ABUNDANCE THEOREM FOR SEMI LOG CANONICAL THREEFOLDS

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0. Introduction. The main purpose of this paper is to prove the abundance theorem for semi log canonical threefolds. The abundance conjecture is a very important problem in the birational classification of algebraic varieties. The abundance theorem for semi log canonical surfaces was proved in [1] and [15] by D. Abramovich, L.-Y. Fong, S. Keel, J. Kollár, and J. Mckernan. The proof uses semiresolution, and so on, and has some combinatorial complexities. We simplify the proof and generalize the theorem to semi divisorial log terminal surfaces (see Corollary 4.10). By our method we can reduce the problem to the irreducible case and the finiteness of some groups. This shows that if the log Minimal Model Program (log MMP, for short), the log abundance conjecture for n-folds, and the finiteness of B-pluricanonical representations (see Section 3) hold for $(n-1)$-folds, then the abundance conjecture for semi log canonical n-folds is true almost automatically (see Theorem A.1 in the appendix). But unfortunately the log MMP and the log abundance conjecture are only conjectures for n-folds with $n \geq 4$. So we prove the following theorem.

Theorem 0.1 (Abundance theorem for slc threefolds). *Let (X, Δ) be a proper semi log canonical (slc, for short) threefold with $K_X + \Delta$ nef. Then $K_X + \Delta$ is semiample.*

This theorem is a generalization of the abundance theorem for log canonical threefolds proved by S. Keel, K. Matsuki, and J. McKernan (see [16]). According to the authors, the abundance theorem for log canonical threefolds is considered to be the first step towards a proof of the abundance conjecture in dimension 4. We believe that the abundance theorem for semi log canonical threefolds is the second step.

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The notion of semi log canonical singularity was first introduced in [22] for the problem of compactifying the moduli of surfaces. For the further development of this direction, we recommend the readers to see [10].

Let us see the scheme proposed in [2], [12], and [20]. The abundance conjecture states: Let X be a minimal n-fold with terminal singularities. Then for sufficiently divisible and large $m \in \mathbb{N}$, the linear system $|mK_X|$ is basepoint-free. After the minimal model program (still conjectural in dimension ≥ 4) produces a minimal n-fold in the birational equivalence class, the abundance conjecture would provide the Iitaka fibration morphism $\Phi_{|mK_X|} : X \to X_{\mathrm{can}}$ onto its canonical model, which is absolutely crucial for the study of the birational properties of algebraic varieties. The cited authors proposed the following inductional scheme toward a proof of the abundance conjecture.

(i) Show that a member $D \in |mK_X|$ exists for sufficiently divisible and large $m \in \mathbb{N}$.

(ii) Apply the log MMP to the pair (X, D_X) (the boundary D_X is constructed from D in (i)) to obtain a log minimal model (Y, D_Y). Observe that by the (generalized) adjunction

$$K_Y + D_Y|_{D_Y} = K_{D_Y} + \mathrm{Diff},$$

where Diff is the supplementary term for the equality to hold, and the pair (D_Y, Diff) is a minimal $(n-1)$-fold with semi log canonical singularities.

(iii) Apply induction on the pair (D_Y, Diff). Lift the global sections of $m(K_{D_Y} + \mathrm{Diff})$ to those of $m(K_Y + D_Y)$, which should then provide "enough" global sections of the original mK_X to prove that the linear system $|mK_X|$ is basepoint-free.

In order to complete the inductional circle of steps, we consider the abundance statement for log pairs.

(iv) Based upon the abundance for minimal n-folds X with terminal singularities, prove the abundance for log pairs (X, D) with log canonical singularities.

(v) Based upon the abundance for log pairs (X, D) with log canonical singularities, prove the abundance for log pairs with semi log canonical singularities.

In [2], [12], and [20], the authors proved the abundance conjecture for threefolds along the line of argument described as above, establishing the inductional step (v) in dimension 2 through some combinatorial discussions. In this paper we capture the essential difficulty in carrying out step (v) in arbitrary dimension, as what we call the finiteness of B-pluricanonical representations. In dimension $n = 2$ or 3 where we can prove this finiteness in dimension $n - 1 = 1$ or 2, respectively, we establish the step (v) in one stroke without going through the complex combinatorial arguments.

We sketch the contents of this paper. Section 1 sets up some basic definitions and facts. In Section 2, we treat the reduced boundaries of dlt n-folds. This is a reformulation of [1, 12.3.2]. Section 3 deals with B-pluricanonical representations (the precise definitions are given in Definition 3.1). We prove their finiteness for curves and surfaces; it plays an important role in our proof of the abundance theorem for slc n-folds. In Section 4, the main section, we prove the abundance theorem for slc threefolds. In the appendix, we reformulate the main theorem under some assumptions such as log MMP for n-folds, and we collect some known results for the reader's convenience.

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Notation. (1) The word *scheme* is used for schemes that are separated and of finite type over $\mathbb{C}$; the term *variety* stands for a reduced and irreducible scheme. A normal scheme consists of the disjoint union of irreducible normal schemes.

(2) We freely use terminology about singularities of the pair (X, Δ), such as *Kawamata log terminal, log terminal, divisorial log terminal, log canonical* (frequently abbreviated as klt, lt, dlt, and lc), and *terminal*. For the definition of this terminology, we refer the reader to [21, Section 2.3]. (See also [27].) In the definition in [21, Section 2.3], Δ is not necessarily effective, but in this paper we assume Δ is an effective $\mathbb{Q}$-divisor.

(3) Let $f : X \dashrightarrow Y$ be a rational map. We say that a $\mathbb{Q}$-divisor D is *horizontal* (resp., *vertical*) if every irreducible component of D is dominating (resp., not dominating) over Y.

(4) The log MMP means the log MMP for $\mathbb{Q}$-factorial dlt pairs.

(5) ν denotes the numerical Kodaira dimension.

(6) We will make use of the standard notation and definitions as in [21].

1. Definitions and preliminaries. In this section, we present the basic notation and recall the necessary results.

Definition 1.1. Let X be a reduced S_2 scheme. We assume that it is pure n-dimensional and normal crossing in codimension 1. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X (cf. [5, 16.2]) such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Let $X = \cup X_i$ be a decomposition into irreducible components, and let $\mu : X' := \amalg X'_i \to X = \cup X_i$ be the normalization. A $\mathbb{Q}$-divisor Θ on X' is defined by $K_{X'} + \Theta := \mu^*(K_X + \Delta)$ and a $\mathbb{Q}$-divisor Θ_i on X'_i by $\Theta_i := \Theta|_{X'_i}$.

We say that (X, Δ) is a *semi log canonical n-fold* (an slc n-fold, for short) if (X', Θ) is lc.

We say that (X, Δ) is a *semi divisorial log terminal n-fold* (an sdlt n-fold, for short) if X_i is normal; that is, X'_i is isomorphic to X_i, and (X', Θ) is dlt.

Remark 1.2. (1) The definition of slc above is equivalent to the one in [1] (see [17, 4.2]).

(2) If (X, Δ) is an slc n-fold, then X is seminormal (see [1, 12.2.1(8)] and [3, Remark 4.7]).

(3) If (X, Δ) is a dlt n-fold, then $(\llcorner \Delta \lrcorner, \mathrm{Diff}(\Delta - \llcorner \Delta \lrcorner))$ is an sdlt $(n-1)$-fold (see [18, 17.5] and [21, 5.52]).

(4) Let (X, Δ) be lc. Then $(\llcorner \Delta \lrcorner, \mathrm{Diff}(\Delta - \llcorner \Delta \lrcorner))$ is not necessarily slc (see [18, 17.5.2 Example]). The scheme $\llcorner \Delta \lrcorner$ is not necessarily S_2. Note that [5, (16.9.1)] is not correct.

(5) Let (X, Δ) be lc. If $(X, 0)$ is $\mathbb{Q}$-factorial and lt, then the pair $(\llcorner \Delta \lrcorner, \mathrm{Diff}(\Delta - \llcorner \Delta \lrcorner))$ is slc. Since X has only rational singularities, especially, X is Cohen-Macaulay, and $\llcorner \Delta \lrcorner$ is $\mathbb{Q}$-Cartier, $\llcorner \Delta \lrcorner$ satisfies S_2 condition.

The following lemma plays an important role in Section 2.

