Recent developments in minimal model theory

Osamu Fujino

Contents

1 Introduction **2**
1.1 The main theorems and their corollaries 4

2 Minimal model theory **6**
2.1 Classical minimal model theory 6
2.2 Terminology and preliminaries 9
2.3 Log minimal model theory 12
2.4 MMP with scaling 14

3 The existence of pl flips **16**

4 Multiplier ideal sheaves **20**
4.1 Multiplier ideal sheaves and applications 20
4.2 The injectivity and vanishing theorems 23

5 The existence of minimal models **24**
5.1 The existence of pl flips 24
5.2 Special termination 25
5.3 Construction of minimal models 26
5.4 Termination of MMP with scaling 27
5.5 Finite number of minimal models 28
5.6 What is still missing 29

6 What is actually proved **30**

7 Problems for the future **31**
7.1 Termination of flips 31
7.2 Minimal model theory for log canonical pairs 33
7.3 The abundance conjecture 35

8 Updates and addenda added in proof **38**
8.1 Update of December 2007 38
8.2 Author's Addendum, December 2009 39

References **39**

1 Introduction

The birational classification of algebraic varieties is a central problem in algebraic geometry. Starting with Riemann's theory of curves in the 19th century and the Italian school's theory of surfaces at the turn of the 20th century, passing through Kodaira's classification of complex analytic surfaces and the work of the Russian school under Shafarevich, a rather satisfactory classification was obtained for algebraic varieties in low dimensions. The first systematic attempt at a birational classification of algebraic varieties in dimension three and above was due to Iitaka [I1]; from the 1970s onwards, he introduced the notion of the Kodaira dimension of a general algebraic variety, thus taking the first step in the direction of birational classification. Iitaka's many contributions to the subject include the definition of log Kodaira dimension and his additivity conjecture for Kodaira dimension [I2]. These ideas can all be summarized as the Iitaka program.

From the 1980s, Mori introduced Mori theory, or minimal model theory as we call it from now on, and this has become the standard approach to birational classification theory. Building on techniques worked out in the course of his solution of the Hartshorne conjecture [M1], Mori proved his Cone Theorem [M2], that encodes information on birational maps between projective algebraic varieties. This epoch-making piece of work made clear the road that minimal model theory for higher dimensional varieties was to follow (compare [M5]). Following on from this, minimal model theory developed as a combination of a general cohomological theory based on Hironaka's resolution of singularities and the Kawamata–Viehweg vanishing theorem (a generalization of Kodaira vanishing, see Theorem 28), together with Mori's extremely detailed results on the classification of singularities. During the second half of the 1980s Mori [M4] succeeded in completing the construction of minimal models in three dimensions, and was awarded the Fields Medal in Kyoto in 1990. During the early 1990s the conjectures concerning minimal model theory in three dimensions were practically all settled in a satisfactory form.

The next problem to be considered was that of extending minimal model theory to higher dimensions. However, Mori's results in three dimensions depended in an essential way on a detailed classification of singularities [M3], [M4], so that the three dimensional methods do not extend as they stand to higher dimensions; the great breakthroughs were followed by a lull. Around 2000, Shokurov, who had contributed many ideas to minimal model theory continuously from its early stages, claimed to complete the construction of four dimensional minimal models [Sh4]. Shokurov's papers [Sh2]–[Sh4] are a treasure trove of ideas, but the difficulty of reading and understanding them is also something of a trademark. A 2002 seminar at the Cambridge Newton Institute was devoted to deciphering Shokurov [Sh4]; this produced the book [Book][1] and stimulated the rapid developments of recent years centered around the work of Hacon and M$^{\text{c}}$Kernan [HM3], and Birkar, Cascini, Hacon and M$^{\text{c}}$Kernan [BCHM]. Conjectures that until just a few years earlier had seemed impossible to resolve fell one after another. My purpose here is to give an introduction to some aspects of these grand developments.

The main cues for the current developments were the ideas of Shokurov over the last 20 years, combined with the ingenious method of Siu's extension theorem [Si1] based on the use of multiplier ideal sheaves. To cut a long story short, let me state at once one of the main results.

Theorem 1 ([BCHM]) *Let X be a nonsingular algebraic variety defined over the field of complex numbers. Then the canonical ring*

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated $\mathbb{C}$-graded algebra.

Here, of course, the dimension of X is arbitrary. The reader having a little experience in studying algebraic geometry should be in a position to appreciate the power of this theorem. In what follows we always consider varieties over the complex number field $\mathbb{C}$; we need the characteristic of the ground field to be zero to make free use of the resolution of singularities and vanishing theorems in cohomology.

The remainder of this introductory Section 1 discusses the main theorems and corollaries of [BCHM]. Section 2 explains the classical theory of minimal models, including log minimal models and minimal models with scaling. 2.2 summarizes the terminology that we need. Section 3 treats the problem of

[1]Footnote 1, p. 37

the existence of a special class of flips called pl flips; their existence is the main theorem of [HM3].

Section 4 changes track slightly to explain multiplier ideals, together with a number of results that are obtained by applying them; this topic forms the background to the recent developments in the theory of minimal models. In Section 5 we discuss the mechanisms of the proof of [BCHM]. Section 6 gathers together the results actually proved in [BCHM]. The final Section 7 discusses the state of play from here onwards and states a number of recent results related to minimal model theory.

1.1 The main theorems and their corollaries

To get started, we just state the main results of [BCHM], leaving the more detailed explanations of the content to appear gradually. We urge the reader who has difficulties understanding the assertions below to press on nevertheless; if the material gets really painful, please move on to 2.1.

Theorem 2 *Suppose that (X, Δ) is a Kawamata log terminal pair; in particular, we assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon X \to U$ be a proper morphism between quasiprojective varieties. Assume either that $K_X + \Delta$ is π-big, or that Δ is π-big and $K_X + \Delta$ is π-pseudoeffective. Then the following hold:*

(1) $K_X + \Delta$ has a log terminal model over U.

(2) If $K_X + \Delta$ is π-big then (X, Δ) has a log canonical model over U.

(3) If $K_X + \Delta$ is $\mathbb{Q}$-Cartier then $\bigoplus_{m\geq 0} \pi_*\mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor)$ is a finitely generated $\mathcal{O}_U$-algebra.

Everything we say is in general dimensions. Since we have stated Theorem 2 in a form that will be hard for a nonexpert to grasp, we explain its main corollaries before proceeding further. Corollary 3 follows easily from Theorem 2 by applying the negativity lemma.

Corollary 3 *Let X be a nonsingular projective variety of general type; that is, assume K_X is big. Then the following hold:*

(1) X has a minimal model. That is, there exists a projective variety X' birational to X such that X' has at worst $\mathbb{Q}$-factorial terminal singularities and $K_{X'}$ is nef.

(2) X has a canonical model. That is, there exists a projective variety X' birational to X such that X' has at worst canonical singularities and $K_{X'}$ is ample.

(3) The canonical ring

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is finitely generated.

In Corollary 3, only (1) needs to be proved, because then (2) and (3) follow from the base point free theorem. This is still a formidable result.

Corollary 4 *Let (X, Δ) be a Kawamata log terminal pair; suppose that X is a projective variety and Δ a $\mathbb{Q}$-divisor. Of course, we also assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then the log canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is finitely generated.

Note that in Corollary 4 we do not need to assume that $K_X + \Delta$ is big. The corollary can be proved by putting together the canonical bundle formula of Fujino and Mori [FM] with Theorem 2, (3). We repeat a particular case of this result for the reader's benefit.

Corollary 5 *Let X be a nonsingular projective variety. Then the canonical ring $\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$ is always finitely generated.*

For X of general type, the statement of Corollary 5 is given in [Si4].

Corollary 6 *Let (X, Δ) be a Kawamata log terminal pair and $\varphi\colon X \to W$ a flipping contraction for $K_X + \Delta$. Then the flip of φ exists.*

Using the results of [BCHM] one can also prove the following theorem. For the more precise statement see Kawamata [K9].

Theorem 7 ([K9]) *Let X and X' be projective varieties having at worst $\mathbb{Q}$-factorial terminal singularities, and suppose that X and X' are birationally equivalent. Assume also that K_X and $K_{X'}$ are both nef. Then a birational map $X \dashrightarrow X'$ is equal to a composite of a finite number of flops.*

The results of [BCHM] have many applications beyond what we discuss here; we mention here an application to moduli theory that follows from [BCHM] using results of Karu [Kr], Kollár [Ko2] and Keel and Mori [KeM].

Theorem 8 *Let $\mathcal{M}_H^{\mathrm{sm}}$ be the moduli functor that sends a scheme S to the set of isomorphism classes of families of stable smoothable polarized n-folds over S having Hilbert polynomial H. Then the coarse moduli space M_H^{sm} of $\mathcal{M}_H^{\mathrm{sm}}$ exists. Moreover, M_H^{sm} is a projective scheme.*

It may not be immediately clear just from reading the statement of the theorem, but this result is a complete generalization to general dimensions of the results surrounding the moduli space of curves of general type and its compactification. However, the construction techniques have nothing whatever to do with Mumford's methods in terms of geometric invariant theory. There are many further applications, but we leave these and press on for the moment.

2 Minimal model theory

2.1 Classical minimal model theory

We now explain minimal model theory in its classical guise. In what follows X is a normal algebraic variety and K_X its canonical divisor. We start by recalling the Cone Theorem. Please refer to 2.2 for the terminology.

Theorem 9 (Cone Theorem) *Let X be a projective algebraic variety with at worst terminal singularities. Then the following hold.*

(1) There exists an at most countably infinite set of rational curves $C_j \subset X$ satisfying $0 < -K_X \cdot C_j \le 2\dim X$ and such that

$$\overline{\mathrm{NE}}\, X = \bigl(\overline{\mathrm{NE}}\, X\bigr)_{K_X \ge 0} + \sum \mathbb{R}_{\ge 0}[C_j].$$

Here $\overline{\mathrm{NE}}\, X$ is the Kleiman–Mori cone of X and $\bigl(\overline{\mathrm{NE}}\, X\bigr)_{K_X \ge 0}$ the part of $\overline{\mathrm{NE}}\, X$ on which K_X is nonnegative.

(2) Let $R \subset \overline{\mathrm{NE}}\, X$ be a K_X-negative extremal ray. Then there exists a unique morphism $\varphi_R \colon X \to Z$ of X to a projective variety Z satisfying $\varphi_{R*}\mathcal{O}_X = \mathcal{O}_Z$ and such that for a curve $C \subset X$

$$\varphi_R(C) = \text{point} \iff [C] \in R.$$

Moreover, $\rho(X) - \rho(Z) = \rho(X/Z) = 1$.

Theorem 9, (2) is often referred to as the Contraction Theorem. For further details, please refer to Kawamata, Matsuda and Matsuki [KMM] (the bible of the early period of minimal model theory), as well as the textbooks Kollár and Mori [KM] and Matsuki [Ma2].

The estimate $0 < -K_X \cdot C_j \leq 2 \dim X$ for the length of an extremal ray is due to Kawamata [K4], based on Miyaoka and Mori [MM]. Thus this part of the argument needs Mori's technique of reduction to positive characteristic. This estimate on the length of an extremal ray will play an important role in the theory of minimal models with scaling that we will explain presently.

Having said this, we can now explain classical minimal model theory. Suppose that X is a projective variety having only $\mathbb{Q}$-factorial terminal singularities. The idea of minimal model theory is to construct a good model of X starting out from $X_0 = X$. In a little more detail, suppose that we have constructed a projective variety X_i birational to X and having at worst $\mathbb{Q}$-factorial terminal singularities. If K_{X_i} is nef then we set $X^* = X_i$, and say that X^* is a minimal model of X. If K_{X_i} is not nef then there exists a K_{X_i}-negative extremal ray R of $\overline{\mathrm{NE}}\, X_i$. Consider the corresponding contraction morphism $\varphi_R \colon X_i \to Y$. If φ_R is not birational, then we again set $X^* = X_i$, and say that X^* is a *Mori fiber space.* From now on, assume that φ_R is birational.

(1) If φ_R contracts a divisor of X_i, we say that φ_R is a *divisorial contraction.* In this case we set $X_{i+1} = Y$ and return to the start.

(2) If φ_R is an isomorphism in codimension 1, we say that φ_R is a *flipping contraction.* In this case, if a flip

$$\begin{tikzcd} X_i \arrow[rr, dashed] \arrow[dr] & & X_i^+ \arrow[dl] \\ & Y & \end{tikzcd}$$

exists, then we set $X_{i+1} = X_i^+$ and return to the starting point. Flips are explained in detail below.

In either of the two cases (1) and (2), one sees that X_{i+1} is a projective variety with at worst $\mathbb{Q}$-factorial terminal singularities. For case (1) of divisorial contraction, one sees at once by arguing on a count of the Picard number that a divisorial contraction can only happen a finite number of times. Therefore, assuming that the following two conjectures can be solved, it follows that after a finite number of steps one obtains either a minimal model or a Mori fiber space birational to X.

Conjecture 10 (Flip Conjecture I: Existence of flips) *Suppose that $\varphi\colon X \to W$ is a flipping contraction. In other words, assume that*

(1) φ is a projective birational morphism, and is an isomorphism in codimension 1*.*

(2) $-K_X$ is φ-ample.

