Introduction to the theory of Enriques surfaces.

Igor Dolgachev

S: proj. sm. surface/k, $k=\bar{k}$.

- $h^i(\mathcal{F}) := \dim_k H^i(S,\mathcal{F})$.
- $p_g = h^2(\mathcal{O}_S)$, $q = h^1(\mathcal{O}_S)$.

- Riemann-Roch
  - D: divisor on S, $[D]\in \mathrm{Pic}(S)$: linear equiv. class of D.
  - $h^i(D) = h^i(\mathcal{O}_S(D))$, K_S: canonical class.
  - $h^0(D) - h^1(D) + h^2(D) = \dfrac{D^2 - DK_S}{2} + 1 - q + p_g$.
- Duality: $h^i(D) = h^{2-i}(K_S - D)$

MMP $\Rightarrow$ $\exists f: S \to S'$; birational morph. s.t.

$$\begin{cases} \text{(a)} & K_{S'}: \text{nef (i.e. } K_{S'}\cdot C \geq 0 \text{ for any curve } C \text{ on } S) \\ \text{or (b)}. & S' = \mathbb{P}^2 \text{ or } \exists \pi: S' \xrightarrow{\mathbb{P}^1} B: \text{sm. proj. } \cancel{\text{curve}} \text{ cone.}\end{cases}$$

- If (b) happens we say that S is ruled and if $S' = \mathbb{P}^2$ or $B = \mathbb{P}^1$, S is rational.

- S: rational $\Rightarrow$ $p_g = q = 0$.

1899 G. Castelnuovo attempted to prove that

$p_g = q = 0 \Rightarrow S$: rational.

- terminates of adjoints.
  $\forall$ curve C $|C + mK_S| = \emptyset$ for $m \gg 0$.
  $C' \in |C + (m-1)K_S| \neq \emptyset \Rightarrow C' \simeq \mathbb{P}^1$

Additional Assumption $p_g = h^0(K_S)$.

$h^0(2K_S) = 0 \Rightarrow S$: rational.

Abundance: K_S is nef $\Rightarrow K_S^2 \geq 0$.

$$\underset{\substack{\parallel \\ h^0(2K_S)}}{h^0(-K_S) + h^2(-K_S)} \geq K^2 + 1 - q + p_g \geq 1 \Rightarrow -K_S \geq 0.$$

$q = 0 \Rightarrow B = \mathbb{P}^1$.

Enriques example.

Q_1, Q_2: quadr. form.

$X: Q_1(X_1X_2X_3, X_0X_2X_3, X_0X_1X_3, X_0X_1X_2)$
$\quad + X_0X_1X_2X_3\, Q_2(X_0, X_1, X_2, X_3) = 0.$

$\rightsquigarrow X$ has "ordinary singurality"

$k[[x,y,z]]/(xy) \quad k[[x,y,z]]/(xyz)$

Pinch points $\quad k[[x,y,z]]/(x^2 - y^2z)$.

$p: \bar{X} \to X$ (normalization) $\rightsquigarrow \bar{X}$: smooth.

- $c_0 := \mathcal{H}om_{\mathcal{O}_X}(p_*\mathcal{O}_X, \mathcal{O}_X)$ conductor ideal $\subset \mathcal{O}_X$.
- $c := c_0\mathcal{O}_{\bar{X}}$. $\quad p_*c = c_0$.
- $c_0 = \mathrm{Ann}(p_*\mathcal{O}_{\bar{X}}/\mathcal{O}_X)$.

- $\omega_{\bar{X}} \cong c \otimes p^*\omega_X = c \otimes p^*(\mathcal{O}_X(d-4))$.

$X \subset \mathbb{P}^3$. $\quad \omega_X = \mathcal{O}_X(d-4)$ $\quad d = \deg X$.

ordinary singularity $\rightsquigarrow$
- $c_0 = \mathcal{I}_\Gamma$, Γ = double curve.
- $c = \mathcal{O}_{\bar{X}}(-\Delta)$ (Δ = (effective) divisor on $\bar{X}$).

$\rightsquigarrow \boxed{\omega_{\bar{X}} = p^*(\omega_X \otimes \mathcal{I}_\Gamma) = \mathcal{O}_{\bar{X}}(-\bar{\Delta}) \otimes p^*(\omega_X).}$

$\mathcal{I} = \mathcal{I} = I.$ $\rightsquigarrow$ • $\mathcal{O}_{\bar{X}}(2K_{\bar{X}}) = \omega_{\bar{X}}^{\otimes 2} = p^*(\mathcal{O}_X(2(d-4))) \otimes \mathcal{I}_\Gamma^{\langle 2 \rangle}$.

