EFFECTIVE BASEPOINT-FREE THEOREM FOR SEMI-LOG CANONICAL SURFACES

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Abstract. The main purpose of this paper is to propose a Fujita-type freeness conjecture for semi-log canonical pairs. We prove it for curves and surfaces by using the theory of quasi-log schemes. We also prove an effective freeness for log surfaces. For the reader's convenience, we give an effective very ampleness theorem for stable pairs and semi-log canonical Fano varieties.

Contents

1. Introduction

We will work over $\mathbb{C}$, the complex number field, throughout this paper. Note that, by the Lefschetz principle, all the results in this paper hold over any algebraically closed field k of characteristic zero.

One of the main purposes of this paper is to propose the following Fujita-type freeness conjecture for projective semi-log canonical pairs.

Conjecture 1.1 (Fujita-type freeness conjecture for semi-log canonical pairs)**.** *Let (X, Δ) be an n-dimensional projective semi-log canonical*

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pair and let D be a Cartier divisor on X. We put $A = D - (K_X + \Delta)$. Assume that

(1) $(A^n \cdot X_i) > n^n$ *for every irreducible component* X_i *of* X*, and*
(2) $(A^d \cdot W) \geq n^d$ *for every* d*-dimensional irreducible subvariety* W *of* X *for* $1 \leq d \leq n-1$*.*

Then the complete linear system $|D|$ *is basepoint-free.*

As a special case of Conjecture 1.1, we have:

Conjecture 1.2 (Fujita-type freeness conjecture for log canonical pairs)**.** *Let* (X, Δ) *be an* n*-dimensional projective irreducible log canonical pair and let* D *be a Cartier divisor on* X*. We put* $A = D - (K_X + \Delta)$*. Assume that*

(1) $A^n > n^n$*, and*
(2) $(A^d \cdot W) \geq n^d$ *for every* d*-dimensional irreducible subvariety* W *of* X *for* $1 \leq d \leq n-1$*.*

Then the complete linear system $|D|$ *is basepoint-free.*

If $A^n > \left(\frac{1}{2}n(n+1)\right)^n$ and $(A^d \cdot W) > \left(\frac{1}{2}n(n+1)\right)^d$ hold true in Conjecture 1.1, then we know that the complete linear system $|D|$ is basepoint-free. It follows from [Liu, Corollary 3.5], which is obviously a generalization of Anghern–Siu's effective freeness (see [AS] and [Fuj2]). For the details, we recommend the reader to see [Liu].

Of course, the above conjectures are some naive generalizations of Fujita's celebrated conjecture:

Conjecture 1.3 (Fujita's freeness conjecture)**.** *Let* X *be a smooth projective variety with* $\dim X = n$ *and let* H *be an ample Cartier divisor on* X*. Then the complete linear system* $|K_X + (n+1)H|$ *is basepoint-free.*

The main theorem of this paper is:

Theorem 1.4 (Main theorem, see Theorem 2.1 and Theorem 5.1)**.** *Conjecture 1.1 holds true in dimension one and two.*

Theorem 1.4 partially supports Conjecture 1.1. As a corollary of Theorem 1.4, we have:

Corollary 1.5 (cf. [LR, Theorem 24])**.** *Let* (X, Δ) *be a* 2*-dimensional stable pair such that* $K_X + \Delta$ *is* $\mathbb{Q}$*-Cartier. Let* I *be the smallest positive integer such that* $I(K_X + \Delta)$ *is Cartier. Then* $|mI(K_X + \Delta)|$ *is basepoint-free for every* $m \geq 4$*. If* $I \geq 2$*, then* $|mI(K_X + \Delta)|$ *is basepoint-free for every* $m \geq 3$*.*

Note that a *stable pair* (X, Δ) is a projective semi-log canonical pair (X, Δ) such that $K_X + \Delta$ is ample. We also have:

Corollary 1.6 (Semi-log canonical Fano surfaces)**.** *Let (X, Δ) be a projective semi-log canonical surface such that $-(K_X + \Delta)$ is an ample $\mathbb{Q}$-divisor. Let I be the smallest positive integer such that $I(K_X + \Delta)$ is Cartier. Then $|-mI(K_X + \Delta)|$ is basepoint-free for every $m \geq 2$.*

For the theory of log surfaces (see [Fuj4]), the following theorem is a reasonable formulation of the Fujita-type freeness theorem.

Theorem 1.7 (Effective freeness for log surfaces)**.** *Let (X, Δ) be a complete irreducible log surface and let D be a Cartier divisor on X. We put $A = D-(K_X+\Delta)$. Assume that A is nef, $A^2 > 4$ and $A\cdot C \geq 2$ for every curve C on X such that $x \in C$. Then $\mathcal{O}_X(D)$ has a global section not vanishing at x.*

We know that the theory of log surfaces initiated in [Fuj4] now holds in characteristic $p > 0$ (see [FT], [Tan1], and [Tan2]). Therefore, it is natural to propose:

Conjecture 1.8. *Theorem 1.7 holds in characteristic $p > 0$.*

Note that the original form of Fujita's freeness conjecture (see Conjecture 1.3) is still open for surfaces in characteristic $p > 0$.

In this paper, we will use the theory of quasi-log schemes (see [Fuj5], [Fuj7], [Fuj8], and so on). Our approach to the Fujita-type freeness conjectures is different from the standard technique based on the Kawamata–Viehweg vanishing theorem (see [EL]). This is because we do not directly apply the Kawamata–Viehweg vanishing theorem to log canonical pairs and semi-log canonical pairs.

Related to Corollary 1.5 and Corollary 1.6, we have the following effective very ampleness theorem for stable pairs and semi-log canonical Fano varieties, which is essentially contained in [Fuj1] and [Fuj5]. In this paper, we will prove it for the reader's convenience.

Theorem 1.9 (see Corollary 7.2 and Corollary 7.4)**.** *Let (X, Δ) be a projective semi-log canonical pair such that $I(K_X+\Delta)$ (resp. $-I(K_X+\Delta)$) is an ample Cartier divisor for some positive integer I. Then there exists a positive integer N depending only on $\dim X$ such that $NI(K_X + \Delta)$ (resp. $-NI(K_X + \Delta)$) is very ample.*

We summarize the contents of this paper. In Section 2, we prove Conjecture 1.1 for semi-log canonical curves using the vanishing theorem obtained in [Fuj5]. This section may help the reader to understand more complicated arguments in the subsequent sections. In Section 3, we collect some basic definitions. In Section 4, we quickly recall the theory of quasi-log schemes. Section 5 is the main part of this paper. In this section, we prove Conjecture 1.1 for semi-log canonical surfaces.

Section 6 is devoted to the proof of Theorem 1.7, which is an effective freeness for log surfaces. In Section 7, which is independent of the other sections, we prove Theorem 1.9.

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For the standard notations and conventions of the log minimal model program, see [Fuj3] and [Fuj8]. For the details of semi-log canonical pairs, see [Fuj5]. In this paper, a *scheme* means a separated scheme of finite type over $\mathbb{C}$ and a *variety* means a reduced scheme.

