) \in \mathbf{C}^2; xy = a\} \ \ (a \neq 0), \quad \text{(closed)}$$
$$O(0, 1) = \{(0, y) \in \mathbf{C}^2; y \neq 0\}, \quad \text{(not closed)}$$
$$O(1, 0) = \{(x, 0) \in \mathbf{C}^2; x \neq 0\}, \quad \text{(not closed)}$$
$$O(0, 0) = \{(0, 0)\} \quad \text{(closed)}$$

Hence

$$\mathbf{C}^2 /\!/ \mathbf{C}^* = \mathbf{C} = \{O(a, 1); a \neq 0; O(0, 0)\} = \{\text{closed orbit}\}$$

The quotient is the space of closed orbits.

Theorem 6.2. (Seshadri-Mumford)
k closed, X : proj. scheme,
G reductive k-group acting on X,
$X_{ss} := \{$semistable point$\}$ open $\subset X$

Then $\exists$ a proj. k-scheme Y
$\exists$ a G-inv. surj. morphism $\pi : X_{ss} \to Y$, such that

(1) *for any k-scheme Z on which G acts,*
for any G-equiv. morph. $\phi : Z \to X$
$\exists$ a unique morphism $\bar{\phi} : Z \to Y$ such that $\bar{\phi} = \pi\phi$,

(2) *For a, b of X_{ss}*

$$\pi(a) = \pi(b) \text{ iff } \overline{O(a)} \cap \overline{O(b)} \neq \emptyset$$

where the closure is taken in X_{ss},

(3) *$Y(k) = \{G$-orbit closed in $X_{ss}\}$.*

Denote Y by $X_{ss}//G$.

Recall

Definition 6.3. Let $p \in X$.

(1) p is *semistable* if $\exists$ a G-inv. homog. polynomial F on X with $F(p) \neq 0$,
(2) p is *Kempf-stable* (or *closed orbit*) if the orbit $O(p)$ is closed in X_{ss},
(3) p is *properly-stable* if p is Kempf-stable and the stab. subgp of p in G is finite.

Remark 6.4. If $a, b \in X_{ps}$,

$$\begin{aligned} \pi(a) = \pi(b) &\Longleftrightarrow \overline{O(a)} \cap \overline{O(b)} \neq \emptyset \\ &\Longleftrightarrow O(a) \cap O(b) \neq \emptyset \\ &\Longleftrightarrow O(a) = O(b). \end{aligned}$$

Hence in particular, the quotient space $X_{ps}//G$ is an ordinary orbit space X_{ps}/G.

Theorem 6.5. ([Gieseker82], [Mumford77]) *For a connected curve C of genus ≥ 2, the following are equivalent:*

1. *C is a stable curve, (moduli-stable)*
2. *Any Hilbert point of C embedded by $|mK_C|$ is GIT-stable,*
3. *Any Chow point of C embedded by $|mK_C|$ is GIT-stable.*

Theorem 6.6. *Let $K = H \oplus H^\vee$, $N = |H|$, k closed,* char $.k \neq N$.
Suppose $e_{\min}(K) \geq 3$, and $(Z, L) \subset (\mathbf{P}(V), O_{\mathbf{P}(V_H)}(1))$.
Suppose that (Z, L) is smoothable into a level-$G(K)$ AV.
Then the following are equiv.:

1. *(Z, L) is a PSQAS, (moduli-stable)*
2. *any Hilbert point of (Z, L) are GIT-stable,*
3. *(Z, L) is stable under (a conjugate of) $G(K)$.*

Proof. (1)→(3) Easy.
(3)→(2) by (Kempf+L very ample).
We prove (2)→(1) : PSQAS has a closed orbit. Assume (2) for (Z, L).

- By assumption $\exists$ $(Q, \mathcal{L})$ over a CDVR R such that
  $(Q_\eta, \mathcal{L}_\eta)$ a level-$G(K)$ AV and $(Q_0, \mathcal{L}_0) = (Z, L) =: a$.
- $O(a)$: closed by assuming (2).
- by base change may assume
  $\exists$ a level-$G(K)$ PSQAS $(Q', \mathcal{L}')$ s.t. $(Q'_\eta, \mathcal{L}'_\eta) = (Q_\eta, \mathcal{L}_\eta)$.
- Let $(Q'_0, \mathcal{L}'_0) =: b$. Then $\pi(a) = \pi(b)$. $\pi : X_{ss} \to X_{ss}/\!/ \mathrm{SL}$.
- Hence by Seshadri-Mumford, $\overline{O(a)} \cap \overline{O(b)} \neq \emptyset$.
- both are closed orbits. $O(a) \cap O(b) \neq \emptyset$.
- Hence $O(a) = O(b)$. This shows $(Z, L) \simeq (Q'_0, \mathcal{L}'_0)$ PSQAS.

□

Corollary 6.7. *For any planar cubic C*

1. $(C, O_C(1))$ *is a PSQAS, (smooth or a 3-gon)*
2. *any Hilbert point of* (Z, L) *are GIT-stable,*
3. $(C, O_C(1))$ *is* $G(3)$*-stable, a Hesse cubic.*

Remark 6.8. (Nakamura75) $\exists$ a 2-dim. PSQAS $(Z < L) := (Q_0, \mathcal{L}_0)$ a union of $2n^2$ copies of $\mathbf{P}^2$ with $X/Y = (\mathbf{Z}/n\mathbf{Z})^{\oplus 2}$.

- $a(x) = q^{x^2 - xy + y^2}$,
- $\exists$ two different embeddings of $K = H \oplus H^\vee \subset Q_0$, A,B
- A, B are translate over $k(\eta)$,
- hence $(Z_\eta \supset A_\eta) = (Z_\eta \supset B_\eta)$.
- $(Z \supset A)$ and $(Z \supset B)$ have diff. limit,
- Thus the moduli will be non-separetd.

References

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