(3) X has at worst $\mathbb{Q}$-factorial terminal singularities, and has relative Picard number $\rho(X/W) = 1$.

Then there exists a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] \\ & Y & \end{tikzcd}$$

such that

(i) X^+ is a normal variety, and is projective over W.

(ii) $\varphi^+\colon X^+ \to W$ is a birational morphism, and is an isomorphism in codimension 1*.*

(iii) K_{X^+} is φ^+-ample.

Then $\varphi^+\colon X^+ \to W$ is called the flip *of $\varphi\colon X \to W$.*

Conjecture 11 (Flip Conjecture II: Termination) *Any chain*

$$\begin{tikzcd} X_0 \arrow[rr, dashed] \arrow[dr] & & X_1 \arrow[dl] \arrow[rr, dashed] \arrow[dr] & & X_2 \arrow[dl] \arrow[r, dashed] & \cdots \\ & W_0 & & W_1 & & \end{tikzcd}$$

of flips necessarily terminates after a finite number of steps. To put it another way, there does not exist any infinite sequence of flips.

In two dimensions, there are no terminal singularities other than the nonsingular points, and no flipping contractions. It follows from this that every step of the minimal model theory simply contracts a -1-curve. For 3-folds, the above two conjectures were solved during the 1980s. Shokurov [Sh1] gave an elegant solution to Flip Conjecture II, the termination, by introducing his notion of *difficulty*; his proof, consisting of just a single marvelous idea, was extremely simple. Flip Conjecture I, the existence, was solved by Mori [M4]. Mori's proof consisted of a detailed and exhaustive analysis of the form of an analytic neighborhood of the curves contracted by a flipping contractions $\varphi_R\colon X \to W$; in a formidable piece of work, he gave separate arguments to confirm the existence of the flip in each case.

Be that as it may, in the first nontrivial case of 3-folds, the theory of minimal models was completed within 10 years of its first appearance. The 1980s was a golden age for minimal model theory. For an overall view and the early development of the theory of minimal models and their classification, we strongly recommend reading Kawamata's survey articles [K10] and [K11] as an introduction.

The first early results on the 4-dimensional case of Flip Conjecture I were obtained by Kawamata [K3], after which the only results were by Kachi [Kc1], [Kc2] and Takagi [Tk1], [Tk3]. The first paper to relate the extension theorem (see Siu [Si1], Nakayama [N2], Kawamata [K8]) to the construction of flips was Takagi [Tk3], possibly a result in advance of its time.

2.2 Terminology and preliminaries

We now spell out some of the terminology used so far without explanation, and also carry out some necessary preliminaries before proceeding to more advanced topics.

1. Divisors A $\mathbb{Q}$-divisor or $\mathbb{R}$-divisor is a formal finite sum $D = \sum d_i D_i$ of prime divisors D_i with rational coefficients d_i (respectively, real). We can define the *round-down* $\lfloor D \rfloor$ of D by taking the integral part of each coefficient d_i; the fractional part is $\{D\} = D - \lfloor D \rfloor$. The *round-up* of D is $\lceil D \rceil = -\lfloor -D \rfloor$. A divisor D is $\mathbb{Q}$-*Cartier* (or $\mathbb{R}$-*Cartier*) if it can be written as a linear combination of Cartier divisors with coefficients in $\mathbb{Q}$ (respectively $\mathbb{R}$).

We say that two $\mathbb{R}$-divisors D and D' are $\mathbb{R}$-linearly equivalent (or $\mathbb{Q}$-linearly equivalent, or linearly equivalent) if there exist finitely many rational functions $f_1, \dots, f_k$ on X and real numbers $r_1, \dots, r_k$ (or rational numbers, or integers) such that the difference $D - D'$ can be written in the form $D - D' = \sum_{i=1}^r k_i \operatorname{div}(f_i)$ (here $\operatorname{div}(f_i)$ is the principal divisor associated with f_i); we denote this linear equivalence by $D \sim_{\mathbb{R}} D'$ (or $D \sim_{\mathbb{Q}} D'$, or $D \sim D'$). The usual notions for a Cartier divisor of ample, semi-ample and nef can be defined in a similar way for $\mathbb{Q}$-Cartier and $\mathbb{R}$-Cartier divisors. (However, there are technical issues that are strangely awkward for $\mathbb{R}$-Cartier divisors.)

The introduction of $\mathbb{Q}$-divisors was an important component in the cohomological methods that led to remarkable progress in the minimal model theory of the 1980s; $\mathbb{R}$-divisors are also required for certain limiting procedures. We say that a normal variety X is $\mathbb{Q}$-*factorial* if every prime divisor on X is $\mathbb{Q}$-Cartier. This condition is more powerful than might appears at first sight and we assume it on many occasions as a magic invocation.

2. Discrepancy coefficients Let X be a normal variety and Δ an $\mathbb{R}$-divisor on X. Assume that $K_X+\Delta$ is an $\mathbb{R}$-Cartier divisor, and let $f\colon Y \to X$ be a birational morphism from a normal variety Y. We can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E,$$

where E runs over all prime divisors of Y; the real coefficient $a(E, X, \Delta)$ is called the *discrepancy* of E with respect to (X, Δ).

Next, let X be a normal variety such that K_X is a $\mathbb{Q}$-Cartier divisor. We say that X has *terminal singularities* (or *canonical singularities*) if for any birational morphism $f\colon Y \to X$ from a normal variety Y, and any f-exceptional prime divisor E we have $a(E, X, 0) > 0$ (respectively $a(E, X, 0) \geq 0$). For a surface X, terminal singularities are exactly the nonsingular points, and canonical singularities are at worst Du Val singularities. See Reid [R] for the situation for 3-fold singularities.

3. Cones Consider a proper morphism $f\colon X \to S$ between normal varieties X and S. A *relative* 1-*cycle* for f is a formal linear combination $C = \sum_j c_j C_j$ of curves C_j on X contracted by f to a single point $f(C_j)$ on S. We say that Cartier divisors D, D' on X are *numerically equivalent* over S if $(D \cdot C) = (D' \cdot C)$ for every relative 1-cycle C. Similarly, we say that relative 1-cycles C, C' are numerically equivalent if $(D \cdot C) = (D \cdot C')$ for every Cartier divisor D. We write $D \equiv D'$ respectively $C \equiv C'$ for numerical equivalence. This defines two dual finite dimensional real vector spaces

$$N^1(X/S) = \big(\{\text{Cartier divisors}\}/\equiv\big) \otimes_{\mathbb{Z}} \mathbb{R},$$
$$N_1(X/S) = \big(\{\text{relative 1-cycles}\}/\equiv\big) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The relative Picard number is defined by $\rho(X/S) = \dim_R N^1(X/S)$.

We write $\overline{\mathrm{NE}}(X/S) \subset N_1(X/S)$ for the closure of the convex cone spanned by the equivalence classes of curves C such that $f(C)$ is a point; this is the Kleiman–Mori cone. An $\mathbb{R}$-Cartier divisor D determines a linear functional h_D on $N_1(X/S)$ by $h_D(C) = (D \cdot C)$. We say that D is f-nef[2]

[2]Footnote 2, p. 37

if h_D is nonnegative on $N_1(X/S)$. When f is a projective morphism, the necessary and sufficient condition for D to be ample on X/S is that h_D be positive on $\overline{\mathrm{NE}}(X/S) \setminus \{0\}$. We write $\mathrm{PE}(X/S)$ for the closure of the cone spanned by the equivalence classes of effective Cartier divisor in $N^1(X/S)$. We say that an $\mathbb{R}$-Cartier divisor is *f-pseudoeffective* if its numerical equivalence class is in $\mathrm{PE}(X/S)$. If D is an $\mathbb{R}$-Cartier divisor whose numerical equivalence class is an interior point of $\mathrm{PE}(X/S)$, we say that D is *big*. When S is a single point, we abbreviate $N_1(X/S)$, $N^1(X/S)$, $\overline{\mathrm{NE}}(X/S)$, $\rho(X/S)$ by $N_1(X)$, $N^1(X)$, $\overline{\mathrm{NE}}(X)$, $\rho(X)$.

4. Pairs A pair (X, B) consisting of a normal variety X together with an $\mathbb{R}$-divisior B on X is *divisorially log terminal* (dlt) if all the coefficients of B are nonnegative real numbers ≤ 1, $K_X + B$ is an $\mathbb{R}$-Cartier divisor, and in addition, there exists a resolution of singularities having the following properties:

(a) $f\colon Y \to X$ is a proper birational morphism from a nonsingular variety;

(b) both $\mathrm{Exc}(f)$ and $\mathrm{Exc}(f) \cup f_*^{-1}(B)$ are simple normal crossing divisors where $f_*^{-1}B$ is the strict transform of B under f^{-1} and $\mathrm{Exc}(f)$ is the exceptional set of f;

(c) if we write
$$K_Y + B_Y = f^*(K_X + B) \quad \text{with} \quad -B_Y = \sum_j b_j B'_j,$$
then $b_j > -1$ for every j for which the component B'_j is exceptional, that is $B'_j \subset \mathrm{Exc}(f)$.

We say that (X, B) is Kawamata log terminal (klt) if (X, B) is dlt and $\lfloor B \rfloor = 0$. Also, we say that (X, B) is purely log terminal (plt) if (X, B) is dlt and $\lfloor B \rfloor$ is normal. Obviously from the definition, klt implies plt and plt implies dlt.

Suppose that X is a nonsingular variety, and that $\sum_i B_i$ is the irreducible decomposition of a normal crossing divisor. Then $(X, \sum_i b_i B_i)$ dlt (respectively, klt) is equivalent to $0 \leq b_i \leq 1$ (respectively $0 \leq b_i < 1$) for all i. Assuming that $(X, \sum_i b_i B_i)$ is dlt, it is equivalent to say that $\sum_{b_i=1} B_i$ is nonsingular or that $(X, \sum_i b_i B_i)$ is plt. The ability to distinguish between these notions of dlt, plt and klt and their usage is a prerequisite for becoming a specialist in minimal model theory. For details, please refer to [Ko4] and [F5].[3]

For later use, we introduce one final definition. Consider a pair (X, B) consisting of a normal variety X and an effective $\mathbb{R}$-divisor B on it such that $K_X + B$ is an $\mathbb{R}$-Cartier divisor. We say that (X, B) is a *log canonical pair* if $a(E, X, B) \geq -1$ for every birational morphism $f\colon Y \to X$ and for every prime divisor E of Y. One checks immediately that a dlt pair (X, B) is log canonical.

Moreover, we define a *log canonical center* in X to be the image under f of a prime divisor E for which $a(E, X, B) = -1$.

2.3 Log minimal model theory

Two extensions to the framework of minimal model theory were already in place well before Mori's proof of the existence of 3-fold flips: the extension to the relative case, which has long been standard in algebraic geometry, and the logarithmic case as influenced by the Iitaka program (see [KMM]). What we have said for classical minimal model theory works in exactly the same way for a $\mathbb{Q}$-factorial dlt pair (X, Δ) projective over a fixed algebraic variety S. The cone theorem and the Contraction Theorem hold for a relative $\mathbb{Q}$-factorial dlt pair (X, Δ), and the remaining problems are Conjectures 12 and 14 below. Once these two conjectures are resolved, as explained in the case of classical minimal model theory, one sees that, starting from a given dlt pair (X, Δ), after a finite number of operations, we obtain either a log minimal model or a log Mori fiber space. Passing to the log case and the relative case are not just generalizations for the sake of generalization, but arise in an unavoidable way in the solution of all kinds of problems by induction on the dimension and such-like. These two generalizations are important points; the proofs of [BCHM] cannot be carried out successfully without log pairs and the relative case.

Conjecture 12 (Conjecture I: existence of log flips) *Let $\varphi\colon X \to W$ be a flipping conjecture. That is,*

(1) φ is a projective birational morphism and is an isomorphism in codimension 1.

(2) $-(K_X + \Delta)$ is φ-ample.

[3] Footnote 3, p. 37

(3) X is $\mathbb{Q}$-factorial and (X, Δ) is a dlt pair; also, the relative Picard number $\rho(X/W) = 1$.

Then there exists the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] \\ & W & \end{tikzcd}$$

where

(i) X^+ is a normal variety, and is projective over W.

(ii) $\varphi^+ \colon X^+ \to W$ is a birational morphism and is an isomorphism in codimension 1.

(iii) $K_{X^+} + \Delta^+$ is φ^+-ample. Here Δ^+ is the strict transform of Δ.

The morphism $\varphi^+ \colon X^+ \to W$ is called the log flip *of $\varphi \colon X \to W$*

If (X^+, Δ^+) exists then one proves that it is divisorially log terminal and X^+ is $\mathbb{Q}$-factorial. The Δ appearing in Flip Conjecture I is a priori a general $\mathbb{R}$-divisor, but one may assume that it is a $\mathbb{Q}$-divisor by jiggling its coefficients.

We prepare the following well known result for subsequent use. As we have just said, in Proposition 13, it is enough to assume that Δ is a $\mathbb{Q}$-divisor.

