Vanishing theorems for projective morphisms between complex analytic spaces

Osamu Fujino

Kyoto University

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Background and Motivation

- I would like to discuss the minimal model program for projective morphisms between complex analytic spaces.
  I will use it for the study of complex analytic singularities, degenerations of projective varieties, and so on.
- The Kawamata–Viehweg vanishing theorem can be formulated and proved for projective morphisms of complex analytic spaces.
  We can use L^2-method for kawamata log terminal pairs.
- In order to treat complex analytic log canonical pairs, quasi-log structures on complex analytic spaces, and so on, we have to establish some vanishing theorems.

Kollár's theorem

Theorem 1 (Kollár's torsion-freeness and vanishing theorem)

Let $f\colon X \to Y$ be a surjective morphism of complex projective varieties such that X is smooth.

(i) (Torsion-freeness). $R^q f_* \omega_X$ is torsion-free for every q.

(ii) (Vanishing theorem). Let $\mathcal{A}$ be an ample line bundle on Y. Then $H^p(Y, \mathcal{A} \otimes R^q f_* \omega_X) = 0$ holds for every $p > 0$ and q.

- Kollár's vanishing theorem is a generalization of the Kodaira vanishing theorem for projective varieties.
- Theorem 1 is known to be equivalent to Kollár's injectivity theorem.
- Theorem 1 holds even when X is a compact Kähler manifold.
- We can prove Theorem 1 by the E_1-degeneration of Hodge-to-de Rham spectral sequence.

Takegoshi's theorem

Theorem 2 (Takegoshi's theorem)

Let $f\colon X \to Y$ be a proper surjective morphism from a connected Kähler manifold X to a complex analytic space Y.

(i) (Torsion-freeness). Then $R^q f_* \omega_X$ is torsion-free for every q.

(ii) (Vanishing theorem). Let $\pi\colon Y \to Z$ be a projective morphism of complex analytic spaces and let $\mathcal{A}$ be a π-ample line bundle on Y. Then $R^p \pi_* (\mathcal{A} \otimes R^q f_* \omega_X) = 0$ holds for every $p > 0$ and every q.

- This is a complex analytic generalization of Kollár's torsion-freeness and vanishing theorem.
- In Theorem 2, X and Y are not necessarily compact.
- Takegoshi's result is much more general than Theorem 2.
- Theorem 2 follows from the theory of harmonic forms.

Simple normal crossing pairs

Definition 3 (Globally embedded simple normal crossing pairs)

Let X be a simple normal crossing divisor on a complex manifold M and let B be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(B+X)$ is a simple normal crossing divisor on M and that B and X have no common components. Then we put $D := B|_X$ and call (X, D) an analytic globally embedded simple normal crossing pair.

Definition 4 (Simple normal crossing pairs)

If the pair (X, D) is locally isomorphic to a globally embedded simple normal crossing pair at any point and the irreducible components of X and D are all smooth, then (X, D) is called an analytic simple normal crossing pair.

Decomposition of (X, D)

Definition 5

Let (X, D) be an analytic simple normal crossing pair such that D is reduced. For any positive integer k, we put

$$X^{[k]} := \{x \in X | \operatorname{mult}_x X \geq k\}^{\sim},$$

where $Z^{\sim}$ denotes the normalization of Z. Then $X^{[k]}$ is the disjoint union of the intersections of k irreducible components of X, and is smooth. We have a reduced simple normal crossing divisor $D^{[k]} \subset X^{[k]}$ defined by the pull-back of D by the natural morphism $X^{[k]} \to X$. For any $l \in \mathbb{Z}_{\geq 0}$, we put

$$D^{[k,l]} := \left\{x \in X^{[k]} \middle| \operatorname{mult}_x D^{[k]} \geq l\right\}^{\sim}.$$

We note $D^{[k,0]} = X^{[k]}$ and $\dim D^{[k,l]} = \dim X + 1 - k - l$.

Spectral sequence

Theorem 6 (Fujino–Fujisawa–Saito, 2014)

Let (X, D) be an analytic simple normal crossing pair such that D is reduced and let $f\colon X \to Y$ be a proper morphism to a complex manifold Y. Assume that f is Kähler on each irreducible component of X. Then there is the weight spectral sequence

$$ {}_F E_1^{-q,i+q} = \bigoplus_{k+l=\dim X+q+1} R^i f_* \omega_{D^{[k,l]}/Y} \Rightarrow R^i f_* \omega_{X/Y}(D), $$

degenerating at E_2, and its E_1-differential d_1 splits so that the ${}_F E_2^{-q,i+q}$ are direct factors of ${}_F E_1^{-q,i+q}$.

- Theorem 6 follows from Saito's theory of mixed Hodge modules.
- The proof of Theorem 6 becomes simpler when f is projective.

