F-equivalence

Osamu Fujino (Nagoya Univ.)

2004. 6. 29.

This note is written for Kusatsu Algebraic Geometry Seminar 2004.

If you have questions or comments, please send e-mails to fujino@math.nagoya-u.ac.jp

Note that I'm not an expert of derived categories.

We will work over $\mathbb{C}$ throughout this note.

§ K-equivalence & D-equivalence.

Def 1. X : smooth proj var $/\mathbb{C}$

$D(X) := D^b(Coh(X))$: the derived category of bounded complexes of coherent sheaves on X.

Def 2. (D-equivalence)

X, Y : smooth proj var.

$X \underset{D}{\sim} Y$ D-equivalent

$\underset{def}{\Longleftrightarrow}$ $D(X)$ is equivalent to $D(Y)$ as triangulated categories.

Def 3 (K-equivalence)

X, Y : smooth proj var

$X \underset{K}{\sim} Y$ K-equivalent

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}Z$: smooth proj var

${}^{\exists}f : Z \longrightarrow X$, ${}^{\exists}g : Z \longrightarrow Y$

birational morphisms.

such that $f^*K_X \sim g^*K_Y$ "

From now on, we mainly treat toric varieties.

<u>Theorem 4</u> X, Y : smooth proj toric var.

$X \underset{D}{\sim} Y \Longrightarrow X \underset{K}{\sim} Y$ "

Let us recall Kawamata's result.

<u>Theorem 5</u> (Kawamata [K, Theorem 1.4]).

X, Y : smooth proj var.

Assume $X \underset{D}{\sim} Y$ & $\kappa(X, -K_X) = \dim X$

$\Longrightarrow X \underset{K}{\sim} Y$ "

Lemma 6 X : smooth proj toric var
with $\dim X = n$.
$\Rightarrow \kappa(X, -K_X) = n$.

Proof X : smooth toric
$\Rightarrow -K_X = \sum_{i \in I} D_i$ in $\mathrm{Pic}(X)$.
D_i : torus invariant prime divisor
$\{D_i\}_{i \in I}$ generates $\mathrm{Pic}(X)$.
$\Rightarrow \exists m \in \mathbb{Z}_{>0}$ such that
$-mK_X \geq \exists$(ample Cartier divisor)
$\Rightarrow \kappa(X, -K_X) = n$. //

Proof of Theorem 4

Theorem 5 + Lemma 6 $\Rightarrow$ Theorem 4 //

Remark 7 X : smooth proj toric var

If $\dim X \leqq 3$, then

D-equivalence $\Leftrightarrow$ K-equivalence.

<u>Conjecture 8</u> D-equivalence $\Leftrightarrow$ K-equivalence

for smooth proj toric varieties.

By Theorem 4, we have to show

K-equivalence $\Longrightarrow$ D-equivalence.

Note that I have no results on

Conjecture 8!

§ F-equivalence for toric varieties

<u>Def 9</u> (F-equivalence)

X, Y : smooth proj toric var.

$X \underset{F}{\sim} Y$ F-equivalent

$\underset{\text{def}}{\Longleftrightarrow}$ $^{\exists}$ a sequence of flops

$$\begin{tikzcd}
X =: X_0 \arrow[rr, dashed] \arrow[dr] \arrow[ddr] & & X_1 \arrow[rr, dashed] \arrow[dl] \arrow[dr] \arrow[ddl] \arrow[ddr] & & X_2 \arrow[r, dashed] \arrow[dl] \arrow[ddl] & \cdots \arrow[r, dashed] & X_{k-1} \arrow[rr, dashed] \arrow[dr] \arrow[ddr] & & X_k =: Y \arrow[dl] \arrow[ddl] \\
& W_0 \arrow[d] & & W_1 \arrow[d] & & & & W_{k-1} \arrow[d] & \\
& S_0 & & S_1 & & & & S_{k-1} &
\end{tikzcd}$$

such that X_i : smooth for $\forall i$

Note that $X_i \longrightarrow S_i$: proj morphism

$\varphi_i : X_i \longrightarrow W_i$: flopping contraction

over S_i

i.e. $\begin{cases} -K_{X_i} \text{ is } \varphi_i \text{ numerically trivial} \\ \rho(X_i/W_i) = 1 \\ \varphi_i : \text{small} \end{cases}$

$X_{i+1} \longrightarrow W_i$: flop of φ_i

Thus X_{i+1} : projective over S_i

However, X_{i+1} is not necessarily proj

over $\mathbb{C}$.

