The foundations of the minimal model program

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1. Today’s plan
2. New framework of vanishing theorems
3. Log canonical pairs
4. Semi-log canonical pairs
Part I: Main Results (English, this slide, unfortunately, somewhat technical)
Today’s plan

- Part I: Main Results (English, this slide, unfortunately, somewhat technical)
- Part II: Ideas, Background, History, and so on (Japanese, no slides, fortunately, not technical)
SNC pairs

- $M$: smooth variety \(/\mathbb{C}\)
- $X$: SNC divisor on $M$
- $B$: $\mathbb{R}$-divisor on $M$ such that $\text{Supp} B$: SNC divisor
- $B$ and $X$ have no common components, $\text{Supp}(B + X)$: SNC divisor
- $D = B|_X$
SNC pairs

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Definition 2.1 (GESNC pair)

$(X, D)$: globally embedded simple normal crossing (GESNC) pair
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**Definition 2.1 (GESNC pair)**

$(X, D)$: globally embedded simple normal crossing (GESNC) pair

**Definition 2.2 (SNC pair)**

$(Y, \Delta)$: simple normal crossing (SNC) pair

\[ \text{def} \quad (Y, \Delta): \text{Zariski locally isomorphic to a GESNC pair} \]
Stratum of SNC pair

- \((X, D)\): SNC pair
- \(D \in [0, 1]\)
- \(\nu : X^\nu \to X\): normalization
- \(K_{X^\nu} + \Theta = \nu^*(K_X + D)\)
Stratum of SNC pair

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**Definition 2.3 (Stratum)**

- $W$: closed subvariety of $X$

$W$: stratum of $(X, D)\quad \overset{\text{def}}{\iff} \quad W = \nu(C)$, where $C$ is a log canonical center of $(X^\nu, \Theta)$, or $W$ is an irreducible component of $X$
Hodge theoretic injectivity theorem

Theorem 2.4 (Relative Hodge theoretic injectivity theorem)

- \((X, \Delta): \text{SNC pair, } \Delta \in [0, 1], \pi : X \to S: \text{proper}\)
- \(L: \text{Cartier divisor on } X\)
- \(D: \text{effective Weil divisor on } X\)
- \(\text{Supp } D \subset \text{Supp } \Delta\)
- \(L \sim_{\mathbb{R}, \pi} K_X + \Delta\)
Today's plan
New framework of vanishing theorems
Log canonical pairs
Semi-log canonical pairs

Hodge theoretic injectivity theorem

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- \(\text{Supp} \; D \subset \text{Supp} \; \Delta\)
- \(L \sim_{\mathbb{R}, \pi} K_X + \Delta\)

Then

\[
R^q\pi_*O_X(L) \to R^q\pi_*O_X(L + D)
\]

is injective for every \(q\).
Injectivity theorem for SNC pair

Theorem 2.5 (Injectivity for SNC pair)

- \((X, \Delta)\): SNC pair, \(\Delta \in [0, 1]\), \(\pi : X \to S\): proper, as before
- \(L\): Cartier divisor on \(X\)
- \(D\): effective Cartier, permissible with respect to \((X, \Delta)\)
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- \(D\): effective Cartier, permissible with respect to \((X, \Delta)\)

We further assume:

(i) \(L \sim_{R, \pi} K_X + \Delta + H\)

(ii) \(H\): \(\pi\)-semi-ample \(R\)-divisor

(iii) \(tH \sim_{R, \pi} D + D'\), \(t \in \mathbb{R}_{>0}\),

\(D'\): effective \(R\)-Cartier \(R\)-divisor, permissible with respect to \((X, \Delta)\)
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Then \(R^q\pi_*O_X(L) \to R^q\pi_*O_X(L + D)\) is injective for every \(q\).
Torsion-freeness and Vanishing for SNC pair

Theorem 2.6 (Torsion-freeness and Vanishing thereom)

- $(Y, \Delta)$: SNC pair, $\Delta \in [0, 1]$, $f: Y \to X$: proper
- $L$: Cartier divisor on $Y$ such that $L - (K_Y + \Delta)$: $f$-semi-ample
Torsion-freeness and Vanishing for SNC pair

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- \((Y, \Delta)\): SNC pair, \(\Delta \in [0, 1]\), \(f : Y \to X\): proper
- \(L\): Cartier divisor on \(Y\) such that \(L - (K_Y + \Delta)\): \(f\)-semi-ample

Then we have:

(i) Every associated prime of \(R^q f_* O_Y(L)\) is the generic point of the \(f\)-image of some stratum of \((Y, \Delta)\).