2. Semi-log canonical curves

In this section, we prove Conjecture 1.1 in dimension one based on [Fuj5]. This section will help the reader to understand the subsequent sections. Precisely speaking, we give a proof of:

Theorem 2.1. *Let (X, Δ) be a projective semi-log canonical curve and let D be a Cartier divisor on X. We put $A = D - (K_X + \Delta)$. Assume that $(A \cdot X_i) > 1$ for every irreducible component X_i of X. Then the complete linear system $|D|$ is basepoint-free.*

If (X, Δ) is log canonical, that is, X is normal, in Theorem 2.1, then the statement is obvious (see Conjecture 1.2). However, Theorem 2.1 seems to be nontrivial when X is not normal.

Proof of Theorem 2.1. We will see that the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(P) \tag{2.1}$$

is surjective for every $P \in X$. Of course, it is sufficient to prove that $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$, where $\mathcal{I}_P$ is the defining ideal sheaf of P on X. If P is a zero-dimensional semi-log canonical center of (X, Δ), then we know that $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$ by [Fuj5, Theorem 1.11]. Therefore, we may assume that P is not a zero-dimensional semi-log canonical center of (X, Δ). Thus, we see that X is normal, that is, smooth, at P (see, for example, [Fuj5, Corollary 3.5]). We put

$$c = 1 - \mathrm{mult}_P \Delta. \tag{2.2}$$

Then we have $0 < c \leq 1$. We consider $(X, \Delta + cP)$. Then $(X, \Delta + cP)$ is semi-log canonical and P is a zero-dimensional semi-log canonical center of $(X, \Delta + cP)$. Since

$$((D - (K_X + \Delta + cP)) \cdot X_i) > 0 \tag{2.3}$$

for every irreducible component X_i of X by the assumption that $(A \cdot X_i) > 1$ and the fact that $c \leq 1$, we obtain that $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$ (see [Fuj5, Theorem 1.11]). Therefore, we see that $H^1(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$ for every $P \in X$. Thus, we have the desired surjection (2.1). ☐

The above proof of Theorem 2.1 heavily depends on the vanishing theorem for semi-log canonical pairs (see [Fuj5, Theorem 1.11]), which follows from the theory of quasi-log schemes based on the theory of mixed Hodge structures on cohomology with compact support. For the details, see [Fuj5] and [Fuj8].

3. Preliminaries

In this section, we collect some basic definitions.

3.1 (Operations for $\mathbb{R}$-divisors)**.** Let D be an $\mathbb{R}$-divisor on an equidimensional variety X, that is, D is a finite formal $\mathbb{R}$-linear combination

$$D = \sum_i d_i D_i \tag{3.1}$$

of irreducible reduced subschemes D_i of codimension one, where $D_i \neq D_j$ for $i \neq j$. We define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where for every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i \quad \text{and} \quad D^{>1} = \sum_{d_i > 1} d_i D_i. \tag{3.2}$$

We call D a *boundary* (resp. *subboundary*) $\mathbb{R}$-divisor if $0 \leq d_i \leq 1$ (resp. $d_i \leq 1$) for every i.

3.2 (Singularities of pairs)**.** Let X be a normal variety and let Δ be an $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a resolution such that $\mathrm{Exc}(f) \cup f_*^{-1}\Delta$, where $\mathrm{Exc}(f)$ is the exceptional locus of f and $f_*^{-1}\Delta$ is the strict transform of Δ on Y, has a simple normal crossing support. We can write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i. \tag{3.3}$$

We say that (X, Δ) is *sub log canonical* (*sub lc*, for short) if $a_i \geq -1$ for every i. We usually write $a_i = a(E_i, X, \Delta)$ and call it the *discrepancy coefficient* of E_i with respect to (X, Δ). Note that we can define $a(E, X, \Delta)$ for every prime divisor E *over* X. If (X, Δ) is sub log canonical and Δ is effective, then (X, Δ) is called *log canonical* (*lc*, for short).

It is well-known that there is the largest Zariski open subset U of X such that $(U, \Delta|_U)$ is sub log canonical. If there exist a resolution $f : Y \to X$ and a divisor E on Y such that $a(E, X, \Delta) = -1$ and $f(E) \cap U \neq \emptyset$, then $f(E)$ is called a *log canonical center* (an *lc center*, for short) with respect to (X, Δ). A closed subset C of X is called a *log canonical stratum* (an *lc stratum*, for short) of (X, Δ) if and only if C is a log canonical center of (X, Δ) or C is an irreducible component of X. We note that the *non-lc locus* of (X, Δ), which is denoted by $\mathrm{Nlc}(X, \Delta)$, is $X \setminus U$.

Let X be a normal variety and let Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. If $a(E, X, \Delta) > -1$ for every divisor E over X, then (X, Δ) is called *klt*. If $a(E, X, \Delta) > -1$ for every exceptional divisor E over X, then (X, Δ) is called *plt*.

Let us recall the definitions around *semi-log canonical pairs*.

3.3 (Semi-log canonical pairs)**.** Let X be an equidimensional variety that satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor whose support does not contain any irreducible components of the conductor of X. The pair (X, Δ) is called a *semi-log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and
(2) (X^ν, Θ) is log canonical, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$.

Let (X, Δ) be a semi-log canonical pair and let $\nu : X^\nu \to X$ be the normalization. We set

$$K_{X^\nu} + \Theta = \nu^*(K_X + \Delta) \tag{3.4}$$

as above. A closed subvariety W of X is called a *semi-log canonical center* (an *slc center*, for short) *with respect to* (X, Δ) if there exist a resolution of singularities $f : Y \to X^\nu$ and a prime divisor E on Y such that the discrepancy coefficient $a(E, X^\nu, \Theta) = -1$ and $\nu \circ f(E) = W$. A closed subvariety W of X is called a *semi-log canonical stratum* (*slc stratum*, for short) of the pair (X, Δ) if W is a semi-log canonical center with respect to (X, Δ) or W is an irreducible component of X.

We close this section with the notion of *log surfaces* (see [Fuj4]).

3.4 (Log surfaces)**.** Let X be a normal surface and let Δ be a boundary $\mathbb{R}$-divisor on X. Assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then the pair (X, Δ) is called a *log surface*. A log surface (X, Δ) is not always assumed to be log canonical.

In [Fuj4], we establish the minimal model program for log surfaces in full generality under the assumption that X is $\mathbb{Q}$-factorial or (X, Δ) has only log canonical singularities. In characteristic $p > 0$, see [FT], [Tan1], and [Tan2].

4. On quasi-log structures

Let us quickly recall the definitions of *globally embedded simple normal crossing pairs* and *quasi-log schemes* for the reader's convenience. For the details, see, for example, [Fuj6, Section 3] and [Fuj8, Chapter 5 and Chapter 6].