LEMMA 1.3 (Connectedness Lemma) [26, 5.7], [18, 17.4], [13, 1.4]. *Let X and Y be normal varieties, and let $f : X \to Y$ be a proper surjective morphism with connected fibers. Let $\Delta = \sum d_i \Delta_i$ be a $\mathbb{Q}$-divisor on X. Let $g : Z \to X$ be a log resolution (cf. [21, Notation 0.4(10)]) such that $h := f \circ g$. Let*

$$K_Z = g^*(K_X + \Delta) + \sum e_i E_i, \qquad \textit{and} \qquad F := -\sum_{i : e_i \le -1} e_i E_i.$$

Assume that

(1) *if $d_i < 0$, then $f(\Delta_i)$ has codimension at least 2 in Y;*

(2) *$-(K_X + \Delta)$ is f-nef and f-big.*

Then $\mathrm{Supp}\, F = \mathrm{Supp} \llcorner F \lrcorner$ *is connected in a neighborhood of any fiber of h. In particular, if (X, Δ) is lc and $(X, \Delta - \llcorner \Delta \lrcorner)$ is klt, then $\llcorner \Delta \lrcorner \cap f^{-1}(y)$ is connected for every point $y \in Y$.*

LEMMA-DEFINITION 1.4 ($\mathbb{Q}$-factorial dlt model) (cf. [15, 8.2.2]). *Let (X, Δ) be an lc n-fold with $n \le 3$. Then there is a projective birational morphism $f : (Y, \Theta) \to (X, \Delta)$ such that (Y, Θ) is $\mathbb{Q}$-factorial dlt and $K_Y + \Theta = f^*(K_X + \Delta)$. Furthermore, if (X, Δ) is dlt, then we may take f a small projective morphism that induces an isomorphism at every generic point of a center of log canonical singularities for the pair (Y, Θ). (For the definition of a center of log canonical singularities, see Definition 4.8.) We say that (Y, Θ) is a $\mathbb{Q}$-factorial dlt model of (X, Δ).*

Definition 1.5. Let $(X, \Delta) = \amalg_{i=1}^n (X_i, \Delta_i)$ and $(X', \Delta') = \amalg_{i=1}^n (X'_i, \Delta'_i)$ be normal schemes with $\mathbb{Q}$-divisor, such that $K_X + \Delta$ and $K_{X'} + \Delta'$ are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

We say that $f : (X, \Delta) \dashrightarrow (X', \Delta')$ is a B-birational map (resp., morphism) if $f : X \dashrightarrow X'$ is a proper birational map (resp., morphism) and there exists a common resolution $\alpha : T \to X$, $\beta : T \to X'$ of $f : X \dashrightarrow X'$ such that $\alpha^*(K_X + \Delta) = \beta^*(K_{X'} + \Delta')$. That is, there exists a permutation σ such that $f_i : X_i \dashrightarrow X'_{\sigma(i)}$ is a proper birational map (resp., morphism) and there exists a common resolution $\alpha_i : T_i \to X_i$, $\beta_i : T_i \to X'_{\sigma(i)}$ of f_i such that $\alpha_i^*(K_{X_i} + \Delta_i) = \beta_i^*(K_{X'_{\sigma(i)}} + \Delta'_{\sigma(i)})$ on

T_i for every i. The last condition means that if we write

$$K_{T_i} = \alpha_i^*(K_{X_i} + \Delta_i) + F \qquad \text{and} \qquad K_{T_i} = \beta_i^*\big(K_{X'_{\sigma(i)}} + \Delta'_{\sigma(i)}\big) + E,$$

then $F = E$.

If there is a B-birational map from (X, Δ) to (X', Δ'), we say that (X, Δ) is B-birationally equivalent to (X', Δ') and write $(X, \Delta) \sim_B (X', \Delta')$. Here the symbol B is the initial of *boundary*.

LEMMA 1.6. *Let (X, Δ) and (Z, Δ') be normal varieties with $\mathbb{Q}$-divisors such that $K_X + \Delta$ and $K_Z + \Delta'$ are $\mathbb{Q}$-Cartier. Let $f : X \to R$ and $h : Z \to R$ be proper surjective morphisms onto a normal variety R and $p : X \dashrightarrow Z$ a birational map such that $f = h \circ p$. Assume that*

(1) *p^{-1} has no exceptional divisors;*

(2) *$\Delta' = p_*\Delta$;*

(3) *$K_X + \Delta \equiv_f 0$, $K_Z + \Delta' \equiv_h 0$.*

Then p is a B-birational map.

Proof. Let $\beta : W \to X$ be a resolution such that the induced rational map $\alpha = p \circ \beta : W \to Z$ is a morphism. Let $m > 0$ be a sufficiently divisible integer. We have linear equivalences

$$-mK_W \sim -\alpha^*\big(m(K_Z + \Delta')\big) - F,$$

$$mK_W \sim \beta^*\big(m(K_X + \Delta)\big) + E.$$

Adding the two we obtain

$$\beta^*(K_X + \Delta) - \alpha^*(K_Z + \Delta') = F - E.$$

By assumption, $F - E$ is α-exceptional and numerically α-trivial. Then $F = E$, that is, $\alpha^*(K_Z + \Delta') = \beta^*(K_X + \Delta)$. □

2. Reduced boundaries of dlt n-folds. The following is a reformulation of [1, 12.3.2], which fits better in our arguments.

PROPOSITION 2.1 (cf. [26, 6.9], [1, 12.3.2]). *Let (X, Θ) be a $\mathbb{Q}$-factorial dlt n-fold with $n \le 3$. Let $f : X \to R$ be a projective surjective morphism onto a normal variety R with connected fibers. Assume that $K_X + \Theta$ is numerically f-trivial. Then one of the following holds.*

(0) $\dim R = 0$.

(0.1) *$\llcorner \Theta \lrcorner$ is connected.*

(0.2) *$\llcorner \Theta \lrcorner$ has two connected components Δ_1 and Δ_2, and there exists a rational map $v : X \dashrightarrow (V, P)$ onto a $\mathbb{Q}$-factorial lc $(n-1)$-fold (V, P) with general fiber $\mathbb{P}^1$. The pair $(V, 0)$ is lt. Furthermore, there exists an irreducible component $D'_i \subset \Delta_i$ such that $v|_{D'_i} : (D'_i, \mathrm{Diff}(\Theta - D'_i)) \dashrightarrow (V, P)$ is a B-birational map for $i = 1, 2$.*

(1) $\dim R \geq 1$.

(1.1) $\llcorner\Theta\lrcorner \cap f^{-1}(r)$ *is connected for every* $r \in R$.

(1.2) *The number of connected components of* $\llcorner\Theta\lrcorner \cap f^{-1}(r)$ *is at most two for every* $r \in R$. *There exists a rational map* $v : X \dashrightarrow (V, P)$ *onto a* $\mathbb{Q}$*-factorial lc* $(n-1)$*-fold* (V, P) *with general fiber* $\mathbb{P}^1$. *The pair* $(V, 0)$ *is lt. The horizontal part* Θ^h *of* $\llcorner\Theta\lrcorner$ *is one of the following:*

(i) $\Theta^h = D'_1$, *which is irreducible, and the mapping degree* $\deg[D'_1 : V] = 2$*; there is also a* B*-birational involution on* $(D'_1, \mathrm{Diff}(\Theta - D'_1))$ *over* V*;*

(ii) $\Theta^h = D'_1 + D'_2$ *such that* D'_i *is irreducible and* $v|_{D'_i} : (D'_i, \mathrm{Diff}(\Theta - D'_i)) \dashrightarrow (V, P)$ *is a* B*-birational map for* $i = 1, 2$.

Remark 2.2. (1) In Proposition 2.1 the assumption $K_X + \Theta \equiv_f 0$ is equivalent to $K_X + \Theta \sim_{\mathbb{Q}, f} 0$. It is because the relative log abundance theorem holds when $\dim X \leq 3$ (see Theorem A.2 in the appendix).

(2) If the log MMP holds for n-folds, then Proposition 2.1 is also true for n-folds.

First, we prove the following lemma.

LEMMA 2.3. *Let* (Z, Δ) *be a* $\mathbb{Q}$*-factorial lc* n*-fold with* $n \geq 2$ *and* $\llcorner\Delta\lrcorner \neq 0$. *Let* $h : Z \to R$ *be a projective surjective morphism onto a normal variety* R *with connected fibers. Assume the following conditions:*

(1) $(Z, \Delta - \varepsilon\llcorner\Delta\lrcorner)$ *is klt, where* ε *is a small positive rational number;*

(2) $K_Z + \Delta \equiv_h 0$*;*

(3) *there is a* $(K_Z + \Delta - \varepsilon\llcorner\Delta\lrcorner)$*-extremal Fano contraction* $u : Z \to V$ *over* R *such that* $\dim V = n - 1$.

Then the horizontal part Δ^h *of* $\llcorner\Delta\lrcorner$ *is one of the following:*

(a) $\Delta^h = D_1$, *which is irreducible, and* $\deg[D_1 : V] = 2$*;*

(b) $\Delta^h = D_1$, *which is irreducible, and* $\deg[D_1 : V] = 1$*;*

(c) $\Delta^h = D_1 + D_2$, *such that* D_i *is irreducible and* $\deg[D_i : V] = 1$ *for* $i = 1, 2$.

In the cases (a) and (c), the number of connected components of $\llcorner\Delta\lrcorner \cap h^{-1}(r)$ *is at most two for every* $r \in R$.

In the case (b), $\llcorner\Delta\lrcorner \cap h^{-1}(r)$ *is connected for every* $r \in R$.

Furthermore, there is a $\mathbb{Q}$*-divisor* P *on* V *such that* (V, P) *is a* $\mathbb{Q}$*-factorial lc* $(n-1)$*-fold and* $K_{D_i} + \mathrm{Diff}(\Delta - D_i) = u|_{D_i}{}^*(K_V + P)$ *for* $i = 1, 2$.

In the case (a), there is a B*-birational involution* ι *over* V*; that is,* $\iota : (D_1, \mathrm{Diff}(\Delta - D_1)) \dashrightarrow (D_1, \mathrm{Diff}(\Delta - D_1))$ *over* V *is a* B*-birational map and* $\iota^2 = \mathrm{id}$.