Remark 10

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

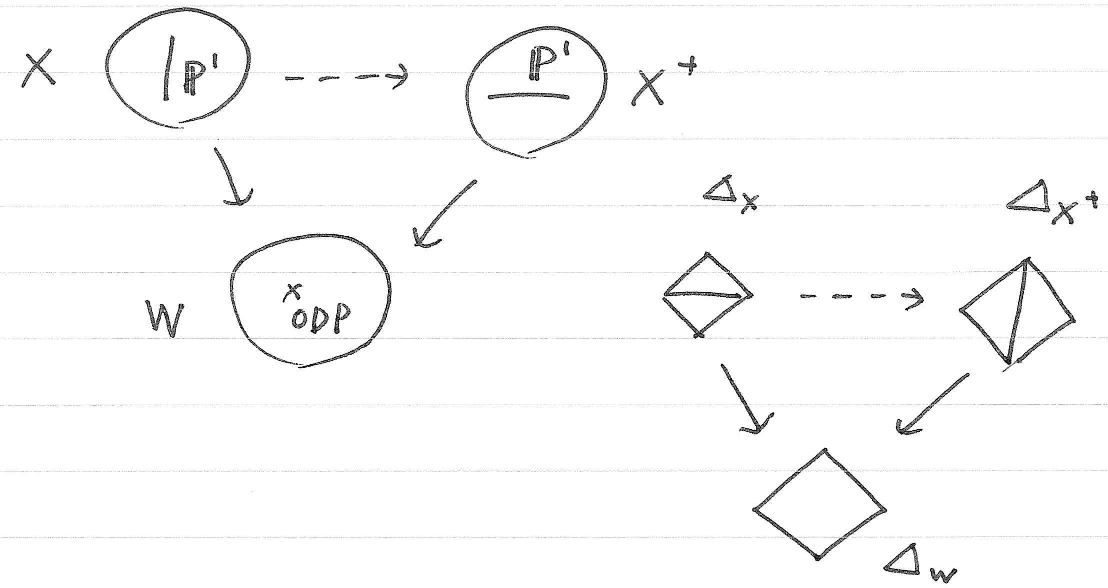
toric flop

$\dim X = 3$.

X : smooth.

$\Longrightarrow$ W has only one ODP.

$X \to W$, $X^+ \to W$ are small resolutions of ODP.



so-called Atiyah's flop.

In this case, X^+ : smooth.

Remark 11

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

toric flop

$\dim X = 4$

X : smooth

$\Longrightarrow$ $\exists$ 2 types of flops

$$\begin{tikzcd} A \arrow[r, phantom, "\subset"] \arrow[ddrr, bend right] & X \arrow[rr, dashed] \arrow[dr, "\varphi"] & & X^+ \arrow[dl, "\varphi^+"'] & A^+ \arrow[l, phantom, "\supset"] \arrow[ddll, bend left] \\ & & W & & \\ & & B \arrow[u, phantom, "\cup"] & & \end{tikzcd}$$

$A = \mathrm{Exc}(\varphi)$
$A^+ = \mathrm{Exc}(\varphi^+)$
$B = \varphi(A)$

① X^+: smooth
$B = \mathbb{P}^1$
$A \to \mathbb{P}^1$: $\mathbb{P}^1$-bundle
$A^+ \longrightarrow \mathbb{P}^1$: $\mathbb{P}^1$-bundle

This case is a family of

Atiyah's flops.

In this case, $X \dashrightarrow X^+$: symmetric

operation.

②
$$\begin{tikzcd} \mathbb{P}^2 = A \arrow[r, phantom, "\subset"] \arrow[ddrr, bend right] & X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] & A^+ \cong \mathbb{P}^1 \arrow[l, phantom, "\supset"] \arrow[ddll, bend left] \\ & & W & & \\ & & pt \arrow[u, phantom, "\cup"] & & \end{tikzcd}$$

X^+: singular.

Proposition 12

$$\begin{tikzcd}
 & X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] & \\
A \arrow[ur, phantom, "\subset"] \arrow[drr] & & W & & A^+ \arrow[ul, phantom, "\supset"] \arrow[dll] \\
 & & B \arrow[u, phantom, "\cup"] & &
\end{tikzcd}$$

toric flop

X. X^+: smooth.

$\dim X = n$.