Definition 4.1 (Globally embedded simple normal crossing pairs)**.** Let Y be a simple normal crossing divisor on a smooth variety M and let D be an $\mathbb{R}$-divisor on M such that $\mathrm{Supp}(D+Y)$ is a simple normal crossing divisor on M and that D and Y have no common irreducible components. We put $B_Y = D|_Y$ and consider the pair (Y, B_Y). We call (Y, B_Y) a *globally embedded simple normal crossing pair* and M the *ambient space* of (Y, B_Y). A *stratum* of (Y, B_Y) is the ν-image of a log canonical stratum of (Y^ν, Θ) where $\nu : Y^\nu \to Y$ is the normalization and $K_{Y^\nu} + \Theta = \nu^*(K_Y + B_Y)$.

In this paper, we adopt the following definition of quasi-log schemes.

Definition 4.2 (Quasi-log schemes)**.** A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a proper closed subscheme $X_{-\infty} \subset X$, and a finite collection $\{C\}$ of reduced and irreducible subschemes of X such that there is a proper morphism $f : (Y, B_Y) \to X$ from a globally embedded simple normal crossing pair satisfying the following properties:

(1) $f^*\omega \sim_{\mathbb{R}} K_Y + B_Y$.
(2) The natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil)$ induces an isomorphism
$$\mathcal{I}_{X_{-\infty}} \xrightarrow{\simeq} f_*\mathcal{O}_Y(\lceil -(B_Y^{<1})\rceil - \lfloor B_Y^{>1}\rfloor),$$
where $\mathcal{I}_{X_{-\infty}}$ is the defining ideal sheaf of $X_{-\infty}$.
(3) The collection of subvarieties $\{C\}$ coincides with the image of (Y, B_Y)-strata that are not included in $X_{-\infty}$.

We simply write $[X, \omega]$ to denote the above data
$$\big(X, \omega, f : (Y, B_Y) \to X\big)$$
if there is no risk of confusion. Note that a quasi-log scheme X is the union of $\{C\}$ and $X_{-\infty}$. We also note that ω is called the *quasi-log canonical class* of $[X, \omega]$, which is defined up to $\mathbb{R}$-linear equivalence.

We sometimes simply say that $[X, \omega]$ is a *quasi-log pair*. The subvarieties C are called the *qlc strata* of $[X, \omega]$, $X_{-\infty}$ is called the *non-qlc locus* of $[X, \omega]$, and $f : (Y, B_Y) \to X$ is called a *quasi-log resolution* of $[X, \omega]$. We sometimes use $\mathrm{Nqlc}(X, \omega)$ to denote $X_{-\infty}$. A closed subvariety C of X is called a *qlc center* of $[X, \omega]$ if C is a qlc stratum of $[X, \omega]$ which is not an irreducible component of X.

Let $[X, \omega]$ be a quasi-log scheme. Assume that $X_{-\infty} = \emptyset$. Then we sometimes simply say that $[X, \omega]$ is a *qlc pair* or $[X, \omega]$ is a quasi-log scheme with only *quasi-log canonical singularities*.

Definition 4.3 (Nef and log big divisors for quasi-log schemes)**.** Let L be an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) on a quasi-log pair $[X, \omega]$ and let $\pi : X \to S$ be a proper morphism between schemes. Then L is *nef and log big over S with respect to* $[X, \omega]$ if L is π-nef and $L|_C$ is π-big for every qlc stratum C of $[X, \omega]$.

The following theorem is a key result for the theory of quasi-log schemes.

Theorem 4.4 (Adjunction and vanishing theorem)**.** *Let $[X, \omega]$ be a quasi-log scheme and let X' be the union of $X_{-\infty}$ with a* (*possibly empty*) *union of some qlc strata of $[X, \omega]$. Then we have the following properties.*

(i) *Assume that $X' \neq X_{-\infty}$. Then X' is a quasi-log scheme with $\omega' = \omega|_{X'}$ and $X'_{-\infty} = X_{-\infty}$. Moreover, the qlc strata of $[X', \omega']$ are exactly the qlc strata of $[X, \omega]$ that are included in X'.*

(ii) *Assume that $\pi : X \to S$ is a proper morphism between schemes. Let L be a Cartier divisor on X such that $L - \omega$ is nef and log big over S with respect to $[X, \omega]$. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

For the proof of Theorem 4.4, see, for example, [Fuj7, Theorem 3.8] and [Fuj8, Theorem 6.3.4]. We can slightly generalize Theorem 4.4 (ii) as follows.

Theorem 4.5. *Let $[X, \omega]$, X', and $\pi : X \to S$ be as in Theorem 4.4. Let L be a Cartier divisor on X such that $L - \omega$ is nef over S and that $(L - \omega)|_W$ is big over S for any qlc stratum W of $[X, \omega]$ which is not contained in X'. Then $R^i\pi_*(\mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0$ for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.*

Theorem 4.5 is obvious by the proof of Theorem 4.4. For a related topic, see [Fuj5, Remark 5.2]. Theorem 4.5 will play a crucial role in the proof of Theorem 1.7 in Section 6.

Finally, we prepare a useful lemma, which is new, for the proof of Theorem 1.4.

Lemma 4.6. *Let* $[X, \omega]$ *be a qlc pair such that* X *is irreducible. Let* E *be an effective* $\mathbb{R}$*-Cartier divisor on* X*. This means that*

$$E = \sum_{i=1}^{k} e_i E_i$$

where E_i *is an effective Cartier divisor on* X *and* e_i *is a positive real number for every* i*. Then we can give a quasi-log structure to* $[X, \omega + E]$*, which coincides with the original quasi-log structure of* $[X, \omega]$ *outside* $\operatorname{Supp} E$*.*

For the details of the quasi-log structure of $[X, \omega + E]$, see the construction in the proof below.

Proof. Let $f : (Z, \Delta_Z) \to [X, \omega]$ be a quasi-log resolution, where (Z, Δ_Z) is a globally embedded simple normal crossing pair. By taking some suitable blow-ups, we may assume that the union of all strata of (Z, Δ_Z) mapped to $\operatorname{Supp} E$, which is denoted by Z'', is a union of some irreducible components of Z (see [Fuj6, Proposition 4.1] and [Fuj8, Proposition 6.3.1]). We put $Z' = Z - Z''$ and $K_{Z'} + \Delta_{Z'} = (K_Z + \Delta_Z)|_{Z'}$. We may further assume that $(Z', \Delta_{Z'} + f'^*E)$ is a globally embedded simple normal crossing pair, where $f' = f|_{Z'} : Z' \to X$. By construction, we have a natural inclusion

$$\mathcal{O}_{Z'}(\lceil -(\Delta_{Z'} + f'^*E)^{<1} \rceil - \lfloor (\Delta_{Z'} + f'^*E)^{>1} \rfloor) \subset \mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil). \tag{4.1}$$

This is because

$$-\lfloor (\Delta_{Z'} + f'^*E)^{>1} \rfloor \leq -Z''|_{Z'} \tag{4.2}$$

and

$$\mathcal{O}_{Z'}(-Z''|_{Z'}) \subset \mathcal{O}_Z. \tag{4.3}$$

Thus, we have

$$\begin{aligned} &f'_*\mathcal{O}_{Z'}(\lceil -(\Delta_{Z'} + f'^*E)^{<1} \rceil - \lfloor (\Delta_{Z'} + f'^*E)^{>1} \rfloor) \\ &\subset f_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_X. \end{aligned} \tag{4.4}$$