(ii) \(\pi : X \to V\): projective
- \(L - (K_Y + \Delta) \sim_{\mathbb{R}} f^* H\), \(H\): \(\pi\)-ample \(\mathbb{R}\)-divisor on \(X\)
- \(\implies R^p \pi_* R^q f_* O_Y(L) = 0\) for every \(p > 0\) and \(q \geq 0\).
Vanishing theorem of Reid–Fukuda type

We can generalize Theorem 2.6 (ii) as follows.
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**Theorem 2.7 (Vanishing theorem of Reid–Fukuda type)**

Use the same notation as in Theorem 2.6
- \((Y, \Delta)\): GESNC, or \(Y\): quasi-projective (extra assumption!)
- \(H\): nef and log big over \(V\) with respect to \(f: (Y, \Delta) \rightarrow X\), that is, \(H\): nef over \(V\) and \(H|_{f(W)}\): big over \(V\) for every stratum \(W\) of \((Y, \Delta)\)
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Then $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$. 
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Then \(R^p \pi_* R^q f_* O_Y(L) = 0\) for every \(p > 0\) and \(q \geq 0\).

We can see that these results contain various classical results as special cases.
Theorem 2.8 (Kawamata–Viehweg)

- $X$: smooth projective variety
- $D$: nef and big $\mathbb{Q}$-divisor
- $\text{Supp}\{D\}$: SNC divisor

Then $H^q(X, \mathcal{O}_X(K_X + [D])) = 0$ for every $q > 0$. 

Kawamata–Viehweg is a generalization of Kodaira.

Theorem 2.9 (Kodaira)

- $X$: smooth projective variety
- $D$: ample Cartier divisor

Then $H^q(X, \mathcal{O}_X(K_X + D)) = 0$ for every $q > 0$. 

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Theorem 2.8 (Kawamata–Viehweg)

- $X$: smooth projective variety
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Then $H^q(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for every $q > 0$.

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- $X$: smooth projective variety
- $D$: ample Cartier divisor

Then $H^q(X, \mathcal{O}_X(K_X + D)) = 0$ for every $q > 0$. 
Theorem 2.10 ((algebraic version of) Nadel)

- $X$: smooth projective variety
- $L$: Cartier divisor
- $D$: effective $\mathbb{Q}$-divisor
- $L - D$: nef and big

Then $H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0$ for every $q > 0$, where $\mathcal{J}(X, D)$: multiplier ideal sheaf of $(X, D)$. 
Theorem 2.11 (Kollár)

- $X$: smooth projective variety
- $Y$: projective variety
- $f : X \to Y$: surjective morphism

Then we have:

(i) $R^qf_*\mathcal{O}_X(K_X)$ is torsion-free

(ii) $H^p(Y; R^qf_*\mathcal{O}_X(K_X)\otimes \mathcal{O}_Y(H)) = 0$ for every $p > 0$ and $q \geq 0$,

where $H$: ample Cartier divisor on $Y$. 

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Theorem 2.11 (Kollár)

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(i) $R^q f_* O_X(K_X)$: torsion-free

(ii) $H^p(Y, R^q f_* O_X(K_X) \otimes O_Y(H)) = 0$ for every $p > 0$ and $q \geq 0$, where $H$: ample Cartier divisor on $Y$. 
Our result for SNC pairs contains Kodaira, Kawamata–Viehweg, Nadel, Kollár, and many other powerful and useful vanishing results as very special cases.
Almost all the classical vanishing theorems (Kawamata–Viehweg, Kollár, etc.) can be proved by the $E_1$-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$
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My idea is to use the $E_1$-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p(\log D) \otimes \mathcal{O}_X(-D)) \Rightarrow H_c^{p+q}(X \setminus D, \mathbb{C}),$$

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where $X$: smooth projective variety, $D$: SNC divisor. In my framework,

$$\mathcal{O}_X(K_X + D) \simeq \mathcal{H}om(\Omega^0_X(\log D) \otimes \mathcal{O}_X(-D), \omega_X).$$

We do not see $\mathcal{O}_X(K_X + D)$ as $\bigwedge^{\dim X} \Omega^1_X(\log D)$. 

