ON THE RATIONAL $K_2$ OF A CURVE OF $GL_2$ TYPE OVER A GLOBAL FIELD OF POSITIVE CHARACTERISTIC

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Abstract. Let $k$ be the function field of a curve $C$ over a finite field. Let $X$ be a proper smooth curve over $k$. Then there exists a regular model $\mathcal{X}$ which is proper and flat over $C$. Assume that the $\mathbb{Q}_p$-coefficient $Gal(k^{sep}/k)$-representation $H'_2(X \times Spec \ k \ Spec k^{sep}, \mathbb{Q}_p)$ is a direct sum of 2 dimensional irreducible representations.

Let $\mathcal{C}_0$ denote the set of closed points of $C$. For a closed point $\wp \in \mathcal{C}_0$, let $\kappa(\wp)$ be the residue field of $C$ at $\wp$. We set $\mathcal{X}_{\kappa(\wp)} = \mathcal{X} \times_C Spec \kappa(\wp)$. Assume that the characteristic of $k$ is greater than 2. We prove that the boundary map $K_2(X) \otimes \mathbb{Q} \rightarrow \bigoplus_{\wp \in \mathcal{C}_0} G_1(\mathcal{X}_{\kappa(\wp)}) \otimes \mathbb{Q}$ in the localization sequence of $G$-theory tensored with $\mathbb{Q}$ is surjective. (We write $G$ for $K'$ in Quillen’s paper [28].)

Examples satisfying the assumption above are non-isotrivial elliptic curves, Drinfeld modular curves, and the moduli of $\mathbb{D}$-elliptic sheaves of rank 2. Given an integer $n$, we also construct a genus two curve that satisfies the condition above and for which the dimension of $\bigoplus_{\wp \in \mathcal{C}_0} G_1(\mathcal{X}_{\kappa(\wp)}) \otimes \mathbb{Q}$ (hence of $K_2(X) \otimes \mathbb{Q}$) is greater than $n$.

1. Introduction

In this section, we state our main result (Theorem 1.1 and Corollary 1.3) and give an outline of the proof.

1.1. Let $\mathbb{F}_q$ be a finite field of $q$ elements of characteristic $p$. Let $C$ be a projective smooth (geometrically connected) curve over $\mathbb{F}_q$ and $k$ be the function field of $C$. Consider the following cartesian diagram

$$
\begin{array}{ccc}
X & \xrightarrow{g} & \mathcal{X} \\
\downarrow f & & \downarrow \square \\
Spec k & \xrightarrow{\eta} & C
\end{array}
$$

where $\eta$ is the generic point, $f$ is proper flat, $g$ is proper smooth, and $\mathcal{X}$ is regular. We remark here that for a given proper smooth curve $X$ over $k$, there exists an $\mathcal{X}$ as above by [25, p.456, Section 10, Proposition 1.8].

Let $\mathcal{C}_0$ denote the set of closed points of $C$. We regard an element in $\mathcal{C}_0$ as a place of the global field $k$ and vice versa. We write $(-)_{\mathbb{Q}} = (-) \otimes \mathbb{Q}$. Let $K_n(-)$ and $G_n(-)$ denote the higher algebraic $K$-theory and $G$-theory of Quillen [28]. (We write $G$ for $K'$ in Quillen’s paper [28].) The localization sequence of $G$-theory tensored by $\mathbb{Q}$ gives the exact sequence

$$
(1.1) \quad \cdots \rightarrow K_2(\mathcal{X})_{\mathbb{Q}} \rightarrow K_2(X)_{\mathbb{Q}} \oplus_{\wp \in \mathcal{C}_0} G_1(\mathcal{X}_{\kappa(\wp)})_{\mathbb{Q}} \rightarrow K_1(\mathcal{X})_{\mathbb{Q}} \rightarrow \cdots,
$$

where $\kappa(\wp)$ is the residue field at $\wp$ and $\mathcal{X}_{\kappa(\wp)} = \mathcal{X} \times_C Spec \kappa(\wp)$. In this article, we consider the following condition $(GL_2)$ on a proper smooth $k$-scheme $Z$. Let us write $G_L = Gal(L^{sep}/L)$ for the absolute Galois group of a field $L$, and $Z_{\mathbb{M}} = Z \times Spec k Spec M$ for the base change, where $M$ is a $k$-algebra. Let $\ell$ be a prime different from $p$. The condition is the following.

$$(GL_2) \quad \text{The $G_L$-representation } H'_2(Z_{\mathbb{M}}, \mathbb{Q}_\ell) \text{ is a direct sum of 2 dimensional irreducible representations.}$$

Our main theorem is as follows.

Theorem 1.1. Assume that the characteristic of $k$ is greater than 2. Let $X$ be a proper smooth curve over $k$. Suppose that $X$ satisfies the condition $(GL_2)$ (with $Z = X$). Let $\mathcal{X}$ be as above. Then the boundary map

$$
K_2(X)_{\mathbb{Q}} \oplus_{\wp \in \mathcal{C}_0} G_1(\mathcal{X}_{\kappa(\wp)})_{\mathbb{Q}}
$$

is surjective.
This result is motivated by Parshin’s conjecture. Recall that Parshin’s conjecture says that \( K_i(Z) \otimes \mathbb{Q} = 0 \) for all \( i \geq 1 \) when \( Z \) is a projective smooth scheme over \( \mathbb{F}_q \). (We were not able to find a written account by Parshin on this conjecture, but see [10].) With our assumptions on \( \mathcal{X} \), the validity of the conjecture then implies that the boundary map \( \otimes_{\ell} \mathcal{O}_\ell \) in Theorem 1.1 is an isomorphism in view of the exact sequence (1.1).

**Remark 1.2.** The assumption that \( p > 2 \) is necessary for the use of the Tate conjecture proved by Zarhin [41]. For the case of \( p = 2 \), there is an unpublished work by Mori (see [27, pp.9–10 and pp.154–161]). Therefore if we use Mori’s result, one can show that our main theorem holds also for \( p = 2 \).

We have the following corollary (see Corollary 6.1 for the precise statement).

**Corollary 1.3.** Assume that the characteristic of \( k \) is greater than 2. The surjectivity statement of Theorem 1.1 holds when \( X \) is

1. an elliptic curve, which is not isotrivial,
2. a Drinfeld modular curve,
3. the moduli of \( \mathfrak{D} \)-elliptic sheaves of rank 2, or
4. one of the genus two curves constructed in Section 7.

In the four cases above, the curve \( X \) satisfies the condition (GL\(_2\)). Hence this corollary follows from Theorem 1.1. In the paper [20], the theorem for Case (1) has been proved.

**1.2.** Let us give a detailed outline of the proof of Theorem 1.1 below. From now on, we always assume that the characteristic of \( k \) is greater than 2. First, we translate the theorem into a statement about étale Chern class maps.

**Proposition 1.4.** Let the notation be as in Theorem 1.1. Suppose that the condition (GL\(_2\)) holds. Then the composite map

\[
H^2_{\mathcal{M}}(X, \mathbb{Q}(2)) \otimes \mathbb{Q}_\ell \to \prod_{\varphi \in \mathcal{C}_n} H^2_{\mathcal{M}}(X_{k_{\varphi}}, \mathbb{Q}(2)) \otimes \mathbb{Q}_\ell
\]

where the first arrow is the product of pullback maps and the second arrow is the product of étale Chern class maps, is surjective.

To prove that Proposition 1.4 implies Theorem 1.1, we will use lemmas in the paper [20] (see Section 5.3 for details).

We remark that we can compute the dimension of target group explicitly (see Section 6) and see that the statement for the target group is an isomorphism in view of the exact sequence (1.1).

**1.3.** To prove Proposition 1.4, we will show the following proposition.

**Proposition 1.5.** Suppose \( X = B \) is a simple abelian variety over \( k \) satisfying the condition (GL\(_2\)). Then the map

\[
H^2_{\mathcal{M}}(h^1(X)) \otimes \mathbb{Q}_\ell \to \prod_{\varphi \in \mathcal{C}_n} H^2_{\mathcal{M}}(h^1(X_{k_{\varphi}})) \otimes \mathbb{Q}_\ell
\]

is surjective. Here we refer to Section 4 for the notation concerning Chow motives such as \( H^1_{\mathcal{M}} \) and \( h^1 \). The first arrow is the pullback map and the second arrow is the étale Chern class map.

To see that Proposition 1.5 implies Proposition 1.4, we use the formalism of Chow motives. In the category of Chow motives (with rational equivalence and rational coefficients), we have an isomorphism

\[
h(X) \cong \bigoplus_{j=0}^2 h^j(X) \cong h^0(X) \oplus h^1(\text{Jac}X) \oplus h^2(X) \cong h^0(X) \oplus \bigoplus_{i=1}^m h^1(B_i) \oplus h^2(X),
\]

where each \( B_i \) is a \( k \)-simple abelian variety satisfying the condition (GL\(_2\)) such that \( \text{Jac}X \) is isogenous (over \( k \)) to \( \prod_{i=1}^m B_i \) (see Section 5.2). Then as étale Chern classes respect the decomposition, and since the statement for the \( h^0 \) and \( h^2 \) is not an essential difficulty, Proposition 1.4 can be reduced to Proposition 1.5.

To prove Proposition 1.5, we proceed as follows. By the condition (GL\(_2\)), we have \( H^1_{\mathcal{M}}(B_{k_{\varphi}}, \mathbb{Q}_\ell) \cong \bigoplus_{j=1}^n \rho_j \) where each \( \rho_j \) is an irreducible 2 dimensional representation of \( G_k \) with coefficients in \( \mathbb{Q}_\ell \). For a nonarchimedean local field \( L \) whose characteristic is prime to \( \ell \), define \( Sp \) to be the 2 dimensional representation of \( G_L \) with coefficients in \( \mathbb{Q}_L \) which is the unique nontrivial extension of \( \mathbb{Q}_\ell \) by \( \mathbb{Q}_\ell(-1) \). By abuse of notation we also write \( Sp \) for the base change \( Sp \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_L \). We consider two cases separately.

