ACC FOR LOG CANONICAL THRESHOLDS FOR COMPLEX ANALYTIC SPACES

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Dedicated to Professor Vyacheslav V. Shokurov on the occasion of his seventieth birthday

Abstract. We show that log canonical thresholds for complex analytic spaces satisfy the ACC.

1. Introduction

As usual, *ACC* stands for the *ascending chain condition* and *DCC* stands for the *descending chain condition*. In [HMX], the ACC for log canonical thresholds, which was conjectured by Shokurov, was completely settled for algebraic varieties. We note that Shokurov raised many conjectures that assert the ascending or descending chain condition for various naturally defined invariants coming from algebraic geometry (see, for example, [S1], [S2], [K, Chapter 18], [K1, Section 8], and so on). In this paper, we generalize it for complex analytic spaces.

Let us start with the definition of *log canonical thresholds* for complex analytic spaces. Note that X is a normal complex analytic space in Definition 1.1. For various aspects of log canonical thresholds, we strongly recommend the reader to see [K1, Sections 8, 9, and 10].

Definition 1.1 (Log canonical thresholds for complex analytic spaces)**.** Let (X, Δ) be a log canonical pair and let M be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let c be a nonnegative real number such that $(X, \Delta + cM)$ is log canonical and that there exists a non-kawamata log terminal center of $(X, \Delta + cM)$ which is contained in $\mathrm{Supp} M$. Then c is called the *log canonical threshold* of M with respect to (X, Δ) and is usually denoted by $\mathrm{lct}(X, \Delta; M)$. When $M = 0$, we put $\mathrm{lct}(X, \Delta; M) = +\infty$.

The following definition and example show the reason why we adopt the above definition of log canonical thresholds for complex analytic

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spaces, which looks slightly different from the usual definition of log canonical thresholds for algebraic varieties.

Definition 1.2. Let X be a normal complex variety. A *prime divisor* on X is an irreducible and reduced closed subvariety of codimension one. An $\mathbb{R}$-*divisor* D on X is a *locally finite* formal sum

$$D = \sum_i a_i D_i,$$

where D_i is a prime divisor on X with $a_i \in \mathbb{R}$ for every i and $D_i \neq D_j$ for $i \neq j$. When $a_i \in \mathbb{Q}$ holds for every i, D is called a $\mathbb{Q}$-*divisor* on X.

Let D be an $\mathbb{R}$-divisor on a normal complex variety X and let x be a point of X. If D is written as a finite $\mathbb{R}$-linear (resp. $\mathbb{Q}$-linear) combination of Cartier divisors on some open neighborhood of x, then D is said to be $\mathbb{R}$-*Cartier at* x (resp. $\mathbb{Q}$-*Cartier at* x). If D is $\mathbb{R}$-Cartier (resp. $\mathbb{Q}$-Cartier) at x for every $x \in X$, then D is said to be $\mathbb{R}$-*Cartier* (resp. $\mathbb{Q}$-*Cartier*).

Example 1.3. We consider $X = \mathbb{C}$. Let $\{P_n\}_{n \in \mathbb{Z}_{>0}}$ be a set of mutually distinct discrete points of X. We put $M = \sum_{n \in \mathbb{Z}_{>0}} \frac{n-1}{n} P_n$. Then M is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. In this case, (X, M) is log canonical and (X, tM) is not log canonical for every positive real number t with $t > 1$. However, there are no non-kawamata log terminal centers of (X, M), that is, (X, M) is kawamata log terminal.

We note an obvious remark.

Remark 1.4. (1) If (X, Δ) and M are both algebraic in Definition 1.1, then it is easy to see that the following equality

$$\mathrm{lct}(X, \Delta; M) = \sup\{t \in \mathbb{R} \,|\, (X, \Delta + tM) \text{ is log canonical}\}$$

holds.

(2) In Definition 1.1, let U be a relatively compact open subset of X. Then we can check that

$$\mathrm{lct}(U, \Delta|_U; M|_U) = \sup\{t \in \mathbb{R} \,|\, (U, \Delta|_U + tM|_U) \text{ is log canonical}\}$$

holds by using the resolution of singularities.

By Remark 1.4 (1), $\mathrm{lct}(X, \Delta; M)$ coincides with the usual one when (X, Δ) and M are all algebraic.

