Rigidity of certain solvable actions on the torus

Masayuki ASAOKA*

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Abstract
An analog of the Baumslag-Solitar group $BS(1, k)$ acts on the torus naturally. The action is not locally rigid in higher dimension, but any perturbation of the action should be homogeneous.

1 Introduction
For integers $n \geq 1$ and $k \geq 2$, let $\Gamma_{n,k}$ be the finitely presented group given by

$$\Gamma_{n,k} = \langle a, b_1, \ldots, b_n \mid ab_i a^{-1} = b_j^k, b_i b_j = b_j b_i \text{ for any } i, j = 1, \ldots, n \rangle.$$  

The group $\Gamma_{1,k}$ is just the Baumslag-Solitar group $BS(1, k) = \langle a, b \mid aba^{-1} = b^k \rangle$. It acts on the projective line $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$ by $a \cdot x = kx$ and $b \cdot x = x + 1$, where we set $c \cdot \infty = \infty$ and $\infty + t = \infty$ for any $c \neq 0$ and $t \in \mathbb{R}$. This action preserves the standard projective structure on $\mathbb{R}P^1$. In [2], Burslem and Wilkinson proved a classification theorem of smooth smooth $BS(1, k)$-action on $\mathbb{R}P^1$. As a corollary, they obtained the following rigidity result.

Theorem 1.1 (Burslem and Wilkinson [2]). Any real analytic $BS(1, k)$-action on $\mathbb{R}P^1$ is locally rigid. In particular, the above projective action is locally rigid.

Recall the definition of local rigidity of a smooth action of a discrete group. Let $\Gamma$ be a discrete group and $M$ a smooth closed manifold. The group $\text{Diff}(M)$ of smooth diffeomorphisms is endowed with the $C^\infty$-topology. A $\Gamma$-action is a homomorphism from $\Gamma$ to $\text{Diff}(M)$. For a $\Gamma$-action $\rho$ and $\gamma \in \Gamma$, we write $\rho^\gamma$ for the diffeomorphism $\rho(\gamma)$. By $\mathcal{A}(\Gamma, M)$, we denote the set of smooth $\Gamma$-actions on $M$. This set is endowed with the topology generated by the open basis

$$\{O_{\gamma, U} = \{\rho \in \mathcal{A}(\Gamma, M) \mid \rho^\gamma \in U\} \},$$

where $\gamma$ and $U$ run over $\Gamma$ and all open subsets of $\text{Diff}(M)$. We say two $\Gamma$-actions $\rho_1$ and $\rho_2$ are smoothly conjugate if there exists a diffeomorphism $h$ of $M$ such

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1The term ‘smooth’ means ‘of $C^\infty$’ in this paper.
that $\rho^\gamma_2 = h \circ \rho^\gamma_1 \circ h^{-1}$ for any $\gamma \in \Gamma$. An $\Gamma$-action $\rho_0$ is locally rigid if it admits a neighborhood in $A(\Gamma, M)$ in which any action is smoothly conjugate to $\rho_0$.

The above projective $BS(1,k)$-action on $\mathbb{R}P^1$ can be generalized to $\Gamma_{n,k}$-actions on the sphere $S^n$. Let $B = (v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$. We define an $BS(n,k)$-action $\hat{\rho}_B$ on $S^n = \mathbb{R}^n \cup \{\infty\}$ by $\hat{\rho}_B^\gamma(x) = k \cdot x$ and $\hat{\rho}_B^{\gamma i}(x) = x + v_i$ for $x \in \mathbb{R}^n$, where $c \cdot \infty = \infty$ and $\infty + v = \infty$ for any $c \neq 0$ and $v \in \mathbb{R}^n$. The sphere $S^n$ admits a natural conformal structure and the action $\hat{\rho}_B$ preserves it. In [1], the author of this paper proved that the action $\hat{\rho}_B$ is not locally rigid but it exhibits a weak form of rigidity.

**Proposition 1.2 ([1]).** $\hat{\rho}_B$ and $\hat{\rho}_B'$ are smoothly conjugate if and only if there exists a conformal linear transformation $T$ of $\mathbb{R}^n$ such that $TB = B'$. In particular, $\hat{\rho}_B$ is not locally rigid if $n \geq 2$.