In the case (c), $u|_{D_i} : D_i \to V$ *is a* B*-birational morphism for* $i = 1, 2$. *In particular,* $(D_1, \mathrm{Diff}(\Delta - D_1)) \sim_B (D_2, \mathrm{Diff}(\Delta - D_2))$.

Note that $(V, 0)$ *is lt.*

Proof. Since $\llcorner\Delta\lrcorner$ is u-ample by the assumptions (2) and (3), we have $\Delta^h \neq 0$. So the general fiber of $Z \to V$ is $\mathbb{P}^1$ and $\deg[\Delta^h : V] \leq 2$, because $K_Z + \Delta$ is numerically h-trivial. Since $u : Z \to V$ is extremal, the vertical component Δ^v of

$\llcorner \Delta \lrcorner$ is a pullback of a $\mathbb{Q}$-divisor on V and $(V, 0)$ is a $\mathbb{Q}$-factorial lt pair (see [7, Corollary 3.5] or [24, Appendix]). Therefore the first part is proved.

Let $H_1, H_2, \ldots, H_{n-2}$ be general hypersurfaces on V. Consider

$$u^{-1}\bigl(H_1 \cap H_2 \cap \cdots \cap H_{n-2}\bigr) \longrightarrow H_1 \cap H_2 \cap \cdots \cap H_{n-2}.$$

By using [12, 3.5.1 and 3.5.2] and [1, 12.3.4] to the above morphism, we have a $\mathbb{Q}$-divisor P on V satisfying $K_{D_i} + \mathrm{Diff}(\Delta - D_i) = u|_{D_i}{}^*(K_V + P)$. The normalization of $(D_1, \mathrm{Diff}(\Delta - D_1))$ is lc and the normalization of (V, P) in the function field $\mathbb{C}(D_1)$ is lc, since $K_{D_1} + \mathrm{Diff}(\Delta - D_1) = u|_{D_1}{}^*(K_V + P)$. Thus (V, P) is lc and $\mathbb{Q}$-factorial, since Z is $\mathbb{Q}$-factorial and u is extremal. $\square$

Proof of Proposition 2.1. If f is birational, then Connectedness Lemma 1.3 implies that we are in the case (0.1) or (1.1). Thus we may assume that $\dim R < \dim X$.

We run the $(K_X + \Theta - \varepsilon \llcorner \Theta \lrcorner)$-MMP on X over R for $0 < \varepsilon \ll 1$. The end result is a birational map $p : X \dashrightarrow Z$ over R. Let $h : Z \to R$ be the induced morphism. Since $K_X + \Theta \equiv_f 0$, we obtain that $K_Z + p_*\Theta \equiv_h 0$. Then $(Z, p_*\Theta)$ is a $\mathbb{Q}$-factorial lc pair (see Lemma 1.6) and $(Z, p_*\Theta - \varepsilon \llcorner p_*\Theta \lrcorner)$ is klt.

Step 1. If $(Z, p_*\Theta - \varepsilon \llcorner p_*\Theta \lrcorner)$ is a minimal model, then $K_Z + p_*\Theta - \varepsilon \llcorner p_*\Theta \lrcorner$ is h-nef and $K_Z + p_*\Theta \equiv_h 0$. So $-\llcorner p_*\Theta \lrcorner$ is h-nef. If $\dim R = 0$, then $\llcorner p_*\Theta \lrcorner = 0$. By Lemma 2.4 below, we have $\llcorner \Theta \lrcorner = 0$. If $0 < \dim R < \dim X$, then $\llcorner p_*\Theta \lrcorner \cap h^{-1}(r)$ is connected for every $r \in R$. By Lemma 2.4 we have the case (1.1).

Step 2. If there exists an extremal Fano contraction $u : Z \to V$ over R, then $-(K_Z + p_*\Theta - \varepsilon \llcorner p_*\Theta \lrcorner)$ is u-ample. Let $g : V \to R$ be the induced morphism. We note $h = g \circ u$.

First, assume that $\dim R = 0$. If $\llcorner p_*\Theta \lrcorner$ is connected, we have the case (0.1) by Lemma 2.4. Thus we can assume that $\llcorner p_*\Theta \lrcorner$ is not connected. Since $\llcorner p_*\Theta \lrcorner$ is relatively ample, we have $\dim V = \dim X - 1$. Then by Lemma 2.3 we have the case (c) in Lemma 2.3 and $u|_{D_i} : D_i \to V$ are finite, where D_i are as in Lemma 2.3(c) for $i = 1, 2$. It is because D_1 intersects D_2 by the relative ampleness of D_1 and D_2 if $u|_{D_1}$ or $u|_{D_2}$ is not finite. We have $D_i \simeq V$ by Zariski's main theorem, because V is normal. By Lemma 1.6, the adjunction, and Lemma 2.4, we have the case (0.2).

Thus, we may assume $\dim R \geq 1$.

If $\dim V = \dim X - 1$, we get case (1.2) by Lemmas 1.6, 2.3, and 2.4 below, and the adjunction.

If $\dim R = 1$, $\dim V = 1$, and $\dim X = 3$, then $V \simeq R$. Since u is extremal, $\llcorner p_*\Theta \lrcorner$ is h-ample and $\rho(Z/R) = 1$. Then every horizontal irreducible component of $\llcorner p_*\Theta \lrcorner$ is h-ample, and every vertical irreducible component of $\llcorner p_*\Theta \lrcorner$ is a pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on R. Then $\llcorner p_*\Theta \lrcorner \cap h^{-1}(r)$ is connected for every $r \in R$. We have the case (1.1). $\square$

The next lemma is used in the proof of Proposition 2.1.

LEMMA 2.4. *If $f : X \to R$, $h : Z \to R$, $p : X \dashrightarrow Z$ are as in the proof of Proposition 2.1, then the number of connected components of $\llcorner \Theta \lrcorner \cap f^{-1}(r)$ is equal* *to the number of connected components of* $\llcorner p_*\Theta\lrcorner \cap h^{-1}(r)$ *for every* $r \in R$.

Proof. $p : X \dashrightarrow Z$ is a composition of flips and divisorial contractions. $X \dashrightarrow X^1 \dashrightarrow X^2 \dashrightarrow \cdots \dashrightarrow X^i \dashrightarrow \cdots \dashrightarrow Z$. Use Connectedness Lemma 1.3 in each step. Let Θ^i be the proper transform of Θ on X^i. Note that $\llcorner\Theta^i\lrcorner$ is relatively ample for each flipping or divisorial contraction and $K_{X^i}+\Theta^i$ is numerically trivial over R. □

3. Finiteness of B**-pluricanonical representations.** We consider the birational automorphism groups of pairs.

Definition 3.1. Let (X, Δ) be a pair of a normal scheme and a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We define

$$\mathrm{Bir}(X,\Delta) := \{\sigma : (X,\Delta) \dashrightarrow (X,\Delta) \mid \sigma \text{ is a } B\text{-birational map}\},$$

$$\mathrm{Aut}(X,\Delta) := \{\sigma : X \to X \mid \sigma \text{ is an automorphism and } \sigma^*\Delta = \Delta\}.$$

Since $\mathrm{Bir}(X,\Delta)$ acts on $H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ for every integer m such that $m(K_X+\Delta)$ is a Cartier divisor, we can define B-pluricanonical representation $\rho_m : \mathrm{Bir}(X,\Delta) \to \mathrm{Aut}\, H^0(X, m(K_X+\Delta))$.

The following conjecture plays an important role when we reduce the problem to the irreducible case.

CONJECTURE 3.2 (Finiteness of B-pluricanonical representations). *Let* (X, Δ) *be an* n*-dimensional (not necessarily connected) proper lc pair. Assume that* $K_X+\Delta$ *is nef. Then there is a positive integer* m_0 *such that* $\rho_{m_1m_0}(\mathrm{Bir}(X,\Delta))$ *is finite for every* $m_1 \in \mathbb{N}$.

For Conjecture 3.2, it is obviously sufficient to prove it under the assumption that X is irreducible. In Theorems 3.3, 3.4, and 3.5, we prove the conjecture for curves and surfaces.

THEOREM 3.3 (Cf. [1, 12.2.11]). *Let* (C, Δ) *be a proper lc curve. Then there is a positive integer* m_0 *such that* $\rho_{m_1m_0}(\mathrm{Aut}(C,\Delta))$ *is finite for every* $m_1 \in \mathbb{N}$.

Proof. If the genus $g(C) \geq 2$, then it is trivial. If $g(C) = 1$ and $\Delta \neq 0$, then it is also true by [23, p. 60, Application 1]. If $g(C) = 1$ and $\Delta = 0$, then we put $m_0 = 12$ and the theorem holds by [1, 12.2.9.1]. So we assume that C is a rational curve. If $\deg(K_C+\Delta) < 0$, then there is nothing to be proved. If $|\mathrm{Supp}\,\Delta| \geq 3$, then $\mathrm{Aut}(C,\Delta)$ is a finite group. So we can reduce to the case where $\Delta = \llcorner\Delta\lrcorner = \{\text{two points}\}$. In this case we can prove easily that $\rho_m(\mathrm{Aut}(C,\Delta))$ is trivial if m is even. □

THEOREM 3.4. *Let* (S, Δ) *be a proper klt surface. Let* $m_0 \geq 2$ *be an integer such that* $m_0(K_S+\Delta)$ *is a Cartier divisor. Then* $\rho_{m_1m_0}(\mathrm{Bir}(S,\Delta))$ *is finite for every* $m_1 \in \mathbb{N}$.