Definition 1.5. Let $\mathfrak{T}^{an} = \mathfrak{T}^{an}_n(I)$ denote the set of log canonical pairs (X, Δ), where X is a normal complex variety of dimension n and the coefficients of Δ belong to a set $I \subset [0, 1]$. We put

$$\mathrm{LCT}^{an}_n(I, J) = \{\mathrm{lct}(X, \Delta; M) \,|\, (X, \Delta) \in \mathfrak{T}^{an}_n(I)\},$$

where the coefficients of M belong to a subset J of the positive real numbers.

The main result of this short paper is the ACC for log canonical thresholds for complex analytic spaces, which is a generalization of [HMX, Theorem 1.1].

Theorem 1.6 (ACC for the log canonical threshold for complex analytic spaces)**.** *We fix a positive integer n, $I \subset [0,1]$, and a subset J of the positive real numbers. If I and J satisfy the DCC, then $\mathrm{LCT}^{an}_n(I,J)$ satisfies the ACC.*

The main ingredient of the proof of Theorem 1.6 is the ACC for numerically trivial pairs, which is nothing but [HMX, Theorem 1.5] (see Theorem 1.7), and the minimal model program for projective morphisms between complex analytic spaces established in [F2]. Note that one of the motivations of [F2] is to make the minimal model program applicable to the study of germs of complex analytic singularities. We also note that a similar result was obtained independently by Das, Hacon, and Păun (see [DHP, Theorem 6.4]).

Theorem 1.7 (ACC for numerically trivial pairs, see [HMX, Theorem 1.5])**.** *Fix a positive integer n and a set $I \subset [0,1]$, which satisfies the DCC. Then there is a finite subset $I_0 \subset I$ with the following property:*

If (X,Δ) is an n-dimensional projective log canonical pair such that $K_X+\Delta$ is numerically trivial and that the coefficients of Δ belong to I, then the coefficients of Δ belong to I_0.

We note that de Fernex, Ein, and Mustaţă established a striking result on Shokurov's ACC conjecture before [HMX]. Here we only explain a very special case. For the details and some related topics, see [dFEM], [K2], [T], and so on.

Definition 1.8 (Log canonical thresholds of holomorphic functions)**.** Let f be a holomorphic function in a neighborhood of $0 \in \mathbb{C}^n$. The *log canonical threshold* of f at 0 is the number $c = \mathrm{lct}_0(f)$ such that

- $|f|^{-s}$ is L^2 in a neighborhood of 0 for $s < c$, and
- $|f|^{-s}$ is not L^2 in a neighborhood of 0 for $s > c$.

Hence, if $f(0) \neq 0$, then $\mathrm{lct}_0(f) = +\infty$.

We put

$$\mathcal{HT}_n := \{\mathrm{lct}_0(f) \mid f \in \mathcal{O}_{\mathbb{C}^n,0}, f(0)=0\} \subset \mathbb{R}.$$

This means that $\mathcal{HT}_n$ is the set of log canonical thresholds of all possible holomorphic functions of n variables vanishing at $0 \in \mathbb{C}^n$.

Then we have:

Theorem 1.9 ([dFEM])**.** *$\mathcal{HT}_n$ satisfies the ACC.*

Note that the following natural inclusion

$$\mathcal{HT}_n \subset \mathrm{LCT}^{an}_n(\{0\}, \mathbb{Z}_{>0})$$

holds. Therefore, Theorem 1.9 is a very special case of Theorem 1.6.

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In this paper, we will freely use [F2]. We always assume that complex analytic spaces are *Hausdorff* and *second-countable*. We use the standard notation of the theory of minimal models as in [KM], [F1], and [F2].

2. Proof

Let us start with the definition of ACC sets and DCC sets.

Definition 2.1 (ACC sets and DCC sets, see [HMX, 3.4. DCC sets])**.** Let I be a set of real numbers. We say that I satisfies the *ascending chain condition* or *ACC* (resp. *descending chain condition* or *DCC*) if it does not contain any infinite strictly increasing (resp. decreasing) sequences.

We take $I \subset [0,1]$. We put

$$I_+ := \{0\} \bigcup \left\{ j \in [0,1] \,\middle|\, j = \sum_{p=1}^{l} i_p \text{ for some } i_1 \ldots, i_l \in I \right\}$$

and

$$D(I) := \left\{ a \in [0,1] \,\middle|\, a = \frac{m-1+f}{m} \text{ for some } m \in \mathbb{Z}_{>0} \text{ and } f \in I_+ \right\}.$$

It is easy to see that I satisfies the DCC if and only if $D(I)$ satisfies the DCC.