1. There exists a pair \((j, \varphi)\) of an integer \( 1 \leq j \leq n \) and a place \( \varphi \) of \( k \) such that \( \rho_j|_{G_{k_{\varphi}}} \) is isomorphic to \( Sp \).
2. The condition in (1) does not hold.
In Case (2), using the computation of Galois cohomology in Section 3, we show that the target of the Chern class map is zero (see Lemma 5.1). Hence the surjectivity becomes a trivial statement. In Case (1), we show that \( B \) is isogenous to a simple factor of the Jacobian of some Drinfeld modular curve such that \( \varphi \) is the fixed place at infinity. (We refer to Section 2.1 for the relevant definitions and properties of Drinfeld modular curves.) This is of course the function field analogue of Shimura’s theory [33]. We note that the condition that there exists a place where the representation at the place \( \varphi \) is isomorphic to \( \text{Sp} \) is the function field analogue of (or rather the Drinfeld modular analogue of) modular forms of weight 2 (which is a condition at the place at infinity). As we did not find the construction in the literature, we give a detailed proof of this fact in Section 2. We note that we do not follow the proof for the case of the moduli of elliptic curves, but we use freely the Tate conjecture (a theorem of Zarhin) to make the exposition simple. With the Tate conjecture at our disposal, the only other necessary ingredient is some Atkin-Lehner theory over general global fields, but we use a theorem of Jacquet and Shalika for simplicity (see the proof of Lemma 2.3).

Thus we are reduced to proving Proposition 1.5 when \( B \) is a factor of the Jacobian of a Drinfeld modular curve. The idea, which does not apply to the case of a number field, and which already appeared in the paper [20], is used at this point. Note that \( B \) may appear as a factor of a Drinfeld modular curve for EVERY \( \varphi \) satisfying the condition in (1), so that we have as many “parametrizations” as such \( \varphi \). When \( B \) is an elliptic curve, this is really the modular parametrization as discussed in detail by Gekeler and Reversat [11]. We then invoke the result of [20]. The setup is that of Proposition 1.4, and \( X \) is a Drinfeld modular curve. The result in [20] says that there exists a certain subspace of \( H^2_M(X,\mathbb{Q}(2)) \) which maps surjectively by the Chern classes at infinity (or \( \varphi \) in this case) and maps to zero at all the other places. By varying \( \varphi \), we obtain Proposition 1.5. This finishes the outline of the proof of Proposition 1.5 and hence of Theorem 1.1.

1.4. To see that the surjectivity statement is a nontrivial one, we will compute (Lemma 6.2) the dimension of the target group explicitly in terms of the \( G_k \)-representation that appears in the condition (GL2). More precisely, if the representation is the direct sum \( \bigoplus_{j=1}^n \rho_j \) of 2 dimensional representations \( \rho_j \), then the dimension of the target equals the number of the pairs \((j,\varphi)\) which appeared in the condition of (1).

In Corollary 1.3, we have four kinds of curves. For an elliptic curve, the condition on \( \varphi \) above is equivalent to that the elliptic curve has split multiplicative reduction at \( \varphi \). For a Drinfeld modular curve, the number of such pairs is bounded below by the genus since \((j,\infty)\) always satisfy the condition of (1). Note that the genus grows as \( \mathbb{K} \) becomes small. The genus 2 curve in (4) is constructed from Bruner-Hashimoto’s family of genus 2 curves. We construct explicit examples within this family for which the number of pairs \((j,\varphi)\) satisfying the condition of (1) is large.

1.5. We give a remark concerning our title and the condition (GL2). Suppose a proper smooth curve \( X \) satisfies the condition (GL2). Then the Jacobian of \( X \) is isogenous to a product of simple abelian varieties of GL2-type if there exists a place \( \varphi_j \) such that \((j,\varphi_j)\) satisfies the condition of (1) for each \( j \). This can be seen using the procedure given in Section 5.1. When there does not exist such \( \varphi_j \)’s, we do not know if this holds true.

1.6. The paper is organized as follows. In Section 2, given an automorphic representation \( \pi \) of \( GL_2(A_k) \) such that \( \pi_{\infty} \cong \text{St} \) at some place \( \infty \) of \( k \) (see Section 2 for the notation), we construct a \( k \)-simple abelian variety \( A \) whose Tate module is related to \( \pi \). Section 3 is devoted to the computation of Galois cohomology. We use the classification of representations of \( GL_2(F) \) (see Lemma 3.4) for a nonarchimedean local field \( F \). The aim is to compute the local Galois cohomology of some two dimensional representations corresponding to the tempered ones via the local Langlands correspondence. In Section 4, we collect some facts concerning Chow motives, their cohomology theories, and the morphism between the cohomology theories. We are interested in the rational motivic cohomology and the absolute \( \ell \)-adic cohomology. We refer to Jannsen’s book [19] for cohomology theories and to Scholl’s article [35] for generalities on Chow motives. In Section 5, we give a proof of Theorem 1.1 following the outline given in this introduction. Sections 5.1, 5.2, 5.3 are devoted to the proof of Propositions 1.5, 1.4 and Theorem 1.1 respectively. Section 7 is independent of the earlier sections aside from some general notations. The aim is to construct an explicit family of genus 2 curves satisfying the condition (GL2) such that the dimension of the rational \( G_1 \) group is arbitrarily large.

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2. THE CONSTRUCTION OF ABELIAN VARIETIES ASSOCIATED TO AUTOMORPHIC REPRESENTATIONS OF GL_2 IN THE FUNCTION FIELD CASE

The aim of this section is to prove Proposition 2.1. The abelian variety $A_π$ will be used throughout this article. This proposition is the function field analogue of a theorem of Shimura [33, p.183, Thms 7.14, 7.15]. The proof is different in that we use the Tate conjecture (a theorem of Zarhin) in this paper. In doing so, we need not use the theory of modular forms (which does not seem to be available in the literature for the function field case) as much as Shimura did. Our construction has a shortcoming that the resulting abelian variety $A_π$ is neither a sub nor a quotient of the Jacobian of a Drinfeld modular curve in a canonical way. Another output of this section is Corollary 2.5. This will be used in the proof of Lemma 5.2.

We fix an isomorphism $\iota : \mathbb{C}_ℓ \cong \mathbb{C}$. We use the results from the theory of automorphic forms with values in $\mathbb{C}$ as those with values in $\mathbb{C}_ℓ$ via this fixed isomorphism.

2.1. We regard a place of $k$ as a closed point of $C$ and vice versa. Let us choose a closed point $\infty \in C_0$ and set $A = H^0(C \setminus \{\infty\}, Ω_C)$. For a place $v$ of $k$, we let $k_v$ denote the completion of $k$ at $v$. If $v \neq \infty$, we denote by the same symbol $v$ the prime ideal of $A$ corresponding to $v$. Then the $v$-adic completion $A_v$ of $A$ is equal to the ring of integers of $k_v$.

Let $I \subset A$ be a non-zero ideal. We let $M_I^∞$ denote the functor which classifies Drinfeld modules of rank 2 with a Drinfeld full level $I$ structure on a $k$-scheme. When $Sp(A/I)$ contains two or more closed points, it is representable by an affine curve over $Spec k$ (see [8, p.576, Proposition 5.3]) for construction of the moduli scheme over $A$; we consider its base change to $k$. We will denote the representing scheme by the same symbol. Let $\hat{M}_I^∞$ denote the smooth compactification of $M_I^∞$. We set $\hat{M}_I^\infty = \varprojlim_M M_I^∞$ where the limit is taken over ideals of $A$ such that $Spec(A/I)$ contains two or more closed points.

Let $A = A_k$ denote the ring of adeles of $k$. We set $\hat{A} = \varprojlim A/I$. Let $A_∞ = \hat{A} \otimes_A k$ denote the ring of finite adeles. The group $GL_2(A_∞)$ acts on $\hat{M}_I^∞$ (see [8, p.577]). Let $K \subset GL_2(\hat{A}) \subset GL_2(A_∞)$ be a compact open subgroup. We let $\hat{M}_I^∞ = \hat{M}_I^\infty$ denote the quotient. For example ([8, p.578]), if $K = K_I = Ker(GL_2(\hat{A}) \to GL_2(\hat{A}/I\hat{A}))$, then we recover $\hat{M}_I^∞ = K_I/\hat{M}_I^\infty$ for $I$ as in the previous paragraph.

If $L$ is a $k$-algebra, we write $\hat{M}_I^∞ = \hat{M}_I^\infty \times_{Spec k} Spec L$.

2.2. Let $\Pi$ denote the set of (isomorphism classes of) cuspidal automorphic representations of $GL_2(\hat{A})$. Let $St$ denote the Steinberg representation of $GL_2$ of a nonarchimedean local field. We have an isomorphism

\begin{equation}
H^1_{\et}(\hat{M}_I^∞, \mathbb{Q}_ℓ) \cong \bigoplus_{\pi \in \Pi, \pi_∞ \neq St} (\otimes_{v \neq \infty} \pi_v) \otimes \rho(\pi)
\end{equation}

of $GL_2(A_∞) \times Gal(k^{sep}/k)$-modules, where $\rho(\pi)$ is the 2-dimensional representation of $Gal(k^{sep}/k)$ over $\mathbb{Q}_ℓ$ which is determined by $\pi$ via the global Langlands correspondence and whose restriction to $Gal(k^{sep}/k_∞)$ is isomorphic to Sp. (See [6, Corollary 2.3].) Given a compact open subgroup $K \subset GL_2(\hat{A})$, we may take the $K$-invariant part of both sides above and obtain (see [11, Section 8] for the description even when $K$ is large)

\begin{equation}
H^1_{\et}(\hat{M}_K^∞, \mathbb{Q}_ℓ) \cong \bigoplus_{\pi \in \Pi, \pi_∞ \neq St} (\otimes_{v \neq \infty} \pi_v)^K \otimes \rho(\pi).
\end{equation}

For any place $v$ of $k$, the Galois representation $\rho(\pi)|_{GL_v(\hat{A})}$ (or the associated Weil-Deligne representation) corresponds to $\pi_v \otimes |\det|^{1/2}$ via the local Langlands correspondence. (See [8, Theorem A]. See also [22, Corollaire VII.5].)

2.3. Let $n$ be a non-zero ideal of $A$. Set

\[ K_{1, v}(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A_v) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ mod \ n \right\} \]

for a place $v \neq \infty$ and

\[ K_1(n) = \prod_{v \neq \infty} K_{1, v}(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{A}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ mod \ n \right\}. \]

Let $π$ be a cuspidal automorphic representation of $GL_2(\hat{A})$ such that $π_∞ \cong St$. By [4, p.302, Theorem 1.4], there exists a non-zero ideal $n_π \subset A$ such that $dim_{k_π} \otimes_{v \neq \infty} \pi_v^{K_{1, v}(n_π)} = 1$. (This is the prime-to-$∞$ part of the conductor of $π$. The conductor is then $n_π∞$ since $π_∞ \cong St$.) We consider the action of the Hecke
Lemma 2.2. such that is an abelian variety. (This fact is well known when the base field is perfect; we refer to [26, p.1, Theorem 29.2, (29.5)]. If b where the first isomorphism is the Tate conjecture (a theorem of Zarhin [41]), and the second is [5, Lemma 2.4.].