**Theorem 1.3 ([1]).** There exists a neighborhood of $\hat{\rho}_B$ in $A(\Gamma_{n,k}, S^n)$ in which any action $\rho$ is smoothly conjugate to $\hat{\rho}_B'$ with some basis $B'$. In particular, any $\Gamma_{n,k}$-action close to $\hat{\rho}_B$ preserves a smooth conformal structure on $S^n$.

In this paper, we prove analogous results for another generalization of the projective $BS(1,k)$-action on $\mathbb{R}P^1$. Let $B = (v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$ with $v_j = (v_{ij})_{i=1}^n$. We define a $\Gamma_{n,k}$-action $\rho_B$ on the $n$-dimensional torus $\mathbb{T}^n = (\mathbb{R} \cup \{\infty\})^n$ by

\[
\rho_B^k(x_1, \ldots, x_n) = (k \cdot x_1, \ldots, k \cdot x_n) \\
\rho_B^{b i}(x_1, \ldots, x_n) = (x_1 + v_{ij}, \ldots, x_n + v_{nj}).
\]

Remark that the point $x_\infty = (\infty, \ldots, \infty) \in \mathbb{T}^n$ is a global fixed point of the action $\rho_B$.

Let $G$ be the subgroup of $GL_n\mathbb{R}$ consisting of linear transformations $f$ which have the form $f(x_1, \ldots, x_n) = (a_1x_{\sigma(1)}, \ldots, a_nx_{\sigma(n)})$ with real numbers $a_1, \ldots, a_n \neq 0$ and a permutation $\sigma$ on $\{1, \ldots, n\}$. The aim of this paper is to show that the action $\rho_B$ is not locally rigid if $n \geq 2$, but it exhibits rigidity like the above $\Gamma_{n,k}$-action on $S^n$.

**Proposition 1.4.** Two actions $\rho_B$ and $\rho_B'$ are smoothly conjugate if and only if $B' = gB$ for some $g \in G$. In particular, $\rho_B$ is not locally rigid if $n \geq 2$.

**Theorem 1.5.** There exists a neighborhood of $\rho_B$ in $A(\Gamma_{n,k}, \mathbb{T}^n)$ in which any action is smoothly conjugate to $\rho_B$ for some basis $B$ of $\mathbb{R}^n$.

The theorem is proved by an application of the method used in [1]. Firstly, we show persistence of the global fixed point $x_\infty$. Next, we reduce the theorem to the corresponding theorem for local actions at the global fixed point. The same argument as in [2], we can see that the theorem for local actions follows from exactness of a finite dimensional linear complex. The exactness can be checked by an elementary computation.
2 Proof of Theorem 1.5

2.1 Reduction from global to local

Let $\Gamma$ be a discrete group and $M$ a smooth closed manifold. We say that a point $x_\ast \in M$ is a global fixed point of a $\Gamma$-action $\rho$ on $M$ if $\rho^\gamma(x) = x$ for any $\gamma \in \Gamma$.

We can apply the following general result on persistence of a global fixed point of $\Gamma_{n,k}$-action to the action $\rho_B$.

**Lemma 2.1** ([1, Lemma 2.10]). Let $M$ be a manifold and $\rho$ be a $\Gamma_{n,k}$-action on $M$. Suppose that $\rho_0$ has a global fixed point $p_0$ such that $(D\rho_i^0)_{p_0} = k^{-1}I$ and $(D\rho_i^0)_{p_0} = I$ for any $i = 1, \ldots, n$. Then, there exists a neighborhood $U \subset \text{Hom}(\Gamma_{n,k}, \text{Diff}(M))$ of $\rho_0$ and a continuous map $\hat{\rho} : U \to M$ such that $\hat{\rho}(p_0) = p_0$ and $\hat{\rho}(\rho)$ is a global fixed point of $\rho$ for any $\rho \in U$.

The action $\rho_B$ and its global fixed point $x_\infty$ satisfy the assumption of the lemma. Hence, any action $\rho$ close to $\rho_B$ admits a global fixed point $x_\rho$ close to $x_\infty$.