By putting

$$\mathcal{I}_{X_{-\infty}} = f'_*\mathcal{O}_{Z'}(\lceil -(\Delta_{Z'} + f'^*E)^{<1} \rceil - \lfloor (\Delta_{Z'} + f'^*E)^{>1} \rfloor), \tag{4.5}$$

$f' : (Z', \Delta_{Z'} + f'^*E) \to [X, \omega + E]$ gives a quasi-log structure to $[X, \omega + E]$. By construction, it coincides with the original quasi-log structure of $[X, \omega]$ outside $\operatorname{Supp} E$. □

5. Semi-log canonical surfaces

In this section, we prove Conjecture 1.1 for surfaces. Equivalently, we have:

Theorem 5.1. *Let (X, Δ) be a projective semi-log canonical surface and let D be a Cartier divisor on X. We put $A = D - (K_X + \Delta)$. Assume that $(A^2 \cdot X_i) > 4$ for every irreducible component X_i of X and that $A \cdot C \geq 2$ for every curve C on X. Then the complete linear system $|D|$ is basepoint-free.*

Remark 5.2. By assumption and Nakai's ampleness criterion for $\mathbb{R}$-divisors (see [CP]), A is ample in Theorem 5.1. However, we do not use the ampleness of A in the proof of Theorem 5.1.

Our proof of Theorem 5.1 uses the theory of quasi-log schemes.

Proof. We will prove that the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(P)$$

is surjective for every $P \in X$.

Step 1 (Quasi-log structure)**.** By [Fuj5, Theorem 1.2], we can take a quasi-log resolution $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$. Precisely speaking, (Z, Δ_Z) is a globally embedded simple normal crossing pair such that Δ_Z is a subboundary $\mathbb{R}$-divisor on Z with the following properties.

- (i) $K_Z + \Delta_Z \sim_{\mathbb{R}} f^*(K_X + \Delta)$.
- (ii) $f_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_X$.
- (iii) $\dim Z = 2$.
- (iv) W is a semi-log canonical stratum of (X, Δ) if and only if $W = f(S)$ for some stratum S of (Z, Δ_Z).

It is worth mentioning that $f : Z \to X$ is not necessarily birational. This step is nothing but [Fuj5, Theorem 1.2].

Step 2. Assume that P is a zero-dimensional semi-log canonical center of (X, Δ). Then $H^i(X, \mathcal{I}_P \otimes \mathcal{O}_X(D)) = 0$ for every $i > 0$, where $\mathcal{I}_P$ is the defining ideal sheaf of P on X (see [Fuj5, Theorem 1.11] and Theorem 4.4). Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(P)$$

is surjective.

From now on, we may assume that P is not a zero-dimensional semi-log canonical center of (X, Δ).

Step 3. Assume that there exists a one-dimensional semi-log canonical center W of (X, Δ) such that $P \in W$. Since P is not a zero-dimensional semi-log canonical center of (X, Δ), W is normal, that is, smooth, at P by [Fuj5, Corollary 3.5]. By adjunction (see Theorem 4.4), $[W, (K_X + \Delta)|_W]$ has a quasi-log structure with only quasi-log canonical singularities induced by the quasi-log structure $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$ constructed in Step 1. Let $g : (Z', \Delta_{Z'}) \to [W, (K_X + \Delta)|_W]$ be the induced quasi-log resolution. We put

$$c = \sup_{t \geq 0} \left\{ t \;\middle|\; \begin{array}{l} \text{the normalization of } (Z', \Delta_{Z'} + tg^*P) \text{ is} \\ \text{sub log canonical.} \end{array} \right\}. \tag{5.1}$$

Then, by [Fuj7, Lemma 3.16], we obtain that $0 < c < 2$. Note that P is a Cartier divisor on W. Let us consider $g : (Z', \Delta_{Z'} + cg^*P) \to [W, (K_X + \Delta)|_W + cP]$, which defines a quasi-log structure. Then, by construction, P is a qlc center of $[W, (K_X + \Delta)|_W + cP]$. Moreover, we see that

$$(D|_W - ((K_X + \Delta)|_W + cP)) = (A \cdot W) - c > 0 \tag{5.2}$$

by assumption. Therefore, we obtain that

$$H^i(W, \mathcal{I}_P \otimes \mathcal{O}_W(D)) = 0 \tag{5.3}$$

for every $i > 0$ by Theorem 4.4, where $\mathcal{I}_P$ is the defining ideal sheaf of P on W. Thus, the restriction map

$$H^0(W, \mathcal{O}_W(D)) \to \mathcal{O}_W(D) \otimes \mathbb{C}(P) \tag{5.4}$$

is surjective. On the other hand, by Theorem 4.4 again, we have that

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(D)) = 0 \tag{5.5}$$

for every $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. This implies that the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(W, \mathcal{O}_W(D)) \tag{5.6}$$

is surjective. By combining (5.4) with (5.6), the desired restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(P) \tag{5.7}$$

is surjective.

Therefore, from now on, we may assume that no one-dimensional semi-log canonical centers of (X, Δ) contain P.

Step 4. In this step, we assume that P is a smooth point of X. Let X_0 be the unique irreducible component of X containing P. By adjunction (see Theorem 4.4), $[X_0, (K_X + \Delta)|_{X_0}]$ has a quasi-log structure with only quasi-log canonical singularities induced by the quasi-log structure $f:(Z,\Delta_Z)\to[X,K_X+\Delta]$ constructed in Step 1. By Theorem 4.4,

$$H^i(X,\mathcal{I}_{X_0}\otimes\mathcal{O}_X(D))=0 \tag{5.8}$$

for every $i>0$, where $\mathcal{I}_{X_0}$ is the defining ideal sheaf of X_0 on X. Therefore, the restriction map

$$H^0(X,\mathcal{O}_X(D))\to H^0(X_0,\mathcal{O}_{X_0}(D)) \tag{5.9}$$

is surjective. Thus, it is sufficient to prove that the natural restriction map

$$H^0(X_0,\mathcal{O}_{X_0}(D))\to\mathcal{O}_{X_0}(D)\otimes\mathbb{C}(P) \tag{5.10}$$

is surjective. We put $A_0=A|_{X_0}$. Since $A_0^2>4$, we can find an effective $\mathbb{R}$-Cartier divisor B on X_0 such that $\operatorname{mult}_P B>2$ and that $B\sim_{\mathbb{R}}A_0$. We put $U=X_0\setminus\operatorname{Sing}X_0$ and define

$$c=\max\{t\geq 0\,|\,(U,\Delta|_U+tB|_U)\text{ is log canonical at }P\}. \tag{5.11}$$

Then we obtain that $0<c<1$ since $\operatorname{mult}_P B>2$. By Lemma 4.6, we have a quasi-log structure on $[X_0,(K_X+\Delta)|_{X_0}+cB]$. By construction, there is a qlc center W of $[X_0,(K_X+\Delta)|_{X_0}+cB]$ passing through P. Let X' be the union of the non-qlc locus of $[X_0,(K_X+\Delta)|_{X_0}+cB]$ and the minimal qlc center W_0 of $[X_0,(K_X+\Delta)|_{X_0}+cB]$ passing through P. Note that $D|_{X_0}-((K_X+\Delta)|_{X_0}+cB)\sim_{\mathbb{R}}(1-c)A_0$. Then, by Theorem 4.4,

$$H^i(X_0,\mathcal{I}_{X'}\otimes\mathcal{O}_{X_0}(D))=0 \tag{5.12}$$

for every $i>0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X_0.

