# The foundations of the minimal model program

Osamu Fujino

Kyoto University

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# Today's plan

- Part I: Main Results (English, this slide, unfortunately, somewhat technical)

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- Part I: Main Results (English, this slide, unfortunately, somewhat technical)
- Part II: Ideas, Backgroud, History, and so on (Japanese, no slides, fortunately, not technical)

# SNC pairs

- $M$: smooth variety $/\mathbb{C}$
- $X$: SNC divisor on $M$
- $B$: $\mathbb{R}$-divisor on $M$ such that $\operatorname{Supp} B$: SNC divisor
- $B$ and $X$ have no common components, $\operatorname{Supp}(B+X)$: SNC divisor
- $D = B|_X$

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**Definition 2.1 (GESNC pair)**

$(X, D)$: globally embedded simple normal crossing (GESNC) pair

**Definition 2.2 (SNC pair)**

$(Y, \Delta)$: simple normal crossing (SNC) pair
$\overset{\text{def}}{\Longleftrightarrow}$ $(Y, \Delta)$: Zariski locally isomorphic to a GESNC pair

## Stratum of SNC pair

- $(X, D)$: SNC pair
- $D \in [0, 1]$
- $\nu : X^\nu \to X$: normalization
- $K_{X^\nu} + \Theta = \nu^*(K_X + D)$

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**Definition 2.3 (Stratum)**

- $W$: closed subvariety of $X$

$W$: stratum of $(X, D)$
$\overset{\text{def}}{\Longleftrightarrow}$ $W = \nu(C)$, where $C$ is a log canonical center of $(X^\nu, \Theta)$, or $W$ is an irreducible component of $X$

## Hodge theoretic injectivity theorem

### Theorem 2.4 (Relative Hodge theoretic injectivity theorem)

- $(X, \Delta)$: SNC pair, $\Delta \in [0, 1]$, $\pi : X \to S$: proper
- $L$: Cartier divisor on $X$
- $D$: effective Weil divisor on $X$
- $\operatorname{Supp} D \subset \operatorname{Supp} \Delta$
- $L \sim_{\mathbb{R},\pi} K_X + \Delta$

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Then

$$R^q\pi_*\mathcal{O}_X(L) \to R^q\pi_*\mathcal{O}_X(L + D)$$

is injective for every $q$.

# Injectivity theorem for SNC pair

## Theorem 2.5 (Injectivity for SNC pair)

- $(X, \Delta)$: SNC pair, $\Delta \in [0, 1]$, $\pi : X \to S$: proper, as before
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  We further assume:

(i) $L \sim_{\mathbb{R},\pi} K_X + \Delta + H$

(ii) $H$: $\pi$-semi-ample $\mathbb{R}$-divisor

(iii) $tH \sim_{\mathbb{R},\pi} D + D'$, $t \in \mathbb{R}_{>0}$,
$D'$: effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, permissible with respect to $(X, \Delta)$

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Then $R^q\pi_*\mathcal{O}_X(L) \to R^q\pi_*\mathcal{O}_X(L + D)$ is injective for every $q$.

# Torsion-freeness and Vanishing for SNC pair

## Theorem 2.6 (Torsion-freeness and Vanishing thereom)

- $(Y, \Delta)$: SNC pair, $\Delta \in [0, 1]$, $f : Y \to X$: proper
- $L$: Cartier divisor on $Y$ such that $L - (K_Y + \Delta)$: $f$-semi-ample

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- $L$: Cartier divisor on $Y$ such that $L - (K_Y + \Delta)$: $f$-semi-ample

Then we have:

(i) Every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the $f$-image of some stratum of $(Y, \Delta)$.

(ii)
- $\pi : X \to V$: projective
- $L - (K_Y + \Delta) \sim_{\mathbb{R}} f^* H$, $H$: $\pi$-ample $\mathbb{R}$-divisor on $X$

$\Longrightarrow R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.

## Vanishing theorem of Reid–Fukuda type

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**Theorem 2.7 (Vanishing theorem of Reid–Fukuda type)**

Use the same notation as in Theorem 2.6

- $(Y, \Delta)$: GESNC, or $Y$: quasi-projective (extra assumption!)
- $H$: nef and log big over $V$ with respect to $f : (Y, \Delta) \to X$, that is, $H$: nef over $V$ and $H|_{f(W)}$: big over $V$ for every stratum $W$ of $(Y, \Delta)$

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Then $R^p\pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.

We can see that these results contain various classical results as special cases.

## Kawamata–Viehweg

### Theorem 2.8 (Kawamata–Viehweg)

- $X$: smooth projective variety
- $D$: nef and big $\mathbb{Q}$-divisor
- $\mathrm{Supp}\{D\}$: SNC divisor

Then $H^q(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for every $q > 0$.

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Kawamata–Viehweg is a generalization of Kodaira.

**Theorem 2.9 (Kodaira)**

- $X$: smooth projective variety
- $D$: ample Cartier divisor

Then $H^q(X, \mathcal{O}_X(K_X + D)) = 0$ for every $q > 0$.