A $\Gamma$-action with a global fixed point induces a local $\Gamma$-action. We define the space of local actions on $\mathbb{R}^n$ as follows. Let $\mathcal{D}$ be the group of germs of local diffeomorphisms of $\mathbb{R}^n$ fixing the origin. For $F \in \mathcal{D}$ and $r \geq 1$, we denote the $r$-th derivative of $F$ at the origin by $D_0^{(r)}F$. It is an element of the vector space $\mathcal{S}^{r,n}$ of symmetric $r$-multilinear maps from $(\mathbb{R}^n)^r$ to $\mathbb{R}^n$. We define a norm $\| \cdot \|^{(r)}$ on $\mathcal{S}^{r,n}$ by

$$\|F\|^{(r)} = \sup\{\|F(\xi_1, \ldots, \xi_r)\| \mid \xi_1, \ldots, \xi_r \in \mathbb{R}^n, \|\xi_i\| \leq 1 \text{ for any } i\},$$

where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^n$. We also define a pseudo-distance $d_r$ on $\mathcal{D}$ by

$$d_r(F, G) = \sum_{i=1}^r \|D_0^{(i)}F - D_0^{(i)}G\|^{(i)}.$$

The pseudo-distance on $\mathcal{D}$ induces a topology on $\mathcal{D}$, which we call the $C^{r,\text{loc}}$-topology. Remark that it is not Hausdorff. Let $\text{Hom}(\Gamma, \mathcal{D})$ the set of homomorphisms from $\Gamma$ to $\mathcal{D}$, which can be regarded as the set of local $\Gamma$-actions on $(\mathbb{R}^n, 0)$. The $C^{r,\text{loc}}$-topology on $\mathcal{D}$ induces a topology on $\text{Hom}(\Gamma, \mathcal{D})$ like $\mathcal{A}(\Gamma, M)$.

We also call this topology on $\text{Hom}(\Gamma, \mathcal{D})$ the $C^{r,\text{loc}}$-topology. We say two local $\Gamma$-actions $P_1$ and $P_2$ are smoothly conjugate if there exists $H \in \mathcal{D}$ such that $P_2^\gamma = H \circ P_1^\gamma \circ H^{-1}$ for any $\gamma \in \Gamma$.

Let $\varphi$ be the local coordinate of $T^n$ at $x_\infty$ given by

$$\varphi(x_1, \ldots, x_n) = \left( \frac{1}{x_1}, \ldots, \frac{1}{x_n} \right),$$

where $1/\infty = 0$. For a basis $B$ of $\mathbb{R}^n$, we define a local $\Gamma_{n,k}$-action $P_B$ by $P_B^\gamma = \varphi \circ \rho_B^\gamma \circ \varphi^{-1}$. For each $\Gamma_{n,k}$-action $\rho$ close to $\rho_B$, we can take a local coordinate $\varphi_\rho$ close to $\varphi$ with $\varphi(x_\rho) = 0$ so that a local $\Gamma_{n,k}$-action given by $P_\rho^\gamma = \varphi_\rho \circ \rho_B^\gamma \circ \varphi_\rho^{-1}$ is $C^{3,\text{loc}}$-close to $\rho_B$. 

3
The following proposition reduces Theorem 1.5 to the corresponding result for local actions.

**Proposition 2.2.** Let $\rho$ be a $\Gamma_{n,k}$-action on $\mathbb{T}^n$ close to $\rho_B$. Suppose that the induced local action $P_{\rho}$ is smoothly conjugate to $P_{\rho_B}$ for some basis $B'$ of $\mathbb{R}^n$. Then, the action $\rho$ is smoothly conjugate to $\rho_B$.