Proposition 2.1. (1) The field $K_\pi$ is a number field. Let $I(\pi)$ denote the set of embeddings $K_\pi \hookrightarrow \overline{Q}_\ell$.

For an abelian variety $B$, we write $T_t B$ for the Tate module, set $V_t B = T_t B \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $\nabla_t B = V_t B \otimes_{\mathbb{Q}_\ell} \overline{Q}_\ell$. For a $\overline{Q}_\ell$-vector space $V$, we let $V^\vee = \text{Hom}_{\overline{Q}_\ell}(V, \overline{Q}_\ell)$ denote the dual.

Proof. The construction is done inductively. Suppose we are given an abelian variety $A_\pi$ such that $n_{\pi'} \equiv \overline{S}_t$. Then the Tate conjecture (a theorem of Zarhin [41]), and the second is [5, Lemma 29.2, (29.5)]. If $b \geq 1$, then we can construct a nontrivial map $\nabla_t B \to \nabla_t A_\pi$ (namely, the projection to $\rho(\pi)$ followed by the canonical inclusion). Hence there exists a nontrivial morphism $B \to A_\pi$, which is surjective since $A_\pi$ is simple. Let $B^0$ denote the connected component of the kernel of this morphism. Then $B^0 = B^0_{\text{red}}$ is an abelian variety. (This fact is well known when the base field is perfect; we refer to [26, p.1, Theorem 1] for the case when the base field is not necessarily perfect.) Note that $\text{Hom}_K(V_t B, V_t A_\pi) \otimes \mathbb{Q}_\ell \cong \text{Hom}_K[V_t B, V_t A_\pi]$ where the first isomorphism is the Tate conjecture (a theorem of Zarhin [41]), and the second is [5, Lemma 29.2, (29.5)]. If $b \geq 1$, then we can construct a nontrivial map $\nabla_t B \to \nabla_t A_\pi$ (namely, the projection to $\rho(\pi)$ followed by the canonical inclusion). Hence there exists a nontrivial morphism $B \to A_\pi$, which is surjective since $A_\pi$ is simple. Let $B^0$ denote the connected component of the kernel of this morphism. Then $B^0 = B^0_{\text{red}}$ is an abelian variety. (This fact is well known when the base field is perfect; we refer to [26, p.1, Theorem 1] for the case when the base field is not necessarily perfect.) Note that $\text{Hom}_K(V_t B, V_t A_\pi)$ is isomorphic to an abelian variety of the form $\mathcal{A}_{\pi,n}$ with $n_{\pi'} \neq n$.

Note that by (c) and the strong multiplicity one theorem, $\rho(\pi)$ is uniquely determined. Note also that by the Tate conjecture, the abelian variety $A_\pi$ is uniquely determined up to isogeny.

2.4. We prove Proposition 2.1 inductively. Fix a nonzero ideal $n \subset A$. The inductive assumption is that

$\text{(*)}$ There exists an abelian variety $A_{\pi'}$ satisfying the conditions in Proposition 2.1 for those $\pi'$ such that $n_{\pi'}$ divides $n$ and $n_{\pi'} \neq n$.

Note that (\text{*)}) for $n = 1$ holds trivially. Assuming (\text{*)}), we prove the assertions for $\pi$ such that $n_{\pi} = n$. Set $d_{\pi,n} = \dim_{\mathbb{Q}_\ell} \otimes_{\mathbb{Q}_\ell} \mathcal{A}_{\pi,n}(\mathbb{Z}_\ell)$. Let $\Pi^{\text{new}}_1(n)$ denote the (finite) set of cuspidal automorphic representations $\pi$ such that $\pi_\infty \cong \overline{S}_t$ and $d_{\pi,n} = 1$.

Lemma 2.2. Assume that (\text{*)}) holds. There exist abelian varieties over $K$, denoted $J^{\infty,\text{new}}_{K_1(n)}$ and $J^{\infty,\text{old}}_{K_1(n)}$, such that

\[
(\nabla_t J^{\infty,\text{new}}_{K_1(n)})^\vee \cong \bigoplus_{\pi \in \Pi^{\text{new}}_1(n)} \rho(\pi),
\]

\[
(\nabla_t J^{\infty,\text{old}}_{K_1(n)})^\vee \cong \bigoplus_{d_{\pi,n} > 1} \rho(\pi)^{d_{\pi,n}}.
\]

Moreover $J^{\infty,\text{old}}_{K_1(n)}$ is isogenous to an abelian variety of the form $\prod_{\pi} A_{\pi}$ (possibly with multiplicity) where $n_{\pi}|n$ and $n_{\pi} \neq n$ hold for each $\pi$ appearing in the product.

Proof. The construction is done inductively. Suppose we are given an abelian variety $B$ over $K$ such that

\[
(\nabla_t B)^\vee \cong \left[ \bigoplus_{d_{\pi,n} = 1} \rho(\pi) \right] \oplus \left[ \bigoplus_{d_{\pi,n} > 1} \rho(\pi)^{d_{\pi,n}} \right]
\]

with $b_{\pi} \geq 0$ for each $\pi$ such that $d_{\pi,n} > 1$. Note that for $\pi$ such that $d_{\pi,n} > 1$, we have $n_{\pi} \neq n$ (using [4, p.306, Corollary]) so the inductive hypothesis applies, that is, there exists an abelian variety $A_{\pi}$ for this $\pi$. We have

\[
\text{Hom}_K(B, A_{\pi}) \otimes \overline{Q}_\ell \cong \text{Hom}_{\mathbb{Q}_\ell}(V_t B, V_t A_{\pi}) \otimes_{\mathbb{Q}_\ell} \overline{Q}_\ell \cong \text{Hom}_{\mathbb{Q}_\ell}(V_t B, V_t A_{\pi})
\]

where the first isomorphism is the Tate conjecture (a theorem of Zarhin [41]), and the second is [5, Lemma 29.2, (29.5)]. If $b_0 \geq 1$, then we can construct a nontrivial map $\nabla_t B \to \nabla_t A_{\pi}$ (namely, the projection to $\rho(\pi)$ followed by the canonical inclusion). Hence there exists a nontrivial morphism $B \to A_{\pi}$, which is surjective since $A_{\pi}$ is simple. Let $B^0$ denote the connected component of the kernel of this morphism. Then $B^0 = B^0_{\text{red}}$ is an abelian variety. (This fact is well known when the base field is perfect; we refer to [26, p.1, Theorem 1] for the case when the base field is not necessarily perfect.) Note that $\text{Hom}_K(V_t B, V_t A_{\pi}) \otimes \mathbb{Q}_\ell \cong \text{Hom}_K(V_t B, V_t A_{\pi})$ with $b_{\pi} \geq 1$ and $B$ is isogenous to $B^0 \times A_{\pi}$.

Now we apply this procedure repeatedly. To prove the lemma, we start from the Jacobian $J^{\infty}_{K_1(n)}$ of $\mathcal{M}^{\infty}_{K_1(n)}$ and proceed until $\sum_{\pi} b_{\pi} = 0$. Then we have $J^{\infty,\text{new}}_{K_1(n)}$ as $B^0$ in the last step of the process. Also one has $J^{\infty,\text{old}}_{K_1(n)}$ as the product of all the $A_{\pi}$’s that appeared in the process. It is clear that $J^{\infty,\text{new}}_{K_1(n)}$ and $J^{\infty,\text{old}}_{K_1(n)}$ satisfy the desired properties.
2.5. We now describe the decomposition of \( J_{K_1(n)}^{\infty} \) into the \( k \)-simple factors (that is, we find an abelian variety over \( k \) isogenous to \( J_{K_1(n)}^{\infty} \), which is a product of \( k \)-simple abelian varieties). Each simple factor will be the desired \( A_\pi \) for some \( \pi \). We define a Hecke algebra \( T \subset \text{End}_k(J_{K_1(n)}^{\infty}) \) as the subalgebra generated by the Hecke operators \( T_\nu \) and \( S_\nu \) for \( \nu \) prime to \( n \). We define \( T_{\text{new}} \subset \text{End}_k(J_{K_1(n)}^{\infty}) \otimes \mathbb{Z} \mathbb{Q} \) as follows. By Lemma 2.2 and the Tate conjecture, we obtain and fix an isogeny \( \psi : J_{K_1(n)}^{\infty} \times J_{K_1(n)}^{\text{old}} \rightarrow J_{K_1(n)}^{\infty} \). If \( T \in \text{End}_k(J_{K_1(n)}^{\infty}) \), we obtain an element \( T' \) in \( \text{End}_k(J_{K_1(n)}^{\infty}) \) as the composite

\[
J_{K_1(n)}^{\infty} \rightarrow J_{K_1(n)}^{\infty} \times J_{K_1(n)}^{\text{old}} \xrightarrow{\psi} J_{K_1(n)}^{\infty} \xrightarrow{T} J_{K_1(n)}^{\infty} \xrightarrow{\psi'} J_{K_1(n)}^{\infty} \times J_{K_1(n)}^{\text{old}} \rightarrow J_{K_1(n)}^{\infty},
\]

where the first arrow is the canonical inclusion, \( \psi' \) is the dual isogeny, and the last arrow is the projection. We define \( T_{\text{new}} \subset \text{End}_k(J_{K_1(n)}^{\infty}) \otimes \mathbb{Z} \mathbb{Q} \) to be the subalgebra generated by the elements of the form \( (\deg \psi)^{-1} T' \). Note that \( T \rightarrow T_{\text{new}} \) is an algebra homomorphism and that \( T_{\text{new}} \) is a commutative algebra since \( T \) is commutative. By abuse of notation, we let \( T_\nu \in T_{\text{new}} \) (resp. \( S_\nu \in T_{\text{new}} \)) denote the element corresponding to \( T_\nu \in T \) (resp. \( S_\nu \in T \)).

From here, we proceed in a manner very similar to the proof of Proposition 4.2 in [37, p.228]. We have

\[
(2.3) \quad \text{End}_k(J_{K_1(n)}^{\infty}) \otimes \mathbb{Z} \mathbb{Q}_\ell \cong \text{End}_{\mathbb{Q}_\ell[G_\ell]}(\nabla \ell J_{K_1(n)}^{\infty}) \cong \text{End}_{\mathbb{Q}_\ell[G_\ell]}(\bigoplus_{\pi \in \Pi_{\text{new}}(n)} \rho(\pi)^{\vee}) \cong \prod_{\pi \in \Pi_{\text{new}}(n)} \mathbb{Q}_\ell
\]

where the first isomorphism is due to the Tate conjecture and the last isomorphism follows from the irreducibility of \( \rho(\pi) \) and the fact that if \( i \neq j \) then \( \rho(\pi_i) \) and \( \rho(\pi_j) \) are not isomorphic.