$S = \bar{X}$. $\rightsquigarrow$ $\rightsquigarrow h^0(\mathcal{O}_S(2K_S)) = 1$. $\quad 2K_S \sim 0$.

- $q = 0, p_g = 0$, but $h^0(2K_S) \neq 0$. $\quad \rightsquigarrow$ not rational.

- $T: (x_0 : x_1 : x_2 : x_3) = (y_2y_3 : y_0y_1 : y_0y_2 : y_0y_3)$ birat. transf. of $\mathbb{P}^3$.

$F = y_0^4 y_0^3 y_3^2\, Q_1(y_0y_1, y_2y_3, y_1y_3, y_1y_2)$
$\quad + y_0^3 y_1 y_2^3 y_3^2\, Q_2(y_0y_1, y_2y_3, y_1y_3, y_1y_2)$.

$G = y_0 Q_1(\cdots) + y_1 Q_2(\cdots) = 0$. Quintic surface $\underset{b.}{\simeq} X$ Enriques exh

- $G = 0$ is a normal quintic surface in $\mathbb{P}^3$. w/ singular pts at: the vertices of the coordinate tetrahedron.

$(1:0:0:0)$ $(0:1:0:0)$: ordinary tacnodes.
$(0:0:1:0), (0:0:0:1)$: ordinary triple pinch.

$z^2 = f_4(x,y)$: tacnode  $\leftarrow$ $\{E_{(-2)}$ $\{E_{-3}$

$z=0$: tacnode tangent plane

$x_0^2 A_3(x_1,x_2,x_3) + 2x_0 B_4(x_1,x_2,x_3) + C_5(x_1,x_2,x_3) = 0$.

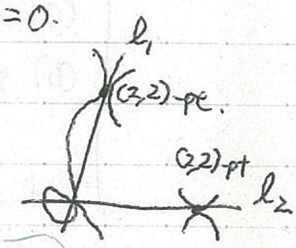
project from $(1:0:0:0)$. $X \overset{2:1}{\dashrightarrow} \mathbb{P}^2$.

$\underline{B_4^2 - C_5A_3 = 0}$: Branch locus

degree 8 curve

$l_1 l_2 A_6(x_1,x_2,x_3)$ $\boxed{z^2 = l_1 l_2 A_6(x_1,x_2,x_3)}$

Enriques sextic.



1894/1896 Castelnuovo: gave another example of a non-rat. surface w/ $q=p_g=0$.

$X \subset \mathbb{P}^3$: degree 7.

- (a) triple line
- (b) double conic separate from line.
- (c) 3 tacnode ~~outside~~ with tacnodal plane contains the triple line.

$\rightsquigarrow X: f_7 = f_3^2 h_1 + \alpha\beta\gamma f_4 = 0$.

• $\bar{X}$ = minimal resol. of X. $q=p_g=0$. $\omega_X = \mathcal{O}_X(7-4) = \mathcal{O}_X(3)$

$p_g=0$. $h^0(2K_{\bar X})=2$ $2K_{\bar X} \not\sim 0$. $2|2|2|$.

$\bar{X} \to \mathbb{P}^1$: fibers is an ellip. curve, $2K_{\bar X} \simeq F$ — fiber.

$\mathrm{Kod}(\bar X)=1$.

Kod(Enrique) = 0. B. Areback? M. Artin

F. Enriques, 1906

$K_X \not\sim 0$, $2K_X \sim 0$, $q=0$, birat. isom. to a sextic passing through the edge of tetrahedron or its degeneration.

• $1-q+p_g = \frac{1}{12}(K_S^2+c_2)$ $c_2 = \sum(-1)^i \underline{b_i(S)}$ Betti # (l-adic coh. or classical coh.)

Classical definition

$q=p_g=0$, $2K_S=0$. $\rightsquigarrow c_2=12$, $b_1=0$, $b_2=10$.