Proof. Let $h : S' \to S$ be a terminal model, that is, (S', Δ') is terminal and $K_{S'} + \Delta' = h^*(K_S + \Delta)$ (see [19, (6.9.4)]). Note that Δ' is effective and $\llcorner \Delta' \lrcorner = 0$ by the construction. The discrepancy of every exceptional divisor over (S', Δ') is positive and that of a nonexceptional divisor is nonpositive. The composite $\sigma' := h^{-1} \circ \sigma \circ h$ does not change discrepancies because σ' is a B-birational map. Thus σ' is a biregular morphism such that $\sigma'^* \Delta' = \Delta'$. We may assume (S, Δ) is terminal such that $\llcorner \Delta \lrcorner = 0$ and replace $\mathrm{Bir}(S, \Delta)$ with $\mathrm{Aut}(S, \Delta)$. Let $f : S' \to S$ be a finite sequence of blowings-up whose centers are over Δ such that $K_{S'} + f_*^{-1}\Delta = f^*(K_S + \Delta) + \sum a_i E_i$ and $\mathrm{Supp}(f_*^{-1}\Delta \cup \sum E_i)$ is a simple normal crossing divisor. Let $m_0 \geq 2$ be an integer such that $m_0(K_S + \Delta)$ is a Cartier divisor. We put $D := f_*^{-1}\Delta + \sum (1/m_0) E_i$. Then we obtain

$$K_{S'} + D = f^*(K_S + \Delta) + \sum \left(a_i + \frac{1}{m_0} \right) E_i .$$

Note that D is effective, $\llcorner D \lrcorner = 0$, and $\mathrm{Supp}\, D$ is a simple normal crossing divisor. Since $S' - D \simeq_f S - \Delta$, $\sigma' := f^{-1} \circ \sigma \circ f$ acts on $H^0(S', \mathcal{O}_{S'}(mK_{S'} + (m-1)\ulcorner D \urcorner))$. (Cf. [25, Theorem 2.1 and Proposition 1.4], [9, §11.1].) Then the image of $\mathrm{Aut}(S, \Delta) \to \mathrm{Aut}\, H^0(S', \mathcal{O}_{S'}(mK_{S'} + (m-1)\ulcorner D \urcorner))$ is a finite group for every m. It was proved by I. Nakamura, K. Ueno, P. Deligne, and F. Sakai. (See [25, Theorem 5.1] and [28, §14].) Since

$$\begin{aligned} H^0\big(S, \mathcal{O}_S\big(m_1 m_0 (K_S + \Delta)\big)\big) &= H^0\big(S', \mathcal{O}_{S'}\big(m_1 m_0 \big(K_{S'} + D\big)\big)\big) \\ &\subset H^0\big(S', \mathcal{O}_{S'}\big(mK_{S'} + (m-1)\ulcorner D \urcorner\big)\big) \end{aligned}$$

for every positive integer m_1 with $m = m_1 m_0$, the image of $\rho_{m_1 m_0} : \mathrm{Aut}(S, \Delta) \to \mathrm{Aut}\, H^0(S, m_1 m_0 (K_S + \Delta))$ is a finite group. □

THEOREM 3.5. *Let (S, Δ) be a projective lc surface. Assume that $K_S + \Delta$ is nef. Then there is a positive integer m_0 such that $\rho_{m_1 m_0}(\mathrm{Bir}(S, \Delta))$ is finite for every $m_1 \in \mathbb{N}$.*

Proof. Let $f := \Phi_{|k(K_S+\Delta)|} : S \to R$ be a morphism with connected fibers for a sufficiently large and divisible integer k by the log abundance theorem.

Case 1: $\nu(S, K_S + \Delta) = 2$.

Then f is a birational morphism and $\mathrm{Bir}(S, \Delta)$ acts on R biregularly. We put $\Xi := f_*\Delta$. Then $K_S + \Delta = f^*(K_R + \Xi)$ and Ξ is $\mathrm{Bir}(S, \Delta)$-invariant. Let $h : S' \to R$ be the unique minimal resolution; so we have $K_{S'} + \Delta' = h^*(K_R + \Xi)$, $\mathrm{Bir}(S, \Delta)$ acts on S' biregularly, and Δ' is $\mathrm{Bir}(S, \Delta)$-invariant. Thus, we may reduce to the case where (S, Δ) is lc, S is smooth and $\mathrm{Bir}(S, \Delta) = \mathrm{Aut}(S, \Delta)$ in Theorem 3.4. Since $K_S + \Delta$ is big, we obtain an effective Cartier divisor D such that $am_0(K_S + \Delta) \sim \llcorner \Delta \lrcorner + D$, where a is a sufficiently large integer and m_0 is a sufficiently divisible integer so that $m_0(K_S + \Delta)$ is a Cartier divisor. Observing

$$(m_1 + a)\left(m_0(K_S + \Delta) - \frac{1}{m_1 + a} \llcorner \Delta \lrcorner \right) = m_1 m_0 (K_S + \Delta) + D,$$

we have

$$H^0\big(S, \mathcal{O}_S\big(m_1m_0(K_S+\Delta)\big)\big) \subset H^0\big(S, \mathcal{O}_S\big((m_1+a)m_0(K_S+\Delta)-\llcorner\Delta\lrcorner\big)\big).$$

By using Theorem 3.4 for the right-hand side, we obtain the result.

Case 2: $\nu(S, K_S+\Delta)=1$.

Let $g := S' \to S$ be a minimal resolution and $\Delta' := f^*(K_S+\Delta)-K_{S'} > 0$. We may replace (S, Δ) with (S', Δ'). By contracting (-1)-curves in the fibers, we may reduce to the case where $f : S \to R$ is a $\mathbb{P}^1$-bundle or a minimal elliptic surface. When the horizontal part $\Delta^h \neq 0$, we take an irreducible component D of Δ^h. By elementary transformations we may assume D is smooth. Then

$$H^0\big(S, \mathcal{O}_S\big(m_1m_0(K_S+\Delta)\big)\big) \subset H^0\big(D, \mathcal{O}_D\big(m_1m_0\big(K_D+\mathrm{Diff}(\Delta-D)\big)\big)\big).$$

So we have the result by Theorem 3.3. Next we may assume $\Delta^h=0$. When $f : S \to R$ is a $\mathbb{P}^1$-bundle, $\llcorner\Delta\lrcorner = \sum f^*p_i$ for some $p_i \in R$. When $f : S \to R$ is a minimal elliptic surface, we have $K_S \sim_{\mathbb{Q},f} 0$ and $\Delta \equiv_f 0$. Then $\Delta = \sum a_i f^*p_i$ for some $p_i \in R$ and positive rational numbers a_i (see [4, VIII.3, VIII.4]). $\mathrm{Bir}(S, \Delta)$ acts on $f(\llcorner\Delta\lrcorner)$. We define $B := \sum_{p_i \in f(\llcorner\Delta\lrcorner)} p_i$. Let H be an ample Cartier divisor on R such that $m_0(K_S+\Delta) \sim f^*H$. Then we have $bH \sim B+D$ for some effective divisor D and some sufficiently large integer b. Then $bm_0(K_S+\Delta) \sim f^*(B+D)$. We put $A := f^*B$. Observing

$$(m_1+b)\left(m_0(K_S+\Delta)-\frac{1}{m_1+b}A\right) = m_1m_0(K_S+\Delta)+f^*D,$$

we have

$$H^0\big(S, \mathcal{O}_S\big(m_1m_0(K_S+\Delta)\big)\big) \subset H^0\big(S, \mathcal{O}_S\big((m_1+b)m_0(K_S+\Delta)-A\big)\big).$$

By using Theorem 3.4 for the right-hand side, we obtain a result.

Case 3: $\nu(S, K_S+\Delta)=0$.

In this case we can show that $\mathrm{Bir}(S, \Delta)$ acts on $H^0(S, m_1m_0(K_S+\Delta))$ trivially, using Lemma 4.9 below. First, we replace (S, Δ) with its $\mathbb{Q}$-factorial dlt model (see Lemma-Definition 1.4). So we may assume that (S, Δ) is dlt. By Proposition 4.5, we can take a preadmissible section s (cf. Definition 4.1). Then $g^*s = s$ for every $g \in \mathrm{Bir}(S, \Delta)$ by Lemma 4.9. Since $\nu(S, K_S+\Delta)=0$, we have the result. □

4. The abundance theorem for slc threefolds. We introduce the notion of preadmissible and admissible sections for the inductive proof of the abundance conjecture for slc n-folds.