The rest of this subsection is devoted to the proof of the proposition. Let $B' = (v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$ such that $P_{\rho}$ is smoothly conjugate to $P_{\rho_B}$. For each $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{\pm 1\}^n$, there exist integers $m_1^\sigma, \ldots, m_n^\sigma$ such that $m_i^\sigma \cdot \sum_{j=1}^n m_j^\sigma v_{ij} > 0$ for any $i = 1, \ldots, n$. Set $b_\sigma = b_1^\sigma \cdots b_n^\sigma$ and $v_\sigma = \sum_{j=1}^n m_j^\sigma v_{ij}$. Then, we have

$$\rho_B^{b_\sigma}(x_1, \ldots, x_n) = (x + v_1^\sigma, \ldots, x_n + v_n^\sigma).$$ (1)

Let $\bar{m}$ be the maximum of $\{|m_i| | \sigma \in \{\pm 1\}^n, i = 1, \ldots, n\}$ and put $S = \{a^{\pm 1} \cup \{b_1^\sigma \cdots b_n^\sigma | |l_i| \leq \bar{m}\} \cup \mathbb{R}^n \setminus S\}$. By the assumption of the proposition, there exists a neighborhood $U$ of $x_\infty$ and a diffeomorphism $h : U \to h(U)$ such that $x_\infty \in U$, \(h(x_\infty) = x_\rho\) and $h \circ \rho^{-1} = \rho \circ h$ on $U \cap (\rho_B^{-1}(U))$ for any $\gamma \in S$. Take an open interval $I \subset \mathbb{R}^{P^n} \setminus \{0\}$ such that $x_\infty \in I^n \subset U \cap \bigcap_{x \in S}(\rho_B^{-1}(U))$. Put $I_1 = \{x \in I | x = \infty \text{ or } x > 0\}$, $I_{-1} = \{x \in I | x = \infty \text{ or } x < 0\}$ and $U_x = I_x \times \cdots \times I_{x_n}$ for $x = (\sigma_1, \ldots, \sigma_n) \in \{\pm 1\}^n$. Equation (1) implies that $\rho_B^{b_\sigma}(x)(U_x) \subset U_x$, $\mathbb{T}^n = \bigcup_{n \geq 0}(\rho_B^{b_\sigma})^{-n}(U_x)$, and $\lim_{n \to \infty}(\rho_B^{b_\sigma})^{n}(x) = x_\infty$ for any $x \in \mathbb{T}^n$.

For $\sigma \in \{\pm 1\}^n$, let $m(x, \sigma)$ be the minimal integer $m$ such that $(\rho_B^{b_\sigma})^m(x)$ is contained in $U_x$. We define a map $h_\sigma : \mathbb{T}^n \to \mathbb{T}^n$ by

$$h_\sigma(x) = (\rho_B^{b_\sigma})^{-m(x, \sigma)} \circ h \circ (\rho_B^{b_\sigma})^{m(x, \sigma)}(x).$$

We will show that $h_\sigma$ does not depend on the choice of $\sigma$ and it is a smooth conjugacy between $\rho_B$ and $\rho$.

**Lemma 2.3.** $h_\sigma(x) = (\rho_B^{b_\sigma})^{-m} \circ h \circ (\rho_B^{b_\sigma})^{m}(x)$ for any $m \geq m(x, \sigma)$.

**Proof.** Proof is done by induction of $m$. Suppose that the equation holds for some $m \geq m(x, \sigma)$. Since $(\rho_B^{b_\sigma})^m(U_x) \subset U_x \subset I^n$, we have

$$(\rho_B^{b_\sigma})^{-(m+1)} \circ h \circ (\rho_B^{b_\sigma})^{m+1}(x) = (\rho_B^{b_\sigma})^{-(m+1)} \circ h \circ (\rho_B^{b_\sigma})^{m}(x)$$

$$= (\rho_B^{b_\sigma})^{-m} \circ h \circ (\rho_B^{b_\sigma})^{m}(x).$$

Hence, the required equation holds for $m + 1$.

**Lemma 2.4.** The map $h_\sigma$ is injective.

**Proof.** Take $x_1, x_2 \in \mathbb{T}^2$ and $m = \max\{m(x_1, \sigma), m(x_2, \sigma)\}$. Then, we have

$$h_\sigma(x_i) = (\rho_B^{b_\sigma})^{-m} \circ h \circ (\rho_B^{b_\sigma})^{m}(x_i).$$

for $i = 1, 2$. The map in the right-hand side is injective.\[\square\]
Lemma 2.5. \( h_\gamma \circ \rho^\gamma_B = \rho^\gamma \circ h_\gamma \) for any \( \gamma \in \Gamma \).