Case 1. If $\dim W_0=0$, then P is isolated in $\operatorname{Supp}\mathcal{O}_{X_0}/\mathcal{I}_{X'}$. Therefore, the restriction map

$$H^0(X_0,\mathcal{O}_{X_0}(D))\to\mathcal{O}_{X_0}(D)\otimes\mathbb{C}(P) \tag{5.13}$$

is surjective.

Case 2. If $\dim W_0=1$, then let us consider the quasi-log structure of $[X',((K_X+\Delta)|_{X_0}+cB)|_{X'}]$ induced by the quasi-log structure of $[X_0,(K_X+\Delta)|_{X_0}+cB]$ constructed above by Lemma 4.6 (see Theorem 4.4 (i)). As in Step 3, we can take $0<c'\leq 1$ such that P is a zero-dimensional qlc center of $[X',((K_X+\Delta)|_{X_0}+cB)|_{X'}+c'P]$. More precisely, by shrinking (X,Δ) around P, we may assume that $(X,\Delta+cB)$ is plt. We put $\operatorname{mult}_P B=2+a$ with $a>0$. We write $\Delta+cB=L+\Delta'$, where L is the unique one-dimensional log canonical center of (X,Δ) passing through P and $\Delta'=\Delta+cB-L$. We put

$\mathrm{mult}_P(\Delta + cB) = 1 + \delta$ with $\delta \geq 0$, equivalently, $\delta = \mathrm{mult}_P \Delta' \geq 0$. Note that

$$1 + \delta = \mathrm{mult}_P(\Delta + cB) = \mathrm{mult}_P \Delta + c(2 + a). \tag{5.14}$$

Therefore, we have

$$c = \frac{1 + \delta - \alpha}{2 + a}, \tag{5.15}$$

where $\alpha = \mathrm{mult}_P \Delta \geq 0$. We also note that

$$\delta \leq \mathrm{mult}_P(\Delta'|_L) < 1. \tag{5.16}$$

Then, we can choose $c' = 1 - \mathrm{mult}_P(\Delta'|_L)$. This is because $(X, \Delta + cB + c'H)$ is log canonical but is not plt at P, where H is a general smooth curve passing through P.

In this situation, we have

$$\begin{aligned}
&\deg(D|_L - (K_X + \Delta + cB)|_L - c'P) \\
&\geq \left(1 - \frac{1 + \delta - \alpha}{2 + a}\right) \cdot 2 - (1 - \delta) \\
&= \frac{1}{2 + a}((2 + a - 1 - \delta + \alpha) \cdot 2 - (2 + a)(1 - \delta)) \\
&= \frac{1}{2 + a}(a + 2\alpha + a\delta) \\
&\geq \frac{a}{2 + a} > 0.
\end{aligned} \tag{5.17}$$

Thus, by Theorem 4.4,

$$H^i(X', \mathcal{I}_{X''} \otimes \mathcal{O}_{X'}(D)) = 0 \tag{5.18}$$

for every $i > 0$, where X'' is the union of the non-qlc locus of $[X', ((K_X + \Delta)|_{X_0} + cB)|_{X'} + c'P]$ and P, and $\mathcal{I}_{X''}$ is the defining ideal sheaf of X'' on X'. Thus, we have that

$$H^0(X', \mathcal{O}_{X'}(D)) \to \mathcal{O}_{X'}(D) \otimes \mathcal{O}_{X'}/\mathcal{I}_{X''} \tag{5.19}$$

is surjective. Note that P is isolated in $\mathrm{Supp}\, \mathcal{O}_{X'}/\mathcal{I}_{X''}$. Therefore, we obtain surjections

$$\begin{aligned}
H^0(X, \mathcal{O}_X(D)) &\twoheadrightarrow H^0(X_0, \mathcal{O}_{X_0}(D)) \\
&\twoheadrightarrow H^0(X', \mathcal{O}_{X'}(D)) \twoheadrightarrow \mathcal{O}_{X'}(D) \otimes \mathbb{C}(P)
\end{aligned} \tag{5.20}$$

by (5.9), (5.12), and (5.19). This is the desired surjection.

Finally, we further assume that P is a singular point of X.

Step 5. Note that (X, Δ) is klt in a neighborhood of P by assumption. We will reduce the problem to the situation as in Step 4. Let $\pi : Y \to X$ be a minimal resolution of P. We put $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$. Since $\mathrm{Bs}\,|\pi^* D| = \pi^{-1}\,\mathrm{Bs}\,|D|$, it is sufficient to prove that $Q \not\subset \mathrm{Bs}\,|\pi^* D|$ for some $Q \in \pi^{-1}(P)$. Without loss of generality, we may assume that $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$ factors through $[Y, K_Y + \Delta_Y]$ and assume that $(Z, \Delta_Z) \to [Y, K_Y + \Delta_Y]$ induces a natural quasi-log structure compatible with the original semi-log canonical structure of (Y, Δ_Y) (see Step 1 and [Fuj5, Theorem 1.2]). We put $Y_0 = \pi^{-1}(X_0)$ where $P \in X_0$ as in Step 4. We can take an effective $\mathbb{R}$-Cartier divisor B' on Y_0 such that $B' \sim_{\mathbb{R}} (\pi|_{Y_0})^* A_0$, $\mathrm{mult}_Q B' > 2$ for some $Q \in \pi^{-1}(P)$, and $B' = (\pi|_{Y_0})^* B$ for some effective $\mathbb{R}$-Cartier divisor B on X_0. We put $U' = Y_0 \setminus \mathrm{Sing}\, Y_0$. We set

$$c = \sup_{t \geq 0} \left\{ t \;\middle|\; \begin{array}{l} (U', (\Delta_Y)|_{U'} + tB'|_{U'}) \text{ is log canonical} \\ \text{at any point of } \pi^{-1}(P). \end{array} \right\}. \tag{5.21}$$

Then we have $0 < c < 1$. By adjunction (see Theorem 4.4) and Lemma 4.6, we can consider a quasi-log structure of $[Y_0, (K_Y + \Delta_Y)|_{Y_0} + cB']$. If there is a one-dimensional qlc center C of $[Y_0, (K_Y + \Delta_Y)|_{Y_0} + cB']$ such that

$$(\pi^* D - ((K_Y + \Delta_Y)|_{Y_0} + cB')) \cdot C = (1 - c)(\pi|_{Y_0})^* A_0 \cdot C = 0. \tag{5.22}$$