Modern Definition

$b_1=0, b_2=10, \mathrm{Kod}(S)=0$ $\quad$ (K_S: num. trivial)

$\Rightarrow c_2=12, K_S^2=0.$ $\rightsquigarrow q=p_g$, $\mathrm{Kod}(S)=0 \rightsquigarrow p_g \leq 1.$

(a) $q=p_g=1$ $\quad K_S=0$ $\quad H^1(S,\mathcal{O}_S)=k.$

(b) $q=p_g=0$ $\quad K_S\neq 0.$

$\mathrm{Kod}(\cancel{K_S})=2$ $\quad$ gen. type

$\mathrm{Kod}(S)=1$ $\quad$ ellip. surf.

$\mathrm{Kod}(S)=0$ $\quad K_S\equiv 0.$ $\quad$ a) Abelian surface $q=2, K_S=0$

b) K3 surface $q=0, K_S=0, b_2=22$

* c) Enriques surface

d) hyperelliptic surface $q=1, p_g=0, b_2=2$

$E\times E'/G$

$\mathrm{Kod}(S)=-\infty$ $\quad S$: ruled.

(a) happens only if $p=\mathrm{char}\, k=2.$

Torsion divisor in $\mathrm{Pic}(S)$

$h^0(D)+h^0(K_S-D)\geq 1 \rightsquigarrow D=0$ or $K_S-D=0.$

$\rightsquigarrow \mathrm{Tor}(\mathrm{Pic}(S))=\langle K_S\rangle.$

(a) $\mathrm{Tors}(\mathrm{Pic}(S))=0$

(b) $\mathrm{Tors}(\mathrm{Pic}(S))=\mathbb{Z}/2$ $\quad$ gen. by $K_S.$

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Noether's Formula $1-q+p_g=\frac{1}{12}(K_S^2+c_2)$

$q=h^1(\mathcal{O}_S)$, $p_g=h^2(\mathcal{O}_S)=h^0(K_S)$, $c_2=\sum(-1)^i b_i(S)$, $b_i(S)=\dim H^i(S,\mathbb{Q}_\ell)$ $p\neq 0$)

Classical def'n of an Enriques surface.

$q=p_g=0$, $2K_S=0$ (in $\mathrm{Pic}(S)$).

- $\underline{\mathrm{Pic}}_{S/k}$ = Picard scheme　$\underline{\mathrm{Pic}}^0_{S/k}$: connected comp. of 1 ; fin. type, proper$/k$.
- $q=h^1(\mathcal{O}_S)=\dim T\underline{\mathrm{Pic}}^0_{S/k}$

  so $q=0 \Rightarrow \underline{\mathrm{Pic}}^0_{S/k}=\{1\}$.
- Standard computation (Kummer theory) $\Rightarrow b_1=0$.

Noether's formula $\Rightarrow$ $c_2=12$, $b_2=10$.

Modern definition of an Enriques surf.

Enriques surfaces : a minimal surface S of $\mathrm{Kod}(S)=0$ & $b_1=0$, $b_2=10$. ($\Rightarrow c_2=12$).

Classification

$\Rightarrow$ other surfaces of $\mathrm{Kod}(S)=0$.

$$\begin{cases}(a)\ K3 : b_2=22,\ b_1=0\\ (b)\ \text{Tori.}\quad c_2=0,\ q=2\\ (c)\ \text{hyperelliptic.}\end{cases}$$

- S : Enriques $\mathrm{Kod}(S)=0 \Rightarrow K_S^2=0$.

  $\therefore h^0(K_S)\le 1$. $\overset{\text{Noether}}{\Rightarrow}$ $q=p_g\le 1$.
- $b_1=0 \Rightarrow (\underline{\mathrm{Pic}}^0_{S/k})_{red}=\{1\}$.

  - $q=1 \Rightarrow T_0\neq 0 \Rightarrow \underline{\mathrm{Pic}}_{S/k}$ is not reduced.

Thm. (Oort)

$$\boxed{\text{All group schemes are reduced if } p=0.}$$

$\therefore q=1 \Rightarrow \mathrm{char}\,k=p>0$.

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Further Analysis (Bombieri–Mumford) $\Rightarrow p=2$.

- $p\neq 2 \Rightarrow q=p_g=0$.

- $p=2$ if $\mathrm{Pic}^{\equiv\tau}_{S/k}$ (Numerically Equivalence) : not reduced.