2.6. Let \( f_\pi^{\infty} \) denote a nonzero element in \( \otimes_{\varphi \neq \varphi_\pi K_1(n)} \). We define a map

\[
\phi : \bigoplus_{\pi \in \Pi_{\text{new}}(n)} \otimes_{\varphi \neq \varphi_\pi K_1(n)} \rightarrow \text{Hom}_\mathbb{Q}(T_{\text{new}}, \mathbb{Q}_\ell)
\]

by \( \phi(f_\pi^{\infty})(T_\nu) = a_\nu(\pi) \) for \( \nu \) prime to \( n \) (this map is independent of the choice of the \( f_\pi^{\infty} \)).

**Lemma 2.3.** The map \( \phi \) is an isomorphism.

**Proof.** We see from (2.3) that \( \dim_{\mathbb{Q}_\ell} T_{\text{new}} \) is smaller than or equal to the dimension of the left hand side. It suffices to prove the injectivity of \( \phi \).

Let \( f = \sum_{\pi \in \Pi_{\text{new}}(n)} c_\pi f_\pi^{\infty} \) with \( c_\pi \in \mathbb{Q}_\ell \) be an element such that \( \phi(f)(T) = 0 \) for all \( T \in T_{\text{new}} \). It is enough to show \( f = 0 \). For each \( \pi \) and each place \( \nu \) such that \( \nu \nmid n \), denote the Satake parameters at \( \nu \) for \( \pi \) by \( \alpha_\nu(\pi) \) and \( \beta_\nu(\pi) \). For each \( \nu \) with \( \nu \nmid n \) and \( r \geq 1 \), it is easy to find an element \( T_{\nu,r} \in T_{\text{new}} \) such that

\[
\phi(f_\pi^{\infty})(T_{\nu,r}) = \alpha_\nu(\pi)^r + \beta_\nu(\pi)^r
\]

for all \( \pi \in \Pi_{\text{new}}(n) \). Then we have

\[
\sum_{\pi \in \Pi_{\text{new}}(n)} c_\pi (\alpha_\nu(\pi)^r + \beta_\nu(\pi)^r) = 0
\]

for any \( r \geq 1 \). This implies that \( c_\pi = 0 \) for all \( \pi \in \Pi_{\text{new}}(n) \) by [18, p.806, THEOREM]. \( \square \)

2.7. Using Lemma 2.3, we obtain \( T_{\text{new}} = \text{End}_k(J_{K_1(n)}^{\text{new}}) \otimes \mathbb{Z} \mathbb{Q} \). Since \( T_{\text{new}} \) is commutative and \( \text{End}_k(J_{K_1(n)}^{\text{new}}) \otimes \mathbb{Z} \mathbb{Q} \) is a semisimple \( \mathbb{Q} \)-algebra, we have \( T_{\text{new}} \cong F_1 \times \cdots \times F_s \) where each \( F_i \) is a number field by Wedderburn’s theorem, for some \( s \in \mathbb{Z} \). We set

\[
J_i = J_{K_1(n)}/\text{Ker}[\text{End}_k(J_{K_1(n)}^{\text{new}}) \rightarrow F_1] \quad J_{K_1(n)}^{\text{new}}
\]

for each \( i \). Then \( J_{K_1(n)}^{\text{new}} \) is isogenous to \( J_1 \times \cdots \times J_s \).

We have

\[
\nabla_\ell J_i \cong \bigoplus_{\pi \in \Pi_i} (\otimes_{\varphi \neq \varphi_\pi K_1(n)} \rho(\pi)) \otimes \rho(\pi)
\]

for some subset \( \Pi_i \subset \Pi_{\text{new}}(n) \). Take a \( \pi \in \Pi_i \). We claim that \( J_i \) satisfies the conditions (1), (2) and (3) of Proposition 2.1 so that we may call this \( A_\pi \).

We see from (2.3) that an element of \( T_{\text{new}} \) acts as a scalar on each \( (\otimes_{\varphi \neq \varphi_\pi K_1(n)} \rho(\pi)) \). Giving this action is equivalent to giving an algebra homomorphism

\[
T_{\text{new}} \xrightarrow{pr_i} F_i \xrightarrow{\varphi_\pi} \mathbb{Q}_\ell
\]

where \( pr_i \) is the projection to the \( i \)-th factor and \( \varphi_\pi \) is some embedding.

From Lemma 2.3, we have \( \dim_{\mathbb{Q}_\ell} \oplus_{\pi \in \Pi_i} (\otimes_{\varphi \neq \varphi_\pi K_1(n)} = \dim_{\mathbb{Q}} F_i \), hence if we fix an element \( \pi \in \Pi_i \), then any other \( \pi' \in \Pi_i \) corresponds to the algebra homomorphism

\[
T_{\text{new}} \xrightarrow{pr_i} F_i \xrightarrow{\varphi_{\pi'}} \mathbb{Q}_\ell
\]
for some embedding \( \varphi_{\pi'} \).

Note that \( K_\pi \subseteq F_i \). If \( K_\pi \neq F_i \), then there exists some \( \varphi_{\pi'} \) such that \( \varphi_{\pi}|_{K_\pi} = \varphi_{\pi'}|_{K_\pi} \). This contradicts that \( \pi \) and \( \pi' \) are not isomorphic using the strong multiplicity one theorem, hence \( K_\pi = F_i \). This completes the proof of Proposition 2.1. \( \square \)

2.8. We will restate the result obtained in this section in the form to be used elsewhere. We will use Corollary 2.5 in the proof of Lemma 5.2.

We define an equivalence relation on the set \( \Pi_{\mathcal{G}} \) of cuspidal automorphic representations such that \( \pi_{\infty} \cong \text{St} \) if and only if \( A_{\pi} \) is isogenous to \( A_{\pi'} \).

**Corollary 2.4.** The Jacobian \( J_{E_i/(n)}^{\infty} \) is isogenous to \( \prod_{n \in \Pi_{\mathcal{G}}} A_{\pi}^{\infty} \).

**Proof.** We see from Lemma 2.2 that \( J_{E_i/(n)}^{\infty} \) is isogenous to some product of the form \( \prod_{n} A_{\pi} \) (possibly with multiplicity). Then the description of \( \nabla A_{\pi} \) in Proposition 2.1(3) and (2.2) leads to the claim. \( \square \)

In the following corollary, we are taking \( \infty \) to be \( \varphi \).

**Corollary 2.5.** Let \( \pi \) be a cuspidal automorphic representation. Let \( \text{St}_{\pi} \) denote the set of places \( v \) such that \( \pi_v \cong \text{St}_v \). Then, for each \( \varphi \in \text{St}_{\pi} \), there exists a Drinfeld modular curve \( \mathcal{M}_{\pi}^\varphi \) such that \( A_{\pi} \) is a simple factor of \( \text{Jac}\mathcal{M}_{\pi}^\varphi \).

**Proof.** Fix a prime \( \varphi \in \text{St}_{\pi} \). As we had done in Section 2.3, we can find an ideal \( \mathfrak{n} \) of the coordinate ring \( \mathcal{H}_0 \setminus \{v\}, \mathcal{O}_C \) such that \( \dim_{\mathfrak{m}} \otimes_{\mathfrak{m}' \neq \mathfrak{m}} \mathcal{H}_0 \mathfrak{n} \geq 1 \). Then by Corollary 2.4, \( A_{\pi} \) is a simple factor of \( \text{Jac}\mathcal{M}_{\pi}^\varphi \). \( \square \)

3. Galois cohomology of two dimensional representations

The aim of this section is to prove Proposition 3.1. The vanishing result will be used in the proof of Lemma 5.1. The nonzero result will be used in the proof of Lemma 6.2. For the application, \( \sigma \) below will be a local component of some \( 2 \)-dimensional direct factor of the Jacobian of a curve over \( k \).

3.1. Let \( F \) be a nonarchimedean local field of characteristic \( p \geq 0 \). Fix a uniformizer \( \varpi \in F \). Let \( G_F = \text{Gal}(F^{\text{sep}}/F) \) denote the absolute Galois group. Let \( \ell \) be a prime number prime to \( p \). Let \( \sigma \) be a two dimensional \( \ell \)-adic representation, that is, a continuous homomorphism \( \sigma : G_F \to \text{GL}(V_\sigma) \) where \( V_\sigma \) is a two dimensional \( E \)-vector space (\( |E| : \mathbb{Q}_l| \leq \infty \)). Let \( V_{\sigma, \mathfrak{m}} := V_{\sigma} \otimes_E \mathfrak{m}_\ell \). Let \( W_F \) denote the Weil group and \( i : W_F \hookrightarrow G_F \) be the canonical inclusion. We also denote \( \sigma \) the representation of \( W_F \) on \( V_{\sigma, \mathfrak{m}} \) obtained by the restriction. We write \( WD(\sigma) \) for the Weil-Deligne representation attached to \( \sigma \) (see Bushnell-Henniart [3, p.206, Theorem]) We let \( \varpi = \varpi(\sigma) \) denote the irreducible admissible representation of \( \text{GL}_2(F) \) which corresponds to \( \sigma \) via the local Langlands correspondence (see Bushnell-Henniart [3, p.212, Langlands correspondence]). We fixed an isomorphism \( i : \mathfrak{m}_\ell \simeq \mathbb{C} \). Let \( St \) denote the Steinberg representation of \( \text{GL}_2(F) \).

**Proposition 3.1.** Let the notations be as above. Suppose that \( \varpi(\sigma) \otimes |\text{det}|^{1/2} \) is tempered. Then

\[
\dim E H^1(G_F, V_\sigma(n)) = \begin{cases} 1 & \text{if } \varpi(\sigma) \otimes |\text{det}|^{1/2} \cong \text{St} \text{ and } n = 0 \text{ or } \varpi(\sigma) \otimes |\text{det}|^{1/2} \cong \text{St} \text{ and } n = 2, \\ 0 & \text{otherwise}. \end{cases}
\]

**Lemma 3.2.** We have

\[
\dim E H^1(G_F, V_\sigma(n)) = -\text{ord}_{s=0}L(s, V_\sigma(n)) - \text{ord}_{s=0}L(s, V_\sigma(n)^*),
\]

where \( \text{ord}_{s=0} \) means the order of zero at \( s = 0 \).