Proof. Fix \( x \in T^n \) and take \( m \geq m(x, \sigma) \) such that \( m \geq m(\rho^\gamma_B(x), \sigma) \) for any \( \gamma \in S \). It is sufficient to show that \( h_\sigma \circ \rho^\gamma_B = \rho^\gamma \circ h_\sigma \) for \( \gamma \in \{a, b_1, \ldots, b_n\} \).

For any \( i = 1, \ldots, n \), the identity \( b_i b_j = b_j b_i \) implies that

\[
\begin{align*}
    h_\sigma \circ \rho^b_i(x) &= (\rho^b)^{-m} \circ h \circ (\rho^b)^m \circ \rho^b_i(x) \\
    &= (\rho^b)^{-m} \circ (h \circ \rho^b_{b_i}) \circ (\rho^b)^m(x) \\
    &= (\rho^b)^{-m} \circ (\rho^b_i \circ h) \circ (\rho^b)^m(x) \\
    &= \rho^b_i \circ (\rho^b)^{-m} \circ h \circ (\rho^b)^m(x) \\
    &= \rho^b_i \circ h_\sigma(x).
\end{align*}
\]

The identity \( ab_i = b_i^a \) also implies

\[
\begin{align*}
    h_\sigma \circ \rho^a_i(x) &= (\rho^a)^{-km} \circ h \circ (\rho^a)^{km} \circ \rho^a_i(x) \\
    &= (\rho^a)^{-km} \circ (h \circ \rho^a_{b_i}) \circ (\rho^a)^m(x) \\
    &= (\rho^a)^{-km} \circ (\rho^a_i \circ h) \circ (\rho^a)^m(x) \\
    &= \rho^a_i \circ (\rho^a)^{-m} \circ h \circ (\rho^a)^m(x) \\
    &= \rho^a_i \circ h_\sigma(x).
\end{align*}
\]

For a diffeomorphism \( f \) on a manifold \( M \) and a hyperbolic fixed point \( p \) of \( f \), we denote the unstable manifold of \( p \) by \( W^u(p, f) \) (see e.g., [3] for the definitions and basic results on hyperbolic dynamics). By \( \text{Fix}(f) \), we denote the set of fixed point of \( f \). For \( l \geq 0 \), let \( \text{Fix}_l(f) \) be the set of hyperbolic fixed point of \( f \) whose unstable manifold is \( l \)-dimensional.

The diffeomorphisms \( \rho^a_B \) and \( \rho^b_B \) are Morse-Smale diffeomorphisms with the fixed point set \( \{0, \infty\}^n \). For each fixed point \( p = (p_1, \ldots, p_n) \in \{0, \infty\}^n \), \( W^u(p, \rho^a_B) = W_1 \times \cdots \times W_n \) with \( W_j = \mathbb{R} \) if \( p_j = 0 \) and \( W_j = \{\infty\} \) if \( p_j = \infty \).

If \( p \) is sufficiently close to \( \rho^a_B \), then \( \rho^a_B \) is a Morse-Smale diffeomorphism and \( \text{Fix}(\rho^a_B) \) has the same cardinality as \( \text{Fix}_l(\rho^a_B) \), and hence, as \( \text{Fix}_l(\rho^a_B) \) for any \( l = 0, \ldots, n \). By Lemma 2.4 and Lemma 2.5, \( h_\sigma \) maps \( \text{Fix}(\rho^a_B) \) to \( \text{Fix}(\rho^a_B) \) bijectively.

Lemma 2.6. For any \( l = 0, \ldots, n \) and \( p \in \text{Fix}_l(\rho^a_B) \), \( h_\sigma(p) \) is a point in \( \text{Fix}_l(\rho^a_B) \). Moreover, the restriction of \( h_\sigma \) to \( W^u(p, \rho^a_B) \) is a diffeomorphism onto \( W^u(h_\sigma(p), \rho^a_B) \).

Remark that \( W^u(q, \rho^a_B) \) is an (embedded) submanifold diffeomorphic to \( \mathbb{R}^l \) for \( q \in \text{Fix}_l(\rho^a_B) \) since \( \rho^a_B \) is Morse-Smale.