Proposition 13 *The necessary and sufficient condition for the flip of a flipping contraction $\varphi \colon X \to W$ to exist is that the graded $\mathcal{O}_W$-algebra $\bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$ is finitely generated. When this holds, X^+ is given by*

$$X^+ = \bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor).$$

In particular, Flip Conjecture I is a local problem in W.

Conjecture 14 (Conjecture II: termination of log flips) *Any chain*

$$\begin{tikzcd} X_0 \arrow[rr, dashed] \arrow[dr] & & X_1 \arrow[dl] \arrow[rr, dashed] \arrow[dr] & & X_2 \arrow[dl] \arrow[r, dashed] & \cdots \\ & W_0 & & W_1 & & \end{tikzcd}$$

of log flips necessarily terminates after a finite number of steps. To put it another way, there does not exist any infinite sequence of log flips.

In Flip Conjecture II, we are not allowed to assume that Δ is a $\mathbb{Q}$-divisor. This point requires some care: whether Δ is a $\mathbb{Q}$-divisor or an $\mathbb{R}$-divisor makes a subtle difference to the difficulty of the problem.

In this discussion, as in the title log minimal model theory, the prefix log is sometimes included and often omitted. Flip Conjecture I is now completely understood in arbitrary dimension and for dlt pairs. For klt pairs, this follows from Corollary 6. Flip Conjecture II is not yet completely solved. The fact that infinite sequences of flips do not exist within a special framework is one of the main topics of [BCHM].

Before proceeding further, we give the rigorous definition of minimal model and log canonical model.

Definition 15 (Log minimal and canonical models) Let $\pi\colon X \to U$ be a projective morphism between quasiprojective varieties. Let (X, Δ) be a dlt pair and $\varphi\colon X \dashrightarrow Y$ a birational map over U such that φ^{-1} does not contract any divisor; here we set $\Gamma = \varphi_*\Delta$. If $K_Y + \Gamma$ is ample over U and $a(E, X, \Delta) \le a(E, Y, \Gamma)$ holds for every φ-exceptional divisor E then we say that Y is a *log canonical model* of (X, Δ).

We say that (Y, Γ) is a *log terminal model* of (X, Δ) if (Y, Γ) is $\mathbb{Q}$-factorial and dlt, $K_Y + \Gamma$ is nef over U and $a(E, X, \Delta) < a(E, Y, \Gamma)$ holds for every φ-exceptional divisor E. We also often call this simply a *(log) minimal model*.

Note that the minimal models that arise as a result of log minimal model theory are log minimal models in the sense of Definition 15.

The 3-fold log flip Conjecture I was proved by Shokurov [Sh2]. This paper was extremely difficult to read, and practically no-one has read it in fine detail. Takagi's paper [Tk2] is based on deciphering [Sh2]. The papers[4] [CK] and [Ko3] showed that the result of [Sh2] can be recovered by putting together the methods of the paper [K6] that solved the 3-fold log flip Conjecture II with Mori's *magnum opus* [M4]. Be that as it may, [Sh2] was a paper that introduced a wealth of new concepts and ideas into the world of minimal model theory.[5]

2.4 MMP with scaling

Here we explain the MMP with scaling. The idea itself was already present in [Sh2], but using it effectively was crucial to the success of [BCHM].

[4]Footnote 4, p. 37
[5]Footnote 5, p. 37

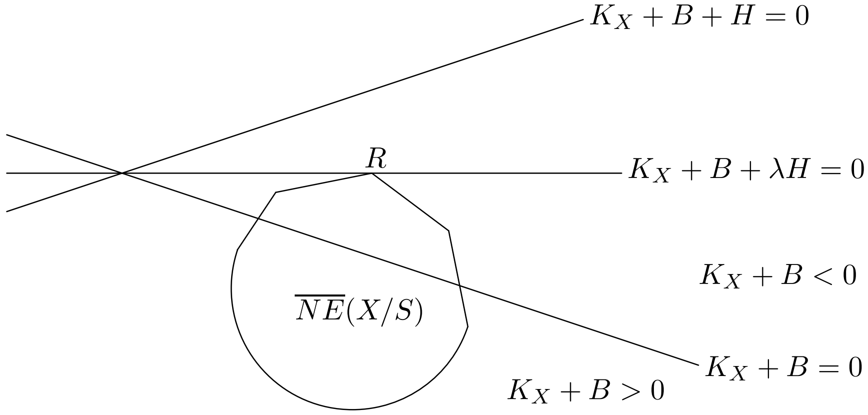

Figure 1: The choice of extremal ray in MMP with scaling.

Let $\pi \colon X \to U$ be a projective morphism between quasiprojective varieties. The reader averse to the relative case may take U to be a single point. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair. Suppose that we also have an effective $\mathbb{R}$-Cartier divisor C such that $K_X + \Delta + C$ is nef over U and $(X, \Delta + C)$ is dlt. If $K_X + \Delta$ is nef then (X, Δ) is itself a log terminal model over U. Thus we can assume that $K_X + \Delta$ is not nef. In this case, there exists a $(K_X + \Delta)$-negative extremal ray $R \subset \overline{\mathrm{NE}}(X/U)$ and a threshold value $\lambda \in \mathbb{R}$ such that $0 < \lambda \leq 1$ and $K_X + \Delta + \lambda C$ is nef but $(K_X + \Delta + \lambda C) \cdot R = 0$. The existence of this R follows from the bound on the length of extremal ray (the statement $\leq 2 \dim X$ of Theorem 9, (1)).

Consider the contraction morphism $\varphi_R \colon X \to Y$ associated with this R. If φ_R is not birational then $\varphi_R \colon (X, \Delta) \to Y$ is the thing that we call a log Mori fiber space. In what follows we assume that φ_R is birational. If $f = \varphi_R$ is a divisorial contraction then we replace X by Y, Δ by $f_*\Delta$ and C by $\lambda f_* C$. If φ_R is a flipping contraction then we apply the flip $X \dashrightarrow Y$, and replace X by X^+, Δ by its strict transform Δ^+ and C by λ times its strict transform λC^+.

After this, one sees that the new $K_X + \Delta + C$ is again nef over U and is $\mathbb{Q}$-factorial and dlt. Now we repeat the above procedure. In the final analysis, what the procedure is doing is just running an ordinary minimal model program for $K_X + \Delta$ over U, but we choose the $(K_X + \Delta)$-negative extremal ray R to be the ray in $N_1(X/U)$ in which the hyperplane $K_X + \Delta + \lambda C = 0$ touches the cone $\overline{\mathrm{NE}}(X/U)$ (see Figure 1). This conditional MMP is called the *MMP with scaling* by C; we run the MMP while successively decreasing

C.

In [BCHM] it is proved, assuming that (X, Δ) is klt and Δ is big, that MMP with scaling works in all dimension. Let us state this as a theorem.

Theorem 16 *Let $\pi\colon X \to U$ be a morphism between normal quasiprojective varieties. Suppose that (X, Δ) is $\mathbb{Q}$-factorial klt and Δ is big. If $K_X + \Delta + C$ is klt and π-nef then the MMP over U with scaling by C works. That is, the flips that are needed in the course of the MMP necessarily exist, and infinite sequences of flips do not occur.*

3 The existence of pl flips

Although it may seem somewhat abrupt, this section consists of commentary on Hacon and McKernan's proof of the existence of pl flips [HM3]; our explanation parallels [HM4]. We first recall the definition of pl flipping contraction, also introduced by Shokurov in [Sh2]. This notion is extremely cunning, and its advantages should gradually become apparent as we proceed through the following sections.

Definition 17 We say that a proper birational morphism $f\colon X \to Z$ between normal varieties is a pl flipping contraction if it satisfies the following conditions:

(1) f is an isomorphism in codimension 1, and has Picard number 1.

(2) X is $\mathbb{Q}$-factorial and Δ is a $\mathbb{Q}$-divisor.

(3) (X, Δ) is purely log terminal (plt) and $S = \lfloor \Delta \rfloor$ is irreducible.

(4) Both $-(K_X + \Delta)$ and $-S$ are f-ample.

The flip $f^+\colon X^+ \to Z$ of a pl flipping contraction $f\colon X \to Z$ is called a *pl flip*.

The main theorem of [HM3] is as follows

Theorem 18 *Assume that the MMP works in dimension $n-1$. That is, assume that Flip Conjectures I and II both hold in dimension $n-1$. Then n dimensional pl flips exist.*

We now outline the proof of this, following [HM4]. We first recall some definitions.

Definition 19 Let D be a divisor on a normal variety X, and assume that the linear system $|D|$ is nonempty. Then we write $F = \operatorname{Fix} D$ for the *fixed part* of $|D|$ and $\operatorname{Mob} D = D - F$ for its *mobile part*.

Since the problem is local in Z (see Proposition 13), in what follows we assume throughout that Z is affine. It will eventually be enough to prove that the $\mathcal{O}_Z$-graded algebra $\mathfrak{R} = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$ is finitely generated. Write $B = \{\Delta\}$ for the fractional part, and write $(K_X + S + B)_{|S} = K_S + B_S$.[6] Consider the restriction map

$$\rho\colon \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(m(K_X + S + B))) \to \bigoplus_{m\geq 0} H^0(S, \mathcal{O}_S(m(K_S + B_S))),$$

and write $\mathfrak{R}_S$ for the image of ρ. One sees easily that the finite generation of $\mathfrak{R}$ is equivalent to the finite generation of $\mathfrak{R}_S$; this fact follows easily from conditions (1) and (4) in the definition of pl flipping contraction. If ρ were surjective then one could show by induction on the dimension that $\mathfrak{R}_S$ is finitely generated. In general however we have no way of knowing whether ρ is surjective or otherwise.

At this point, the real proof is performed along the following main lines. First, consider a birational projective morphism $g\colon Y \to X$ from a nonsingular variety Y and set $K_Y + \Gamma = g^*(K_X + \Delta) + E$, where $g_*\Gamma = \Delta$ and E is an effective exceptional divisor. If we assume that $k(K_X + \Delta)$ is Cartier then

$$H^0(X, \mathcal{O}_X(mk(K_X + \Delta))) \simeq H^0(Y, \mathcal{O}_Y(mk(K_Y + \Gamma)))$$

holds for every positive integer m. We now set

$$G_m = \frac{1}{mk} \operatorname{Fix}(mk(K_Y + \Gamma)) \wedge \Gamma,$$

where $\wedge$ denotes the greatest common divisor. In other words, if

$$\frac{1}{mk} \operatorname{Fix}(mk(K_Y + \Gamma)) = \sum_j a_j D_j \quad \text{and} \quad \Gamma = \sum_j b_j D_j$$

we set $G_m = \sum_j \min\{a_j, b_j\} D_j$. Then

$$H^0(Y, \mathcal{O}_Y(mk(K_Y + \Gamma))) \simeq H^0(Y, \mathcal{O}_Y(mk(K_Y + \Gamma_m)))$$

[6] Footnote 6, p. 37

where we have set $\Gamma_m = \Gamma - G_m$. So far we haven't said anything beyond the obvious. Now by taking a suitable choice of $g\colon Y \to X$, we can arrange that the restriction map

$$H^0(Y, \mathcal{O}_Y(mK_Y + \Gamma_m))) \to H^0(T, \mathcal{O}_T(mK_T + \Theta_m)))$$

is surjective. Here T is the strict transform of S and $\Theta_m = (\Gamma_m - T)_{|T}$. At this point we make ingenious use of the extension theorem based on multiplier ideals that start with Siu's paper [Si1]; compare Theorem 20 below. Here we must observe that we must make an appropriate choice of $g\colon Y \to X$ depending on m. That is, Y depends on the index m. However, T can be chosen independently of m.

Theorem 20 (Extension theorem) *Let Y be a nonsingular variety and $T \subset Y$ a nonsingular divisor on Y. Suppose that $\pi\colon Y \to Z$ is a projective morphism to a normal affine variety Z. Let L be a Cartier divisor on Y and suppose that $L \sim_{\mathbb{Q}} m(K_Y + T + B)$ for a positive integer m. We make the following assumptions:*

(1) The support of $T + B$ is a simple normal crossing divisor, and T and B have no common components.

(2) B is an effective $\mathbb{Q}$-divisor with $\lfloor B \rfloor = 0$.

(3) We can write B in the form $B \sim_{\mathbb{Q}} A + C$, with A an ample $\mathbb{Q}$-divisor and C an effective $\mathbb{Q}$-divisor whose support does not contain T.

(4) There exists a positive integer p such that the base locus of $|pL|$ does not contain any log canonical center of $(Y, T + \lceil B \rceil)$.

Then the natural restriction map $H^0(Y, \mathcal{O}_Y(L)) \to H^0(T, \mathcal{O}_T(L))$ is surjective.

We make some observations. The case that we really need is when π is a birational morphism. In this case, assumptions (1–4) are easy to satisfy, and they do not present any problems in applications. At first sight they may seem completely artificial conditions, but one has to bear with them. One could say that minimal model theory was sadly deficient until the appearance of this type of theorem in Siu's paper [Si1] in the late 1990s. For more details on multiplier ideal sheaves, see Section 4.