**Proof.** We have

\[
\dim E H^1(G_F, V_\sigma(n)) = \dim E H^0(G_F, V_\sigma(n)) + \dim E H^2(G_F, V_\sigma(n)) = \dim E H^0(G_F, V_\sigma(n)) + \dim E H^0(G_F, (V_\sigma(n))^*) = -\text{ord}_{s=0}L(s, V_\sigma(n)) - \text{ord}_{s=0}L(s, (V_\sigma(n))^*).
\]

The first equality uses that the Euler-Poincare characteristic is zero ([34, p.101, II.5.7. exercise 4]), the second equality uses the local Tate duality (see [32, Theorem 1.4.1]), and the third equality follows from the definition of the \( L \)-factor. \( \square \)

**Lemma 3.3.** Let \( \chi : F^\times \to \mathbb{C}^\times \) be a unitary character. Let \( n \in \mathbb{Z} \). Then we have

\[
\text{ord}_{s=0}L(s + \frac{n}{2}, \chi) = \begin{cases} -1 & \chi = 1 \text{ and } n = 0, \\ 0 & \text{otherwise}. \end{cases}
\]
Proof. The $L$-factor has a pole at $s = 0$ if and only if $\chi(\varpi^{-1})|\varpi^{-1}|^{1/2} = 1$ where $\varpi$ is a uniformizer. If this holds then $\chi$ must be unramified. Since $\chi$ is unitary, $n = 0$ and $\chi(\varpi^{-1}) = 1$. □

We compute the $L$-factor in terms of $\pi(\sigma)$ as follows. We have

$$L(s, V_\sigma(n)) = L(s, \pi(V_\sigma(n)))$$
$$= L(s, \pi(\sigma) \otimes |\det|^{1/2} \otimes |\det|^{n-1/2})$$
$$= L(s + n - \frac{1}{2}, \pi(\sigma) \otimes |\det|^{1/2})$$

and

$$L(s, V_\sigma(n)^*) = L(s, \pi(V_\sigma(n)^*))$$
$$= L(s, \pi(V_\sigma(n)^*))(1)$$
$$= L(s, (\pi(\sigma) \otimes |\det|^{1/2} \otimes |\det|^{-n+1/2})$$
$$= L(s - n + \frac{3}{2}, (\pi(\sigma) \otimes |\det|^{1/2}))$$

We use the following classification.

**Lemma 3.4.** The tempered representations of $GL_2(F)$ are (i) supercuspidals, (ii) unitary twists of the Steinberg representation, and (iii) unitary principal series representation $\text{Ind}_{B}^{GL_2}(\chi_1, \chi_2)$ with unitary characters $\chi_1, \chi_2$.

**Proof.** See Jacquet [16]. See also Kudla [21, p.375, Example 2.5]. □

### 3.2. Proof of Proposition 3.1

It suffices to consider the three cases.

**Case (i)** $|\pi(\sigma) \otimes |\det|^{1/2}$ is supercuspidal: Since the $L$-factor of a supercuspidal representation is 1, we have $L(s, V_\sigma(n)) = L(s, V_\sigma(n)^*) = 1$ in this case. Hence the proposition follows using Lemma 3.2.

**Case (ii)** $|\pi(\sigma) \otimes |\det|^{1/2}$ is a unitary twist $St \otimes \chi$ of the Steinberg representation: We have

$$L(s, V_\sigma(n)) = L(s + n - \frac{1}{2}, St \otimes \chi)$$
$$= L(s + n - \frac{1}{2}, St \otimes \chi)$$

$$L(s, V_\sigma(n)^*) = L(s - n + \frac{3}{2}, St^\vee \otimes \chi^{-1})$$
$$= L(s - n + \frac{3}{2}, St \otimes \chi^{-1})$$
$$= L(s - n + \frac{3}{2}, St \otimes \chi^{-1})$$

where we used that $St \cong St^\vee$ (See Bushnell-Henniart [3, p.68, (9.10.5)]) and formulas in [21, p.377, 3.1]. The claim follows then from Lemmas 3.2 and 3.3.

**Case (iii)** $|\pi(\sigma) \otimes |\det|^{1/2}$ is a unitary principal series representation $\text{Ind}_{B}^{GL_2}(\chi_1, \chi_2)$ with unitary characters $\chi_1, \chi_2$: The $L$-factors are

$$L(s, V_\sigma(n)) = L(s + n - \frac{1}{2}, \text{Ind}_{B}^{GL_2}(\chi_1, \chi_2))$$
$$= L(s + n - \frac{1}{2}, \chi_1) L(s + n - \frac{1}{2}, \chi_2)$$

$$L(s, V_\sigma(n)^*) = L(s - n + \frac{3}{2}, \text{Ind}_{B}^{GL_2}(\chi_1^{-1}, \chi_2^{-1}))$$
$$= L(s - n + \frac{3}{2}, \chi_1^{-1}) L(s - n + \frac{3}{2}, \chi_2^{-1})$$

The claim follows then from Lemmas 3.2 and 3.3.

### 4. Motivic cohomology and $\ell$-adic cohomology for Chow motives

We collect some definitions and properties concerning Chow motives and cohomology theories from Jannsen [19] and Scholl [35]. For the application, we take the base field $L$ below to be either $k$ or $k_\wp$. The Chern class or the map $r$ below will be used only with the base field $k_\wp$.

4.1. We first recall some definitions and properties of twisted Poincaré duality theories ([1], see [19, Section 6]). For a quasi-projective scheme $X$ over a field $L$ and a closed subscheme $Z \subset X$, the (rational) motivic cohomology of $X$ with support on $Z$ and the (rational) motivic homology of $X$ are defined to be

$$H_{\text{mot}}^{i}(X, \mathbb{Q}(j)) = \text{Ker}(\text{Ker}(X)^{(j)})$$
$$H_{\text{mix}}^{i}(X, \mathbb{Q}(b)) = \text{Ker}(\text{Ker}(X)^{(b)})$$

We refer to [19, p.104, 6.12] for the precise definitions.

Let $\ell$ be a prime different from the characteristic of $L$. We will use the (absolute) $\ell$-adic cohomology with support and the (absolute) $\ell$-adic homology:

$$H_{\text{mot}}^{i}(X, \mathbb{Q}_\ell(j)),$$
$$H_{\text{mix}}^{i}(X, \mathbb{Q}_\ell(j)) = H^{-i}(X, \mathbb{Q}_\ell(-j))$$

where $\pi_{X} : X \to \text{Spec} L$ is the structure morphism. (The reader may find convenient the torsion coefficient case given in [1, p.193, 2.1], and the geometric case with rational coefficient given in [19, p.87, 6.8].)
In [19], Jannsen considered the motivic theory and the geometric (as opposed to the absolute) \( \ell \)-adic theory, and the morphism between the two twisted Poincaré duality theories. In a similar manner, one can define a morphism from the motivic theory to the absolute \( \ell \)-adic theory above. We write

\[
\begin{align*}
\rho : H^1_{\text{sha}}(X, Q(j)) &\to H^2_{\text{et}}(X, Q\ell(j)), \\
\rho' : H^2_{\text{et}}(X, Q(b)) &\to H^2_{\text{et}}(X, Q\ell(b))
\end{align*}
\]

for the induced maps. We refer to [19, p.126, (8.4.1)] and the remarks preceding [19, p.125, 8.3 a)] for the details (on the geometric case).

4.2. We will use the setting for Chow motives given by Scholl [35, Section 1]. We recall some of the notations.

Let \( L \) be a field. We let \( V_L \) denote the category of projective smooth \( L \)-schemes. We let \( \mathcal{M}_L \) denote the category of \( L \)-motives for rational equivalence ([35, p.165, 1.4]). Take \( X, Y \in V_L \). Suppose \( X \) is connected and of pure dimension \( d \). We let

\[
\text{Corr}'(X, Y) = \text{CH}^{r+d}(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}
\]

denote the correspondences of degree \( r \). For the definition of correspondences for general \( X \) and the composition \( \circ \) of correspondences, we refer to [35, 1.3]. Recall that an object \( (X, p, m) \in \mathcal{M}_L \) is a triple with \( X \in V_L \), an idempotent \( p = p^2 \in \text{Corr}'(X, X) \), and an integer \( m \). The hom group \( \text{Hom}_{\mathcal{M}_L}(X, p, m), (Y, q, n) \) is by definition \( q \circ \text{Corr}^{\dim X}(X, Y) \circ p \).

We extend the definition of the cohomology groups to objects in \( \mathcal{M}_L \). Let \( (X, p, m) \in \mathcal{M}_L \). Note that \( p \in \text{Corr}'(X, X) \) defines a map \( p : H^*_\text{et}(X, Q(n)) \to H^*_\text{et}(X, Q(\ell(n))) \) and a map \( \rho \otimes \) : \( H^2_{\text{et}}(X, Q(\ell(n))) \to H^2_{\text{et}}(X, Q\ell(n)) \). We refer to [36, Section 1] for the construction when the cohomology theory is the \( K \)-cohomology theory. The procedure works for the two cohomology theories considered here. For integers \( a, b \), we set

\[
\begin{align*}
H^*_\text{et}^{a,b}(X, p, m) &:= \text{Im}[p : H^*_\text{et}(X, Q(m+b)) \to H^*_\text{et}(X, Q(m+b))], \\
H^2_{\text{et}}^{a,b}(X, p, m) &:= \text{Im}[\rho \otimes : H^2_{\text{et}}(X, Q(\ell(m+b)) \to H^2_{\text{et}}(X, Q\ell(m+b))].
\end{align*}
\]

Then the \( r \) and \( r' \) defined in the previous section induce a map

\[
\begin{align*}
r : H^*_\text{et}^{a,b}(X, p, m) &\to H^2_{\text{et}}^{a,b}(X, p, m).
\end{align*}
\]

5. PROOF OF THEOREM 1.1

We give a proof of Theorem 1.1. We refer to Section 1 for the outline of proof.

5.1. We begin with the proof of Proposition 1.5.

By Assumption (GL2), we have \( H^2_{\text{et}}(B_{kerv}, Q_\ell) \cong \bigoplus_{j=1}^n \rho_j \) where each \( \rho_j \) is an irreducible two dimensional representation of \( G_k \) with coefficients in \( Q_\ell \).

We consider the case when there does not exist a pair \( (j, \varphi) \) such that \( \rho_j|_{G_{k_{\varphi}}} \cong \text{Sp} \).

Lemma 5.1. In this case, we have \( H^2_{\text{et}}^{2,2}(h^1(B_{k_{\varphi}})) = 0 \) for any \( \varphi \).