  $\Rightarrow \mathrm{Pic}^{\tau}_{S/k} \cong \mu_2, \alpha_2$ (No k-valued pt $\neq 1$)

  $\mu_2 \to$ μ_2-surfaces, $\alpha_2 \to$ α_2-surfaces

  ($\exists k$-val. pt.

  if $\mathrm{Pic}_{S/k}$: reduced $\Rightarrow \mathrm{Pic}^{\tau}_{S/k} = (\mathbb{Z}/2)_k \rightsquigarrow$ Classical Enriques Surfaces.

- D: torsion divisor, $nD\sim 0$.

  $\Rightarrow h^0(D)+h^0(K-D)\geq 1$.

  either $\begin{cases} \cdot\ D\geq 0 \Rightarrow D=0. \\ \text{or} \cdot\ K-D\geq 0 \Rightarrow K_S=D. \end{cases}$ $\therefore \mathrm{Tors}(\mathrm{Pic}(S))=\langle K_S\rangle$.

- Classical : $\Rightarrow K_S\neq 0$, $\mathrm{Tor}=\mathbb{Z}/2$.

  Non-classical $\Rightarrow \mathrm{Tors}=\{1\}$.

- $\mathrm{Kod}(S)=0 \Rightarrow K_S\equiv 0$ (numerically equiv.).

  $2K_S=0$: always true for Enriques surfaces.

- Classical Enriques Surfaces is an Enriques Surfaces in its classical def'n.
  $K_S\neq 0$, $p\neq 2$.

- $K_S\neq 0$ $p\neq 2$

  $\Rightarrow \exists X \xrightarrow{2:1} S$; étale cover corr. to K_S.

  $\Rightarrow X$ is a K3 surface.

★ Same is true if S is a μ_2-surface.

- $\exists \sigma : X\to X$; fixed-pt. free involution, $S=X/\langle\sigma\rangle$.

- $\underset{\text{num.equiv.}}{\overset{\mathrm{Num}(S)}{\mathrm{Pic}(S)/\equiv}} \cong \mathbb{Z}^{\rho}$, $\langle\cdot,\cdot\rangle$: intersection product.

  $k=\mathbb{C} \Rightarrow \rho=h_2=10$, $\mathrm{Pic}(S)=H^2(S,\mathbb{Z})$.

  Hodge decomp: $H^2(\mathbb{C}) = \underset{0}{H^{2,0}} + H^{1,1} + \underset{0}{H^{0,2}} = H^{1,1}$.

- $H^{1,1} \cap H^2(S,\mathbb{Z}) = H^2(S,\mathbb{Z})$.

  $H^2(S,\mathbb{Z})/_{\mathrm{Tors}} = \mathrm{Num}(S) = \mathbb{Z}^{\rho}$.

- Poincaré $\Rightarrow$ $\mathrm{Num}(S)$ is unimodular.
- CCS: curve $\rightsquigarrow$ $p_a(C) = \frac{C^2 + CK}{2} + 1 = \frac{C^2}{2} + 1$. $\therefore C^2 \in 2\mathbb{Z}$.

  $\rightsquigarrow \mathrm{Num}(S)$: also even ~~Hodge str.~~
- Hodge Index thm. $\Rightarrow$ $\mathrm{sign}(\mathrm{Num}(S)_{\mathbb{R}}) = (1, \rho - 1) = (1, 9)$.
- Milnor $\Rightarrow$ $\mathrm{Num}(S) = U \oplus E_8$ $\left(U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$.

  : E_8: unimodular neg. def. lattice of rank 8.

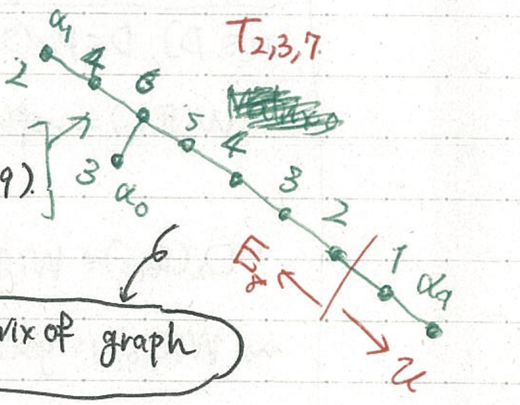
$p=2$: $\rho \le 10$. $\mathrm{Num}(S) \otimes \mathbb{Z}_\ell = \begin{cases} H^2_{et}(S, \mathbb{Z}_\ell) & (\ell \ne p) \\ H^2_{fl}(S, \mathbb{Z}_\ell) & (\ell = p) \end{cases}$

if $\rho = 10$ $\mathrm{Num}(S) = U \oplus E_8$. $\rightsquigarrow$ $\boxed{\rho = 10\,?}$

Mumford, W. Lang

- $E_{10} := U \oplus E_8 \hookrightarrow \mathbb{Z}^{1,10} = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{10}$
  - $e_0^2 = 1$, $e_i^2 = \cancel{0}\ -1$, $e_i e_j = 0$ ~~for~~ $j \ne i$.
  - $K_{10} = 3e_0 - \sum_{i=1}^{10} e_i$.