Definition 4.1. Let (X, Δ) be a proper sdlt n-fold, and let m be a sufficiently large and divisible integer. We define admissible and preadmissible sections inductively on dimension:

- $s \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ is *preadmissible* if the restriction $s|_{(\amalg_i \llcorner\Theta_i\lrcorner)} \in H^0(\amalg_i \llcorner\Theta_i\lrcorner, \mathcal{O}_{(\amalg_i \llcorner\Theta_i\lrcorner)}(m(K_{X'}+\Theta)|_{(\amalg_i \llcorner\Theta_i\lrcorner)}))$ is admissible;
- $s \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ is *admissible* if s is preadmissible and $g^*(s|_{X_j}) = s|_{X_i}$ for every B-birational map $g : (X_i, \Theta_i) \dashrightarrow (X_j, \Theta_j)$ for every i, j.

Note that if $s \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ is admissible, then $s|_{X_i}$ is $\mathrm{Bir}(X_i, \Theta_i)$-invariant for every i.

We define linear subspaces of $H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$ as follows:

$$\mathrm{PA}\big(X, m(K_X+\Delta)\big) := \{s \text{ is preadmissible}\}$$

and

$$\mathrm{A}\big(X, m(K_X+\Delta)\big) := \{s \text{ is admissible}\}.$$

When $\dim X = 1$, the preadmissible section is a slight generalization of the *normalized* section (see [1, 12.2.9]). But in the higher dimensional case, the admissible and preadmissible sections behave much better in the inductive proof of the abundance conjecture for slc n-folds.

LEMMA 4.2. *Let (X, Δ) be a proper slc n-fold, $\mu : X' \to X$ the normalization, and $K_{X'}+\Theta := \mu^*(K_X+\Delta)$. Let $f : (Y, \Xi) \to (X', \Theta)$ be a proper birational morphism such that (Y, Ξ) is dlt with $K_Y+\Xi = f^*(K_{X'}+\Theta)$. Then* $\mathrm{PA}(Y, m(K_Y+\Xi))$ *descends to sections on (X, Δ).*

Proof. By the definition of slc, X is S_2 and normal crossing in codimension 1. So, this lemma is obvious by the definition of preadmissible sections. □

LEMMA 4.3. *Let (X, Δ) be a proper $\mathbb{Q}$-factorial dlt n-fold, $K_X+\Delta$ nef, and $S = \llcorner\Delta\lrcorner \neq 0$. Assume that $f = \Phi_{|k(K_X+\Delta)|} : X \to R$ is a proper surjective morphism onto a normal variety R with connected fibers for a sufficiently large and divisible integer k and $f(\llcorner\Delta\lrcorner) \subsetneq R$. If there exist sections $\{s_i\}_{i=1}^p \subset H^0(S, \mathcal{O}_S(m(K_X+\Delta)|_S))$ without common zeros, then there exist sections $\{u_i\}_{i=1}^l \subset H^0(X, \mathcal{O}_X(rm(K_X+\Delta)))$ for some integer r such that*

(1) $u_i|_S = \begin{cases} s_i^r & \textit{for } 1 \le i \le p, \\ 0 & \textit{for } p < i \le l; \end{cases}$

(2) $\{u_i\}_{i=1}^l$ *have no common zeros.*

Proof. There is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H on R such that $K_X+\Delta \sim_{\mathbb{Q}} f^*H$. We consider the following commutative diagram:

$$\begin{array}{ccc} H^0\big(X, \mathcal{O}_X\big(rm(K_X+\Delta)\big)\big) & \longrightarrow & H^0\big(S, \mathcal{O}_S\big(rm(K_X+\Delta)|_S\big)\big) \\ \uparrow{\cong} & & \uparrow{\cong} \\ H^0\big(R, \mathcal{O}_R(rmH)\big) & \longrightarrow & H^0\big(T, \mathcal{O}_T(rmH|_T)\big). \end{array}$$

Let $T := f(S)$ and I_T the defining ideal of T. By Lemma 4.4 below, $(f|_S)_*\mathcal{O}_S = \mathcal{O}_T$. So the vertical arrows are isomorphisms. Since H is ample, the horizontal arrows are surjective and $\mathcal{O}_R(rmH) \otimes I_T$ is generated by global sections for a sufficiently large integer r. So we can get sections $\{u_i\}_{i=1}^{l}$ with required properties. $\square$

We use the next lemma in Proposition 4.5.

LEMMA 4.4. *Let $f : X \to R$ be a proper surjective morphism between normal varieties with connected fibers. Assume that*

(1) (X, Θ) *is a $\mathbb{Q}$-factorial dlt n-fold;*
(2) $f(\llcorner \Theta \lrcorner) \subsetneq R$;
(3) $K_X + \Theta \sim_{\mathbb{Q}, f} 0$.

Then there is an exact sequence

$$0 \longrightarrow f_*\mathcal{O}_X(-\llcorner \Theta \lrcorner) \longrightarrow \mathcal{O}_R \longrightarrow f_*\mathcal{O}_{\llcorner \Theta \lrcorner} \longrightarrow 0.$$

Proof. There is a positive integer m and a Cartier divisor D on R such that $m(K_X + \Theta) = f^*D$. Observe that

$$m(K_X + \Theta) - \llcorner \Theta \lrcorner - (K_X + \{\Theta\}) = (m-1)(K_X + \Theta).$$

Since $K_X + \{\Theta\}$ is klt and $(m-1)(K_X + \Theta)$ is f-semiample by the assumptions (1) and (3), $R^1 f_*\mathcal{O}_X(m(K_X + \Theta) - \llcorner \Theta \lrcorner)$ is torsion-free (see [14, 1-2-7]). By the assumption (2), $f_*\mathcal{O}_{\llcorner \Theta \lrcorner}(m(K_X + \Theta))$ is a torsion sheaf. Then we have an exact sequence

$$\begin{aligned} 0 &\longrightarrow f_*\mathcal{O}_X\big(m(K_X + \Theta) - \llcorner \Theta \lrcorner\big) \\ &\longrightarrow f_*\mathcal{O}_X\big(m(K_X + \Theta)\big) \\ &\longrightarrow f_*\mathcal{O}_{\llcorner \Theta \lrcorner}\big(m(K_X + \Theta)\big) \longrightarrow 0. \end{aligned}$$

Tensoring $\mathcal{O}_R(-D)$ gives the result. $\square$

The next proposition is the main part of the proof of the abundance theorem for slc n-folds.

PROPOSITION 4.5. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial dlt pair with $n \le 3$. Let m be a sufficiently large and divisible integer, especially $m \in 2\mathbb{Z}$. Assume that*

(1) $K_X + \Delta$ *is nef;*
(2) $\mathrm{A}(\llcorner \Delta \lrcorner, m(K_X + \Delta)|_{\llcorner \Delta \lrcorner})$ *generates* $\mathcal{O}_{\llcorner \Delta \lrcorner}(m(K_{\llcorner \Delta \lrcorner} + \mathrm{Diff}(\Delta - \llcorner \Delta \lrcorner)))$.

Then

$$\mathrm{PA}\big(X, m(K_X + \Delta)\big) \longrightarrow \mathrm{A}\big(\llcorner \Delta \lrcorner, m(K_X + \Delta)|_{\llcorner \Delta \lrcorner}\big)$$

is surjective, and $\mathrm{PA}(X, m(K_X + \Delta))$ *generates* $\mathcal{O}_X(m(K_X + \Delta))$.

Proof. It is sufficient to prove this proposition for each connected component. So we can assume that (X, Δ) is a projective $\mathbb{Q}$-factorial irreducible dlt pair.

Apply the log abundance theorem. We get $f := \Phi_{|k(K_X+\Delta)|} : X \to R$ for a sufficiently large and divisible integer k. If $\llcorner\Delta\lrcorner = 0$, then the proposition is trivial. Thus, we may assume $\llcorner\Delta\lrcorner \neq 0$. We have the following possibilities by Proposition 2.1:

(1) $\nu(X, K_X+\Delta) = 0$ and $\llcorner\Delta\lrcorner$ is connected;
(2) $\nu(X, K_X+\Delta) \geq 1$ and $f^{-1}(r)\cap\llcorner\Delta\lrcorner$ is connected for every $r \in R$ and $f(\llcorner\Delta\lrcorner) = R$;
(3) $\nu(X, K_X+\Delta) \geq 1$ and $f^{-1}(r)\cap\llcorner\Delta\lrcorner$ is connected for every $r \in R$ and $f(\llcorner\Delta\lrcorner) \subsetneq R$;
(4) $f^{-1}(r)\cap\llcorner\Delta\lrcorner$ is not connected for some $r \in R$.

Case 1. Consider the exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0\big(X, \mathcal{O}_X\big(m(K_X+\Delta)-\llcorner\Delta\lrcorner\big)\big) \\ &\longrightarrow H^0\big(X, \mathcal{O}_X\big(m(K_X+\Delta)\big)\big) \\ &\longrightarrow H^0\big(\llcorner\Delta\lrcorner, \mathcal{O}_{\llcorner\Delta\lrcorner}\big(m(K_X+\Delta)|_{\llcorner\Delta\lrcorner}\big)\big) \longrightarrow \cdots . \end{aligned}$$

Since $\nu(X, K_X+\Delta) = 0$, $H^0(X, \mathcal{O}_X(m(K_X+\Delta)-\llcorner\Delta\lrcorner)) = 0$ and the second and third terms are one-dimensional. Thus, we get the result.