To return to the point. If we set $\mathfrak{R}_T = \bigoplus_{m \geq 0} H^0(T, \mathcal{O}_T(mk(K_T + \Theta_m)))$, we see that $\mathfrak{R}_T = \mathfrak{R}_S$, so the question is to prove that $\mathfrak{R}_T$ is finitely generated. We choose suitable positive integers k and s and set $l = ks$, and consider $\mathfrak{R}_T(s) = \bigoplus_{m\geq 0} H^0(T, \mathcal{O}_T(ml(K_T + \Theta_m))$. Then $\mathfrak{R}_T(s)$ has the following properties:

(1) The inequality $i\Theta_i + j\Theta_j \leq (i+j)\Theta_{i+j}$ holds for every i, j; this condition is called *convexity*. In addition, the limit $\Theta = \lim\limits_{i\to\infty} \Theta_i$ exists; this condition is called boundedness.

(2) (T, Θ) is klt. In general Θ is an $\mathbb{R}$-divisor.[7]

(3) We set $M_m = \mathrm{Mob}(ml(K_T + \Theta_m))$ and $D_m = M_m/m$. Then the limit $D = \lim\limits_{m\to\infty} D_m$ is a semiample $\mathbb{R}$-divisor.

(4) There exists a $\mathbb{Q}$-divisor F on T that satisfies $\lceil F \rceil \geq 0$, and such that

$$\mathrm{Mob}\,\lceil jD_{is} + F\rceil \leq jD_{js}$$

holds for every $i \geq j \gg 0$.

We give additional commentary on these points one by one. (1) and (2) are fairly clear by construction. Condition (4), called *asymptotic saturation*, is one of the marvelous discoveries contained in [Sh4]. The involved notation in (4) makes it hard to understand, but the proof involves nothing more than Kawamata–Viehweg vanishing. (3) follows from the MMP in $n-1$ dimensions, using the finiteness of the set of minimal models. This point will turn up later (see 5.5), where we discuss it in more detail. Roughly speaking, it uses the fact that we can choose the minimal model of (T, Θ_m) independently of m. The problem of the finiteness of minimal models appears naturally here. Given (3) and (4), one sees that there exists a positive m_0 such that $D = D_{m_0}$. At this point we need some Diophantine approximation (see [K2]). In particular, D turns out to be a semiample $\mathbb{Q}$-divisor. Once this point is understood, the finite generation of $\bigoplus_{m\geq 0} H^0(T, \mathcal{O}_T(mD))$ implies the finite generation of $\mathfrak{R}_T(s)$. One needs a little argument for this, but we obtain the finite generation of $\mathfrak{R}_T = \mathfrak{R}_S$. Therefore this proves the existence of pl flips.

Finally, we examine the asymptotic saturation condition (4) for Y an affine curve. When we study the flip problem, Y has dimension ≥ 3, but the ideas become completely transparent in the case of curves. The inequality we consider is $\mathrm{Mob}\,\lceil jD_{is} + F\rceil \leq jD_{js}$, but let us set $s = 1$ for simplicity. Write $D_i = \sum d_{m,i}P_m$ and $F = \sum a_m P_m$; by assumption $a_m > -1$. Set

[7] Footnote 7, p. 37

$D = \lim_{i\to\infty} D_i = \sum d_m P_m$. The condition Mob $\lceil jD_i + F\rceil \le jD_j$ then just means that $\lceil jd_{m,i} + a_m\rceil \le jd_{m,j}$ for each m. Letting $i \to \infty$, we get

$$\lceil jd_m + a_m\rceil \le jd_{m,j} \le jd_m.$$

This holds for every j, and it follows that d_m is a rational number. We also see that $d_{m,j_0} = d_m$ for some j_0. Therefore $D_{j_0} = D$. The general case is slightly more involved, but the mechanism is the same.

4 Multiplier ideal sheaves

4.1 Multiplier ideal sheaves and applications

We now change the subject somewhat, to examine through the multiplier ideal sheaves used in the proof of the existence of pl flips, together with their applications. The notion of multiplier ideal was introduced by Nadel [Nd] in the course of studying Kähler–Einstein metrics on Fano manifolds. We note that the idea of multiplier ideals themselves was introduced in Kohn's study of the $\overline{\partial}$-Neumann problem, although the setup was different (see [Kh] and [Si3]). After this, multiplier ideals are applied systematically by Demailly, Siu and Tsuji in the problem of base points freedom of linear systems. Please consult [D2] for the definition of singular Hermitian metrics and their associated multiplier ideal sheaves. The majority of singular Hermitian metrics that are used in application to algebraic geometry are those associated to $\mathbb{Q}$-divisors. In this context, the following definition should be sufficient.

Definition 21 Let X be a nonsingular variety and D an effective $\mathbb{Q}$-divisor. Let $f\colon Y \to X$ be a proper birational morphism from a nonsingular algebraic variety such that $\operatorname{Supp}(f^*D) \cup \operatorname{Exc} f$ is a simple normal crossing divisor. Then the *multiplier ideal sheaf* associated to D is defined by

$$\mathcal{J}(D) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*D\rfloor) \subset \mathcal{O}_X,$$

where $K_{Y/X} = K_Y - f^*K_X$.

Since we are omitting all explanations of the analytic approach, we are unable to give any details, but generalizing the Hermitian metrics that appear in the Kodaira vanishing theorem to singular Hermitian metrics gives the result known as Nadel's vanishing theorem, and the Kawamata–Viehweg vanishing theorem is a particular case of this (see Theorem 28). If we restrict attention to the singular Hermitian metrics associated to a $\mathbb{Q}$-divisors then

Nadel's vanishing theorem is nothing other than the Kawamata–Viehweg vanishing theorem (see [D1] and [D2]). The most important results from the first period of applications of multiplier ideals to algebraic geometry are those of Anghern and Siu [AS]. One of their results is the following.

Theorem 22 *Let X be a nonsingular n-fold and L an ample Cartier divisor. Then $K_X + mL$ is generated by its global sections for all $m > \binom{n+1}{2}$.*

The particular importance of the paper [AS], in addition to its marvelous results, is as the first application of Ohsawa–Takegoshi extension theorem [OT] to problems of algebraic geometry. In the final analysis, as far as this part is concerned, it can be replaced by a purely algebraic argument (see [Ko4] and [L]), using the inversion of adjunction that is a corollary of the Kawamata–Viehweg vanishing theorem. Siu, who observed the importance of the Ohsawa–Takegoshi extension theorem, proved the following major result, the invariance of plurigenera [Si1].

Theorem 23 *Let $f\colon X \to S$ be a smooth proper morphism between nonsingular quasiprojective varieties.*[8] *Suppose in addition that every fiber $X_s = f^{-1}(s)$ of f is of general tpe. Then for every positive integer m, the plurigenus $P_m(X_s) = H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$ does not depend on s.*

It was known [N1] that in the event that minimal model theory could be completely established, this theorem would appear as a corollary, but Siu gave a direct proof. His paper [Si1] made use of results from complex analysis such as the Ohsawa–Takegoshi extension and Skoda division theorems, but Kawamata [K7] and Nakayama [N2] succeeded in making the proof algebraic and generalized it in a number of directions. Kawamata [K7] proved the deformation invariance of canonical singularities and Nakayama [N2] proved the deformation invariance of terminal singularities. We put their results together as follows:

Theorem 24 *Let $f\colon X \to S$ be a flat morphism from a germ of an algebraic variety to a nonsingular germ of an algebraic variety, and suppose that the central fiber $X_0 = f^{-1}(0)$ has at worst canonical singularities (respectively, terminal singularities). Then X itself has at worst canonical singularities (respectively, terminal singularities). In particular, every fiber $X_s = f^{-1}(s)$ has at worst canonical singularities (respectively, terminal singularities).*

[8] Footnote 8, p. 38

This subject is discussed in detail in Kawamata [K8] and Lazarsfeld [L]. Theorem 20, which was used effectively in the course of the proof of pl flips is also obtained by a generalization of this method. After this, Siu finally obtained the following result [Si2].

Theorem 25 *Let $f\colon X \to S$ be a smooth proper morphism between nonsingular quasiprojective varieties. Then for any positive m, the plurigenus $P_m(X_s) = h^0(X_s, \mathcal{O}_{X_s}(mK_{X_s})$ is independent of $s \in S$.*

In other words, he obtained a complete solution to the deformation invariance of plurigenera, getting rid of the assumption that the fibers are of general type. Takayama generalized Theorem 25, covering also the case that the fibers have canonical singularities and the case of a reducible central fiber; we refer to [Ty2] for the precise statements. It should be noted that when the fibers are not restricted to be of general type, the only known proofs of Theorem 25 are analytic. Most recently, Păun [P] gave a remarkable simplification of Siu's proof. As a somewhat grandiose overview, this consists simply of a clever use of the Ohsawa–Takegoshi extension theorem, and does not involve the Skoda division theorem, or difficult vanishing theorems, or Hőrmander style $\overline{\partial}$-equations. It uses only an assertion of Ohsawa–Takegoshi extension theorem type, that sections can be extended under L^2 estimates. If Siu had in the first place solved the deformation invariance of plurigenera directly by Păun's method, then history might well have taken a different turn, with none of [K7], [K8] or [N2] coming into existence.

As another application, Hacon and M$^{\text{c}}$Kernan [HM1] and Takayama [Ty1] obtained the following marvelous result. Both papers use an argument that turns the argument of Tsuji [Ts1] and [Ts2] on its head. It is no coincidence that the two papers appeared at the same time, use the same kind of method and passing through the same intermediate results. In constrast to Tsuji [Ts1] and [Ts2], both [HM1] and [Ty1] give purely algebraic proofs.

It is reasonable to describe Theorem 20 as constructed specifically for application to the proof of the following result.

Theorem 26 *Let X be an n-dimensional nonsingular projective variety of general type. Then there exists a positive integer m_n depending only on n such that the linear system $|mK_X|$ gives a birational map for every $m \geq m_n$.*

Several of the results discussed in this section were first obtained using analytic proofs; but except for the result of [Si2], [Ty2] and [P], algebraic proofs are now known for all the results. However, if we consider, say, Enoki's proof [E] of the Kollár injectivity theorem [Ko1], it sometimes happens that the analytic proofs subsequently obtained seem to be superior in some respects.

4.2 The injectivity and vanishing theorems

As the reader will already have noticed, the proofs of practically all the results given so far use the Kawamata–Viehweg vanishing theorem. We now recall Kollár's injectivity theorem, which generalizes this. In this section we work in the following setup. Let X be a nonsingular projective variety, L a Cartier divisor on X and D an effective $\mathbb{Q}$-divisor on X.

Theorem 27 (Kollár's injectivity theorem) *Assume that $H \sim_{\mathbb{Q}} L - D$ is semiample. Then there exists a positive integer m such that mH is Cartier; let $s \in H^0(X, \mathcal{O}_X(mH))$ be a nonzero global section. Then multiplication by s induces maps*

$$\times s\colon H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) \to H^i(X, \mathcal{O}_X(K_X + L + mH) \otimes \mathcal{J}(D))$$

that are injective for every i.

More general assertions are given in Ohsawa [O] and Fujino [F7] and [F8]. One of the ultimate generalizations of Theorem 27 is obtained in [F9] (see Theorem 41). We obtain the following result as a corollary of Theorem 27.

Theorem 28 (Kawamata–Viehweg–Nadel vanishing) *Suppose $L - D$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0 \quad \textit{for every } i > 0.$$

We discuss Theorem 28 first. Using Kodaira's lemma and simple properties of multiplier ideals, we may assume that $L - D$ is ample. Theorem 28 then follows from Theorem 27 and Serre vanishing.

Next, we look at the proof of Theorem 27 following [F7] (compare [F12]). Suppose that $kL \sim kH + kD$ for k a positive integer, where both kH and kD are Cartier divisors. Consider the singular Hermitian metric h_1 on $\mathcal{O}_X(kD)$ naturally associated to the effective divisor kD, and let h_2 be a smooth Hermitian metric on $\mathcal{O}_X(kH)$; define the metric h_L on $\mathcal{O}_X(L)$ by $(h_1 h_2)^{1/k}$. This is a smooth Hermitian metric on the complement $Y = X \setminus \operatorname{Supp} D$. We can construct an appropriate complete Kähler metric on Y, and develop the theory of harmonic integrals on Y with respect to these metrics.

Then

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D))$$

(or $H^i(X, \mathcal{O}_X(K_X + L + mH) \otimes \mathcal{J}(D))$) can be realized as the vector space $\mathcal{H}^{n,i}(Y, L)$ of $\mathcal{O}_X(L)$-valued harmonic (n, i)-forms on Y (resp., the vector space $\mathcal{H}^{n,i}(Y, L + mH)$ of $\mathcal{O}_X(L + mH)$-valued forms). Using Nakano's formula, we see that $\times s$ maps $\mathcal{H}^{n,i}(Y, L)$ to $\mathcal{H}^{n,i}(Y, L + mH)$; this uses the condition on the curvature of h_L. Now it is clear that $\times s \colon \mathcal{H}^{n,i}(Y, L) \to \mathcal{H}^{n,i}(Y, L + mH)$ is injective, and we obtain the result.