Then we obtain that $C \subset \pi^{-1}(P)$. This means that P is a qlc center of $[X_0, (K_X + \Delta)|_{X_0} + cB]$. In this case, we obtain surjections

$$H^0(X, \mathcal{O}_X(D)) \twoheadrightarrow H^0(X_0, \mathcal{O}_{X_0}(D)) \twoheadrightarrow \mathcal{O}_{X_0}(D) \otimes \mathbb{C}(P) \tag{5.23}$$

as in Case 1 in Step 4 (see (5.9) and (5.13)). Therefore, we may assume that

$$(\pi^* D - ((K_Y + \Delta_Y)|_{Y_0} + cB')) \cdot C > 0 \tag{5.24}$$

for every one-dimensional qlc center C of $[Y_0, (K_Y + \Delta_Y)|_{Y_0} + cB']$. Note that

$$(\pi^* D - (K_Y + \Delta_Y)) \cdot C = (D - (K_X + \Delta)) \cdot \pi_* C = A \cdot \pi_* C \geq 2 \tag{5.25}$$

when $\pi_* C \neq 0$, equivalently, C is not a component of $\pi^{-1}(P)$. Then we can apply the arguments in Step 4 to $[Y_0, (K_Y + \Delta_Y)|_{Y_0} + cB']$ and $\pi^* D$. Thus, we obtain that $Q \not\subset \mathrm{Bs}\,|\pi^* D|$ for some $Q \in \pi^{-1}(P)$. This means that $P \not\subset \mathrm{Bs}\,|D|$.

Anyway, we obtain that $P \not\subset \mathrm{Bs}\,|D|$. □

By Theorem 5.1, we can quickly prove Corollary 1.5 as follows.

Proof of Corollary 1.5. We put $D = mI(K_X+\Delta)$ and $A = D-(K_X+\Delta) = (m-1/I)I(K_X+\Delta)$. Then we obtain that $A\cdot C \geq m-1/I$ for every curve C on X and that $(A^2\cdot X_i) \geq (m-1/I)^2$ for every irreducible component X_i of X. By Theorem 5.1, we obtain the desired freeness of $|mI(K_X+\Delta)|$. □

Remark 5.3. In Corollary 1.5, Δ is not necessarily reduced. If Δ is reduced, then Corollary 1.5 is a special case of [LR, Theorem 24]. We note that Δ is always assumed to be reduced in [LR].

As a special case of Corollary 1.5, we can recover Kodaira's celebrated result (see [Kod]). We state it explicitly for the reader's convenience.

Corollary 5.4 (Kodaira)**.** *Let X be a smooth projective surface such that K_X is nef and big. Then $|mK_X|$ is basepoint-free for every $m \geq 4$.*

Proof of Corollary 5.4. Apply Corollary 1.5 to the canonical model of X. Then we obtain the desired freeness. □

We close this section with the proof of Corollary 1.6.

Proof of Corollary 1.6. We put $D = -mI(K_X+\Delta)$ and $A = D-(K_X+\Delta) = -(m+1/I)I(K_X+\Delta)$. Then we obtain that $A\cdot C \geq m+1/I$ for every curve C on X and that $(A^2\cdot X_i) \geq (m+1/I)^2$ for every irreducible component X_i of X. By Theorem 5.1, we obtain the desired freeness of $|-mI(K_X+\Delta)|$. □

6. Log surfaces

In this section, we prove Theorem 1.7.

Proof of Theorem 1.7. The proof is essentially the same as that of Theorem 5.1. However, there are some technical differences. We will have to use Theorem 4.5 instead of Theorem 4.4 (ii). So, we describe it for the reader's convenience.

Step 1. We take a resolution of singularities $f : Z \to X$ such that $\operatorname{Supp} f_*^{-1}\Delta \cup \operatorname{Exc}(f)$ is a simple normal crossing divisor on Z, where $\operatorname{Exc}(f)$ is the exceptional locus of f. We put $K_Z+\Delta_Z = f^*(K_X+\Delta)$. Then, (Z,Δ_Z) gives a natural quasi-log structure on $[X, K_X+\Delta]$.

Step 2. Assume that (X,Δ) is not log canonical at x. We put

$$X' = \operatorname{Nlc}(X,\Delta) \cup \bigcup W, \qquad (6.1)$$

where W runs over the one-dimensional log canonical centers of (X, Δ) such that $A \cdot W = 0$. Then, by Theorem 4.5, we obtain

$$H^i(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0 \tag{6.2}$$

for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X. Note that x is isolated in $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}_{X'}$. Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(x) \tag{6.3}$$

is surjective. Thus, we obtain $x \not\subset \operatorname{Bs}|D|$.

From now on, we may assume that (X, Δ) is log canonical at x.

Step 3. Assume that x is a zero-dimensional log canonical center of (X, Δ). We put

$$X' = \operatorname{Nlc}(X, \Delta) \cup \bigcup W \cup \{x\}, \tag{6.4}$$

where W runs over the one-dimensional log canonical centers of (X, Δ) such that $A \cdot W = 0$. Then, by Theorem 4.5, we obtain

$$H^i(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0 \tag{6.5}$$

for every $i > 0$. Note that x is isolated in $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}_{X'}$. Therefore, we obtain $x \not\subset \operatorname{Bs}|D|$ as in Step 2.

From now on, we may assume that (X, Δ) is plt at x.

Step 4. Assume that (X, Δ) is plt but is not klt at x. Let L be the unique one-dimensional log canonical center of (X, Δ) passing through x. We put

$$X' = \operatorname{Nlc}(X, \Delta) \cup \bigcup W \cup L \tag{6.6}$$

where W runs over the one-dimensional log canonical centers of (X, Δ) such that $A \cdot W = 0$. By Theorem 4.5, we obtain that

$$H^i(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0 \tag{6.7}$$

for every $i > 0$, as usual. Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to H^0(X', \mathcal{O}_{X'}(D)) \tag{6.8}$$

is surjective. By adjunction (see Theorem 4.4), $[X', (K_X + \Delta)|_{X'}]$ has a quasi-log structure induced by the quasi-log structure $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$ constructed in Step 1. Let $g : (Z', \Delta_{Z'}) \to [X', (K_X + \Delta)|_{X'}]$ be the induced quasi-log resolution. We put

$$c = \sup_{t \geq 0} \left\{ t \,\middle|\, \begin{array}{l} \text{the normalization of } (Z', \Delta_{Z'} + tg^*x) \text{ is sub} \\ \text{log canonical over } X' \setminus \operatorname{Nqlc}((K_X + \Delta)|_{X'}). \end{array} \right\}. \tag{6.9}$$