$\rightsquigarrow E_{10} = K_{10}^{\perp}$ $\left[\begin{array}{l} \alpha_0 = e_0 - e_1 - e_2 - e_3 \\ \alpha_i = e_i - e_{i+1} \quad (1 \le i \le 9) \end{array}\right]$

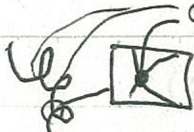


- Matrix of $E_{10} = -2I_{10} + A$

  ~~incidence~~ incidence matrix of graph

- $X \xrightarrow{\pi} \mathbb{P}^2$: blow-up 10 pts.
  - $\mathrm{Pic}\, X = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{10}$. $\quad e_0 = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$

    $= \mathbb{Z}^{1,10}$ $\quad e_i = [E_i]$: exceptional divisor class
  - $K_{10} = -K_X$, $E_{10} = K_X^{\perp} \subset \mathrm{Pic}(X)$.

Take special points  9 pts. blow up $\rightsquigarrow$ Coble surfaces

sextic curve on $\mathbb{P}^2$.

Moduli space exists only as a stack.

Thm. (C. Liedtke)

$$\boxed{\begin{array}{lll} p\neq 2. & \mathcal{M}_{En.} & \text{is a smooth irreducible unirat. of } \dim = 10. \\ p=2 & \mathcal{M}_{En} & \text{is union of 2 is. irr. div. of } \dim = 10 \\ & & \begin{cases} \mathcal{M}_1 : \text{classical Enriques surf.} \\ \mathcal{M}_2 : \mu_2\text{-surfaces} \\ \mathcal{M}_1 \cap \mathcal{M}_2 : \alpha\text{-surfaces}, \quad \dim = 9. \end{cases} \end{array}}$$

- $k = \mathbb{C}$.

  $\mathcal{M}_{En}$ has a moduli. space.

  = a. coarse moduli space comes from the period thm.

  = arithmetic quotient of a 10-dim sym. domain of type K3.

Polarized Enriques Surfaces

- (S, D), $D \in \mathrm{Pic}(S)$, D is nef.. $|D|$: base pt. free
- $W(\mathbb{E}_{10})$ = reflection group of $\mathbb{E}_{10}$.

  generates by $s_\alpha : X \longmapsto X + (X, \alpha)\alpha$, $\alpha^2 = -2$.
- $O_2(\mathbb{E}_{10}) = W(\mathbb{E}_{10}) \times \{\pm 1\}$.

$\rightsquigarrow$ $W(\mathbb{E}_{10})$ is gen'd by s_{α_i} $(i = 0, 1, \cdots, 9)$.

- $W^1(S) = W(\mathbb{E}_{10})$.

  $\cup$

  $W^n(S)$ = gen. by s_α $(\alpha = [R], R \cong \mathbb{P}^1, R^2 = -2)$.

• $D^2 \geq 0$ $(\Leftrightarrow |D| \neq \emptyset)$ ⇝ $\exists\, w \in W^n(S)$, $w(D)$: nef.

$D = D_0 + \sum R_i$: D_0: nef., $R_i \simeq \mathbb{P}^1$.

$$\phi(D) := \min_{\substack{E^2=0 \\ E:\text{nef}}} \{D \cdot E\}$$

⇝ • $\phi(D)=1$ $\Leftrightarrow$ $|D|$ has base pts.

• $\phi(D)=2$ $\Leftrightarrow$ $|D|$ defines deg=2 map onto a rational surf. or bir. morph. onto a non-normal surface.

• $\phi(D)\geq 3$ $\Leftrightarrow$ $|D|$ defines a birat. morph. onto a normal surfaces w/ ADE-singular pt.

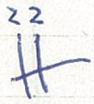
E: nef. and $E^2=0$.