This proof clarifies the assumptions of the theorem, and is much simpler than the original proof that makes repeated use of ramified covers and resolution of singularities. Moreover, working on Y instead of X also obviates the need for approximations of singular Hermitian metrics that are commonly used in the L^2 theory. When $D = 0$ the above proof becomes extremely simple: $Y = X$, so that we also don't need singular Hermitian metrics. In this case Theorem 27 is contained in Enoki's theorem [E]. A commentary in Japanese is given in [F12].[9]

5 The existence of minimal models

In this section we explain the general strategy of the proof of Theorem 2. Let (X, Δ) be an n-dimensional projective klt pair; we wish to construct a minimal model of (X, Δ) in the case that Δ is big and $K_X + \Delta$ is pseudo-effective. The proof proceeds by induction on the dimension. For reasons of space, we only give a detailed discussion of the argument that MMP with scaling in dimension $(n - 1)$ implies the existence of n-dimensional minimal models. This is the material around [BCHM], Sections 4–5.

5.1 The existence of pl flips

This point has already been explained; however, we revisit the argument from the viewpoint of induction on the dimension. In Section 3 we proved the existence of pl flips assuming minimal model theory in dimension $(n-1)$. In fact on reexamining the proof of Section 3, one sees that it is enough to have MMP with scaling in dimension $(n-1)$ for a $\mathbb{Q}$-factorial klt pair (X, Δ) where Δ is a big $\mathbb{R}$-divisor. Therefore in what follows we may assume freely the existence of pl flips in dimension n.

[9] Footnote 9, p. 38

5.2 Special termination

We now explain an important theorem called *special termination*. In what follows, we assume that (X, Δ) is a divisorially log terminal pair and $S = \lfloor \Delta \rfloor$.

Theorem 29 (Special termination) *Assume MMP with scaling holds in dimension $\leq n-1$. Suppose that $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow X_{i+1} \dashrightarrow$ is a MMP with scaling for (X, Δ). Then after a finite number of steps, $X_i \dashrightarrow X_{i+1}$ is an isomorphism in a neighborhood of S_i. This means that MMP with scaling stops in a neighborhood of $\lfloor \Delta \rfloor$.*

This theorem also has its origins in Shokurov [Sh2]. The assertion in general dimension is the starting point of [Sh4]. The rigorous proof was given in Fujino [F6].[10] Of course, these papers were written without using the framework of MMP with scaling, but the proof in [F6] applies without problem in this setup. We discuss here just the main issue. Let $(X, S+B)$ be a dlt pair; here S is an irreducible prime divisor, and we write $B = \sum_j b_j B_j$ with $0 < b_j \leq 1$. Define the divisor B_S on S by $(K_X + S + B)_{|S} = K_S + B_S$. Then the coefficients of B_S that are not equal to 1 belong to the set

$$\mathcal{S}(B) = \left\{ 1 - \frac{1}{m} + \sum_j \frac{r_j b_j}{m} \,\middle|\, m \in \mathbb{Z}_{>0}, r_j \in \mathbb{Z}_{\geq 0} \right\}.$$

This is the so-called adjunction formula (see [F5]) originating with Shokurov [Sh2]. It takes account of the influence of the singularities in codimension 1 on S, that is, in codimension 2 on X; here we use classification results on the singularities of dlt surface pairs. The pair (S, B_S) is $n-1$-dimensional, and by induction we can apply MMP with scaling to it. The assertion we are aiming for is proved using the properties of the set $\mathcal{S}(B)$ containing the coefficients of B_S and the theory of MMP with scaling. In conclusion, if the theory of MMP with scaling holds in dimensions up to $n-1$, the MMP with scaling in dimension n terminates in a neighborhood of $\lfloor \Delta \rfloor$.

This special termination theorem is more powerful than it appears, and termination in this form is enough to construct minimal models. We examine this below. Once again, this idea appears first in [Sh2], and is reproduced in Kollár [FA] and Kollár and Mori [KM]. What [F6] calls the reduction theorem, assuming the theorem on special termination, corresponds to the following step. The main idea of [Sh2] was that the existence of general flips can be proved if we have pl flips and special termination.

[10] Footnote 10, p. 38

5.3 Construction of minimal models

We start by preparing some simple lemmas.

Proposition 30 *Suppose that the special termination theorem Theorem 29 holds in dimension n. Assume in addition that the following conditions hold:*

(1) (X, Θ) *is an n-fold $\mathbb{Q}$-factorial dlt pair.*

(2) There exists a positive real number c and effective $\mathbb{R}$-divisors H and F so that we have the expression $K_X + \Theta \sim_{\mathbb{R}} cH + F$.

(3) $(X, \Theta + H)$ *is a dlt pair and* $K_X + \Theta + H$ *is nef.*

(4) $\operatorname{Supp} F \subset \lfloor \Theta \rfloor$.

Then a minimal model of $(X, \Theta + tH)$ *exists for any t with* $0 \le t \le 1$.

The proof consists simply of running an MMP with scaling. Condition (3) allows us to run an MMP with scaling by H. There is absolutely no problem with the Cone Theorem and the Contraction Theorem, which hold in any dimension. According to the assumptions of MMP with scaling, the extremal ray R we choose at each step satisfies $H \cdot R > 0$ and $(K_X + \Theta) \cdot R < 0$. Now by condition (2) it follows that $F \cdot R < 0$. Using condition (4), we see that every flipping contraction is a pl flipping contraction. The existence of pl flips is already known. so that we can carry out the MMP.

The only remaining issue is to check that this procedure stops after finitely many steps. If this MMP does not terminate, we deduce from $F \cdot R < 0$ and $\operatorname{Supp} F \subset \lfloor \Theta \rfloor$ that the modification that occurs at every step happens inside $\lfloor \Theta \rfloor$. This would contradict special termination. Therefore it does terminate.

In fact the construction of minimal models is as follows. We write it as a theorem.[11] Theorem 31 also gives the solution to Flip Conjecture I for n-fold log pairs.

Theorem 31 *Let (X, Δ) be an n-fold klt pair, and assume that Δ is big and $K_X + \Delta \sim_{\mathbb{R}} D \ge 0$. Assume that special termination holds in dimension n.*

Then a minimal model of (X, Δ) *exists.*

The proof is as follows.

First, a slight tedious point: the divisors we consider are all $\mathbb{Q}$-divisors; in actuality, the proof of finiteness of minimal models does not go through

[11] Footnote 11, p. 38

properly without generalizing everything to $\mathbb{R}$-divisor. By using the theorem on resolution of singularities, we can make the following assumptions:

(1) X is nonsingular and $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$. Here $\mathrm{Supp}(\Delta + D)$ is a simple normal crossing divisor.

(2) There exists an effective ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor B such that Δ can be expressed $\Delta = A + B$.

(3) We can write $D = rM + F$, where M is a mobile effective divisor, and F is an effective $\mathbb{Q}$-divisor every irreducible component of which is contained in the stable base locus of D. In other word, the components of F are contained in the base locus $\mathrm{Fix}\,|mD|$ for every positive integer m.

(4) Δ and M have no common components.

Let $F = \sum_{i=1}^{k} a_i \Delta_i$ where $\Delta = \sum_{i=1}^{l} b_i \Delta_i$; here $k \leq l$. Then setting $\Delta' = \sum_{i=1}^{k}(1 - b_i)\Delta_i$, we define $F' = F + \Delta'$ and $\Theta = \Delta + \Delta'$. By construction $\mathrm{Supp}\, F' \subset \lfloor \Theta \rfloor$. Taking H to be a suitable ample divisor, we can arrange that $(X, \Theta + M + H)$ is a dlt pair, $K + \Theta + M \sim_{\mathbb{Q}} 0 \cdot H + (r+1)M + F'$, $K + \Theta + M + H$ is nef and $\mathrm{Supp}(M + F') \subset \lfloor \Theta + M \rfloor$.

Applying MMP with scaling by H to $(X, \Theta + M)$, by Proposition 30, we conclude that there exists a minimal model of $(X, \Theta + M)$. Hence we may assume from the start that $(X, \Theta + M)$ itself is already a minimal model. Thus $K + \Theta \sim_{\mathbb{Q}} rM + F'$, $K + \Theta + M$ is nef, and moreover $\mathrm{Supp}\, F' \subset \lfloor \Theta \rfloor$. Now applying MMP with scaling by M, again by Proposition 30, we conclude that a minimal model of (X, Θ) exists. We need a small argument for this, but one can see that a minimal model of (X, Θ) is also a minimal model of (X, Δ).[12] This allows us to conclude that minimal models exist. The difficulty is that to complete the induction on dimension, we must also prove the termination problem for the n-dimensional MMP with scaling. In addition, in Theorem 31 we assumed that $K_X + \Delta \sim_{\mathbb{Q}} D \geq 0$; however, in Theorem 2 we only assumed $K_X + \Delta$ to be pseudoeffective. This subtle little difference is actually technically an extremely awkward point.

5.4 Termination of MMP with scaling

We now want to consider the termination problem for the n-fold MMP with scaling. We start by fixing the setup.

[12]Footnote 12, p. 38

Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair, and suppose that $K_X + \Delta + H$ is a nef divisor and is dlt. Then we can run an MMP with scaling by H.

Theorem 32 *If we assume that there are only finitely many minimal models of $(X, \Delta + tH)$ for $0 \leq t \leq 1$, then an MMP scaled by H must terminate.*

This is pretty clear. At each stage of an MMP $X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow X_{i+1} \dashrightarrow$, there exists a decreasing sequence of real numbers $1 \geq t_1 \geq t_2 \geq \cdots$ such that $K_{X_i} + \Delta_i + t_i H_i$ is a minimal model for $K_X + \Delta + t_i H$. Thus it is clear from the assumption that an infinite series of flips cannot exist; indeed, there are only finitely many possibilities for the models in the first place. Here the finiteness of minimal models arises as an important issue.

5.5 Finite number of minimal models

We consider here the finiteness of minimal models in an extremely simple case; the assertion in this case is sufficient for the application to the proof of the existence of pl flips. The complete proof of [BCHM] requires a more involved assertion (cf. Theorem 36). First, some preparations. Let V be a finite dimensional affine subspace in the real vector space of Weil divisors on X, and assume that V is defined over the field of rational numbers. Let A be an $\mathbb{R}$-divisor on X that we suppose not to have common component with any divisors in V. Set $V_A = \{\Delta = A + B \mid B \in V\}$ and

$$\mathcal{L}_A = \{\Delta \in V_A \mid (X, \Delta) \text{ is a log canonical pair}\}.$$

One sees at once that $\mathcal{L}_A$ is a compact polytope.[13]

Theorem 33 *Assume that for every n-fold klt pair (X, Δ) with Δ big and $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$, a minimal model of (X, Δ) always exists. Let X be an n-fold normal projective variety and A a general ample $\mathbb{Q}$ divisor on X. In what follows, A is always fixed.*

Let $\mathcal{C} \subset \mathcal{L}_A$ be a rational polytope, and suppose that $K_X + \Delta$ is klt for any $\Delta \in \mathcal{C}$ and $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$ holds.

Then there exists a finite number of rational maps $\varphi_i \colon X \dashrightarrow Y_i$ for $1 \leq i \leq k$ such that for any $\Delta \in \mathcal{C}$ one of $(Y_i, \varphi_{i*}\Delta)$ is a minimal model of (X, Δ).

[13]Footnote 13, p. 38

The proof is as follows. First pick a $\Delta_0 \in \mathcal{C}$. By compactness of $\mathcal{C}$, it is enough to prove the theorem in a neighborhood of Δ_0. In what follows, we shrink $\mathcal{C}$ to a neighborhood of Δ_0 as necessary. Using the assumption, choose a minimal model $\varphi\colon X \dashrightarrow Y$ of $K_X + \Delta_0$. A simple argument shows that we can replace (X, Δ_0) by $(Y, \varphi_*\Delta_0)$. So we argue assuming that (X, Δ_0) is replaced from the start by $(Y, \varphi_*\Delta_0)$. Then we can use the base-point free theorem.[14] There exists a morphism $f\colon X \to Z$ such that $K_X + \Delta_0 \sim_{\mathbb{R},Z} 0$. This means that there exists a Cartier divisor C on Z such that $K_X + \Delta_0 \sim_{\mathbb{R}} \varphi^* C$. Then considering a sufficiently small neighborhood of Δ_0, one sees that a relative minimal model of (X, Δ) over Z is a minimal model in the usual sense. So we work over Z from now on.

Pick some $\Theta \in \mathcal{C}$. We can take a divisor Δ in the boundary of $\mathcal{C}$ so that Θ is on the line segment joining Δ_0 and Δ, and we can write $\Theta - \Delta_0 = \lambda(\Delta - \Delta_0)$. Noting that $K_X + \Theta \sim_{\mathbb{R},Z} \lambda(K_X + \Delta)$, one sees that $\varphi\colon X \dashrightarrow Y$ a minimal model of (X, Θ) over Z and $\varphi\colon X \dashrightarrow Y$ a minimal model of (X, Δ) over Z are equivalent conditions. Δ is a divisor in the boundary of $\mathcal{C}$, so that the theorem follows by induction on the dimension of $\mathcal{C}$.

I hope that this explains the significance of the finiteness of minimal models. Once finiteness of minimal models is established, this settles the termination of MMP with scaling for n-folds, and completes the induction by the dimension.