Proof. Take \( l = 0, \ldots, n \) and \( p \in \text{Fix}_l(\rho^a_B) \). Notice that \( W^u(p, \rho^a_B) \cap U_\sigma \) is an open subset of \( W^u(p, \rho^a_B) \). Thus, there exists a neighborhood \( V^u(\rho^a_B) \) such that \( (\rho^a_B)^{m(x, \sigma)}(V^u) \subset U_\sigma \). We have \( m(y, \sigma) \leq m(p, \sigma) \) for
any \( y \in V^u \). This implies that \( h_\sigma = (\rho^b)^{-m(p, \sigma)} \circ h \circ (\rho^b)^m(p, \sigma) \) on \( V^u \). In particular, the restriction of \( h_\sigma \) to \( V^u \) is a diffeomorphism onto \( h_\sigma(V^u) \). Since \( W^u(p, \rho^b) = \bigcup_{m \geq 0} (\rho^b)^m(V^u) \), \( h_\sigma \circ \rho^b = \rho^a \circ h_\sigma \), and \( h_\sigma \) is injective, the restriction of \( h_\sigma \) to \( W^u(p, \rho^b) \) is a diffeomorphism onto \( h_\sigma(W^u(p, \rho^b)) \).

For \( x \in W^u(p, \rho^b) \), we have

\[
(\rho^b)^{-m}(h_\sigma(x)) = h_\sigma \circ (\rho^b)^m(x) \xrightarrow{m \to \infty} h_\sigma(p).
\]

This implies that \( h_\sigma(W^u(p, \rho^b)) \) is a subset of \( W^u(h_\sigma(p), \rho^a) \). In particular, the dimension of \( W^u(h_\sigma(p), \rho^a) \) is at least \( l \). Since \( h_\sigma \) maps the finite set Fix\((\rho^b)\) to Fix\((\rho^a)\) bijectively and the sets Fix\((\rho^b)\) and Fix\((\rho^a)\) have the same cardinality for each \( j = 0, \ldots, n \), we obtain that \( h_\sigma \) maps Fix\((\rho^b)\) to Fix\((\rho^a)\) bijectively. The set \( h_\sigma(W^u(p, \rho^b)) \) is a \( \rho^a \)-invariant open subset of \( W^u(h_\sigma(p), \rho^a) \) which contains \( h_\sigma(p) \). It should coincide with \( W^u(h_\sigma(p), \rho^a) \), and hence, the restriction of \( h_\sigma \) to \( W^u(p, \rho^b) \) is a diffeomorphism onto \( W^u(h_\sigma(p), \rho^a) \).

\[\text{Lemma 2.7.} \ h_\sigma(p) \text{ does not depend on the choice of } \sigma \text{ for any } p \in \text{Fix}(\rho^b).
\]

\[\text{Proof.} \text{ Take } l = 0, \ldots, n \text{ and } p = (p_1, \ldots, p_n) \in \text{Fix}(\rho^b). \text{ Put } b_p = \prod_{p_i = \infty} b_i.
\]

Then, \( p \) is the unique element in Fix\((\rho^b)\) which is fixed by \( b_p \). By the identity \( \rho^b \circ h_\sigma = h_\sigma \circ b_p \), \( h_\sigma(p) \) is the unique element in Fix\((\rho^a) = h_\sigma(\text{Fix}(\rho^b)) \) which is fixed by \( \rho^a \).

\[\text{Lemma 2.8.} \ h_\sigma \text{ does not depend on the choice of } \sigma.
\]

\[\text{Proof.} \text{ Take } \sigma, \sigma' \in \{\pm 1\}^n \text{ and put } g = h_\sigma^{-1} \circ h_\sigma. \text{ It is sufficient to show that the restriction } g_p \text{ of } g \text{ to } W^u(p, \rho^b) \text{ is the identity map for each } p = (p_1, \ldots, p_j) \in \text{Fix}(\rho^b), \{0, \infty\}^n. \text{ By the above lemmas, } g_p(p) = p \text{ and the restriction of } g_p \text{ is a diffeomorphism of } W^u(p, \rho^b) \text{ which commutes with } \rho^b. \text{ Recall that } \rho^b(x) = kx \text{ and } W^u(p, \rho^b) \text{ is naturally identified with a vector space } \oplus_{i=0} \mathbb{R}. \text{ Under the identification, we have}
\]

\[
(Dg_p)_0 \cdot x = \lim_{m \to \infty} \frac{g_p(k^{-m}x)}{k^m} = \rho^b \circ g_p \circ (\rho^b)^{-1}(x) = g_p(x).
\]