$\Longrightarrow$ this flop is a symmetric operator

More precisely,

$$2 \leqq \alpha \leqq \beta \leqq n-1.$$
$$\alpha,\ \beta \in \mathbb{Z}, \quad n = \dim X.$$

$$\operatorname{codim} A = \operatorname{codim} A^+ = \alpha$$
$$\dim B = \beta - \alpha$$
$$\alpha = n+1-\beta.$$

$A \longrightarrow B$
$A^+ \longrightarrow B$ have $\mathbb{P}^{n-\beta}$-bundle structures.

such that $N_{A/X}|_F \simeq \mathcal{O}_{\mathbb{P}^{n-\beta}}(-1)^{\oplus \alpha}$

where F: ${}^\forall$fiber of $A \longrightarrow B$.

Therefore, this is a family of higher dimensional generalizations of

Atiyah's flop.

Proof. Please check it by yourself. //

Theorem 13 (Orlov)

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] \\ & W & \end{tikzcd}$$ toric flop

X, X^+ : smooth complete toric var.

$\Longrightarrow X \sim_D X^+$

i.e. $D(X) \simeq D(X^+)$ as triangulated categories.

Proof. See [O, page 544].

Please check it by yourself. //

Cor 14 X, Y : smooth proj toric var.

$X \sim_F Y \Longrightarrow X \sim_D Y$ //

We summarize :

<u>Theorem 15</u> X. Y : smooth proj toric var

$$X \underset{F}{\sim} Y \Rightarrow X \underset{D}{\sim} Y \Rightarrow X \underset{K}{\sim} Y$$ //

Therefore,

<u>Conjecture 16</u> X. Y as above.

$$X \underset{K}{\sim} Y \underset{?}{\Longrightarrow} X \underset{F}{\sim} Y.$$ //

<u>Remark 17</u> Assume that $\dim X = 3$.

Then, it is not difficult to see that $X \underset{K}{\sim} Y \Rightarrow X \underset{F}{\sim} Y$ //

<u>Lemma 18</u> X. Y : proj toric var with $\mathbb{Q}$-factorial terminal sing.

If $^{\exists} f : Z \to X$, $^{\exists} g : Z \to Y$: common resolution s.t $f^* K_X \sim g^* K_Y$, then $^{\exists}$ a sequence of flips, flops, and inverse flips.

$$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_l := Y$$

over $\mathbb{C}$.

Proof $\quad f^* K_X \sim g^* K_Y$

X, Y have only terminal sing

$\Longrightarrow$ X, Y isom in codim one.

well-known

D: very ample div on Y

D': the strict transform of D on X.

If D' is nef $\Longrightarrow X \simeq Y$.

If D' is not nef

$\Longrightarrow$ ${}^\exists D'$-negative extremal ray R of $NE(X)$.

$$\Longrightarrow \quad \begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\varphi_R"'] & & X_1 \arrow[dl] \\ & W & \end{tikzcd}$$

φ_R: small

$-D'$: φ_R-ample

$\rho(X/W) = 1$.

Repeat this process. Q.E.D.

Conjecture 19 X, Y : smooth proj toric var.

$X \underset{K}{\sim} Y$

$\underset{?}{\Longrightarrow}$ $^\exists$a sequence of flops

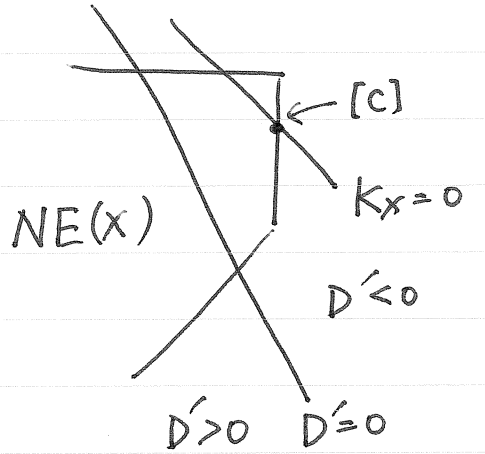
$X =: X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_m := Y$

over $\mathbb{C}$?

Remark 20 We can check that $^\exists$curve C on X such that $K_X \cdot C = 0$ & $C \cdot D' < 0$.