5.6 What is still missing

I hope that the treatment so far explains the general mechanism. However, a number of pieces are still missing. The biggest of these, is the step showing that $K_X + \Delta$ pseudoeffective implies that $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. We have completely omitted this. This result, known as a nonvanishing theorem, is in some sense the newest piece in [BCHM]; for this, see Theorem 37. Next, our treatment of the finiteness of minimal models only covered klt pairs, whereas in actuality we need to set things up slightly more generally in order for induction on the dimension to go through.

In addition, to settle the problem of termination of flips, we need finiteness in a stronger sense than that of Theorem 33 (compare Theorem 36). One proves the finiteness of log canonical models using the argument of Theorem 33, and finally we show the finiteness of weakly log canonical models. This part of the argument is extremely technical. It turns out to be important to observe the fact that when (X, Δ) is klt and Δ is big, $\mathrm{NE}(X)$ has

[14]Footnote 14, p. 38

only finitely many $(K_X + \Delta)$-negative extremal rays. This material is not really appropriate for a survey article, and we refer the reader to [BCHM], Sections 6–7. Or to put it the other way around, this survey covers practically the whole argument of [BCHM] except for the approximately 10 pages of Sections 6–7.

6 What is actually proved

Here we discuss what [BCHM] actually proved. We hope that the meaning and function of the theorems should already be clear in overall terms from the discussion in Section 5. We follow the statements with a brief explanation of the proof of the nonvanishing theorem.

Theorem 34 (Existence of pl flips) *Let $f\colon X \to Z$ be a pl flipping contraction from an n-fold purely log terminal pair (X, Δ). Then the flip $f^+\colon X^+ \to Z$ exists.*

Theorem 35 (Existence of log terminal models) *Let $\pi\colon X \to U$ be a projective morphism between normal quasiprojective varieties, where X is an n-fold. Let (X, Δ) be a klt pair, and assume that Δ is big over U. If there exists an effective $\mathbb{R}$-divisor D with $K_X + \Delta \sim_{\mathbb{R},U} D$ then $K_X + \Delta$ has a log terminal model.*

Theorem 36 (Finite number of models) *Let $\pi\colon X \to U$ be a projective morphism between normal quasiprojective varieties, with X an n-fold. Fix a general ample $\mathbb{Q}$-divisor A on X over U.*

Assume that there exists Δ_0 such that $K_X + \Delta_0$ is klt. Let $\mathcal{C} \subset \mathcal{L}_A$ be a finite convex rational polytope. We assume that one of the two following conditions hold:

(1) $K_X + \Delta$ is big for any $\Delta \in \mathcal{C}$, or

(2) $\mathcal{C} = \mathcal{L}_A$.

Then there exist finitely many birational maps $\psi_j\colon X \dashrightarrow Z_j$ over U (for $1 \le j \le l$) such that for any $\Delta \in \mathcal{C}$ and any weak log canonical model $\psi\colon X \dashrightarrow Z$ of (X, Δ) over U there is a j and an isomorphism $\xi\colon Z_j \to Z$ such that $\psi = \xi \circ \psi_j$.

Theorem 37 (Nonvanishing) *Consider a projective morphism $\pi\colon X \to U$ between normal quasiprojective varieties, with X an n-fold. If $K_X + \Delta$ is π-pseudoeffective and Δ is π-big then there exists an effective $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R},U} D$.*

As proved in Section 3 and 5.1 if we assume MMP with scaling in dimension $n-1$ then we can prove the existence of n-fold pl flips. This is Theorem 34. Then one can use the argument of 5.3 to prove the existence of minimal models, giving Theorem 35. Next one proves the finiteness of models under assumption (1) of Theorem 36. The existence of an effective $\mathbb{R}$-divisor D such that $K_X + \Delta \sim_{\mathbb{R},U} D \geq 0$ follows from the fact that $K + \Delta$ is π-big, so that we can use Theorem 35, which has already been proved. One obtains Theorem 36 under assumption (1) using the argument explained in 5.5.

Note in passing that it is not enough to consider Theorem 36 only for klt pairs, so that the setup is extremely artificial. The assertion itself concerns the finiteness not only of minimal models, but also of weakly log canonical models. We refer to the original paper [BCHM] for the subtleties surrounding this material.

The proof of the Nonvanishing Theorem 37 uses case (1) of Theorem 36. It is proved by an ingenious combination of the argument of Nakayama [N3], Shokurov's classical argument for the Nonvanishing Theorem [Sh1] and the finiteness of minimal models. This is possibly one of the main innovations of [BCHM]. The proof is made more difficult by the fact that we cannot just work with $\mathbb{Q}$-divisors, but have to extend the argument to $\mathbb{R}$-divisors. Once the Nonvanishing Theorem 37 is proved, one sees that Theorem 35 holds under the weaker condition that $K_X + \Delta$ is π-pseudoeffective. Finally, one proves the remaining case of Theorem 36 using this. The crux of the argument here was essentially proved in 5.5. Here we finish the proof by induction on dimension: knowing the finiteness of models Theorem 36, one can solve the problem of termination in the the n-fold MMP with scaling (see 5.4).

7 Problems for the future

To conclude, we consider a number of related unsolved problems

7.1 Termination of flips

The termination of flips is one of the most important unsolved problems. Up to now, the 3-fold case is completely solved; see [Sh1], [K6], [Sh3] and compare [Ko3]. The 4-fold case was started in [KMM], and continued in [Ma1], [F2], [F3], [F4], with the best current result probably due to Alexeev, Hacon and Kawamata [AHK]. However, the problem is still not completely solved. The above papers all rely on generalizations of the notion of *difficulty* introduced by Shokurov [Sh1].

Completely different approaches have been suggested by Birkar [B1] and Shokurov [Sh5]. Here we explain the approach of [Sh5] to solving Flip Conjecture II. Let X be a normal n-fold and Δ an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Write $\Gamma \subset [0,1]$ for the set of coefficients of Δ. Define the *minimal log discrepancy* function $\operatorname{mld}\colon X \to \mathbb{R} \cup \{-\infty\}$ by

$$\operatorname{mld} x = \inf_E a(E, X, \Delta) + 1.$$

Here x is a scheme theoretic point of X, and the minimum runs over all E a divisor on a normal projective variety Y having a birational morphism $f\colon Y \to X$ such that $f(E) = x$. One also writes $\operatorname{mld} x = a(E, X, \Delta) + 1$. Then there are two fundamental conjectures.[15]

Conjecture 38 (Ascending chain condition) *Define the set*

$$A(\Gamma, m) = \big\{a(y, Y, B) + 1\big\}$$

in terms of the totality of all Y, B, y satisfying the following conditions: Y is a normal m-fold, B an $\mathbb{R}$-divisor with all its coefficients contained in Γ and such that $K_Y + B$ is $\mathbb{R}$-Cartier, and $y \in Y$ a closed point.

Then $A(\Gamma, m)$ satisfies the ascending chain condition. That is, for any sequence $a_1 \le a_2 \le \cdots \le a_k \le \cdots$ with $a_i \in A(\Gamma, m)$, there exists k_0 such that $a_k = a_{k_0}$ for every $k \ge k_0$.

Conjecture 39 (Lower semicontinuity) *Let Y be a normal m-fold and B an $\mathbb{R}$-divisor all of whose coefficients are contained in Γ and such that $K_Y + B$ is $\mathbb{R}$-Cartier. Then for any point $y \in Y$ there exists a neighborhood $U \subset Y$ of y such that* $\operatorname{mld}(y) = \inf_{y' \in U} \operatorname{mld}(y')$*; here $y' \in U$ is a closed point, and* $\operatorname{mld}(y) = a(y, Y, B) + 1$, $\operatorname{mld}(y') = a(y', Y, B) + 1$ *(see also Ambro [A1]).*

These two conjectures imply the following theorem.

Theorem 40 *Let $X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow X_{i+1} \dashrightarrow \cdots$ be a chain of flips starting from (X, Δ), and suppose that every flip $X_i \dashrightarrow X_{i+1}$ is projective over a fixed variety S. Suppose that Conjectures 38 and 39 hold in dimension up to* $\dim X$*. Then the given sequence of flips terminates after finitely many steps.*

[15] Footnote 15, p. 38

It follows that solving Conjectures 38 and 39 would complete minimal model theory. In the case that Y has only locally complete intersection singularities, Conjecture 39 is proved using the theory of jet schemes (see Ein, Mustaţă and Yasuda [EMY] and [EM]). In general it is still unsolved. Conjecture 38 remains unsolved at present despite a number of attempts to solve it by people around Shokurov (see Birkar and Shokurov [BS] and [Sh5]).

7.2 Minimal model theory for log canonical pairs

One believes that minimal model theory will eventually hold for lc pairs. To accomplish this, one must be able to extend the Cone Theorem and the Contraction Theorem, the starting point of minimal model theory, to lc pairs. Thinking back through the proof of the Cone Theorem leads us back inevitably to the Kawamata–Viehweg vanishing theorem. In the world of klt pairs, the Kawamata–Viehweg vanishing theorem holds, which enables us to carry out induction on the dimension. This is the method used repeatedly by Kawamata, known as the X-method.

Ambro [A2] observed that proving a generalization of Kollár vanishing and torsion-free theorem (see [Ko1]) for embedded normal crossing pairs would allow the X-method to be applied successfully in the world of quasilog varieties, so that finally we would be able to prove the Cone Theorem and the Contraction Theorem for lc pairs. We note that generalizing the Kawamata–Viehweg vanishing theorem to lc pairs (see [F9]) is insufficient for induction. Thus in fact the crux of the problem is concentrated around proving a generalization of Kollár's theorems[16]. Here, following [F9], we prove the generalization of Kollár's theorem needed for minimal model theory for lc pairs.

Let M be a nonsingular projective variety and Y a simple normal crossing divisor on Y. We write an $\mathbb{R}$-divisor D on M as $D = \sum d_i D_i$; suppose that $0 \leq d_i \leq 1$ holds for all i. Suppose also that D and Y have no common components, and that $\mathrm{Supp}(D+Y)$ is a simple normal crossing divisor on M. Write $B = D_{|Y}$. In what follows, we consider the pair (Y, B). Write $\nu\colon Y' \to Y$ for the normalization of Y, and set $K_{Y'} + B_{Y'} = \nu^*(K_Y + B)$; then (Y', B') is a lc pair. We define a *stratum* of (Y, B) to be an irreducible component of Y or the image under ν of a lc center of (Y', B'). In addition, say that an $\mathbb{R}$-Cartier divisor A on Y is *admissible* for (Y, B) if its support does not contain any stratum of (Y, B). We then obtain the following generalization of Kollár's injectivity theorem [Ko1].

[16] Footnote 16, p. 38

Theorem 41 *Suppose that Y is complete. Let L be a Cartier divisor of Y and A an effective Cartier divisor admissible for (Y, B). Assume also the following:*

(1) $L \sim_{\mathbb{R}} K_Y + B + H$.

(2) H is a semiample $\mathbb{R}$-Cartier divisor.

(3) One can write $tH \sim_{\mathbb{R}} A + A'$ with t a positive real number and A' an effective $\mathbb{R}$-Cartier divisor that is admissible for (Y, B).

Then the map $H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + A))$ induced by the inclusion homomorphism $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y(A)$ is injective for all q.

From Theorem 41 we deduce the following theorem. Here (i) generalizes Kollár's torsion free theorem, and (ii) generalizes Kollár's vanishing theorem.

Theorem 42 *Let $f\colon Y \to X$ be a proper morphism and L a Cartier divisor on Y. Suppose also that $H \sim_{\mathbb{R}} L - (K_Y + B)$ is f-semiample. Then we obtain the following two assertions:*

(i) Every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the image under f of some stratum of (Y, B).

(ii) Let $\pi\colon X \to V$ be a projective morphism. Suppose also that H can be written in the form $H \sim_{\mathbb{R}} f^* H'$ where H' is a π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X.

Then $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ holds for all $p > 0$ and $q \geq 0$.

We refer to [F9] for the detailed proof, which is extremely cumbersome: we have what could be called a noncompact normal crossing V-variety, and cohomology groups having compact support on it; the proof involves analyzing a mixed Hodge structure introduced on this. Be that as it may, this theorem allows us to establish the basic framework of minimal model theory for lc pairs. To complete the minimal model theory we still need to resolve Flip Conjectures I and II. As far as Flip Conjecture II is concerned, one sees that if we can prove it for klt pairs then the result also follows for lc pairs. For details, we refer to [F6] and [F10]. The problem is thus the existence of flips. The 3-fold case was proved by Keel and Kollár [KK]. In the 4-fold case, we used the result of [F1] to give the proof in [F10], although this turned into a hugely elaborate proof, involving the use of the Abundance Theorem for reducible 3-fold.

Let's end by giving a discussion of the Abundunce Conjecture.

7.3 The abundance conjecture

The Abundunce Conjecture is the following statement. There are several different versions, and our statement is probably the most general. The conjecture has been around for more than 20 years, but there has been little progress so far in dimension ≥ 4.