Proof. We use the definition of \( h^1 \) in [35, p.176, Theorem 4.4]. It says in particular that there exists a curve \( Z \to B \) such that the projector to \( h^1 \) factors as

\[
H^2_{\text{et}}^{2,2}(B_{k_{\varphi}}) \to H^2_{\text{et}}^{2,2}(Z_{k_{\varphi}}) \to H^2_{\text{et}}^{2,2}(h^1(B_{k_{\varphi}}))
\]

where the first map is the pullback. Using the Leray spectral sequences for \( B_{k_{\varphi}} \) and \( Z_{k_{\varphi}} \), we see that it suffices to prove that

\[
\text{Im}[H^p(G_{k_{\varphi}}, H^2_{\text{et}}(B_{k_{\varphi}}^\text{cusp}, Q\ell(2))) \to H^p(G_{k_{\varphi}}, H^2_{\text{et}}(Z_{k_{\varphi}}^\text{cusp}, Q\ell(2)))]
\]

where the map is that induced by the pullback map, is zero for each pair \( (p, q) \) with \( p + q = 2 \). It is easy to see that the target is zero when \( (p, q) = (2, 0) \). When \( (p, q) = (0, 2) \), using the fact that \( Z \) is a curve, it follows from the weight argument that the target is zero. Suppose \( (p, q) = (1, 1) \). Let \( \pi_j \) denote the cuspidal automorphic representation corresponding to \( \rho_j \) by the global Langlands correspondence [9, p.566, Theorem A]. By the Petersson-Ramanujan conjecture (see [9, p.566, Theorem B]), we have that each \( \pi_j \otimes \| \text{det}^{\frac{1}{2}} \) is tempered. Since we know that \( \text{Sp} \) corresponds to \( St \) via the local Langlands correspondence, we obtain

\[
H^1(G_{k_{\varphi}}, H^2_{\text{et}}(B_{k_{\varphi}}^\text{cusp}, Q\ell(2))) = 0 \quad \text{from Proposition 3.1.}
\]

Now suppose that there exists a pair \( (j, \varphi) \) for which \( \rho_j|_{G_{k_{\varphi}}} \cong \text{Sp} \). Let \( \pi \) be the unitary cuspidal automorphic representation associated to \( \rho_j \) via the global Langlands correspondence. Since we have \( \pi|_{G_{k_{\varphi}}} \equiv \text{St} \), we can apply Proposition 2.1 and find an abelian variety \( A_{\varphi} \). We use the argument in the proof of Lemma 2.2. We have

\[
\text{Hom}_{k}(B, A_{\varphi}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \cong \text{Hom}_{\mathbb{Q}_\ell(G_k)}(\nabla B, \nabla A_{\varphi}) \neq 0.
\]
Therefore there is a nontrivial homomorphism $B \to A_\pi$ which is an isogeny since both $B$ and $A_\pi$ are simple. We may and will assume that $B = A_\pi$.

**Lemma 5.2.** For any place $\varphi$ of $k$ there exists a finite dimensional $\mathbb{Q}$-vector space $V_{\pi,\varphi} \subset H_{\text{et}}^{2,2}(h^1(A_\pi))$ such that

$$r_{\varphi'}(V_{\pi,\varphi}) \otimes \mathbb{Q}_\ell = \begin{cases} H_{\text{et}}^{2,2}(h^1(A_{\pi,k_\varphi})) & \text{if } \varphi' = \varphi, \\ 0 & \text{if } \varphi' \neq \varphi. \end{cases}$$

**Proof.** Define $\text{St}_\pi$ to be the set of places $\varphi$ of $k$ such that $\pi_\varphi \cong \text{St}$. We set $V_{\pi,\varphi} = 0$. Then it suffices to show that $H_{\text{et}}^{2,2}(h^1(A_{\pi,k_\varphi})) = 0$ for $\varphi \notin \text{St}_\pi$. This can be proved in the same manner as in the proof of Lemma 5.1, hence is omitted.

(2) Case $\varphi \in \text{St}_\pi$. By Corollary 2.5, $A_\pi$ is a $k$-simple factor of $\text{Jac}M_\pi^\varphi$ for some compact open subgroup $K \subset \text{GL}_2(A^\varphi)$. Here $A^\varphi$ is the finite adèles as defined in Section 2.1 with $\varphi$ in place of $\infty$ there.

By [20, p.1090, Proposition 9.1], there exists a finite dimensional $\mathbb{Q}$-vector space

$$V_{\pi,\varphi}^\varphi \subset H_{\text{et}}^{2,2}(h^1(M_\pi^\varphi))$$

such that

$$(5.1) \quad r_{\varphi'}(V_{\pi,\varphi}^\varphi) \otimes \mathbb{Q}_\ell = \begin{cases} H_{\text{et}}^{2,2}(h^1(M_\pi^\varphi)) & \text{if } \varphi' = \varphi, \\ 0 & \text{if } \varphi' \neq \varphi. \end{cases}$$

Since $r_{\varphi'}$ are compatible with the decomposition of a Chow motive, if we set $V_{\pi,\varphi}$ to be image of $V_{\pi,\varphi}^\varphi$ by the projection to $H_{\text{et}}^{2,2}(h^1(A_\pi))$, the desired conditions are satisfied. □

**Remark 5.3.** In [20], they use the étale Chern class map $c_{2,2}$ (introduced in [12]) instead of $r_{\varphi'}$. However, it is known that $r_{\varphi'}$ is two times $c_{2,2}$ (See [12, p.233, Proof of Proposition 2.35]). Therefore, we can apply the result of [20] to our situation.

This completes the proof of Proposition 1.5. □

**5.2.** We prove Proposition 1.4. To prove this, we decompose the Chow motive $h(X)$ into smaller pieces as follows. Let us use the formalism of Chow motives as explained in [35].

By Poincaré reducibility theorem (see [26, p.1, Theorem 1] for the case where the base field is imperfect), the Jacobian $\text{Jac}X$ of $X$ is canonically (up to isogeny) isogenous to a product of $k$-simple abelian varieties: $\text{Jac}X \to B_1 \times \cdots \times B_s$. Then we have

$$h(X) \cong h^0(X) \oplus h^1(X) \oplus h^2(X) \cong h^0(X) \oplus h^1(\text{Jac}X) \oplus h^2(X) \cong h^0(X) \oplus h^1(\bigoplus_{i=1}^s B_i) \oplus h^2(X) \cong h^0(X) \oplus h^1(\bigcap_{i=1}^s B_i) \oplus h^2(X),$$

The first isomorphism is that in [35, p.172, 3.2]. The second isomorphism follows from [35, p.178, Proposition 4.5]. The third isomorphism follows from [35, p.182, Corollary 5.10]. For the fourth isomorphism, recall (see [7, p.216, Theorem 3.1]) that the projector corresponding to the $h^i$ part of an abelian variety is characterized by the fact that the multiplication-by-$n$ map acts as $n^i$ on it. Now observe that $h(\bigcap_{i=1}^s B_i) \cong \bigoplus_{i=1}^s h(B_i)$, since $h(\bigcap_{i=1}^s B_i) \cong h(\text{Spec }k)$ for all $i$, the fourth isomorphism holds true.

Let us write $Y_L = Y \times_{\text{Spec }k} \text{Spec }L$ for a $k$-scheme $Y$ and a $k$-algebra $L$. To prove Proposition 1.4, it suffices to show the following three statements:

1. The map $\oplus_{\varphi} r_{\varphi'}: H_{\text{et}}^{2,2}(h^0(\text{Spec }k_\varphi)) \to \oplus_1^s H_{\text{et}}^{2,2}(h^0(X_{k_\varphi}))$ induces a surjection when the source is tensored with $\mathbb{Q}_\ell$.
2. The map $\oplus_1^s r_{\varphi'}: H_{\text{et}}^{2,2}(h^1(B_i)) \to \oplus_1^s H_{\text{et}}^{2,2}(h^1(B_i,k_\varphi))$ induces a surjection when the source is tensored with $\mathbb{Q}_\ell$ for each $1 \leq i \leq s$.
3. The map $\oplus_1^s r_{\varphi'}: H_{\text{et}}^{2,2}(h^2(X)) \to \oplus_1^s H_{\text{et}}^{2,2}(h^2(X_{k_\varphi}))$ induces a surjection when the source is tensored with $\mathbb{Q}_\ell$.

Note that $h^0(X_{k_\varphi}) \cong h(\text{Spec }k_\varphi)$ and $h^2(X_{k_\varphi}) \cong (\text{Spec }k_\varphi, \text{id}, -1) = \mathbb{L}$ (the Lefschetz motive). Hence the target group of the map in (1) and in (3) are zero, so the surjectivity follows trivially. The statement (2) follows from Proposition 1.5.

This finishes the proof of Proposition 1.4. □
5.3. Proof of Theorem 1.1. Let

$$\partial_{p, Q_v} : K_2(X_{k_v})_{Q_v} \to G_1(\mathcal{X}_{q(v)})_{Q_v}$$

be the boundary map at $p$ tensored with $Q_v$ and

$$c_{2,2,p} : K_2(X_{k_v})_{Q_v} \to H^2_{et}(\mathcal{X}_{k_v}, Q_v(2))$$

the étale Chern class map (defined in [12]) at $p$. By [20, p.1059, Proposition 3.1], (1) holds true. Let $V \subset K_2(X_{k_v})_{Q_v}$ be a finite dimensional vector space. Then, using (1), we have

$$\dim_{Q_v} H^1_{et}(\mathcal{X}_{k_v}, Q_v(2)) = \dim_{Q_v} G_1(\mathcal{X}_{q(v)})_{Q_v}$$

(2.2) in Case (2). For Case (3), let $\Pi_B$ denote the set of cuspidal automorphic representations of $(B \otimes_k \mathbb{A}_k)^{\times}$. Then we have

$$H^1_{et}(\mathcal{X}_{k_v}^{\operatorname{sep}}, Q_v(2)) \cong \bigoplus_{\pi \in \Pi_B, \pi \not\in \mathcal{S}_{\operatorname{St}}} \left(\otimes_{\nu \neq \infty} \pi_{\nu}\right)^{\mathbb{K}} \otimes \rho(\pi)$$

where $\rho(\pi)$ is an irreducible two dimensional representation of $G_k$. To see this, one proceeds as in the proof of [8, p.590, Proposition 10.3]. (See also [11, Theorem 4.13.1].) The key ingredient is the uniformization of the moduli space, which, in the $\mathcal{D}$-elliptic sheaf case, is provided by [2, p.171, Theorem 4.4.11]. The irreducibility of $\pi(\pi)$ follows from the $L$-factor computation given in [24, Theorem 14.9(ii)]. For Case (4), it follows from Corollary 7.2.

6. PROOF OF COROLLARY 1.3 AND THE TARGET GROUP

We give a precise form of Corollary 1.3 and its proof in this section. We also compute the dimension of the target group of the boundary map in Theorem 1.1.

6.1.

Corollary 6.1. Theorem 1.1 holds in all of the following cases.