However, I do not know if there exists R : extremal ray of $NE(X)$ s.t $K_X \cdot R = 0$ & $R \cdot D' < 0$ or not.

The existence of C does not directly imply the existence of R.



<u>Conjecture 21</u> X, Y : smooth toric var.

$X =: X_0 \dashrightarrow \cdots\cdots \dashrightarrow X_m := Y$ a sequence of flops over a fixed toric var S.

$\underset{?}{\Longrightarrow}$ Is there a sequence of flops.

$$\begin{tikzcd}[column sep=small]
X =: V_0 \arrow[rr, dashed] \arrow[dr] \arrow[ddr] & & V_1 \arrow[rr, dashed] \arrow[dl] \arrow[ddl] \arrow[dr] \arrow[ddr] & & V_2 \arrow[dl] \arrow[ddl] \arrow[r, dashed] & \cdots \arrow[r, dashed] & V_{p-1} \arrow[rr, dashed] \arrow[dr] \arrow[ddr] & & V_p := Y \arrow[dl] \arrow[ddl] \\
& W_0 \arrow[d] & & W_1 \arrow[d] & & & & W_{p-1} \arrow[d] & \\
& S_0 & & S_1 & & & & S_{p-1} &
\end{tikzcd}$$

such that each step is a flop over S_i.

V_i : smooth for ${}^\forall i$.

i.e. $X \underset{F}{\sim} Y$? //

Therefore,

Claim 22 Conjectures 19, 21 imply

K-equiv $\Rightarrow$ F-equiv.

Thus Conj 19 + 21 implies

$$\begin{tikzcd} \text{F-equiv} \arrow[r, Leftrightarrow] \arrow[dr, Leftrightarrow] & \text{K-equiv} \arrow[d, Leftrightarrow] \\ & \text{D-equiv} \end{tikzcd}$$

//

§ Comments

Prop 23 $X \underset{K}{\sim} Y$

X, Y : smooth proj toric var.

$\Rightarrow d_k(X) = d_k(Y)$ for $\forall k$.

where $d_k(X) = \#$ of k-dimensional cones in Δ_X

Proof $X \underset{K}{\sim} Y \Rightarrow {}^{\exists}Z$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$ s.t

$f^*K_X \sim g^*K_Y$

$$\overset{\uparrow}{\Longrightarrow} \quad h^{p,q}(X) = h^{p,q}(Y) \quad \text{for } {}^\forall p, q.$$

e.g. motivic integration

$$\overset{\uparrow}{\Longrightarrow} \quad d_k(X) = d_k(Y) \quad \text{for } {}^\forall k.$$

X: disjoint union of tori
Consider Hodge structures on cpt support cohomologies.

<u>Remark 24</u> $\dim X = 4$
X : smooth

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \ \ \swarrow & \\ & W & \end{array} \qquad \text{toric flop}$$

If X^+ is singular, then

$$d_4(X) > d_4(X^+)$$ //

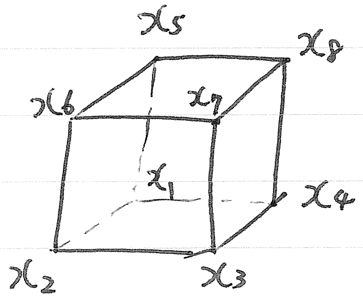
§ Example

I believed Conj 21 is not true.

I constructed an example to check it.

However, Doctor Hiroshi Sato proved my example is not a counter-example of Conj 21.

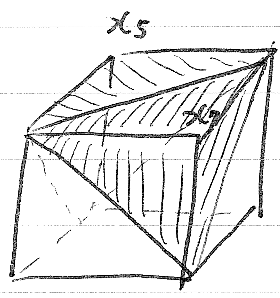
<u>Example 25</u> $\mathbb{R}^3 \times \{1\} \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$



We consider a cone spanned by ☐ $\times \{1\}$ and $\{0\}$.

$x_1 = (0.0.0.1)$ $x_2 = (1.0.0.1)$
$x_3 = (1.1.0.1)$ $x_4 = (0.1.0.1)$
$x_5 = (0.0.1.1)$ $x_6 = (1.0.1.1)$
$x_7 = (1.1.1.1)$ $x_8 = (0.1.1.1)$

Remove 2 cones $\langle x_5 . x_6 . x_8 . x_1 \rangle$
$\langle x_6 . x_7 . x_8 . x_3 \rangle$.