Then, by [Fuj7, Lemma 3.16], we obtain that $0 < c < 2$. Note that x is a Cartier divisor on X'. Let us consider $g : (Z', \Delta_{Z'} + cg^*x) \to [X', (K_X + \Delta)|_{X'} + cx]$, which defines a quasi-log structure. Then, by construction, x is a qlc center of $[X', (K_X + \Delta)|_{X'} + cx]$. Moreover, we see that

$$\deg(D|_L - (K_X + \Delta)|_L - cx) = (A \cdot L) - c > 0 \tag{6.10}$$

by assumption. We put

$$X'' = \mathrm{Nqlc}(X', (K_X + \Delta)|_{X'} + cx) \cup \bigcup W \cup \{x\}, \tag{6.11}$$

where W runs over the one-dimensional qlc centers of $[X', (K_X+\Delta)|_{X'}+cx]$ such that $W \neq L$. Then, by Theorem 4.5, we obtain

$$H^i(X', \mathcal{I}_{X''} \otimes \mathcal{O}_{X'}(D)) = 0 \tag{6.12}$$

for every $i > 0$. Note that x is isolated in $\mathrm{Supp}\, \mathcal{O}_{X'}/\mathcal{I}_{X''}$. Therefore, the restriction map

$$H^0(X', \mathcal{O}_{X'}(D)) \to \mathcal{O}_{X'}(D) \otimes \mathbb{C}(x) \tag{6.13}$$

is surjective. By combining (6.8) with (6.13), the desired restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(x) \tag{6.14}$$

is surjective. This means that $x \not\subset \mathrm{Bs}\,|D|$.

Thus, from now on, we may assume that (X, Δ) is klt at x.

Step 5. In this step, we assume that x is a smooth point of X. Since $A^2 > 4$, we can find an effective $\mathbb{R}$-Cartier divisor B on X such that $\mathrm{mult}_x B > 2$ and that $B \sim_{\mathbb{R}} A$. We put

$$c = \max\{t \geq 0 \,|\, (X, \Delta + tB) \text{ is log canonical at } x.\}. \tag{6.15}$$

Then we obtain that $0 < c < 1$ since $\mathrm{mult}_x B > 2$. We have a natural quasi-log structure on $[X, K_X + \Delta + cB]$ as in Step 1. By construction, there is a log canonical center of $[X, K_X + \Delta + cB]$ passing through x. We put

$$X' = \mathrm{Nlc}(X, \Delta + cB) \cup \bigcup W \cup W_0, \tag{6.16}$$

where W_0 is the minimal log canonical center of $(X, \Delta + cB)$ passing through x and W runs over the one-dimensional log canonical centers of $(X, \Delta+cB)$ such that $A \cdot W = 0$. We note that $D-(K_X+\Delta+cB) \sim_{\mathbb{R}} (1-c)A$. Then, by Theorem 4.5,

$$H^i(X, \mathcal{I}_{X'} \otimes \mathcal{O}_X(D)) = 0 \tag{6.17}$$

for every $i > 0$, where $\mathcal{I}_{X'}$ is the defining ideal sheaf of X' on X.

Case 1. If $\dim_x X' = 0$, then x is isolated in $\operatorname{Supp} \mathcal{O}_X/\mathcal{I}_{X'}$. Therefore, the restriction map

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(x) \tag{6.18}$$

is surjective. Thus, we obtain that $x \not\subset \operatorname{Bs}|D|$.

Case 2. If $\dim_x X' = 1$, then $(X, \Delta + cB)$ is plt at x. We write $\Delta + cB = L + \Delta'$, where L is the unique one-dimensional log canonical center of (X, Δ) passing through x and $\Delta' = \Delta + cB - L$. We put

$$c' = 1 - \operatorname{mult}_x(\Delta'|_L). \tag{6.19}$$

Then $[X', (K_X + \Delta + cB)|_{X'} + c'x]$ has a quasi-log structure such that x is a qlc center of this quasi-log structure as in Case 2 in Step 4 in the proof of Theorem 5.1. We put

$$X'' = \operatorname{Nqlc}(X', (K_X + \Delta + cB)|_{X'} + c'x) \cup \bigcup W \cup \{x\}, \tag{6.20}$$

where W runs over the one-dimensional qlc centers of $[X', (K_X + \Delta + cB)|_{X'} + c'x]$ such that $W \neq L$. By (5.17) in the proof of Theorem 5.1, we obtain that

$$\deg(D|_L - (K_X + \Delta + cB)|_L - c'x) > 0. \tag{6.21}$$

Then, by (6.21) and Theorem 4.5,

$$H^i(X', \mathcal{I}_{X''} \otimes \mathcal{O}_{X'}(D)) = 0 \tag{6.22}$$

for every $i > 0$, where $\mathcal{I}_{X''}$ is the defining ideal sheaf of X'' on X'. Thus, we have that

$$H^0(X', \mathcal{O}_{X'}(D)) \to \mathcal{O}_{X'}(D) \otimes \mathcal{O}_{X'}/\mathcal{I}_{X''} \tag{6.23}$$

is surjective. Note that x is isolated in $\operatorname{Supp} \mathcal{O}_{X'}/\mathcal{I}_{X''}$. Therefore, we obtain surjections

$$H^0(X, \mathcal{O}_X(D)) \twoheadrightarrow H^0(X', \mathcal{O}_{X'}(D)) \twoheadrightarrow \mathcal{O}_{X'}(D) \otimes \mathbb{C}(x) \tag{6.24}$$

by (6.17) and (6.23). This is the desired surjection.

Finally, we further assume that x is a singular point of X.

Step 6. Let $\pi : Y \to X$ be a minimal resolution of P. We put $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$. Since $\operatorname{Bs}|\pi^* D| = \pi^{-1} \operatorname{Bs}|D|$, it is sufficient to prove that $y \not\subset \operatorname{Bs}|\pi^* D|$ for some $y \in \pi^{-1}(x)$. Without loss of generality, we may assume that $f : (Z, \Delta_Z) \to [X, K_X + \Delta]$ factors through $[Y, K_Y + \Delta_Y]$ and assume that $(Z, \Delta_Z) \to [Y, K_Y + \Delta_Y]$ induces a natural quasi-log structure on $[Y, K_Y + \Delta_Y]$. We can take an effective $\mathbb{R}$-Cartier divisor B' on Y such that $B' \sim_{\mathbb{R}} \pi^* A$, $\operatorname{mult}_y B' > 2$ for some

$y \in \pi^{-1}(x)$, and $B' = \pi^* B$ for some effective $\mathbb{R}$-Cartier divisor B on X. We set

$$c = \sup_{t \geq 0} \left\{ t \;\middle|\; \begin{array}{l} (Y, \Delta_Y + tB') \text{ is log canonical} \\ \text{at any point of } \pi^{-1}(x). \end{array} \right\}. \tag{6.25}$$

Then we have $0 < c < 1$. As in Step 1, we can consider a natural quasi-log structure of $[Y, K_Y + \Delta_Y + cB']$. If there is a one-dimensional qlc center C of $[Y, K_Y + \Delta_Y + cB']$ such that $C \cap \pi^{-1}(x) \neq \emptyset$ and that

$$(\pi^* D - (K_Y + \Delta_Y + cB')) \cdot C = (1-c)\pi^* A \cdot C = 0. \tag{6.26}$$