Case 2. We construct a morphism $\varphi : \llcorner\Delta\lrcorner \to C$ by the linear system corresponding to $\mathrm{A}(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$. Since every curve in any fiber of $f|_{\llcorner\Delta\lrcorner}$ goes to a point by φ, there exists a morphism $\psi : R \to C$ such that $\psi\circ(f|_{\llcorner\Delta\lrcorner}) = \varphi$. For $s \in \mathrm{A}(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$, there exists a section t on C such that $s = \varphi^* t$. Thus we obtain $u := f^*\psi^* t \in \mathrm{PA}(X, m(K_X+\Delta))$ such that $u|_{\llcorner\Delta\lrcorner} = s$. Note that if we write $K_X+\Delta \sim_{\mathbb{Q}} f^*H$ as in Lemma 4.3, we have $H^0(X, \mathcal{O}_X(m(K_X+\Delta))) \simeq H^0(R, \mathcal{O}_R(mH)) \simeq H^0(\llcorner\Delta\lrcorner, \mathcal{O}_{\llcorner\Delta\lrcorner}(m(K_X+\Delta)|_{\llcorner\Delta\lrcorner}))$. Since $\mathrm{A}(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$ generates $\mathcal{O}_{\llcorner\Delta\lrcorner}(m(K_{\llcorner\Delta\lrcorner}+\mathrm{Diff}(\Delta-\llcorner\Delta\lrcorner)))$, $\mathrm{PA}(X, m(K_X+\Delta))$ generates $\mathcal{O}_X(m(K_X+\Delta))$.

Case 3. We put $T := f(\llcorner\Delta\lrcorner) \subsetneq R$. By Lemma 4.4 we obtain $\mathcal{O}_T = f_*\mathcal{O}_{\llcorner\Delta\lrcorner}$. Then T is seminormal (see [3, Proposition 4.5]). As in case 2, we construct $\varphi : \llcorner\Delta\lrcorner \to C$ by $\mathrm{A}(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$ and get $\psi : T \to C$. By Lemma 4.3, we can pull back $s \in \mathrm{A}(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$ to $u \in \mathrm{PA}(X, m(K_X+\Delta))$ if m is a sufficiently large and divisible integer. By Lemma 4.3 we can also check that $\mathrm{PA}(X, m(K_X+\Delta))$ generates $\mathcal{O}_X(m(K_X+\Delta))$ if m is a sufficiently large and divisible integer.

Case 4. In this case, X is generically a $\mathbb{P}^1$-bundle over (V, P) by Proposition 2.1. Let $f : (X, \Delta) \to R$ be Iitaka fiber space and $u : (X', \Delta') \to (V, P)$ be the last step of the log MMP over R as in the proof of Proposition 2.1. In this case u is a Fano contraction to an $(n-1)$-dimensional lc pair (V, P). By Lemma 1.6, we have that

$$H^0\big(X, \mathcal{O}_X\big(m(K_X+\Delta)\big)\big) \simeq H^0\big(X', \mathcal{O}_{X'}\big(m(K_{X'}+\Delta')\big)\big).$$

Let $\alpha : (Y, \Theta) \to (X, \Delta)$ and $\beta : (Y, \Theta) \to (X', \Delta')$ be a common log resolution of a B-birational map $p : X \dashrightarrow X'$ such that $K_Y+\Theta = \alpha^*(K_X+\Delta)$ and

$K_Y+\Theta=\beta^*(K_{X'}+\Delta')$. We define $\Theta^B:=\sum_{d_i=1}\Theta_i$, where $\Theta=\sum d_i\Theta_i$ (see Definition 4.8). Then

$$H^0\big(\llcorner\Delta\lrcorner,\mathcal{O}_{\llcorner\Delta\lrcorner}\big(m(K_X+\Delta)|_{\llcorner\Delta\lrcorner}\big)\big)\simeq H^0\big(\Theta^B,\mathcal{O}_{\Theta^B}\big(m(K_Y+\Theta)|_{\Theta^B}\big)\big)$$
$$\simeq H^0\big(\llcorner\Delta'\lrcorner,\mathcal{O}_{\llcorner\Delta'\lrcorner}\big(m(K_{X'}+\Delta')|_{\llcorner\Delta'\lrcorner}\big)\big)$$

by Lemma 1.3 (see also the proof of Lemma 4.9). We note that $\llcorner\Delta\lrcorner$ and $\llcorner\Delta'\lrcorner$ are seminormal (see Remark 1.2). So it is sufficient to treat (X',Δ') instead of (X,Δ). Let s be a section in $\mathrm{A}(\llcorner\Delta\lrcorner,m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$. By the above isomorphism, we can assume that the section s is in $H^0(\llcorner\Delta'\lrcorner,\mathcal{O}_{\llcorner\Delta'\lrcorner}(m(K_{X'}+\Delta')|_{\llcorner\Delta'\lrcorner}))$. We have the decomposition $\llcorner\Delta'\lrcorner=\Delta'^h\cup\Delta'^v$, where Δ'^h (resp., Δ'^v) is the horizontal (resp., vertical) part of $\llcorner\Delta'\lrcorner$ with respect to the morphism u. Since $s|_{\Delta'^h}$ is B-birational involution invariant over (V,P), it descends to a section t on (V,P). By the isomorphism $H^0(X,\mathcal{O}_X(m(K_X+\Delta)))\simeq H^0(X',\mathcal{O}_{X'}(m(K_{X'}+\Delta')))\simeq H^0(V,\mathcal{O}_V(m(K_V+P)))$, we can pull the section t back to the section $w\in H^0(X,\mathcal{O}_X(m(K_X+\Delta)))\simeq H^0(X',\mathcal{O}_{X'}(m(K_{X'}+\Delta')))$.

First, we prove that $s|_{\Delta'^h}=w|_{\Delta'^h}$. It is true because there is a small analytic open set U in V such that $u^{-1}(U)$ is biholomorphic to $\mathbb{P}^1\times U$ and $u|_{u^{-1}(U)}:\mathbb{P}^1\times U\to U$ is a second projection. By the same argument as in [1, 12.3.4], the difference between $s|_{\Delta'^h}$ and $w|_{\Delta'^h}$ is at most $(-1)^m$. Since we assume that m is even, we have that $s|_{\Delta'^h}=u|_{\Delta'^h}$.

Next, we check that $s|_{\Delta'^v}=w|_{\Delta'^v}$. By Lemma 2.3, $\Delta'^v=\sum u^*D_i$, where $D_i\subset\llcorner P\lrcorner$ is an irreducible divisor. We define $E_i:=u^*D_i$. It is sufficient to check that $s|_{E_i}=w|_{E_i}$ for every i. Let Ξ_i be an irreducible component of $E_i\cap\Delta'^h$ such that $u|_{\Xi_i}:\Xi_i\to D_i$ is dominant. Since $\Delta'^h\cap\Delta'^v\neq\emptyset$, we can always take such Ξ_i. We consider the commutative diagram

$$\begin{tikzcd}
H^0\big(E_i,\mathcal{O}_{E_i}\big(m(K_{X'}+\Delta')|_{E_i}\big)\big) \arrow[r] & H^0\big(\Xi_i,\mathcal{O}_{\Xi_i}\big(m(K_{X'}+\Delta')|_{\Xi_i}\big)\big) \\
H^0\big(D_i,\mathcal{O}_{D_i}\big(m(K_V+P)|_{D_i}\big)\big) \arrow[u, "\cong"'] \arrow[r, "\mathrm{id}"] & H^0\big(D_i,\mathcal{O}_{D_i}\big(m(K_V+P)|_{D_i}\big)\big). \arrow[u, "\iota"']
\end{tikzcd}$$

The map ι is injective since $u|_{\Xi_i}:\Xi_i\to D_i$ is dominant and the left vertical map is an isomorphism since D_i is seminormal (see [3, Proposition 3.2]) and $u|_{E_i}$ has connected fibers. As $s|_{\Xi_i}=w|_{\Xi_i}$, so we get $s|_{E_i}=w|_{E_i}$ for every i. Note that $\Xi_i\subset\Delta'^h$. Thus we have $s=w|_{\llcorner\Delta'\lrcorner}$.

Finally, since $\mathrm{A}(\llcorner\Delta\lrcorner,m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$ generates $\mathcal{O}_{\llcorner\Delta\lrcorner}(m(K_{\llcorner\Delta\lrcorner}+\mathrm{Diff}(\Delta-\llcorner\Delta\lrcorner)))$ by the assumption, the restriction to Δ'^h, which descends to sections on (V,P), generates $\mathcal{O}_{\Delta'^h}(m(K_{X'}+\Delta')|_{\Delta'^h})$. Therefore $\mathrm{PA}(X,m(K_X+\Delta))$ generates $\mathcal{O}_X(m(K_X+\Delta))$. This completes the proof. $\square$

Remark 4.6. In Proposition 4.5 the assumption $n\le 3$ is used for the log MMP and the log abundance theorem, which are used in Proposition 2.1, too.

LEMMA 4.7. *In Proposition 4.5, if* $\dim X \leq 2$, *then we can replace* $\mathrm{PA}(X, m(K_X+\Delta))$ *with* $\mathrm{A}(X, m(K_X+\Delta))$.

