Vanishing theorems for complex projective varieties

Osamu Fujino

Osaka University

July 26, 2016

Kodaira

The Kodaira vanishing theorem is one of the most famous and most important vanishing theorems for complex projective varieties.

Kodaira

The Kodaira vanishing theorem is one of the most famous and most important vanishing theorems for complex projective varieties.

Theorem 1.1 (Kodaira)

- X: smooth projective variety
- D: ample Cartier divisor

Then $H^q(X, \mathcal{O}_X(K_X + D)) = 0$ for every $q > 0$.

Kodaira

The Kodaira vanishing theorem is one of the most famous and most important vanishing theorems for complex projective varieties.

Theorem 1.1 (Kodaira)

- X: smooth projective variety
- D: ample Cartier divisor

Then $H^q(X, \mathcal{O}_X(K_X + D)) = 0$ for every $q > 0$.

Note:

- K_X: canonical divisor of X
- $\wedge^{\dim X}\Omega_X \simeq \mathcal{O}_X(K_X)$

Kawamata–Viehweg

As is well-known, the Kawamata–Viehweg vanishing theorem is a powerful generalization of the Kodaira vanishing theorem.

Kawamata–Viehweg

As is well-known, the Kawamata–Viehweg vanishing theorem is a powerful generalization of the Kodaira vanishing theorem.

Theorem 1.2 (Kawamata–Viehweg)

- X: smooth projective variety
- D: nef and big $\mathbb{Q}$-divisor
- $\mathrm{Supp}\{D\}$: SNC divisor

Then $H^q(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for every $q > 0$.

Kawamata–Viehweg

As is well-known, the Kawamata–Viehweg vanishing theorem is a powerful generalization of the Kodaira vanishing theorem.

Theorem 1.2 (Kawamata–Viehweg)

- X: smooth projective variety
- D: nef and big $\mathbb{Q}$-divisor
- $\mathrm{Supp}\{D\}$: SNC divisor

Then $H^q(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$ for every $q > 0$.

Note:

- $\{D\}$: the fractional part of D, $\lceil D \rceil$: the round-up of D
- D: nef and big $\iff$ $D^{\dim X} > 0$ and $D \cdot C \geq 0$ for any curve C

Nadel

The (algebraic version of) Nadel vanishing theorem is also a useful generalization of Kodaira.

Nadel

The (algebraic version of) Nadel vanishing theorem is also a useful generalization of Kodaira.

Theorem 1.3 (Nadel)

- X: smooth projective variety
- L: Cartier divisor
- D: effective $\mathbb{Q}$-divisor
- $L - D$: nef and big

Then $H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, D)) = 0$ for every $q > 0$, where $\mathcal{J}(X, D)$: multiplier ideal sheaf of (X, D).

Kollár

Kollár’s results contain Kodaira as a special case.

Kollár

Kollár's results contain Kodaira as a special case.

Theorem 1.4 (Kollár)

- X: smooth projective variety
- Y: projective variety
- $f : X \to Y$: surjective morphism

Kollár

Kollár's results contain Kodaira as a special case.

Theorem 1.4 (Kollár)

- X: smooth projective variety
- Y: projective variety
- $f : X \to Y$: surjective morphism

Then we have:

(i) $R^q f_*\mathcal{O}_X(K_X)$: torsion-free for every q

(ii) $H^p(Y, R^q f_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_Y(H)) = 0$ for every $p > 0$ and $q \geq 0$, where H: ample Cartier divisor on Y.

We need more general vanishing theorems for the minimal model program (MMP, for short) for higher-dimensional algebraic varieties.

SNC pairs

- M: smooth variety $/\mathbb{C}$
- X: SNC divisor on M
- B: $\mathbb{R}$-divisor on M such that $\operatorname{Supp} B$: SNC divisor
- B and X have no common components, $\operatorname{Supp}(B + X)$: SNC divisor
- $D = B|_X$

SNC pairs

- M: smooth variety $/\mathbb{C}$
- X: SNC divisor on M
- B: $\mathbb{R}$-divisor on M such that $\operatorname{Supp} B$: SNC divisor
- B and X have no common components, $\operatorname{Supp}(B+X)$: SNC divisor
- $D = B|_X$

Definition 2.1 (GESNC pair)

(X, D): globally embedded simple normal crossing (GESNC) pair

SNC pairs

- M: smooth variety $/\mathbb{C}$
- X: SNC divisor on M
- B: $\mathbb{R}$-divisor on M such that $\operatorname{Supp} B$: SNC divisor
- B and X have no common components, $\operatorname{Supp}(B+X)$: SNC divisor
- $D = B|_X$

Definition 2.1 (GESNC pair)

(X, D): globally embedded simple normal crossing (GESNC) pair

Definition 2.2 (SNC pair)

(Y, Δ): simple normal crossing (SNC) pair

$\overset{\text{def}}{\Longleftrightarrow}$ (Y, Δ): Zariski locally isomorphic to a GESNC pair

Stratum of SNC pair

- (X, D): SNC pair
- $D \in [0, 1]$
- $\nu : X^{\nu} \to X$: normalization
- $K_{X^{\nu}} + \Theta = \nu^{*}(K_X + D)$

Stratum of SNC pair

- (X, D): SNC pair
- $D \in [0, 1]$
- $\nu : X^\nu \to X$: normalization
- $K_{X^\nu} + \Theta = \nu^*(K_X + D)$

Definition 2.3 (Stratum)

- W: closed subvariety of X

W: stratum of (X, D)
$\overset{\text{def}}{\Longleftrightarrow}$ $W = \nu(C)$, where C is a log canonical center of (X^ν, Θ), or W is an irreducible component of X