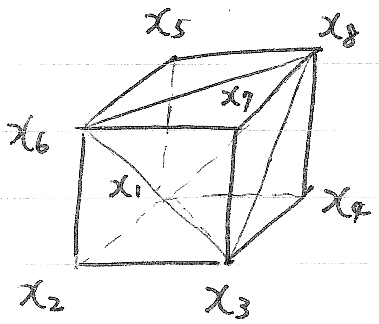
We write it Δ_W.

$W = W(\Delta_W)$: affine Gorenstein terminal 4-fold.

Divide Δ_W into
$$\Delta_Y := \left\{ \begin{array}{l} \langle x_1, x_2, x_3, x_6 \rangle \\ \langle x_1, x_3, x_4, x_8 \rangle \\ \langle x_1, x_3, x_6, x_8 \rangle \end{array} \right.$$

$Y = Y(\Delta_Y) \longrightarrow W$: small proj

$\rho(Y/W) = 2$.



Y: Gor. terminal sing

Y : singular.

$NE(Y/W)$ has 2 rays R_1, R_2.

$\Updownarrow$

$^\exists$ 2 walls $\langle x_1, x_3, x_6 \rangle$
$\langle x_1, x_3, x_8 \rangle$.

R_1 : $2x_2 + x_8 = x_1 + x_3 + x_6$

R_2 : $2x_4 + x_6 = x_1 + x_3 + x_8$

It is obvious that φ_{R_1} and φ_{R_2}

are flopping contractions over W.

$$\begin{tikzcd}
X' \arrow[d] & & Y \arrow[ll, dashed] \arrow[rr, dashed] \arrow[dl, "\varphi_{R_2}"] \arrow[dr, "\varphi_{R_1}"'] & & X \arrow[dl] \\
 & Z' & & Z &
\end{tikzcd}$$

$Y \dashrightarrow X$, $Y \dashrightarrow X'$: flops $/W$.

X. X' : smooth.

More precisely,

$$\Delta_X = \left\{ \begin{array}{l} \langle x_8, x_2, x_3, x_6 \rangle \\ \langle x_8, x_2, x_6, x_1 \rangle \\ \langle x_8, x_2, x_1, x_3 \rangle \\ \langle x_8, x_4, x_1, x_3 \rangle \\ \text{and their faces} \end{array} \right\}$$

$$\rho(Y/W) = 2 \implies \rho(X/W) = \rho(X'/W) = 2.$$

Thus X and X' are connected by flops

Y : singular $\quad X' \dashrightarrow Y \dashrightarrow X$.

Apply 2 ray game.

$$\begin{array}{c} Y \dashrightarrow X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow X_3 \\ \hphantom{Y \dashrightarrow X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow} \parallel \\ \hphantom{Y \dashrightarrow X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow} X' \end{array}$$

We can check that $X_3 \simeq X' / W$.

$X_1, X_2, X_3 (= X')$: smooth.

$X \dashrightarrow X_1$, $X_1 \dashrightarrow X_2$, $X_2 \dashrightarrow X_3$

: toric flops.

Therefore, $X \underset{F}{\sim} X'$ //

Conclusion 26 It seems very hard to construct counter-examples to Conjecture 21. //

Remark 27 In the above example, it took one night to carry out 2 ray game (i.e. to construct X_1, X_2, X_3).

This messy computation was carried out by H. Sato. I omit the details here. //

We summarize:

Conclusion 28 In the category of smooth

projective toric varieties,

if $\dim = 3$.

$$\begin{tikzcd}[column sep=small]
K\text{-equiv} \arrow[rr, Leftrightarrow] \arrow[dr, Leftrightarrow] & & F\text{-equiv} \arrow[dl, Leftrightarrow] \\
& D\text{-equiv} &
\end{tikzcd}$$

if $\dim \geqq 4$

$$\begin{tikzcd}[column sep=small]
K\text{-equiv} \arrow[rr, Rightarrow, "?"'] & & F\text{-equiv} \arrow[dl, Rightarrow] \\
& D\text{-equiv} \arrow[ul, Rightarrow] &
\end{tikzcd}$$

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References

[K] Y. Kawamata, D-equivalence and K-equivalence, JDG 61 (2002), 147~171.

[O]. D.O. Orlov, Derived categories of coherent sheaves and equivalences between them, Russ. Math. Surveys 58 (2003), 511~591.

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