Proof. For every $s \in \mathrm{A}(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner})$, we can take a section $s' \in \mathrm{PA}(X, m(K_X+\Delta))$ such that $s'|_{\llcorner\Delta\lrcorner} = s$ by Proposition 4.5. If we put $t := 1/|G| \sum_{g\in G} g^* s' \in \mathrm{A}(X, m(K_X+\Delta))$, where $G = \rho_m(\mathrm{Bir}(X, \Delta))$, then $t|_{\llcorner\Delta\lrcorner} = s$ by Lemma 4.9. Therefore,

$$\mathrm{A}\big(X, m(K_X+\Delta)\big) \longrightarrow \mathrm{A}\big(\llcorner\Delta\lrcorner, m(K_X+\Delta)|_{\llcorner\Delta\lrcorner}\big)$$

is surjective. Let k be a sufficiently large and divisible integer. Then $\mathrm{Bir}(X, \Delta)$ acts on

$$\bigoplus_{l\geq 0} \mathrm{PA}\big(X, lk(K_X+\Delta)\big).$$

Let $N := |\rho_k(\mathrm{Bir}(X, \Delta))| < \infty$ by Section 3, and let σ_i be the ith elementary symmetric polynomial. We obtain

$$\{s = 0\} \supset \bigcap_{j=1}^{N} \big\{g_j^* s = 0\big\} = \bigcap_{i=1}^{N} \big\{\sigma_i\big(g_1^* s, \ldots, g_N^* s\big) = 0\big\}.$$

If $s \in \mathrm{PA}(X, k(K_X+\Delta))$, then

$$\sigma_i^{N!/i}\big(g_1^* s, \ldots, g_N^* s\big) \in \mathrm{A}\big(X, N!k(K_X+\Delta)\big).$$

The vector space $\mathrm{PA}(X, k(K_X+\Delta))$ generates $\mathcal{O}_X(k(K_X+\Delta))$ by Proposition 4.5. Thus we can prove that $\mathrm{A}(X, N!k(K_X+\Delta))$ generates $\mathcal{O}_X(N!k(K_X+\Delta))$ by using Proposition 4.5 and Lemma 4.9. $\square$

In order to prove Lemma 4.9 we need the following definition.

Definition 4.8. Assume that X is nonsingular, $\mathrm{Supp}\,\Delta$ is a simple normal crossing divisor, and $\Delta = \sum_i d_i \Delta_i$ is a $\mathbb{Q}$-divisor such that $d_i \leq 1$ (d_i may be negative) for every i. In this case we say that (X, Δ) is *B-smooth*.

Let (X, Δ) be dlt or B-smooth. A subvariety W of X is said to be a *center of log canonical singularities* if there is a proper birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y with the discrepancy $a(E, X, \Delta) = -1$ such that $\mu(E) = W$ (cf. [13, Definition 1.3]).

Let (X, Δ) be dlt or B-smooth. We write $\Delta = \sum_i d_i \Delta_i$ such that Δ_i are distinct prime divisors. Then the *B-part* of Δ is defined by $\Delta^B := \sum_{d_i=1} \Delta_i$.

If (X, Δ) is dlt or B-smooth, then a center of log canonical singularities is an irreducible component of an intersection of some B-part divisors. (See the Divisorial Log Terminal Theorem of [27] and [21, Section 2.3].) When we consider a center of log canonical singularities W, we always consider the pair (W, Ξ) such that $K_W + \Xi = (K_X+\Delta)|_W$, where Ξ is defined by using the adjunction repeatedly. Note that if (X, Δ) is dlt (resp., B-smooth), then (W, Ξ) is dlt (resp., B-smooth) by the adjunction.

If (X, Δ) is dlt pair or B-smooth and W is a center of log canonical singularities, then we write $W \Subset X$.

LEMMA 4.9. *Let* (X,Δ) *be a pure* n*-dimensional proper dlt pair with* $K_X+\Delta$ *nef and let* m *be a sufficiently large and divisible integer. We write* $G=\rho_m(\mathrm{Bir}(X,\Delta))$*. If* $s\in\mathrm{PA}(X,m(K_X+\Delta))$*, then* $g^*s|_{\llcorner\Delta\lrcorner}=s|_{\llcorner\Delta\lrcorner}$ *and* $g^*s\in\mathrm{PA}(X,m(K_X+\Delta))$ *for every* $g\in G$*.*

In particular if $|G|$ *is finite, then*

$$\sum_{g\in G}g^*s\in\mathrm{A}\big(X,m(K_X+\Delta)\big),$$

$$\prod_{g\in G}g^*s\in\mathrm{A}\big(X,m|G|(K_X+\Delta)\big),$$

and

$$\prod_{g\in G}g^*s|_{\llcorner\Delta\lrcorner}=(s|_{\llcorner\Delta\lrcorner})^{|G|}.$$

Proof. In this proof we omit the restriction symbols such as $|_{\Theta^B}$ when there is no confusion. Let $\alpha,\beta:(Y,\Theta)\to(X,\Delta)$ be a common log resolution of a B-birational map $\sigma:X\dashrightarrow X$ such that $\alpha=\sigma\circ\beta$ and $\rho_m(\sigma)=g$. Since $\llcorner\Delta\lrcorner$ is seminormal and $\Theta^B\to\llcorner\Delta\lrcorner$ has connected fibers by Connectedness Lemma 1.3, we have $\alpha_*\mathcal{O}_{\Theta^B}=\mathcal{O}_{\llcorner\Delta\lrcorner}$ and $\beta_*\mathcal{O}_{\Theta^B}=\mathcal{O}_{\llcorner\Delta\lrcorner}$. Then α^* and β^* induce the isomorphisms between $H^0(\llcorner\Delta\lrcorner,\mathcal{O}_{\llcorner\Delta\lrcorner}(m(K_X+\Delta)|_{\llcorner\Delta\lrcorner}))$ and $H^0(\Theta^B,\mathcal{O}_{\Theta^B}(m(K_Y+\Theta)|_{\Theta^B}))$. So in order to prove $g^*s|_{\llcorner\Delta\lrcorner}=s|_{\llcorner\Delta\lrcorner}$ it is sufficient to check that $\alpha^*(s|_{\llcorner\Delta\lrcorner})=\beta^*(s|_{\llcorner\Delta\lrcorner})$ in $H^0(\Theta^B,\mathcal{O}_{\Theta^B}(m(K_Y+\Theta)|_{\Theta^B}))$ for some common log resolution (Y,Θ).

For this purpose we prove the next two claims.

CLAIM (A_n). *Let* (T,Θ) *and* (S,Ξ) *be* n*-dimensional* B*-smooth pairs and* $h:S\to T$ *a* B*-birational morphism. If* $W\Subset T$*, then there is a* $W'\Subset S$ *such that* $W'\to W$ *is a* B*-birational morphism.*

CLAIM (B_n). *Let* (T,Θ) *and* (S,Ξ) *be* n*-dimensional* B*-smooth pairs and* $h:S\to T$ *a* B*-birational morphism. Assume that* $W\Subset S$*. If* $W\to h(W)$ *is not* B*-birational, then there is a* $W'\Subset W$ *such that* $W'\to h(W)$ *is a* B*-birational morphism and the inclusion* $W'\to W$ *induces the isomorphism* $H^0(W,\mathcal{O}_W(m(K_S+\Xi)|_W))\simeq H^0(W',\mathcal{O}_{W'}(m(K_S+\Xi)|_{W'}))$*.*

Proof of Claims. We prove Claim (A_n) and Claim (B_n) by induction on n. If $n=1$, then (A_1) and (B_1) are trivial.

First, we treat (A_n). If W is a divisor, then we can take the proper transform of W as W'. Otherwise, take a divisor $V\Subset T$ such that $W\Subset V$. We define $U\Subset S$ as the proper transform of V. By using the (A_{n-1}) to the B-birational morphism $U\to V$, we obtain $W'\Subset U$ such that $W'\to W$ is a B-birational morphism. Thus, we have (A_n).