Hodge theoretic injectivity theorem

Theorem 2.4 (Relative Hodge theoretic injectivity theorem)

- (X, Δ): SNC pair, $\Delta \in [0, 1]$, $\pi : X \to S$: proper
- L: Cartier divisor on X
- D: effective Weil divisor on X
- $\operatorname{Supp} D \subset \operatorname{Supp} \Delta$
- $L \sim_{\mathbb{R},\pi} K_X + \Delta$

Hodge theoretic injectivity theorem

Theorem 2.4 (Relative Hodge theoretic injectivity theorem)

- (X, Δ): SNC pair, $\Delta \in [0, 1]$, $\pi : X \to S$: proper
- L: Cartier divisor on X
- D: effective Weil divisor on X
- $\operatorname{Supp} D \subset \operatorname{Supp} \Delta$
- $L \sim_{\mathbb{R},\pi} K_X + \Delta$

Then

$$R^q\pi_*\mathcal{O}_X(L) \to R^q\pi_*\mathcal{O}_X(L + D)$$

is injective for every q.

Injectivity theorem for SNC pair

Theorem 2.5 (Injectivity for SNC pair)

- (X, Δ): SNC pair, $\Delta \in [0, 1]$, $\pi : X \to S$: proper, as before
- L: Cartier divisor on X
- D: effective Cartier, permissible with respect to (X, Δ)

Injectivity theorem for SNC pair

Theorem 2.5 (Injectivity for SNC pair)

- (X, Δ): SNC pair, $\Delta \in [0, 1]$, $\pi : X \to S$: proper, as before
- L: Cartier divisor on X
- D: effective Cartier, permissible with respect to (X, Δ)
  We further assume:

(i) $L \sim_{\mathbb{R},\pi} K_X + \Delta + H$

(ii) H: π-semi-ample $\mathbb{R}$-divisor

(iii) $tH \sim_{\mathbb{R},\pi} D + D'$, $t \in \mathbb{R}_{>0}$,
D': effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, permissible with respect to (X, Δ)

Injectivity theorem for SNC pair

Theorem 2.5 (Injectivity for SNC pair)

- (X, Δ): SNC pair, $\Delta \in [0, 1]$, $\pi : X \to S$: proper, as before
- L: Cartier divisor on X
- D: effective Cartier, permissible with respect to (X, Δ)
  We further assume:

(i) $L \sim_{\mathbb{R},\pi} K_X + \Delta + H$

(ii) H: π-semi-ample $\mathbb{R}$-divisor

(iii) $tH \sim_{\mathbb{R},\pi} D + D'$, $t \in \mathbb{R}_{>0}$,
D': effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, permissible with respect to (X, Δ)

Then $R^q\pi_*\mathcal{O}_X(L) \to R^q\pi_*\mathcal{O}_X(L + D)$ is injective for every q.

Torsion-freeness and Vanishing for SNC pair

Theorem 2.6 (Torsion-freeness and Vanishing thereom)

- (Y, Δ): SNC pair, $\Delta \in [0, 1]$, $f : Y \to X$: proper
- L: Cartier divisor on Y such that $L - (K_Y + \Delta)$: f-semi-ample

Torsion-freeness and Vanishing for SNC pair

Theorem 2.6 (Torsion-freeness and Vanishing thereom)

- (Y, Δ): SNC pair, $\Delta \in [0, 1]$, $f : Y \to X$: proper
- L: Cartier divisor on Y such that $L - (K_Y + \Delta)$: f-semi-ample

Then we have:

(i) Every associated prime of $R^q f_* \mathcal{O}_Y(L)$ is the generic point of the f-image of some stratum of (Y, Δ).

(ii)
- $\pi : X \to V$: projective
- $L - (K_Y + \Delta) \sim_{\mathbb{R}} f^* H$, H: π-ample $\mathbb{R}$-divisor on X

$\Longrightarrow R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.

Main statement

Our result for SNC pairs contains Kodaira, Kawamata–Viehweg, Nadel, Kollár, and many other powerful and useful vanishing results as very special cases.

MHS for cohomology with compact support

Almost all the classical vanishing theorems (Kawamata–Viehweg, Kollár, etc.) can be proved by the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

MHS for cohomology with compact support

Almost all the classical vanishing theorems (Kawamata–Viehweg, Kollár, etc.) can be proved by the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

My idea is to use the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p(\log D) \otimes \mathcal{O}_X(-D)) \Rightarrow H_c^{p+q}(X \setminus D, \mathbb{C}),$$

where X: smooth projective variety, D: SNC divisor.

MHS for cohomology with compact support

Almost all the classical vanishing theorems (Kawamata–Viehweg, Kollár, etc.) can be proved by the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C}).$$

My idea is to use the E_1-degeneration of

$$E_1^{pq} = H^q(X, \Omega_X^p(\log D) \otimes \mathcal{O}_X(-D)) \Rightarrow H_c^{p+q}(X \setminus D, \mathbb{C}),$$

where X: smooth projective variety, D: SNC divisor.
In my framework,

$$\mathcal{O}_X(K_X + D) \simeq \mathcal{H}om(\Omega_X^0(\log D) \otimes \mathcal{O}_X(-D), \omega_X).$$

We do not see $\mathcal{O}_X(K_X + D)$ as $\bigwedge^{\dim X} \Omega_X^1(\log D)$.

Final remarks

Precisely speaking:

- (X, D): SNC pair, or finite cyclic cover of SNC pair

We have to consider MHS on

$$H^k_c(X \setminus \lfloor D \rfloor, \mathbb{C}).$$

Thank you

Thank you very much!