Conjecture 43 *Let (X, Δ) be a log canonical pair and $\pi\colon X \to S$ a proper morphism. If $K_X + \Delta$ is nef over S then $K_X + \Delta$ is semiample over S.*

After Kawamata's survey [K11], the 3-fold case of the Abundance Conjecture was settled in Keel, Matsuki and M^{c}Kernan [KKMc1] (see also [KKMc2]). At present, the conjecture has been generalized to the case of reducible 3-fold semi log canonical pairs (see [F1]). This can be viewed as a first step towards the 4-fold case; in fact, as we mentioned above, it was used in the proof of the Flip Conjecture for 4-fold lc pairs. It seems that the Abundance Conjecture is a much deeper statement than the other conjectures, but actually, we don't really understand too much. We write out a special case of the above conjecture.

Conjecture 44 *Let X be a projective variety with at worst terminal singularities. If K_X is nef then K_X is semiample.*

This conjecture asserts that a minimal model has a natural Iitaka fibration. The following conjecture, which should serve as a possible starting point for Conjecture 44 is still unsolved.

Conjecture 45 *Let X be a nonsingular projective variety and assume that K_X is pseudoeffective. Then there exists a positive integer m for which $H^0(X, \mathcal{O}_X(mK_X)) \neq 0$. In other words, X has nonnegative Kodaira dimension $\kappa(X)$.*

Boucksom, Demailly, Păun and Peternell [BDPP] obtained a characterization of the pseudoeffective cone $\mathrm{PE}(X)$ of a projective variety X (see also Lazarsfeld [L]). Putting this together with the result of Mori and Miyaoka [MM], gives the following result.

Theorem 46 *Let X be a nonsingular projective variety. Then K_X is not pseudoeffective if and only if X is uniruled.*

Thus Conjecture 45 asserts that if X is not uniruled then $\kappa(X) \geq 0$.

Although not as it stands providing direct progress towards Conjecture 45, a possible tool in solving it is the considerable simplification due to Kebekus, Solá Conde and Toma [KST] of Miyaoka's theorem (see [Mi] and [S-B]); their main result depends on Graber, Harris and Starr [GHS]. Recently, the [GHS] results have led to a considerable clarification of the status of rationally connected varieties (see also Hacon and M^{c}Kernan [HM2]).

Here we add a remark concerning Kawamata's result [K1]. The following is well known as a direct corollary of the main theorem of [K1].

Theorem 47 *Let (X, Δ) be a klt pair with Δ a $\mathbb{Q}$-divisor, and suppose that $K_X + \Delta$ is nef. If the Kodaira dimension $\kappa(K_X + \Delta)$ equals the numerical Kodaira dimension $\nu(K_X + \Delta)$ then $K_X + \Delta$ is semiample.*

Thus for a klt pair (X, Δ), the abundance conjecture asserts the equality $\kappa(K_X + \Delta) = \nu(K_X + \Delta)$ of the Kodaira dimension and the numerical Kodaira dimension. The proof of [K1] depends on the so-called X-method of induction on the dimension. However, in contrast to the usual base point free theorem, we are in a situation where the Kawamata–Viehweg vanishing theorem does not hold, and we use instead a generalization of Kollár's injectivity theorem. To be able to apply induction on the dimension successfully, we need to prove something like an extension of Kollár's result to generalized normal crossing variety. This is close to the situation considered in Ambro [A2]; or rather, Ambro [A2] appears as an adaptation of Kawamata's result [K1] to the new situation. An alternative proof of the main result Kawamata [K1] is given in Fujino [F11]; this uses the canonical bundle formula to reduce to the usual well known base point free theorem. Whereas Kawamata's proof [K1] is based on an appeal to mixed Hodge structures for reducible varieties, the proof of [F11] depends instead on a reduction to variation of Hodge structures using the theory of canonical extensions of Hodge bundles.

Acknowledgments

My work is supported in part by the Grant-in-Aid for Young Scientists (A) 17684001 from JSPS and by the Inamori Foundation, and I thank them. I received a lot of help in understanding [BCHM] from TAKAGI Hiromichi (University of Tokyo) and KAWAKITA Masayuki (RIMS, Kyoto University), and I owe deep gratitude to both of them. I thank SAITO Natsuo (Hiroshima City University) for help with graphics. Finally, I am grateful to KAWAKITA Masayuki, TAKAGI Hiromichi, TAKAYAMA Shigeharu (Uni-

versity of Tokyo), YASUDA Takehiko (RIMS) and two reviewers for valuable comments.

Footnotes

1. The seven authors of [Book] were Florin Ambro, Alessio Corti, FUJINO Osamu, Christopher Hacon, János Kollár, James M^{c}Kernan and TAKAGI Hiromichi. It would possibly be better described as a collection of articles. The actual seminar participants were Corti playing the central role, together with Ambro, Fujino, Kawakita, M^{c}Kernan and Takagi. Before this, a seminar deciphering [Sh2] led by Kollár was held at Utah in 1992, resulting in the collection [FA]; Corti and M^{c}Kernan took part in the Utah seminar, while the other members were youngsters.

2. Nef is an acronym for numerically effective or numerically eventually free.

3. It should be noted that the main references [KMM], [FA], [KM], [Ma2], all use slightly different definitions of log terminal singularities. For details, see [F5].

4. In writing this survey, we reworked Corti and Kollár [CK]. In doing so, we observed that the condition in the second half of [CK], (5.1.2) should be that the centre of E is a curve contained in $\lfloor D \rfloor$.

5. Shokurov [Sh2] introduced many of the notions of log terminal pair, most importantly divisorially log terminal and purely log terminal; at the same time, he introduced the ideas discussed below: inversion of adjunction, pl flips and special termination. [Sh2] also contains the famous Shokurov connectedness lemma.

6. If $(X, S+B)$ is plt then (S, B_S) is klt. The converse statement, that (S, B_S) klt implies that $(X, S+B)$ is plt is a neighborhood of S is the inversion of adjunction.

7. This is the first appearance of $\mathbb{R}$-divisors. The problem of the finite number of minimal models appears naturally in the proof of the next property (3); in this argument, it is insufficient to work with rational numbers, the argument needing the continuity property of real numbers.

8. In Theorem 23 and Theorem 25 it is enough to take S to be the open disk.

9. This was completely unknown to most algebraic geometry experts. Following Enoki [E], Takegoshi [Tg] and Ohsawa [O] continued this direction of research, which seems to have been well known among complex geometers in Japan.

10. A particular 3-dimensional case of special termination was proved in Shokurov [Sh2]. The general statement and proof for 3-folds is contained in Kollár and Matsuki [KMa]. Shokurov [Sh4] contains an extremely general assertion in general dimension; however, this only gives a sketch in the context of a special 4-dimensional case. [F6] develops a formalism allowing an induction on the dimension to be carried through successfully, and contains a rigorous proof in arbitrary dimension.

11. This theorem was simplified and generalized in Birkar [B2]. In this, the assumption Δ big is not necessary. However, the assumption that Δ is big is needed in what follows for the finiteness of minimal models. Here we follow [BCHM]. See also Kollár's commentary in the lecture notes [CHKLM].

12. This part requires that Δ is big. Under this assumption, we can deduce that $K + \Theta$ nef implies it is semiample.

13. Here $\mathbb{Q}$-divisors are not sufficient; it is for this reason that we must consider $\mathbb{R}$-divisors.

14. Here the assumption that Δ_0 is big is very effective.

15. Shokurov [Sh5] treats this in a more general setup.

16. Ambro [A2] gives a proof of a generalization of Kollá's theorem; unfortunately, I am unable to follow this proof even in the case of a nonsingular projective variety.

8 Updates and addenda added in proof

8.1 Update of December 2007

As of December 2007, the preprint of Birkar, Cascini, Hacon and M$^{\mathrm{c}}$Kernan [BCHM] is in the course of a major revision and yet to be submitted; this is an epoch-making paper, that must be published after appropriate laundry work. According to Hacon and M^{c}Kernan, the Sarkisov program, which studies birational maps between two different Mori fiber spaces, can also be completely generalized to higher dimensional as an application of the theorem on finiteness of models (Theorem 36); their result has subsequently appeared [HM5].

Shokurov [Sh6] is an attempt to construct minimal models (or Mori fiber spaces) without imposing conditions on $K + \Delta$.

8.2 Author's Addendum, December 2009

This survey was written in the summer 2007. Since then, [BCHM] has given new treatments of Special termination (see 5.2) and Finite number of models (Theorem 36), thereby considerably simplifying the arguments of [BCHM]. We refer the reader to [BCHM] for details, and also to two new survey articles [D] and [K12].

Birkar's papers [B2], [B3], [B4], and [B6] treat the problem of the existence of log minimal models. In [BP], the nonvanishing theorem (Theorem 37) was proved without running the MMP with scaling. The proof relates more closely to Siu's extension theorem. We note that the termination of 4-fold log flips is still open [B5]. The finite generation of log canonical ring for 4-fold log canonical pairs has been settled in [F18]. The paper [F18] also contains a partial answer to the 4-fold abundance conjecture.

The contents of 7.2 has been greatly expanded in [F16], a completely revised and expanded version of [F9] and [F10]. In [F17] and [F20], we apply [BCHM] to give a short and quick proof of the Cone and Contraction Theorems for log canonical pairs. This new approach is described in detail in [F21] in full generality. In a series of papers [F13], [F14], and [F15], we give various applications of our new vanishing and torsion-free theorem (see Theorem 41). The paper [F19] is an elementary introduction to the theory of quasi-log varieties by Ambro [A2].

References

[AHK] V. Alexeev, C. Hacon and Y. Kawamata, Termination of (many) 4-dimensional log flips, Invent. Math. **168** (2007), 433–448

[A1] F. Ambro, On minimal log discrepancies, Math. Res. Lett. **6** (1999), 573–580

[A2] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233

[AS] U. Anghern and Y.-T. Siu, Effective freeness and point separation for adjoint bundles, Invent. Math. **122** (1995), 291–308

[B1] C. Birkar, Ascending chain condition for log canonical thresholds and termination of log flips, Duke Math. J. **136** (2007), 173–180

[B2] C. Birkar, On existence of log minimal models, Compositio Math., to appear

[B3] C. Birkar, On existence of log minimal models II, arXiv:0907.4170, 18 pp.

[B4] C. Birkar, On existence of log minimal models and weak Zariski decompositions, arXiv:0907.5182, 15 pp.

[B5] C. Birkar, On termination of log flips in dimension four, Math. Ann. **346** no. 2 (2009), 251–257.

[B6] C. Birkar, Log minimal models according to Shokurov, to appear in Algebra Number Theory.

[BCHM] C. Birkar, P. Cascini, C. Hacon and J. M^{c}Kernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc., to appear

[BP] C. Birkar and M. Păun, Minimal models, flips and finite generation: a tribute to V. V. Shokurov and Y.-T. Siu, arXiv:0904.2936, 40 pp.

[BS] C. Birkar and V.V. Shokurov, Mld's vs thresholds and flips, arXiv:math/0609539, 41 pp.

[BDPP] S. Boucksom, J.-P. Demailly, M. Păun and T. Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, arXiv:math/0405285, 39 pp.

[Book] A. Corti (Ed.), Flips for 3-folds and 4-folds, Oxford Lecture Ser. Math. Appl. **35**, Oxford University Press, Oxford, 2007

[C] A. Corti, 3-fold flips after Shokurov, in [Book], 18–48

[CHKLM] A. Corti, P. Hacking, J. Kollár, R. Lazarsfeld and M. Mustaţă, Lectures on flips and minimal models, arXiv:0706.0494, 29 pp.

[CK] A. Corti, J. Kollár, Existence of canonical flips, in book [FA], pp. 69–73

[CKS] J. Kollár, A. Corti and K.E. Smith, Rational and nearly rational varieties. Cambridge Studies in Advanced Mathematics, **92**, Cambridge University Press, Cambridge, 2004

[D1] J.-P. Demailly, Transcendental proof of a generalized Kawamata–Viehweg vanishing theorem. Geometrical and algebraical aspects in several complex variables (Cetraro, 1989), 81–94, Sem. Conf. **8**, EditEl, Rende, 1991

[D2] J.-P. Demailly, Multiplier ideal sheaves and analytic methods in algebraic geometry. School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes **6**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001

[D] S. Druel, Existence de modèles minimaux pour les variétés de type général, Exposé 982, Séminaire Bourbaki, 2007/08.

[EM] L. Ein and M. Mustaţă, Inversion of adjunction for local complete intersection varieties, Amer. J. Math. **126** (2004), 1355–1365

[EMY] L. Ein, M. Mustaţă and T. Yasuda, Jet schemes, log discrepancies and inversion of adjunction, Invent. Math. **153** (2003), 519–535

[E] I. Enoki, Kawamata–Viehweg vanishing theorem for compact Kähler manifolds. Einstein metrics and Yang–Mills connections (Sanda, 1990), 59–68, Lecture Notes in Pure and Appl. Math. **145**, Dekker, New York, 1993

[FA] J. Kollár et al., Flips and abundance for algebraic threefolds. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991. Astérisque **211** (1992). Société Mathématique de France, Paris, 1992. pp. 1–258

[F1] O. Fujino, Osamu Abundance theorem for semi log canonical threefolds, Duke Math. J. **102** (2000), 513–532

[F2] O. Fujino, Termination of 4-fold canonical flips, Publ. Res. Inst. Math. Sci. **40** (2004), 231–237

[F3] O. Fujino, Addendum to paper [F2], Publ. Res. Inst. Math. Sci. **41** (2005), 251–257

[F4] O. Fujino, On termination of 4-fold semi-stable log flips, Publ. Res. Inst. Math. Sci. **41** (2005), 281–294

[F5] O. Fujino, What is log terminal? in [Book], 49–62

[F6] O. Fujino, Special termination and reduction to pl flips, in [Book], 63–75

[F7] O. Fujino, A transcendental approach to Kollár's injectivity theorem, I. arXiv:0704.0073, pp. 19

[F8] O. Fujino, A transcendental approach to Kollár's injectivity theorem, II. arXiv:0705.1199, pp. 20

[F9] O. Fujino, Vanishing and injectivity theorems for LMMP, arXiv:0705.2075, 30 pp.