# Nadel

**Theorem 2.10 ((algebraic version of) Nadel)**

- $X$: smooth projective variety
- $L$: Cartier divisor
- $D$: effective $\mathbb{Q}$-divisor
- $L - D$: nef and big

Then $H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0$ for every $q > 0$, where $\mathcal{J}(X, D)$: multiplier ideal sheaf of $(X, D)$.

# Kollár

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- $Y$: projective variety
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Then we have:

(i) $R^q f_* \mathcal{O}_X(K_X)$: torsion-free

(ii) $H^p(Y, R^q f_* \mathcal{O}_X(K_X) \otimes \mathcal{O}_Y(H)) = 0$ for every $p > 0$ and $q \geq 0$, where $H$: ample Cartier divisor on $Y$.

## Main statement

Our result for SNC pairs contains Kodaira, Kawamata–Viehweg, Nadel, Kollár, and many other powerful and useful vanishing results as very special cases.

## MHS for cohomology with compact support

Almost all the classical vanishing theorems (Kawamata–Viehweg, Kollár, etc.) can be proved by the $E_1$-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

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My idea is to use the $E_1$-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p(\log D) \otimes \mathcal{O}_X(-D)) \Rightarrow H_c^{p+q}(X \setminus D, \mathbb{C}),$$

where $X$: smooth projective variety, $D$: SNC divisor.

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where $X$: smooth projective variety, $D$: SNC divisor.
In my framework,

$$\mathcal{O}_X(K_X + D) \simeq \mathcal{H}om(\Omega_X^0(\log D) \otimes \mathcal{O}_X(-D), \omega_X).$$

We do not see $\mathcal{O}_X(K_X + D)$ as $\bigwedge^{\dim X} \Omega_X^1(\log D)$.

## Some remarks

Precisely speaking:

- $(X, D)$: SNC pair, or finite cyclic cover of SNC pair

We have to consider MHS on

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By considering VMHS, we have various semipositivity theorems (Fujino–Fujisawa, Fujino–Fujisawa–Saito).

- $(X, D)$: projective SNC pair, $D$: reduced
- $f : X \to Y$: surjective, $Y$: smooth projective variety

Under some suitable assumptions, we obtain that

$$R^q f_* \mathcal{O}_X(K_{X/Y} + D)$$

is a semipositive locally free sheaf for every $q$.

# Cone Theorem

## Theorem 3.1 (Cone and contraction theorem)

- $(X, \Delta)$: projective log canonical pair

Then

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum_j R_j.$$

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- $R_j$: $(K_X + \Delta)$-negative extremal ray

Then there is a contraction morphism

$$\varphi_{R_j} : X \to Y$$

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It was classically well-know for "log terminal" pairs.

## MMP for log canonical pairs

- $(X, \Delta)$: projective log canonical pair

Then we can run the minimal model program (MMP) (with scaling). Thus we obtain a sequence of flips and divisorial contractions.

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**Conjecture 3.2 (Flip Conjecture II)**

There are no infinite sequences of flips.

This conjecture is widely open. It is well-known that it is sufficient to prove that Conjecture 3.2 holds for kawamata log terminal pairs.

## Open problem for log canonical pairs

### Conjecture 3.3 (Finite generation of log canonical ring)

- $X$: smooth projective variety
- $\Delta$: $\mathbb{Q}$-divisor, $\operatorname{Supp}\Delta$: SNC divisor, $\Delta \in [0, 1]$

Then

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$$

is a finitely generated $\mathbb{C}$-algebra.

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- Conjecture 3.3 was completely solved for kawamata log terminal (KLT) pairs by BCHM
- It also holds for KLT pairs in Fujiki's class $\mathcal{C}$ (Fujino).
- It implies the existence of good minimal models, abundance conjecture, and so on (Fujino–Gongyo).

# SLC pairs

- $X$: equidimensional variety, Serre's $S_2$ condition, normal crossing in codimension one
- $\Delta$: effective $\mathbb{R}$-divisor on $X$, no components of $\Delta$ are contained in $\operatorname{Sing} X$.
- $K_X + \Delta$: $\mathbb{R}$-Cartier
- $\nu : X^\nu \to X$: normalization, $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$

**Definition 4.1 (SLC pair)**

$(X, \Delta)$: semi-log canonical (SLC) pair $\overset{\text{def}}{\Longleftrightarrow}$ $(X^\nu, \Theta)$: log canonical pair

# Why SLC?

**Example 4.2**

nodal pointed curve is SLC

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nodal pointed curve is SLC

- We need the notion of SLC pairs in order to compactify some moduli spaces (Kollár–Shepherd-Barron, Alexeev, ...)

# Cone Theorem for SLC pairs

## Theorem 4.3 (Cone and contraction theorem)

- $(X, \Delta)$: projective SLC pair

Then

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta \geq 0} + \sum_j R_j.$$

- $R_j$: $(K_X + \Delta)$-negative extremal ray

Then there is a contraction morphism

$$\varphi_{R_j} : X \to Y$$

associated to $R_j$.

## Remarks on SLC pairs

- quasi-projective SLC pair has a natural quasi-log structure (Ambro, Fujino)
- Kodaira-type vanishing theorems hold for SLC pairs !
- We can generalize many results for kawamata log terminal pairs to SLC pairs !!
- Unfortunately, we can not run the minimal model program for SLC pairs (Kollár, Fujino)

## Thank you

Thank you very much!