⇝ either (a) $\frac{1}{2}[E]$ doesn't exist in $\mathrm{Num}(S)$ $\Rightarrow$ $|E| = \{E\}$.

or (b) $\frac{1}{2}[E]$ exists. " $\Rightarrow$ $|E|$ is a pencil.

$S \xrightarrow{f} \mathbb{P}^1$: an elliptic fibration

• $p \neq 2$: 2 double fiber.



(conversely any ell. fib. w/ 2 double fiber is Enriques.)

$p=2$: 2 or 1 double fiber

$\boxed{\text{classical}}$ $\boxed{\text{non-classical.}}$

D ⇝ $\phi(D)^2 \leq D^2$.

• $\boxed{D^2=4}$ ⇝ $\begin{cases} \phi(D)=2 \\ \text{if } \phi(D)=1 \Rightarrow 2 \text{ base pt.} \end{cases}$ $S \xrightarrow{4:1} \mathbb{P}^2$.

$\dim|D| = 2$.

• $\boxed{D^2=6}$ ⇝ $S \xrightarrow{\text{bir.}} \mathbb{P}^3$. Enriques sextic. $D = |E_1+E_2+E_3|$.

• $\boxed{D^2=8}$ • $\phi(D)=2$ $S \to \mathbb{P}^4$; image is non-normal octic.

$|D| = |E_1+E_2|$. $E_2^2 = E_1^2 = 0$, $E_1E_2 = 1$.

$S \xrightarrow{2:1} \mathcal{D} \subset \mathbb{P}^4$

4 nodes $\mathbb{P}^1$

DP surf.

FANO MODEL

$\boxed{D^2=10}$ $\phi(D)=3.$

$3\Delta \sim E_1+\cdots+E_{10}$. $E_iE_j=\overset{1}{\cancel{0}}$, $E_i^2=0$.

$\Delta^2=10$, $\Delta \cdot E_i=3 \rightsquigarrow |\Delta| : S \hookrightarrow \mathbb{P}^5$: deg 10

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(S,D) $|D|$: b.pt. free

- $\underline{D^2=4}$; $S \xrightarrow{4:1} \mathbb{P}^2$
- $\underline{D^2=6}$; Enriques sextic.
- $\underline{D^2=8}$;  $DP_4 = Q_1 \cap Q_2 \subset \mathbb{P}^4$. $(\phi(D) \geq 2)$.
  $B = DP_4 \cap Q_3$.
- $\underline{D^2=10,\ \phi(D) \geq 3}$;
  Fano model in $\mathbb{P}^5$.

$\boxed{\text{Birational type of } \mathcal{M}_{En,2n}\text{?}}$

- Known: $\underline{n=2,3,4}$ $\mathcal{M}_{En,2n}$: rational.

Automorphisms of Enriques Surfaces (started by G. Fano)

- $\underline{\mathrm{Aut}}(S)$ proper loc. fin. type group scheme$/k$.
  $\mathrm{Aut}^0(S)$: of fin. type gp. sch.
- $\underline{\mathrm{Aut}}(S)/\underline{\mathrm{Aut}}^0(S)$: discrete group.

- $\mathrm{Kod}(S) \geq 0 \rightsquigarrow \dim \underline{\mathrm{Aut}}^0(S) = 0$.
- $q=0 \rightsquigarrow \mathrm{Aut}(S)$: discrete group. $(= \underline{\mathrm{Aut}}(S)(k))$.
- $T(\underline{\mathrm{Aut}}(S))$ may be non-trivial.
  $H^0(S, \Theta_S) \neq 0$ for classical Enriques if $p=2$.

- $\rho: \mathrm{Aut}(S) \longrightarrow O(\mathrm{Num}(S)) = O(\mathbb{E}_{10}) = W(\mathbb{E}_{10}) \times \{\pm 1\}$

$$\begin{array}{ccc} \mathrm{Aut}(S) & \longrightarrow & O(\mathbb{E}_{10}) \\ & \searrow & \downarrow \\ & & W(\mathbb{E}_{10}) \end{array}$$

- S: unnodal if S does not contain (-2)-curve $(\cong \mathbb{P}^1)$.
- $2\mathbb{E}_{10} \subset \mathbb{E}_{10}$ $\qquad \mathbb{E}_{10}/2\mathbb{E}_{10} \cong \mathbb{F}_2^{10} \xrightarrow{q} \mathbb{F}_2$.
  $\mathbb{E}_{10} \cong \mathbb{Z}^{10}$ $\qquad \mathbb{E}_{10}/2\mathbb{E}_{10} = \bar{\mathbb{E}}_{10}$ $\qquad q$: quadr. form; nondeg. of even type.
  $q(x + 2\mathbb{E}_{10}) = x^2/2 \mod 2$.