[F10] O. Fujino, Notes on the log minimal model program, arXiv:0705.2076v2, 26 pp.

[F11] O. Fujino, On Kawamata's theorem. preprint (2007), revised version, arXiv:0910.1156, 12 pp.

[F12] O. Fujino, On Kollár's injectivity theorem (Japanese), Sûrikaiseki kenkyûsho Kôkyûroku, no. 1550 (2007), 131–140

[F13] O. Fujino, Effective base point free theorem for log canonical pairs—Kollár type theorem, Tohoku Math. J. **61** (2009), 475–481.

[F14] O. Fujino, Effective base point free theorem for log canonical pairs II—Angehrn–Siu type theorems, Michigan Math. J., to appear

[F15] O. Fujino, Theory of non-lc ideal sheaves—basic properties, Kyoto J. of Math., to appear

[F16] O. Fujino, Introduction to the log minimal model program for log canonical pairs, arXiv:0907.1506, 158 pp.

[F17] O. Fujino, On injectivity, vanishing and torsion-free theorems for algebraic varieties, Proc. Japan Acad. Ser. A Math. Sci. **85** (2009), no. 8, 95–100.

[F18] O. Fujino, Finite generation of the log canonical ring in dimension four, Kyoto J. of Math., to appear

[F19] O. Fujino, Introduction to the theory of quasi-log varieties, arXiv:0711.1203v8, 17 pp.

[F20] O. Fujino, Nonvanishing theorem for log canonical pairs, arXiv:0901.0409v3, 13 pp.

[F21] O. Fujino, Fundamental theorems for the log minimal model program, arXiv:0909.4445v2, 60 pp.

[FM] O. Fujino and S. Mori, A canonical bundle formula, J. Differential Geom. **56** (2000), 167–188

[GHS] T. Graber, J. Harris and J. Starr, Families of rationally connected varieties, J. Amer. Math. Soc. **16** (2003), 57–67

[HM1] C. Hacon and J. M^{c}Kernan, Boundedness of pluricanonical maps of varieties of general type, Invent. Math. **166** (2006), 1–25

[HM2] C. Hacon and J. M^{c}Kernan, On Shokurov's rational connectedness conjecture, Duke Math. J. **138** (2007), 119–136

[HM3] C. Hacon and J. M^{c}Kernan, On the existence of flips, arXiv:math/0507597, 28 pp.

[HM4] C. Hacon and J. M^{c}Kernan, Extension theorems and the existence of flips, in [Book], 76–110

[HM5] C. Hacon and J. M^{c}Kernan, The Sarkisov program, arXiv:0905.0946, 16 pp.

[HM6] C. Hacon, J. M^{c}Kernan, Existence of minimal models for varieties of log general type II, J. Amer. Math. Soc., to appear

[I1] S. Iitaka, Genus and classification of algebraic varieties, I. (Japanese) Sûgaku **24** (1972), 14–27

[I2] S. Iitaka, Algebra and geometry—classification and enumeration of algebraic varieties, II. (Japanese) Sûgaku **29** (1977), 334–349 (1978)

[Kc1] Y. Kachi, Flips of semi-stable 4-folds whose degenerate fibers are unions of Cartier divisors which are terminal factorial 3-folds, Math. Ann. **307** (1997), 647–662

[Kc2] Y. Kachi, Flips from 4-folds with isolated complete intersection singularities, Amer. J. Math. **120** (1998), 43–102

[Kr] K. Karu, Minimal models and boundedness of stable varieties, J. Algebraic Geom. **9** (2000), 93–109

[K1] Y. Kawamata, Pluricanonical systems on minimal algebraic varieties, Invent. Math. **79** (1985), 567–588

[K2] Y. Kawamata, The Zariski decomposition of log canonical divisors, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 425–433, Proc. Sympos. Pure Math. **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987

[K3] Y. Kawamata, Small contractions of four dimensional algebraic manifolds, Math. Ann. **284** (1989), 595–600

[K4] Y. Kawamata, On the length of an extremal rational curve, Invent. Math. **105** (1991), 609–611

[K5] Y. Kawamata, Abundance theorem for minimal threefolds, Invent. Math. **108** (1992), 229–246

[K6] Y. Kawamata, Termination of log flips for algebraic 3-folds, Internat. J. Math. **3** (1992), 653–659

[K7] Y. Kawamata, Deformations of canonical singularities, J. Amer. Math. Soc. **12** (1999), 85–92

[K8] Y. Kawamata, On the extension problem of pluricanonical forms, In Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 193–207, Contemp. Math. **241**, Amer. Math. Soc., Providence, RI, 1999

[K9] Y. Kawamata, Flops connect minimal models, Publ. Res. Inst. Math. Sci. **44** (2008), 419–423

[K10] Y. Kawamata, A classification theory of higher-dimensional algebraic varieties—towards a theory of minimal models, Sûgaku **40** (1988), 97–114. Translated in Sugaku Expositions **3** (1990), 1–24.

[K11] Y. Kawamata, Recent developments in the theory of minimal models, Sûgaku **45** (1993), 330–345. Translated in Sugaku Expositions **8** (1995), 197–216

[K12] Y. Kawamata, Finite generation of a canonical ring, *Current developments in mathematics, 2007*, 43–76, Int. Press, Somerville, MA, 2009.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, In Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987.

[KST] S. Kebekus, L. Solá Conde and M. Toma, Rationally connected foliations after Bogomolov and McQuillan, J. Algebraic Geom. **16** (2007), 65–81

[KK] S. Keel and J. Kollár, Log canonical flips, in book [FA], pp. 95-101

[KKMc1] S. Keel, K. Matsuki and J. M^{c}Kernan, Log abundance theorem for threefolds, Duke Math. J. **75** (1994), 99–119

[KKMc2] S. Keel, K. Matsuki and J. M^{c}Kernan, Corrections to: "Log abundance theorem for threefolds", Duke Math. J. **122** (2004), 625–630

[KeM] S. Keel and S. Mori, Quotients by groupoids, Ann. of Math. (2) **145** (1997), 193–213

[Kh] J.J. Kohn, Subellipticity of the $\overline{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions, Acta Math. **142** (1979), 79–122

[Ko1] J. Kollár, Higher direct images of dualizing sheaves, I. Ann. of Math. (2) **123** (1986), 11–42. and II. Ann. of Math. (2) **124** (1986), 171–202.

[Ko2] J. Kollár, Projectivity of complete moduli, J. Differential Geom. **32** (1990), 235–268

[Ko3] J. Kollár, Crepant descent, In book [FA], pp. 75-87

[Ko4] J. Kollár, Singularities of pairs, In Algebraic geometry—Santa Cruz 1995, 221–287, Proc. Sympos. Pure Math. **62**, Part 1, Amer. Math. Soc., Providence, RI, 1997.

[KMa] J. Kollár and K. Matsuki, Termination of 3-fold log flips near the reduced boundary, In book [FA], pp. 89-93

[KM] J. Kollár and S. Mori, Birational geometry of algebraic varieties (Japanese), Iwanami 1998. Translated as Birational geometry of algebraic varieties With the collaboration of C. H. Clemens and A. Corti. Cambridge Tracts in Mathematics **134**. Cambridge University Press, Cambridge, 1998

[L] R. Lazarsfeld, Positivity in algebraic geometry, II. Positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge Springer-Verlag, Berlin, 2004

[Ma1] K. Matsuki, Termination of flops for 4-folds, Amer. J. Math. **113** (1991), 835–859

[Ma2] K. Matsuki, Introduction to the Mori program, Universitext. Springer-Verlag, New York, 2002

[Mi] Y. Miyaoka, Deformations of a morphism along a foliation and applications, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 245–268, Proc. Sympos. Pure Math. **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987

[MM] Y. Miyaoka and S. Mori, A numerical criterion for uniruledness, Ann. of Math. (2) **124** (1986), 65–69

[M1] S. Mori, Projective manifolds with ample tangent bundles, Ann. of Math. (2) **110** (1979), 593–606

[M2] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math. (2) **116** (1982), 133–176

[M3] S. Mori, On 3-dimensional terminal singularities, Nagoya Math. J. **98** (1985), 43–66

[M4] S. Mori, Flip theorem and the existence of minimal models for 3-folds, J. Amer. Math. Soc. **1** (1988), 117–253

[M5] S. Mori, Hartshorne conjecture and extremal rays, (Japanese) Sûgaku **35** (1983), 193–209. Translated in Sugaku Expositions **1** (1988), 15–37

[Nd] A. Nadel, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. (2) **132** (1990), 549–596

[N1] N. Nakayama, Invariance of the plurigenera of algebraic varieties under minimal model conjectures, Topology **25** (1986), 237–251

[N2] N. Nakayama, Invariance of the plurigenera of algebraic varieties, Preprint RIMS-1191, 1998

[N3] N. Nakayama, Zariski-decomposition and abundance. MSJ Memoirs **14**. Mathematical Society of Japan, Tokyo, 2004

[O] T. Ohsawa, On a curvature condition that implies a cohomology injectivity theorem of Kollár–Skoda type, Publ. Res. Inst. Math. Sci. **41** (2005), 565–577.

[OT] T. Ohsawa and K. Takegoshi, On the extension of L^2 holomorphic functions, Math. Z. **195** (1987), 197–204

[P] M. Păun, Siu's invariance of plurigenera: a one-tower proof, J. Differential Geom. **76** (2007), 485–493

[R] M. Reid, Young person's guide to canonical singularities, In Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math. **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414

[S-B] N. Shepherd-Barron, Miyaoka's theorems on the generic seminegativity of T_X and on the Kodaira dimension of minimal regular threefolds, in book [FA], pp. 103–114

[Sh1] V. V. Shokurov, A nonvanishing theorem, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), 635–651. English translation: Math. USSR-Izv. **26** (1985), 591–604.

[Sh2] V. V. Shokurov, Three-dimensional log flips, With an appendix in English by Yujiro Kawamata, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), 105–203; translation in Russian Acad. Sci. Izv. Math. **40** (1993), 95–202

[Sh3] V. V. Shokurov, 3-fold log models, Algebraic geometry, 4. J. Math. Sci. **81** (1996), 2667–2699

[Sh4] V. V. Shokurov, Prelimiting flips, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 82–219; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 75–213

[Sh5] V. V. Shokurov, Letters of a bi-rationalist, IV. Geometry of log flips, Algebraic geometry, 313–328, de Gruyter, Berlin, 2002

[Sh6] V. V. Shokurov, Letters of a bi-rationalist: VII, Ordered termination, Tr. Mat. Inst. Steklova, 2009, **264**, 184–208.

[Si1] Y.-T. Siu, Invariance of plurigenera, Invent. Math. **134** (1998), 661–673

[Si2] Y.-T. Siu, Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type, Complex geometry (Göttingen, 2000), 223–277, Springer, Berlin, 2002

[Si3] Y.-T. Siu, Multiplier ideal sheaves in complex and algebraic geometry, Sci. China Ser. A **48** (2005), suppl., 1–31

[Si4] Y.-T. Siu, A general non-vanishing theorem and an analytic proof of the finite generation of the canonical ring, arXiv:math/0610740

[Tk1] H. Takagi, Remarks on Gorenstein terminal fourfold flips, J. Math. Sci. Univ. Tokyo **5** (1998), 149–164

[Tk2] H. Takagi, 3-fold log flips according to V. V. Shokurov, arXiv:math/9803145, 61 pp.

[Tk3] H. Takagi, The existence of Gorenstein terminal fourfold flips, arXiv:math/9905067v2

[Ty1] S. Takayama, Pluricanonical systems on algebraic varieties of general type, Invent. Math. **165** (2006), 551–587

[Ty2] S. Takayama, On the invariance and the lower semi-continuity of plurigenera of algebraic varieties, J. Algebraic Geom. **16** (2007), 1–18

[Tg] K. Takegoshi, Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms, Math. Ann. **303** (1995), 389–416

[Ts1] H. Tsuji, Pluricanonical systems of projective varieties of general type. I, Osaka J. Math. **43** (2006), no. 4, 967–995.

[Ts2] H. Tsuji, Pluricanonical systems of projective varieties of general type. II, Osaka J. Math. **44** (2007), no. 3, 723–764.

Osamu Fujino
Department of Mathematics, Faculty of Science,
Kyoto University, Kyoto 606-8502, Japan
`fujino@math.kyoto-u.ac.jp`

Translated by M. Reid