1. $X$ is an elliptic curve, which is not isotrivial.
2. $X$ is a (compactified) Drinfeld modular curve. That is, when $X = \overline{M}_K$, for some $\infty$ and $K$ as in Section 2.1.
3. $X = M_{\infty, k}^{\mathcal{D}}$ is the moduli scheme of $\mathcal{D}$-elliptic sheaves of rank 2 with level $K$-structure as defined in [24]. Here, $B$ is an infinite quaternion algebra over $k$, $\mathcal{D}$ is a maximal order of $B$ and $K$ is an open compact subgroup of $(B \otimes_k \mathbb{A}_k)^{\times}$.
4. $X$ is the curve $X_s$ of Section 7 for $s \in k$ such that $D(s) \neq 0$.

Proof. We check that the condition (GL2) holds in each case. It holds trivially in Case (1). It follows from (2.2) in Case (2). For Case (3), let $\Pi_B$ denote the set of cuspidal automorphic representations of $(B \otimes_k \mathbb{A}_k)^{\times}$. Then we have

$$H^1_{et}(M_{\infty, k}^{\mathcal{D}}, Q_v(2)) \cong \bigoplus_{\pi \in \Pi_B, \pi \not\in \mathcal{S}_{\operatorname{St}}} \left(\otimes_{\nu \neq \infty} \pi_{\nu}\right)^{\mathbb{K}} \otimes \rho(\pi)$$

where $\rho(\pi)$ is an irreducible two dimensional representation of $G_k$. To see this, one proceeds as in the proof of [8, p.590, Proposition 10.3]. (See also [11, Theorem 4.13.1].) The key ingredient is the uniformization of the moduli space, which, in the $\mathcal{D}$-elliptic sheaf case, is provided by [2, p.171, Theorem 4.4.11]. The irreducibility of $\rho(\pi)$ follows from the $L$-factor computation given in [24, Theorem 14.9(ii)]. For Case (4), it follows from Corollary 7.2.

6.2. Let $X$ be as in Theorem 1.1. Write $H^1_{et}(X_{k_v}^{\operatorname{sep}}, Q_v(2)) \cong \bigoplus_{j=1}^n \rho_j$ where each $\rho_j$ is a 2 dimensional irreducible representation of $G_k$. Let $\text{Sp}_X$ denote the number of pairs $(j, \varphi)$ such that $\rho_j|_{G_{k_{\varphi}}} \cong \text{Sp}$.

Lemma 6.2. We have

$$\dim_{Q_v} \bigoplus_{\varphi \in C_0} H^1_{et}(h^1(X_{k_v})) = \text{Sp}_X.$$ 

Proof. It suffices to prove this for $X = B$ a simple abelian variety. If $\text{Sp}_B = 0$, this holds by Lemma 5.1. When $\text{Sp}_B \neq 0$, as we have seen in the discussion after Lemma 5.1 that $B = A_{2\pi}$ for some $\pi$. In this case, the claim follows from Proposition 3.1.

Corollary 6.3. Let $X$ be as in Theorem 1.1. Then

- $\dim_{Q_v} K_2(X)_{Q_v} \geq \text{Sp}_X$,
- $\dim_{Q_v} H^2_{et}(X, Q_v(2)) \geq \text{Sp}_X$.

Proof. This follows from Lemma 6.2, using Theorem 1.1 and Proposition 1.4 respectively.

Consider Case (4) in Corollary 6.1. We show (Proposition 7.1) that, given an integer $n$, there exists an element $s \in k$ such that $D(s) \neq 0$ and $\text{Sp}_{X_s} \geq 2n$. Hence we have

Corollary 6.4. Suppose the characteristic of $k$ is greater than 5. For any $n \in \mathbb{Z}$, there exists a genus two curve $X$ such that $K_2(X)_{Q_v} \geq n$.

Proof. This follows from Corollary 6.3 and Proposition 7.1 immediately.
Note that in [20, p.1054, Corollary 1.3], a similar statement where genus two is replaced by genus one is proved.

### 7. Abelian Varieties of GL$_2$-type

We keep the notations in Section 2. In this section we will be concerned with some classes of abelian varieties over $k$ which admit many endomorphisms. Such abelian varieties are originally considered by Ribet [31] in which the ground fields are $\mathbb{Q}$ and they are called abelian varieties of GL$_2$-type. We will define an analogue of such objects in the function fields setting and construct examples of 2 dimensional abelian varieties over $k$ of GL$_2$-type which has multiplicative reduction at given finite places.

Let $B$ be an abelian variety over $k$ of dimension $g$. We say that $B$ is of GL$_2$-type if there exist a number field $F$ of degree $g$ over $\mathbb{Q}$ and an embedding $\iota: F \hookrightarrow \text{End}_k(B) \otimes \mathbb{Q}$ as $\mathbb{Q}$-algebras. The abelian varieties $A_i$, of Proposition 2.1 (3) are examples of abelian varieties over $k$ of GL$_2$-type.

Let $B$ be an abelian variety over $k$ of GL$_2$-type endowed with an embedding $\iota: F \hookrightarrow \text{End}_k(B) \otimes \mathbb{Q}$. Let $\ell$ be a prime number which is different from the characteristic of $k$. Via $\iota$ and the natural embedding $\text{End}_k(B) \otimes \mathbb{Q} \hookrightarrow \text{End}_{\mathbb{Q}(G)}(V_i/B)$, the field $F$ acts on $V_i/B$ and its action is non-trivial and faithful because $F$ is a field and it contains $1$. Since $[F: \mathbb{Q}] = g$ and rank$_{\mathbb{Q}} V_i/B = 2g$, we can view $V_i/B$ as a free $F \otimes_{\mathbb{Q}} Q_\ell$-module of rank 2 by [37, Lemma 2.1(i)]. Then we have the 2 dimensional Galois representation attached to $B$:

$$\rho: G_k \longrightarrow \text{Aut}_{F \otimes \mathbb{Q}}(V_i/B) \simeq \text{GL}_2(F \otimes \mathbb{Q} \otimes \mathbb{Q}_\ell) = \prod_{v | \ell} \text{GL}_2(F_v)$$

where $F_v$ is the completion of $F$ at a finite place $v$ of $F$ dividing $\ell$. We denote by $\rho_v$ the composition of $\rho$ and the projection $\prod_{v | \ell} \text{GL}_2(F_v) \stackrel{\rho_v}{\longrightarrow} \text{GL}_2(F_v)$. It is easy to see that $L(s, V_i/B) = \prod_{v | \ell} L(s, \rho_v)$.

We next construct examples of 2 dimensional abelian varieties over $k$ of GL$_2$-type by using Brumer-Hashimoto’s family.

Let $p > 5$ be a prime number and $q$ be a power of $p$. Let $k$ and $A$ be as in Section 1.2. Let $H(X; a, b, c)$ be the polynomial defined by [15, p.479, (10)]. For any $s \in k$, we define the following polynomial by using $H(X; a, b, c)$:

$$(1.1) \quad H(x; \frac{7}{4}, s + 1, -1) = x^6 - (2s + 3)x^5 + (s^2 + 4s + \frac{13}{4})x^4 + (2s^2 - 2s - \frac{3}{2})x^3 + (s^2 + 2s + \frac{1}{4})x^2 - 2sx$$

with the discriminant

$$D(s) := s^8(9 - 1160s + 1512s^2 + 992s^3 + 48s^4)^2 \neq 0$$

with respect to $x$. For $s \in k$ so that $D(s) \neq 0$, gluing the two finite surjective morphism $f_1 : \text{Spec} k[x, y]/(y^2 - H(x; \frac{7}{4}, s + 1, -1)) \longrightarrow \mathcal{A}_{k, x, y}^1$, $f_2 : \text{Spec} k[x, y]/(y^2 - x^6 H(\frac{1}{x_1}; \frac{7}{4}, s, 1, 1)) \longrightarrow \mathcal{A}_{k, x, y}^1$, $(x_1, y_1) \mapsto x_1$ by the isomorphism $(x, y) \leftrightarrow (x_1, y_1) = (\frac{1}{x_1}, \frac{y}{x_1})$, we have a finite surjective morphism $X_s \longrightarrow \mathbb{P}^1$. The resulting variety $X_s$ is a projective smooth curve of genus 2 over $k$. Put $J_s = \text{Jac} X_s$. Then by [15] we have an embedding $\iota: F = \mathbb{Q}(\sqrt{5}) \hookrightarrow \text{End}_{\mathbb{Q}}(J_s) \otimes \mathbb{Q}$. Actually the results of [15] stated for which ground fields is the rational function fields over $Q$, but it is easy to check that the algebraic operations performed there work over any $\mathbb{Z}[[\frac{1}{p}, \frac{1}{q}]]$-algebra. Hence we can apply the results in [15].

Let $\wp_1, \ldots, \wp_n$ be the non-zero prime ideals of $A$. Take a non-zero element $s \in \cap_{i=1}^n \wp_i$. Then it is easy to see $D(s) \neq 0$. For such $s$ and each finite place $v | \ell$ of $F = \mathbb{Q}(\sqrt{5})$ we attach the 2 dimensional Galois representation $\rho_{s, v}: G_k \longrightarrow \text{GL}_2(F_v)$ from $J_s$ as above. Then the local behaviour of $\rho_{s, v}$ at each $\wp_i$ is as follows.

**Proposition 7.1.** For each $i \in \{1, \ldots, n\}$, $\rho_{s, v(G_{k_{\wp_i}})} \simeq \text{Sp}$. 

**Proof.** For simplicity put $\wp = \wp_i$. Let $A_{\wp}$ (resp. $k_{\wp}$) be the completion of $A$ (resp. $k$) at $\wp$. Fix a uniformizer $w$ of $A_{\wp}$, put $\kappa := A_{\wp}/w$ and $\varphi = \wp k_{\wp}$. Let $W_{k_{\wp}}$ be the Weil group of $k_{\wp}$ (see [39] for basic properties). Fix a lift Frobl_{\wp} \in W_{k_{\wp}}$ of the geometric Frobenius element of $G_{k_{\wp}}$. Then each element $\sigma \in W_{k_{\wp}}$ can be written uniquely as $\sigma = \tau \cdot \text{Frob}^{a}_{\wp}$, $\tau \in I_{k_{\wp}}$, $n \in \mathbb{Z}$. Put $n(\sigma) = n$ for such $\sigma \in W_{k_{\wp}}$. Since $W_{k_{\wp}}$ is dense in $G_{k_{\wp}}$, the restriction map from the category of continuous $G_{k_{\wp}}$-representations to the category of $W_{k_{\wp}}$-representations is injective. So it suffices to prove $\text{WD}(\rho_{s, v(G_{k_{\wp}})}) \simeq \text{WD}(\text{Sp})$.