Then we obtain that $C \subset \pi^{-1}(x)$. This means that x is a qlc center of $[X, K_X + \Delta + cB]$. In this case, we have that

$$H^0(X, \mathcal{O}_X(D)) \to \mathcal{O}_X(D) \otimes \mathbb{C}(x) \tag{6.27}$$

is surjective as in Case 1 in Step 5. Therefore, we may assume that

$$(\pi^* D - (K_Y + \Delta_Y + cB')) \cdot C > 0 \tag{6.28}$$

for every one-dimensional qlc center C of $[Y, K_Y + \Delta_Y + cB']$ with $C \cap \pi^{-1}(x) \neq \emptyset$. We note that

$$(\pi^* D - (K_Y + \Delta_Y)) \cdot C = (D - (K_X + \Delta)) \cdot \pi_* C = A \cdot \pi_* C \geq 2. \tag{6.29}$$

Then we can apply the arguments in Step 5 to $[Y, K_Y + \Delta_Y + cB']$ and $\pi^* D$. Thus, we obtain that $y \not\subset \operatorname{Bs} |\pi^* D|$ for some $y \in \pi^{-1}(x)$. This means that $x \not\subset \operatorname{Bs} |D|$.

Anyway, we obtain that $x \not\subset \operatorname{Bs} |D|$. □

7. Effective very ampleness

In this section, we prove effective very ampleness theorems for stable pairs and semi-log canonical Fano varieties. This section is independent of the other sections.

The statement and the proof of [Kol, 1.2 Lemma] do not seem to be true as stated. János Kollár and the author think that we need some modifications. So, we prepare the following lemma.

Lemma 7.1. *Let (X, Δ) be a projective semi-log canonical pair with $\dim X = n$. Let D be an ample Cartier divisor on X such that $|D|$ is basepoint-free. Assume that $L = D - (K_X + \Delta)$ is nef and log big with respect to (X, Δ), that is, L is nef and $L|_W$ is big for every slc stratum W of (X, Δ). Then $(n+1)D$ is very ample.*

We give a detailed proof of Lemma 7.1 for the reader's convenience.

Proof. By the vanishing theorem (see [Fuj5, Theorem 1.10]), we obtain that $H^i(X, \mathcal{O}_X((n+1-i)D)) = 0$ for every $i > 0$. Then, by the Castelnuovo–Mumford regularity, we see that

$$H^0(X, \mathcal{O}_X(D)) \otimes H^0(X, \mathcal{O}_X(mD)) \to H^0(X, \mathcal{O}_X((m+1)D)) \tag{7.1}$$

is surjective for every $m \geq n+1$ (see, for example, [Kle, Chapter II. Proposition 1]). Therefore, we obtain that

$$\mathrm{Sym}^k H^0(X, \mathcal{O}_X((n+1)D)) \to H^0(X, \mathcal{O}_X(k(n+1)D)) \tag{7.2}$$

is surjective for every $k \geq 1$. We put $A = (n+1)D$ and consider $f = \Phi_{|A|} : X \to Y$. Then there is a very ample Cartier divisor H on Y such that $A \sim f^*H$. By construction and the surjection (7.2), we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^k H^0(Y, \mathcal{O}_Y(H)) & \twoheadrightarrow & \mathrm{Sym}^k H^0(X, \mathcal{O}_X(A)) \\ \downarrow & & \downarrow\!\!\!\downarrow \\ H^0(Y, \mathcal{O}_Y(kH)) & \hookrightarrow & H^0(X, \mathcal{O}_X(kA)) \end{array} \tag{7.3}$$

for every $k \geq 1$. This implies that $H^0(Y, \mathcal{O}_Y(kH)) \simeq H^0(X, \mathcal{O}_X(kA))$ for every $k \geq 1$. Note that $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$ by

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \delta \to 0 \tag{7.4}$$

and

$$\begin{aligned} &0 \to H^0(Y, \mathcal{O}_Y(kH)) \to H^0(X, \mathcal{O}_X(kA)) \\ &\quad \to H^0(Y, \delta \otimes \mathcal{O}_Y(kH)) \to H^1(Y, \mathcal{O}_Y(kH)) \to \cdots \end{aligned} \tag{7.5}$$

for $k \gg 0$. By the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle \Phi_{|kA|}} \searrow & \downarrow {\scriptstyle \Phi_{|kH|}} \\ & & \mathbb{P}^N, \end{array} \tag{7.6}$$

where k is a sufficiently large positive integer such that kA and kH are very ample, we obtain that f is an isomorphism. This means that $A = (n+1)D$ is very ample. □

As an easy application of Lemma 7.1 and [Fuj5, Theorem 6.3], we have:

Corollary 7.2 (Stable pairs)**.** *Let (X, Δ) be a projective semi-log canonical pair such that $K_X + \Delta$ is ample. Assume that $I(K_X+\Delta)$ is Cartier for some positive integer I. Then there exists a positive integer N depending only on* $\dim X$ *such that $NI(K_X+\Delta)$ is very ample.*

Remark 7.3. In Corollary 7.2, we can choose $N = 2^{n+1}(n+2)!(n+1)$, where $n = \dim X$. When $\dim X = 2$, we can take $N = 12$ by Corollary 1.5. Moreover, by Corollary 1.5, we can take $N = 9$ when $\dim X = 2$ and $I \geq 2$. For some related results, see [LR].

Proof of Corollary 7.2. We put $L = I(K_X + \Delta)$. Then $2L - (K_X + \Delta)$ is always ample. By [Fuj5, Theorem 6.3], we obtain that $|mL|$ is basepoint-free, where $m = 2^{n+1}(n+1)!(2+n)$. We put $D = mL$ and apply Lemma 7.1. Then $NI(K_X + \Delta)$ is very ample, where $N = (n+1)2^{n+1}(n+1)!(2+n) = 2^{n+1}(n+2)!(n+1)$. □

We also have:

Corollary 7.4 (Semi-log canonical Fano varieties)**.** *Let* (X, Δ) *be a projective semi-log canonical pair such that* $-(K_X + \Delta)$ *is ample. Assume that* $I(K_X + \Delta)$ *is Cartier for some positive integer* I*. Then there exists a positive integer* N *depending only on* $\dim X$ *such that* $-NI(K_X + \Delta)$ *is very ample.*

Remark 7.5. In Corollary 7.4, we can choose $N = 2^{n+1}(n+1)^3 n!$, where $n = \dim X$. By Corollary 1.6, we can take $N = 6$ when $\dim X = 2$.

Proof of Corollary 7.4. We put $L = -I(K_X + \Delta)$. Then $L - (K_X + \Delta)$ is always ample. By [Fuj5, Theorem 6.3], we obtain that $|mL|$ is basepoint-free, where $m = 2^{n+1}(n+1)!(1+n)$. We put $D = mL$ and apply Lemma 7.1. Then $-NI(K_X + \Delta)$ is very ample, where $N = (n+1)2^{n+1}(n+1)!(1+n) = 2^{n+1}(n+1)^3 n!$. □

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