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- $q = x_0x_1 + \cdots + x_9x_{10}$.
  $O(\mathbb{F}_2^{10}, q) = O^+(10, \mathbb{F}_2)$
- $W(\mathbb{E}_{10}) \twoheadrightarrow O^+(10, \mathbb{F}_2) \rightsquigarrow \mathrm{Ker} =: W(\mathbb{E}_{10})(2)$.
  $= \{w : w(x) - x \in 2\mathbb{E}_{10}\ {}^\forall x \in \mathbb{E}_{10}\}$.

Thm. If S: unnodal, then $\mathrm{Ker}(\rho) = \{1\}$ and $\rho(\mathrm{Aut}(S)) \supset W(\mathbb{E}_{10})(2)$.
(in particular, $\rho(\mathrm{Aut}(S))$: finite index).

Pf. (A. Coble)

- $S \xrightarrow{2:1} \mathcal{D}_4$ $\quad |2E_1 + 2E_2|$ $\quad E_1 \cdot E_2 = 1$; $\langle E_1, E_2 \rangle \simeq U$.
  $\sigma = \mathrm{Aut}(S/\mathcal{D}_4)$ $\quad \sigma_* := \rho(\sigma) =$
  $\mathbb{E}_{10} = U \oplus \underset{\underset{\mathbb{E}_8}{\cong}}{U^\perp}$. $\ni (1_U, -1_{\mathbb{E}_8}) \in W(\mathbb{E}_{10})(2)$.

Lemma $W(\mathbb{E}_{10})(2) = \langle\langle \sigma_* \rangle\rangle$ = minimal normal subgp. containing σ_*
$\forall U \hookrightarrow \mathbb{E}_{10}$,

//

(Barth, Peters, Nikulin using periods $k = \mathbb{C}$)

Q: $G := \mathrm{Aut}(S)$, S: unnodal. $\rightsquigarrow$ $\underline{G/W(\mathbb{E}_{10})(2)} \hookrightarrow O^+(10, \mathbb{F}_2)$.
What could it be?

- S: nodal. i.e. $\exists$ a $\mathbb{P}^1$ on S
  $\overline{\mathbb{E}}_{10} \ni x = R \bmod 2\mathbb{E}_{10}$ $\quad (R \simeq \mathbb{P}^1)$.

  $\Delta \subset \mathbb{F}_2^{10} \xrightarrow{q} \mathbb{F}_2$. $\quad x^2 = 2$. $\rightsquigarrow q(x) = 1 \bmod 2$.
  $\overset{\|}{q^{-1}(1)}$ $\rightsquigarrow \Delta$: -invariant of S (V. Nikulin)
  $R := p^{-1}(\langle \Delta \rangle^\perp)$. $\quad (p : \mathbb{E}_{10} \to \overline{\mathbb{E}}_{10})$

$\mathbb{R} = \{x \in \mathrm{Num}(S) : x \cdot R \equiv 0 \mod 2. \ (\text{for any } R \simeq \mathbb{P}^1)\}.$

$\rightsquigarrow$ $\mathbb{R} \subset \mathrm{Num}(X)$ of index 2^d, $d = \dim\langle\Delta\rangle$.

$\rho : \mathrm{Aut}(S) \longrightarrow O(\mathbb{R})$.

• S is <u>general nodal</u> if $\#\Delta = 1$.

i.e. all smooth rat. curves are congruent mod $2E_{10}$.

$\rightsquigarrow$ $R_1 - R_2 = 2x$ for some $x \in \mathrm{Num}(X)$.

• $\mathrm{Aut}\, S$ is also known.

• $\mathbb{R} \subset E_{10}$: index 2. $\rightsquigarrow$ $\mathbb{R} \simeq U \oplus E_7 \oplus A_1$. Reye lattice.

$\mathbb{R} = L_1 \simeq E_7 \oplus L_1^{\perp} \simeq U \oplus A_1$.