Without any difficulties, we can prove a slight modification of [HMX, Lemma 5.1] for complex analytic spaces by using [F2].

Lemma 2.2. *We fix a positive integer n and a set $1 \in I \subset [0,1]$. Assume that $(X, \Delta + \Delta')$ is an $(n+1)$-dimensional log canonical pair such that $\Delta \geq 0$, $\Delta' \geq 0$ is $\mathbb{R}$-Cartier, and the coefficients of Δ, Δ'*

and $\Delta+\Delta'$ belong to I. We further assume that there exists a non-kawamata log terminal center V of $(X, \Delta+\Delta')$ such that $V \subset \mathrm{Supp}\Delta'$ with $\dim V \leq \dim X - 2$.

Then we can construct a log canonical pair (S, Θ), where S is a projective variety of dimension at most n, the coefficients of Θ belong to $D(I)$, $K_S+\Theta$ is numerically trivial, and some component of Θ has coefficient

$$\frac{m-1+f+kc}{m},$$

where m and k are positive integers, $f \in I_+$, and $c \in I$ is the coefficient of some component of Δ'.

The proof of [HMX, Lemma 5.1] works with only some minor modifications since we can always construct dlt blow-ups by [F2] in the complex analytic setting.

Proof of Lemma 2.2. We can replace V with a maximal (with respect to inclusion) non-kawamata log terminal center of $(X, \Delta+\Delta')$ satisfying $\dim V \leq \dim X - 2$ and $V \subset \mathrm{Supp}\Delta'$. We take an analytically sufficiently general point P of V. Then we take an open neighborhood U of P and a Stein compact subset W of X such that $U \subset W$ and that $\Gamma(W, \mathcal{O}_X)$ is noetherian. By [F2, Theorem 1.21], after shrinking X around W suitably, we can construct a projective bimeromorphic morphism $\pi\colon Y \to X$ with $K_Y+\Delta_Y = \pi^*(K_X+\Delta+\Delta')$ such that

(a) (Y, Δ_Y) is divisorial log terminal,
(b) Y is $\mathbb{Q}$-factorial over W,
(c) $a(E, X, \Delta+\Delta') = -1$ holds for every π-exceptional divisor E, and
(d) there exists a π-exceptional divisor F on Y such that $\pi(F)=V$.

Since Δ' is $\mathbb{R}$-Cartier by assumption, $\pi^*\Delta'$ is well-defined and is π-numerically trivial. Hence we can find B, which is an irreducible component of $\mathrm{Supp}\pi_*^{-1}\Delta'$, and a π-exceptional divisor S with $S \cap B \neq \emptyset$, $\pi(S)=V$, and $\pi(S\cap B)=V$. By adjunction, we obtain

$$K_S+\Theta := (K_Y+\Delta_Y)|_S$$

such that the coefficients of Θ belong to $D(I)$ and some component of Θ has a coefficient of the form

$$\frac{m-1+f+kc}{m},$$

where m and k are positive integers, $f \in I_+$, and $c \in I$ is the coefficient of B in $\pi_*^{-1}\Delta'$. We take an analytically sufficiently general point $v \in V \cap U$ and consider the fiber over v. Then we obtain (S_v, Θ_v), which is divisorial log terminal with $\dim S_v \le n$, such that the coefficients of Θ_v belong to $D(I)$, some component of Θ_v has a coefficient of the form

$$\frac{m-1+f+kc}{m}$$

as desired, and $K_{S_v}+\Theta_v$ is numerically trivial. This is what we wanted. $\square$

Let us prove Theorem 1.6.

Proof of Theorem 1.6. We assume that $c_1, c_2, \ldots \in \mathrm{LCT}^{an}_m(I, J)$ such that $c_i \le c_{i+1}$ holds for every i. It is sufficient to prove that $c_i = c_{i+1}$ holds for every sufficiently large i. By definition, we can take an n-dimensional log canonical pair (X_i, Δ_i) and an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor M_i on X_i such that the coefficients of Δ_i belong to I, the coefficients of M_i belong to J, $(X_i, \Delta_i + c_iM_i)$ is log canonical, and there exists a non-kawamata log terminal center V_i of $(X_i, \Delta_i + c_iM_i)$ with $V_i \subset \mathrm{Supp} M_i$ for every i.