Gluing the two finite surjective morphism

$$\tilde{f}_1 : \mathcal{X}_{s, 1} := \text{Spec} A_{\wp}[x, y]/(y^2 - H(x; \frac{7}{4}, s + 1, -1)) \longrightarrow \mathcal{A}_{k_{\wp}, x, y}^1, (x, y) \mapsto x$$

and

$$\tilde{f}_2 : \mathcal{X}_{s, 2} := \text{Spec} A_{\wp}[x_1, y_1]/(y_1^2 - x_1^6 H(\frac{1}{x_1}; \frac{7}{4}, s, 1, 1)) \longrightarrow \mathcal{A}_{k_{\wp}, x_1, y_1}^1, (x_1, y_1) \mapsto x_1$$

Note that in [20, p.1054, Corollary 1.3], a similar statement where genus two is replaced by genus one is proved.
by the isomorphism \((x, y) \mapsto (x_1, y_1) = \left( \frac{1}{2}, \frac{y}{x} \right)\), we have a finite surjective morphism \(\mathcal{X}_s \to \mathbb{P}^1_{A_0}\). The resulting variety \(\mathcal{X}_s\) is a projective flat scheme over \(A_0\) with relative dimension one. The projectivity follows from the ampleness of the invertible sheaf \(\mathcal{O}(1)\) on \(\mathbb{P}^1_{A_0}\) and [14, III, Exercise 5.8-(d)]. Since the special fiber of \(\mathcal{X}_s\) is obtained by gluing the affine models:

\[
\mathcal{X}_{s,1} \otimes_{A_0} \kappa = \text{Spec} \kappa[x,y]/(y^2 - x^2 - x^2(2y - \frac{1}{2})), \quad \mathcal{X}_{s,2} \otimes_{A_0} \kappa = \text{Spec} \kappa[x_1,y_1]/(y_1^2 - (1 - x_1)^2(2y_1))^2),
\]

the possible singularities are at the points \(P_i \in \mathcal{X}_{s,1}\) corresponding to the ideals \(I(x_i, y, \varpi)\) for \(i \in \{0, 1, 1/2\}\). It is easy to see that \(\mathcal{X}_s\) is regular at \(P_1\). We denote by \(\xi\) the blowing up of \(\mathcal{X}_s\) along to the ideals \((x, y, \varpi)\) and \((x - 1, y, \varpi)\). Then \(\xi\) is a proper strictly semistable model of \(\mathcal{X}_s\) over \(A_0\) and its special fiber is obtained by gluing two copies of \(\mathbb{P}^1_{\kappa}\) corresponding to two irreducible components of \(\mathcal{X}_s \otimes \kappa\) at three points \(P_0, P_1, P_2\) and then replacing \(P_1\) by a chain of \(i + 1\) projective lines for \(i = 0, 1\). Hence \(\mathcal{X}_s \otimes \kappa\) is the union of five divisors \(D_1, \ldots, D_5\) which are isomorphic to \(\mathbb{P}^1_{\kappa}\) over \(\kappa\). Here \(D_1\) and \(D_2\) correspond to irreducible components of \(\mathcal{X}_s \otimes \kappa\) and \(D_3\) (resp. \(D_4, i = 4, 5\)) is one chain (resp. two chains) between \(D_1\) and \(D_2\). These components intersect at \(\kappa\)-rational points. Then by [13, Section 12], we have the claim by Picard-Lefschetz formula which is well-known for experts. However we give a proof here in details for the reader’s convenience and further use the weight spectral sequence of Rapoport and Zink instead of Picard-Lefschetz formula.

Henceforth \(H^*\) means the étale cohomology. To compute the action of Weil group \(W_{k_v}\) of \(k_v\) on \(\mathcal{V}_sJ_s\) we use the isomorphism \(\mathcal{V}_sJ_s \simeq H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) and the weight spectral sequence of Rapoport and Zink for the semistable curve \(X_s\). Put \(D^{(0)} = \prod_i D_i\) and \(D^{(1)} = \prod_{i < j} D_i \cap D_j\). Note that \(D^{(1)}\) consists of six \(\kappa\)-rational points. Then by [30] we have the following spectral sequence:

\[
E_1^{r,s} = H^r\left(\mathcal{I}^s, \mathbb{Q}_\ell\right) \Rightarrow H^s(X_s, \kappa_{sep}, \mathbb{Q}_\ell).
\]

This gives the weight filtration \(\text{Fil}_W^1\) on \(H^s(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\). The essential \(E_1\)-terms are as follows:

\[
E_1^{1,2} = E_1^{0,2} = E_1^{0,1} = E_1^{0,0} = E_1^1.
\]

We can compute the \(E_1\)-terms as follows:

\[
E_1^{1,2} = H^3(\mathcal{I}_w^0, \mathbb{Q}_\ell(-1)) \simeq \mathbb{Q}_\ell(-1)^{\oplus6}, \quad E_1^{0,2} = H^2(\mathcal{I}_w^0, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-1)^{\oplus5}, \quad E_1^{1,0} = H^1(\mathcal{I}_w^0, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^{\oplus5}, \quad E_1^{0,1} = H^0(\mathcal{I}_w^0, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^{\oplus6}.
\]

For each \(\sigma \in W_{k_v}\), since

\[
\sum_w (-1)^w \text{tr}(\sigma|H^s(X_s, \kappa_{sep}, \mathbb{Q}_\ell)) = \sum_r \sum_w (-1)^w \text{tr}(\sigma|E_1^{r,s}) = 5 + 6q_\sigma(-1) - 6q_\sigma^0 = -1 - q_\sigma^0,
\]

we have that

\[
\text{tr}(\sigma|H^0(X_s, \kappa_{sep}, \mathbb{Q}_\ell)) = 1, \quad \text{and} \quad \text{tr}(\sigma|H^2(X_s, \kappa_{sep}, \mathbb{Q}_\ell)) = q_\sigma^0.
\]

Since \(H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) is a free \(F_\ell := F \otimes \mathbb{Q}_\ell\)-module of rank 2 and \(\det(\sigma|H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)) = q_\sigma^{2n}\), the eigenvalues of \(\sigma\) on \(F_\ell\)-modules \(H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) are \(q_\sigma^{n}\). Therefore as \(F_\ell[\mathcal{W}_{k_v}]\)-modules we have that

\[
H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)^{\mathbb{Q}_\ell} \simeq F_\ell \oplus F_\ell(-1).
\]

We next compute the monodromy operator \(N\) on \(H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) which is defined by Grothendieck’s monodromy theorem. By weight monodromy theorem for curves, we have that

1. \(N(\text{Fil}_W^1 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)) \subset \text{Fil}_W^0 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell),\)

2. \(N : \text{gr}_W^1 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-1)^{\oplus2} \to \text{gr}_W^0 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^{\oplus2}.
\]

Recall that \(F\) acts on \(H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) via the embeddings \(F \xleftarrow{\iota} \End_k(J_s) \otimes \mathbb{Q} \to \End_k[G_1](V_sJ_s)\) and \(V_sJ_s \simeq H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) as \(Q\)-modules. Therefore the action of \(F = \mathbb{Q}(\sqrt{5})\) on \(H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) commutes with the Galois action, hence \(F\) acts also on \(\text{gr}_W^0 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell) = \text{Fil}_W^0 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell) \simeq \text{Ker}(N)\) and \(\text{gr}_W^1 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\). Note that \(\text{gr}_W^0 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) is a free \(F_\ell\)-module of rank one for \(i = 0, 2\). So take a basis \(e_2\) of \(\text{gr}_W^0 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell) = \text{Fil}_W^0 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) over \(F_\ell\). Then by (1) as above, \(N(e_2) = 0\) since \(\text{Fil}_W^0 H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell) = 0\). Take a lift \(e_1 \in H^1(X_s, \kappa_{sep}, \mathbb{Q}_\ell)\) of the element corresponding to \(e_2\) via the
isomorphism of (2). Then we have $N(e_1) = e_2$. Clearly $\{e_1, e_2\}$ forms a basis of $H^1(X_s, \mathcal{O}_X)$ over $F_s$. Therefore the matrix representation of $N$ with respect to this basis is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Hence we have $WD(p_v|G_{k_v}) \simeq WD(Sp)$.

**Corollary 7.2.** The notations are same as above. The curve $X_s$ satisfies the condition $(GL_2)$.  

*Proof.* Recall that we have assumed that $p > 5$ for the construction of $X_s$. Then by Tate conjecture [41], $V_f J_s \otimes \mathbb{Q}_l = \bigoplus_{v|l} \mathbb{Q}_l \otimes F_s \mathbb{Q}_l$ is a semi-simple $\mathbb{Q}_l[G_{k_v}]$-module. If $\rho_v \otimes F_s \mathbb{Q}_l$ is reducible for some $v|l$, then the rank of the monodromy operator $N$ on $V_f J_s$ is 0 or 1. However by Proposition 7.1 the rank of $N$ has to be 2 which gives a contradiction. Hence $\rho_v \otimes F_s \mathbb{Q}_l$ is irreducible for any $v|l$. \hfill \square

**Remark 7.3.** It seems to be meaningful to discuss about $k$-simpleness of $J_s$ because if $J_s$ is not $k$-simple, then $J_s$ is isogenous to the product of two elliptic curves over $k$ which have split multiplicative reduction at each $\wp_i$. In this case, Corollary 6.4 comes down to [20, Corollary 1.3] and therefore does not create essentially a new example.

In general, it seems to be difficult to check the $k$-simpleness of $J_s$ for any $k$. However for a given $k$ we can check the $k$-simpleness of $J_s$ as follows. Assume that $X_s$ has good reduction at some non-zero $\wp \in \text{Spec } A$. We have the maps

$$\text{End}_k(J_s) \otimes \mathbb{Q} \xrightarrow{\alpha} \text{End}_{k_v}(J_{s,k_v}) \otimes \mathbb{Q} \xrightarrow{\phi_{k_v}} \text{End}_{k_\wp}(J_{s,\wp}) \otimes \mathbb{Q}.$$ 

The homomorphism $\alpha$ is a natural embedding and the second map $\phi$ is the specialization map which is known to be injective (cf. [23, p.45, Theorem 3.2]). Hence the composition of $\alpha$ and $\phi$ is injective. If $L(s, V_f J_{s,k_v})^{-1}$ is an irreducible polynomial in $\mathbb{Q}[q^{-s}]$, then $J_{s,k_v}$ is $k$-simple by [38], hence $\text{End}_{k_\wp}(J_{s,\wp}) \otimes \mathbb{Q}$ is a division field. Hence $J_s$ is also $k$-simple because of the injectivity of $\phi \circ \alpha$ as above. From this we can easily check the $k$-simpleness of $J_s$ when $k = \mathbb{F}_p(s)$ and $5 < p < 1000$. Here the calculation is done by using Mathematica version 8.0.

### References

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