Next, we treat (B_n) by using (A_l) with $l<n$. Let $u:(S',\Xi')\to S$ be the blowing-up of S with center W. If we prove (B_n) for the pair S' and the exceptional divisor $E\Subset S'$, then we can prove (B_n) for S. It is because $G\Subset E$ implies $u(G)$ is the required center of log canonical singularities. So we may assume that W is a divisor. By

[21, 2.45] we have a sequence of blowings-up $T_0 \to T_1 \to \cdots \to T_k = T$ whose centers are the centers associated to the valuation W such that the rational map $S \dashrightarrow T_0$ is an isomorphism at the generic point of W. We can replace (S, Ξ) with the elimination of indeterminacy (S', Ξ') because W and its proper transform are B-birational. So we can assume that $f_0 : S \to T_0$ is a B-birational morphism. We can apply (A_{n-1}) to the B-birational morphism $f_0 : W \to f_0(W)$. So we only have to prove (B_n) to the B-birational morphism $T_0 \to T$. So we may assume that $(S, \Xi) = (T_0, \Theta_0)$. We use the induction on the number k of the blowings-up. When $k = 1$, we take a divisor $D \Subset T_1$ such that $h(W) \Subset D$ and its proper transform $D' \Subset T_0$. By applying (B_{n-1}) to $D' \cap W \Subset D'$ and $h(D' \cap W) = h(W) \Subset D$, we have the result. Next we apply the induction hypothesis to $a : T_0 \to T_{k-1}$. We have $U \Subset W$ such that $a : U \to a(W)$ is a B-birational morphism. Since $b : T_{k-1} \to T_k$ is one blowing-up whose center is the center associated to the valuation W, we can take a divisor V on T_{k-1} such that $a(U) = a(W) \Subset V$ and $b : V \to b(V)$ is B-birational. By using (B_{n-1}) to $b : V \to b(V)$ we have a $V' \Subset a(U)$ such that $V' \to (b \circ a)(W) = h(W)$ is B-birational. By using (A_l) with $l < n$ to $U \to a(U)$ we obtain a $W' \Subset U \Subset W$ such that $W' \to V'$ is B-birational. So $W' \to h(W)$ is B-birational. By the construction of W', the inclusion $W' \to W$ induces the isomorphism $H^0(W, \mathcal{O}_W(m(K_S + \Xi)|_W)) \simeq H^0(W', \mathcal{O}_{W'}(m(K_S + \Xi)|_{W'}))$. This proves (B_n). □

Proof of Lemma 4.9 continued. We return to the proof of the lemma. Let $f : (X', \Delta') \to (X, \Delta)$ be Szabó's resolution such that $K_{X'} + \Delta' = f^*(K_X + \Delta)$, that is, a log resolution whose discrepancy is greater than -1 (see Resolution Lemma in [27]). Let $\alpha = \amalg \alpha_i$ and $\beta = \amalg \beta_i : (Y, \Theta) = \amalg (Y_i, \Theta_i) \to (X, \Delta)$ be a common log resolution of a B-birational map $\sigma : X \dashrightarrow X$ passing through (X', Δ') such that $\alpha = \sigma \circ \beta$ and $\rho_m(\sigma) = g$. We write $\alpha', \beta' : Y \to X'$ such that $\alpha = f \circ \alpha'$, $\beta = f \circ \beta'$. We take an irreducible divisor $E \subset \Theta^B$. Apply (B_n) to the $\alpha'_i : Y_i \to \alpha'_i(Y_i)$ such that $E \Subset Y_i$. Then we obtain an $E' \Subset E$ or $E' = E$ such that $E' \to \alpha'(E)$ is B-birational. Apply (B_n) to the $\beta'_i : Y_i \to \beta'_i(Y_i)$. Then we obtain an $E'' \Subset E'$ or $E'' = E'$ such that $E'' \to \beta'(E')$ is B-birational. By repeating the above construction, we obtain $F \Subset E$ or $F = E$ such that $\alpha' : F \to \alpha'(F)$ and $\beta' : F \to \beta'(F)$ are B-birational. The morphism $f : \alpha'(F) \to \alpha(F)$ and $f : \beta'(F) \to \beta(F)$ are B-birational since f is Szabó's resolution. Then $\alpha : F \to \alpha(F)$ and $\beta : F \to \beta(F)$ are B-birational. Since $s \in \mathrm{PA}(X, m(K_X + \Delta))$, $\alpha^*(s|_{\alpha(F)}) = \beta^*(s|_{\beta(F)})$ on F. Since $H^0(E, m(K_Y + \Theta)|_E) \simeq H^0(F, m(K_Y + \Theta)|_F)$ by (B_n), we have $\alpha^* s = \beta^* s$ on E. Since E is an arbitrary irreducible component of Θ^B, we have $\alpha^*(s|_{\llcorner \Delta \lrcorner}) = \beta^*(s|_{\llcorner \Delta \lrcorner})$. Thus $g^* s|_{\llcorner \Delta \lrcorner} = s|_{\llcorner \Delta \lrcorner}$.

The latter part is trivial. □

The next corollary is the main theorem of this paper, which includes Theorem 0.1 (for proper pairs, see Remark 4.11) and a generalization of [1, 12.1.1] and [15, 8.5].

COROLLARY 4.10. *Let (X, Δ) be a projective slc n-fold such that $K_X + \Delta$ is nef, where $n \leq 3$. Then $|m(K_X + \Delta)|$ is free for a sufficiently large and divisible integer*

m. In particular, if (X, Δ) is projective sdlt and $n \leq 2$, then $\mathrm{A}(X, m(K_X + \Delta))$ *generates* $\mathcal{O}_X(m(K_X + \Delta))$.

Proof. We prove this corollary inductively on $\dim X$. We take the normalization of (X, Δ) and take a $\mathbb{Q}$-factorial dlt model $(Y, \Xi) = \amalg(Y_i, \Xi_i)$ for each irreducible component by Lemma-Definition 1.4. By the assumption of induction, $\mathrm{A}(\llcorner \Xi \lrcorner, m(K_Y + \Xi)|_{\llcorner \Xi \lrcorner})$ generates $\mathcal{O}_{\llcorner \Xi \lrcorner}(m(K_{\llcorner \Xi \lrcorner} + \mathrm{Diff}(\Xi - \llcorner \Xi \lrcorner)))$. By Proposition 4.5,

$$\mathrm{PA}\big(Y, m(K_Y + \Xi)\big) \longrightarrow \mathrm{A}\big(\llcorner \Xi \lrcorner, m(K_Y + \Xi)|_{\llcorner \Xi \lrcorner}\big)$$

is surjective, and $\mathrm{PA}(Y, m(K_Y + \Xi))$ generates $\mathcal{O}_Y(m(K_Y + \Xi))$. So, by Lemma 4.2, $|m(K_X + \Delta)|$ is free. If $\dim X \leq 2$, then we have the finiteness of B-pluricanonical representation (see Section 3). Therefore, we can replace $\mathrm{PA}(Y, m(K_Y + \Xi))$ with $\mathrm{A}(Y, m(K_Y + \Xi))$ by Lemma 4.7. So we get the latter part of this corollary. We need the latter part for the inductional treatment. In order to prove the abundance for slc n-folds (X, Δ), we use Proposition 4.5, which demands that $\mathcal{O}_{\llcorner \Xi \lrcorner}(m(K_{\llcorner \Xi \lrcorner} + \mathrm{Diff}(\Xi - \llcorner \Xi \lrcorner)))$ is generated by not only $\mathrm{PA}(\llcorner \Xi \lrcorner, m(K_Y + \Xi)|_{\llcorner \Xi \lrcorner})$ but also $\mathrm{A}(\llcorner \Xi \lrcorner, m(K_Y + \Xi)|_{\llcorner \Xi \lrcorner})$. □

Remark 4.11. Proposition 4.5, and hence Theorem 3.5 and Corollary 4.10, hold true even for proper pairs. This can be checked, as in [16, 7.1]. Indeed, let (X, D) be a proper dlt pair such that $K_X + D$ is nef. By the log MMP, we can find a $\mathbb{Q}$-factorial projective dlt pair (X', D') with nef $K_{X'} + D'$. By [26, 1.5], (X', D') is B-birationally equivalent to (X, D). By using the arguments similar to those used in Lemma 4.9 and in case 4 in the proof of Proposition 4.5, we can reduce Proposition 4.5 for (X, D) to that for the projective (X', D'). Therefore, we get Proposition 4.5 for proper dlt pairs. Thus, Theorem 3.5 and Corollary 4.10 hold true for proper pairs.

Appendix

We can state the results similar to Corollary 4.10 in arbitrary dimension, if we list all the necessary results (e.g., log MMP) yet to be established as the assumption.

Theorem A.1. *Assume the log MMP for dimension n.*

(1) *If the abundance conjecture holds for lc n-folds and if the finiteness of B-pluricanonical representations (see Conjecture 3.2) is true for dimension $(n-1)$, then the abundance conjecture is true for slc n-folds.*

(2) *If the abundance conjecture holds for klt n-folds and slc $(n-1)$-folds, then the abundance conjecture is true for lc n-folds.*

Proof. For the proof of (1), see Remarks 2.2 and 4.6, and the proof of Corollary 4.10. One can prove (2) by using the same argument as in [16, Section 7]. □

We list the following two results for the reader's convenience.

Theorem A.2 (Relative log abundance theorem). *Let (X, Δ) be lc and* $\dim X \leq 3$. *Let $f : X \to S$ be a proper surjective morphism onto a variety S. Assume that*

$K_X + \Delta$ is f-nef. Then $K_X + \Delta$ is f-semiample.

Proof. If $\dim S = 0$, then this is nothing but the log abundance theorem (see [6], [8], and [16]). So we may assume $\dim S \geq 1$. If (X, Δ) is klt, the proof is given, for example, in [14, 6-1-11], [11]. When (X, Δ) is lc, we can use the arguments in [16, Section 7] in the relative setting. (See also [15, 8.5].) □

COROLLARY A.3 (Threefold log canonical flips) (cf. [15, 8.1]). *Threefold log canonical flips exist.*

Proof. Let (X, Δ) be an lc pair and $f : (X, \Delta) \to S$ a flipping contraction. We take a $\mathbb{Q}$-factorial dlt model (X', Δ') (see Lemma-Definition 1.4) and run the log MMP over S. Then we obtain a relative minimal model (X'', Δ'') over S. By using Theorem A.2 we have a relative canonical model (X^+, Δ^+). It is easy to check that (X^+, Δ^+) is the flip of f. □

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