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Some remarks

Precisely speaking:

- \((X, D)\): SNC pair, or finite cyclic cover of SNC pair

We have to consider MHS on

\[ H^k_c(X \setminus [D], \mathbb{C}). \]
Some remarks

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We have to consider MHS on
\[
H^k_c(X \setminus [D], \mathbb{C}).
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By considering VMHS, we have various semipositivity theorems (Fujino–Fujisawa, Fujino–Fujisawa–Saito).
- \((X, D)\): projective SNC pair, \(D\): reduced
- \(f : X \to Y\): surjective, \(Y\): smooth projective variety

Under some suitable assumptions, we obtain that
\[
R^q f_* O_X(K_{X/Y} + D)
\]
is a semipositive locally free sheaf for every \(q\).
Theorem 3.1 (Cone and contraction theorem)

Let \((X, \Delta)\) be a projective log canonical pair. Then

\[ \overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta} + \sum_j R_j. \]
Theorem 3.1 (Cone and contraction theorem)

- $(X, \Delta)$: projective log canonical pair

Then

$$\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta \geq 0} + \sum_j R_j.$$ 

- $R_j$: $(K_X + \Delta)$-negative extremal ray

Then there is a contraction morphism

$$\varphi_{R_j}: X \to Y$$

associated to $R_j$. 
Theorem 3.1 (Cone and contraction theorem)

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It was classically well-know for “log terminal” pairs.
(X, Δ): projective log canonical pair

Then we can run the minimal model program (MMP) (with scaling). Thus we obtain a sequence of flips and divisorial contractions.

\[(X, Δ) = (X_0, Δ_0) \rightarrow (X_1, Δ_1) \rightarrow \cdots \rightarrow (X_k, Δ_k) \rightarrow\]
MMP for log canonical pairs

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**Conjecture 3.2 (Flip Conjecture II)**

There are no infinite sequences of flips.

This conjecture is widely open. It is well-known that it is sufficient to prove that Conjecture 3.2 holds for kawamata log terminal pairs.
Open problem for log canonical pairs

Conjecture 3.3 (Finite generation of log canonical ring)

- $X$: smooth projective variety
- $\Delta$: $\mathbb{Q}$-divisor, $\text{Supp} \Delta$: SNC divisor, $\Delta \in [0, 1]$

Then

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$$

is a finitely generated $\mathbb{C}$-algebra.
Open problem for log canonical pairs

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- Conjecture 3.3 was completely solved for kawamata log terminal (KLT) pairs by BCHM
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Conjecture 3.3 was completely solved for Kawamata log terminal (KLT) pairs by BCHM.

It also holds for KLT pairs in Fujiki’s class $\mathcal{C}$ (Fujino).
Open problem for log canonical pairs

Conjecture 3.3 (Finite generation of log canonical ring)

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- Conjecture 3.3 was completely solved for kawamata log terminal (KLT) pairs by BCHM.
- It also holds for KLT pairs in Fujiki’s class $C$ (Fujino).
- It implies the existence of good minimal models, abundance conjecture, and so on (Fujino–Gongyo).
SLC pairs

- $X$: equidimensional variety, Serre’s $S_2$ condition, normal crossing in codimension one
- $\Delta$: effective $\mathbb{R}$-divisor on $X$, no components of $\Delta$ are contained in $\text{Sing } X$.
- $K_X + \Delta$: $\mathbb{R}$-Cartier
- $\nu : X^\nu \rightarrow X$: normalization, $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$

Definition 4.1 (SLC pair)

$(X, \Delta)$: semi-log canonical (SLC) pair $\overset{\text{def}}{\iff} (X^\nu, \Theta)$: log canonical pair
Example 4.2

nodal pointed curve is SLC
Why SLC?

Example 4.2

nodal pointed curve is SLC

- We need the notion of SLC pairs in order to compactify some moduli spaces (Kollár–Shepherd-Barron, Alexeev, ...)
Theorem 4.3 (Cone and contraction theorem)

- \((X, \Delta)\): projective SLC pair

Then

\[
\text{NE}(X) = \text{NE}(X)_{K_X + \Delta \geq 0} + \sum_j R_j.
\]

- \(R_j\): \((K_X + \Delta)\)-negative extremal ray

Then there is a contraction morphism

\[
\varphi_{R_j} : X \to Y
\]

associated to \(R_j\).
Remarks on SLC pairs

- quasi-projective SLC pair has a natural quasi-log structure (Ambro, Fujino)
- Kodaira-type vanishing theorems hold for SLC pairs!
- We can generalize many results for kawamata log terminal pairs to SLC pairs!!
- Unfortunately, we can not run the minimal model program for SLC pairs (Kollár, Fujino)
Thank you very much!