Strata of (X, D)

Definition 7 (Strata)

Let (X, D) be an analytic simple normal crossing pair. Let $\nu\colon X^\nu \to X$ be the normalization. We put

$$K_{X^\nu} + \Theta = \nu^*(K_X + D),$$

that is, Θ is the union of $\nu_*^{-1}D$ and the inverse image of the singular locus of X. If W is an irreducible component of X or the ν-image of some log canonical center of (X^ν, Θ), then W is called a stratum of (X, D).

When D is reduced, W is a stratum of (X, D) if and only if W is the image of an irreducible component of $D^{[k,l]}$ for some $k > 0$ and $l \geq 0$.

Standard setting

Theorem 8 (Fujino, 2022)

Let (X, D) be an analytic simple normal crossing pair such that D is reduced and let $f\colon X \to Y$ be a proper morphism of complex analytic spaces. Assume that f is Kähler on each irreducible component of X. Then

(i) (Strict support condition). Every associated subvariety of $R^q f_*\omega_X(D)$ is the f-image of some stratum of (X, D) for every q.

(ii) (Vanishing theorem). Let $\pi\colon Y \to Z$ be a projective morphism of complex analytic spaces and let $\mathcal{A}$ be a π-ample line bundle on Y. Then

$$R^p\pi_*\left(\mathcal{A} \otimes R^q f_*\omega_X(D)\right) = 0$$

holds for every $p > 0$ and every q.

- Theorem 8 (i) is almost obvious by Theorem 6.
- Theorem 8 (i) is a consequence of the strict support condition of polarizable pure Hodge modules.
- Theorem 8 (ii) follows from Takegoshi's vanishing theorem with the aid of Theorem 6.

Injectivity theorem

Theorem 9 (Injectivity theorem, Fujino, 2022)

Let (X, D) and $f\colon X \to Y$ be as in Theorem 8. Let $\mathcal{L}$ be an f-semiample line bundle on X. Let s be a nonzero element of $H^0(X, \mathcal{L}^{\otimes k})$ for some $k \in \mathbb{Z}_{\geq 0}$ such that the zero locus of s does not contain any strata of (X, D). Then, for every q, the map

$$\times s\colon R^q f_*\left(\omega_X(D) \otimes \mathcal{L}^{\otimes l}\right) \to R^q f_*\left(\omega_X(D) \otimes \mathcal{L}^{\otimes k+l}\right)$$

induced by $\otimes s$ is injective for every $l \in \mathbb{Z}_{>0}$.

- Theorem 9 is a generalization of Kollár's injectivity theorem.
- Theorem 9 can be proved by Theorem 8 without difficulties.
- For various geometric applications, Theorem 8 seems to be more useful than Theorem 9.

Main theorem

Theorem 10 (Main theorem, Fujino 2022)

Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $f\colon X \to Y$ be a projective morphism to a complex analytic space Y and let $\mathcal{L}$ be a line bundle on X. Let q be an arbitrary nonnegative integer. Then

(i) (Strict support condition). If $\mathcal{L} - (\omega_X + \Delta)$ is f-semiample, then every associated subvariety of $R^q f_*\mathcal{L}$ is the f-image of some stratum of (X, Δ).

(ii) (Vanishing theorem). If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^*\mathcal{H}$ holds for some π-ample $\mathbb{R}$-line bundle $\mathcal{H}$ on Y, where $\pi\colon Y \to Z$ is a projective morphism to a complex analytic space Z, then we have

$$R^p \pi_* R^q f_* \mathcal{L} = 0$$

for every $p > 0$.

- When $f\colon X \to Y$ and $\pi\colon Y \to Z$ are algebraic, Theorem 10 is well known and has already played a crucial role for the study of log canonical pairs and quasi-log schemes.
- Theorem 10 will play a crucial role for the study of analytic log canonical pairs and quasi-log structures for complex analytic spaces.

Reid–Fukuda type

Theorem 11 (Vanishing theorem of Reid–Fukuda type)

Let (X, Δ) be an analytic simple normal crossing pair such that Δ is a boundary $\mathbb{R}$-divisor on X. Let $f\colon X \to Y$ and $\pi\colon Y \to Z$ be projective morphisms of complex analytic spaces and let $\mathcal{L}$ be a line bundle on X. If $\mathcal{L} - (\omega_X + \Delta) \sim_{\mathbb{R}} f^*\mathcal{H}$ holds such that $\mathcal{H}$ is an $\mathbb{R}$-line bundle, which is nef and log big over Z with respect to $f\colon (X, \Delta) \to Y$, on Y, then

$$R^p\pi_*R^qf_*\mathcal{L} = 0$$

holds for every $p > 0$ and every q.

Thank you very much!