We put

$$\begin{aligned} K = I &\cup \{c_i\alpha \in [0,1] \,|\, i \in \mathbb{Z}_{>0}, \alpha \in J\} \\ &\cup \{\beta + c_i\gamma \in [0,1] \,|\, i \in \mathbb{Z}_{>0}, \beta \in I, \gamma \in J\} \cup \{1\}. \end{aligned}$$

Then the coefficient of Δ_i, c_iM_i, and $\Delta_i + c_iM_i$ belong to K. It is easy to see that K satisfies the DCC. We also put

$$L = \{1-\alpha \,|\, \alpha \in I\}.$$

Then L obviously satisfies the ACC. Hence $L \cap K$ is a finite set since K satisfies the DCC.

If $\dim V_i = n-1$, then the coefficient of V_i in c_iM_i is in the finite set $L \cap K$. Therefore, it is sufficient to treat the case when $\dim V_i \le n-2$ holds for every i. Hence, from now on, we assume that $\dim V_i \le n-2$ holds for every i. By Lemma 2.2, for every i, we can construct a projective log canonical pair (S_i, Θ_i) such that $\dim S_i \le n-1$, the coefficients of Θ_i belong to $D(K)$, $K_{S_i} + \Theta_i$ is numerically trivial, and some component of Θ_i has coefficient

$$\frac{m_i - 1 + f_i + k_ic_i\alpha_i}{m_i},$$

where m_i and k_i are positive integers, $f_i \in K_+$, and $\alpha_i \in J$. By Theorem 1.7, which is nothing but [HMX, Theorem 1.5], there exists a finite subset $K_0 \subset D(K)$ such that

$$\frac{m_i - 1 + f_i + k_ic_i\alpha_i}{m_i} \in K_0.$$

Then, by [HMX, Lemma 5.2],

$$\{c_i\alpha_i\}_{i\in\mathbb{Z}_{>0}}$$

is a finite set. This implies that $c_i = c_{i+1}$ holds for every sufficiently large i since $\alpha_i \in J$ for every i.

This is what we wanted, that is, $\mathrm{LCT}^{an}_n(I, J)$ satisfies the ACC. $\square$

References

[DHP] O. Das, C. Hacon, M. Păun, On the 4-dimensional minimal model program for Kähler varieties, preprint (2022). arXiv:2205.12205 [math.AG]

[dFEM] T. de Fernex, L. Ein, M. Mustaţă, Shokurov's ACC conjecture for log canonical thresholds on smooth varieties, Duke Math. J. **152** (2010), no. 1, 93–114.

[F1] O. Fujino, *Foundations of the minimal model program*, MSJ Memoirs, **35**. Mathematical Society of Japan, Tokyo, 2017.

[F2] O. Fujino, Minimal model program for projective morphisms between complex analytic spaces, preprint (2022). arXiv:2201.11315 [math.AG]

[HMX] C. D. Hacon, J. McKernan, C. Xu, ACC for log canonical thresholds, Ann. of Math. (2) **180** (2014), no. 2, 523–571.

[K1] J. Kollár, Singularities of pairs, *Algebraic geometry—Santa Cruz 1995*, 221–287, Proc. Sympos. Pure Math., **62**, Part 1, Amer. Math. Soc., Providence, RI, 1997.

[K2] J. Kollár, Which powers of holomorphic functions are integrable?, preprint (2008). arXiv:0805.0756 [math.AG]

[K] J. Kollár, et al., *Flips and abundance for algebraic threefolds*, Société Mathématique de France, Paris, 1992 Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. **211**, 1992.

[KM] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, **134**. Cambridge University Press, Cambridge, 1998.

[S1] V. V. Shokurov, Problems about Fano varieties, *Birational Geometry of Algebraic Varieties: Open Problems*, The XXIIIrd International Symposium, Division of Mathematics, The Taniguchi Foundation (1988), 30–32.

[S2] V. V. Shokurov, Three-dimensional log perestroikas, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 1, 105–203.

[T] B. Totaro, The ACC conjecture for log canonical thresholds (after de Fernex, Ein, Mustaţă, Kollár), Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026. Astérisque No. **339** (2011), Exp. No. 1025, ix, 371–385.

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