$L_1 = \langle \beta_0, \beta_1, \cdots, \beta_6 \rangle$

$= A_1 \quad L_2 = \langle \beta_0, \beta_2, \cdots, \beta_7 \rangle \quad U \qquad L_2 \oplus U(2) \oplus A_1$

$= A_1 \quad L_3 = \langle \beta_0, \beta_2, \cdots, \beta_8 \rangle \quad A_1. \qquad L_3 \oplus U(2)$

• $O(T_{2,4,6}) = W(T_{2,4,6}) \times \{\pm 1\}$.

$\rho(\mathrm{Aut}\, S) \subset W(T_{2,4,6})$. $\qquad T_{2,4,6}/2T_{2,4,6} = \mathbb{F}_2^{10}$.

$W(T_{2,4,6})(2) \triangleleft W(T_{2,4,6})$; quot. $W(T_{2,4,6}) / W(T_{2,4,6})(2) \simeq 2^8 \cdot Sp(8, \mathbb{F}_2)$.

$W(T_{2,4,6})(2)' :=$ preimage of 2^8 $\qquad W(T_{2,4,6})' / W(T_{2,4,6}) \simeq 2^8$.

$\boxed{\text{Thm (A. Coble)} \quad \bullet\ \mathrm{Ker}(\rho) = \{1\}. \quad \bullet\ \rho(\mathrm{Aut}(S)) \supset W(T_{2,4,6})(2)'.}$

Pf. $B = -1_{L_1} \oplus 1_{L_1^{\perp}}$, $G = -1_{L_2} \circ \delta_{pq}$, $K = -1_{L_3} \oplus 1_{L_3^{\perp}}$.

$$\underset{W(T_{2,4,6})(2)}{\rotatebox{90}{$\in$}} \qquad \underset{W(T_{2,4,6})(2)}{\rotatebox{90}{$\in$}} \qquad \underset{W(T_{2,4,6})(2)}{\rotatebox{90}{$\in$}}$$

Lemma

$$\boxed{W(T_{2,4,6})(2)' = \langle\!\langle B, K, G \rangle\!\rangle}$$

$h_* = B$, $g_* = G$, $\kappa_* = K$.

B: E. Bertini

- $|4E_1 + 2R|$ $(R \simeq \mathbb{P}^1, R \cdot E_1 = 1)$.
  $S \xrightarrow{2:1} \underline{D_4}$. ⇝ $\underline{\underline{h}}$: deck transformation of this covering.
  quartic DP, $\underline{2A_1, A_3}$

K: S. Kantor.

- $|2E_1 + 2E_2 - R|$ $(E_1 \cdot E_2 = 1$, $R \cdot E_1 = R \cdot E_2 = 0$
  $S \xrightarrow{2:1} C_3$
  cubic in $\mathbb{P}^3$ ⇝ $\underline{\kappa}$: deck tr. of this covering.
  $\underline{\underline{\text{4-nodal}}}$

G: A. Geiser.

- $|2E_1 + 2E_2|$
  $S \xrightarrow{2:1} D_4$ ⇝ $\underline{g}$: deck tr. of this.

- $p \neq 2$ ⇝
  - $\underset{\underset{K3}{\vdots}}{X} \xrightarrow{2:1} S$; étale
  - $\exists\, X \underset{\Pi}{\xrightarrow{\text{bir.}}} \underline{Y_4} \subset \mathbb{P}^3$
    quartic symmetroid of Cayley.
    $\det_{4\times 4} \boxed{a_{ij}}$ (a_{ij}): bilinear.

    ⇝ Y_4 has 10 nodes.
    $Q_1, \dots, Q_{10}$: excep. curves.
    $H = \mathcal{O}_{Y_4}(1)$.
    ⇝ $2H' = 3H - \sum_{i=1}^{10} Q_i$

  - Q_i, H, H': generates $\mathrm{Pic}\, X$.

$\beta_0 = -H + Q_1 + Q_2 + Q_3 + Q_4$

$\beta_i = Q_i - Q_{i+1}$ $(i = 1, \dots, 3)$.

◦ $\pi^*(\mathrm{Num}(S)) = \mathbb{R}(2)$.

◦ $K : p_1, \dots, p_7 \in \mathbb{P}^3$

$\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$: elliptic fibration.

$|4H - 2p_1 - \cdots - 2p_7| : \mathbb{P}^3 \xrightarrow{2:1} \underset{\text{cone of } v_2(\mathbb{P}^2)}{\underset{\shortparallel}{\Sigma}} \subset \mathbb{P}^6$.

$\downarrow$

K = deck trans. of this map.

No.

Date