In particular, the map \( g_p \) is an linear isomorphism. The linear map \( g_p \) commutes with \( \rho^b_j \) for any \( j = 1, \ldots, n \). This implies that \( g_p(\pi_p(v_j)) = \pi_p(v_j) \), where \( \pi_p : \mathbb{R}^n \to \oplus_{i=0} \mathbb{R} \) is the natural projection. Since \( (\pi_p(v_j))_{j=1}^n \) spans \( \oplus_{i=0} \mathbb{R} \), the map \( g_p \) is the identity map on \( W^u(p, \rho^b) \) for each \( p \in \text{Fix}(\rho^b) \).

Since \( I^n = \bigcup_{\sigma \in \{\pm 1\}} I_\sigma \) and \( h_\sigma = h \) on \( I_\sigma \), the above lemma implies that \( h_\sigma = h \) on \( I^n \). For any \( x \in \mathbb{T}^n \) and \( \sigma \in \{\pm 1\}^n \), the point \( (\rho^b)^{m(x, \sigma)}(x) \) is contained in \( I^n \). Take a neighborhood \( U_x \) of \( x \) such that \( (\rho^b)^{m(x, \sigma)}(U_x) \subset I^n \).

By Lemma 2.5, we have \( h_\sigma(y) = (\rho^b)^{m(x, \sigma)}h_\sigma \circ (\rho^b)^{m(x, \sigma)}(y) \) for any \( y \in \mathbb{T}^n \). This implies that \( h_\sigma(y) = (\rho^b)^{m(x, \sigma)}h_\sigma \circ (\rho^b)^{m(x, \sigma)}(y) \) for any \( y \in U_x \). Hence, \( h_\sigma \) is a local diffeomorphism. Since \( h_\sigma \) is injective, it is a diffeomorphism of \( h_\sigma \).

By Lemma 2.5, it is a smooth conjugacy between two actions \( \rho^b \) and \( \rho \).
2.2 Rigidity of local actions

Fix a basis $B = (v_1, \ldots, v_n)$ of $\mathbb{R}^n$ with $v_j = (v_j)_i^{n}$. Let $P_B$ be the local $\Gamma_{n,k}$-action defined in the previous section. In this subsection, we show the local version of Theorem 1.5.

Theorem 2.9. If a local action $P \in \text{Hom}(\Gamma_{n,k}, D)$ is sufficiently close to $P_B$ in $C^3_{\text{loc}}$-topology, then it is smoothly conjugate to $P_{B'}$ for some basis $B'$ of $\mathbb{R}^n$.

Combined with Proposition 2.2, the theorem implies Theorem 1.5.

The above theorem follows from the same argument as in [1]. Firstly, we prove the stability of the linear part of the local action. Secondly, we show exactness of a linear complex and see that existence of $B'$ follow from it.

For $w = (w_1, \ldots, w_n)_{i=1}^{n} \in \mathbb{R}^n$, we define a map $Q_w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$Q_w(x, y) = \sum_{j=1}^{n}(w_j x_j y_j)e_j$$

for $x = (x_i)_{i=1}^{n}$ and $y = (y_i)_{i=1}^{n}$, where $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$. Then, the local action $P_B$ satisfies that

$$P_B^a(x_1, \ldots, x_n) = k^{-1} \cdot x$$

$$P_B^b_j(x_1, \ldots, x_n) = x - Q_{b_j}(x, x) + O(||x||^3).$$

Let $I$ be the identity matrix of size $n$. We recall a lemma in [1] concerning stability of the linear part of $P^b_i$.

Lemma 2.10 ([1, Lemma 2.2]). Let $P_*$ be a local action in $\text{Hom}(\Gamma_{n,k}, D)$. Suppose that $D_0^{(1)}P_*^a = k^{-1}I$ and $D_0^{(1)}P_*^b = I$ for any $i = 1, \ldots, n$. Then, there exists a $C^3_{\text{loc}}$-neighborhood $U$ of $P_*$ in $\text{Hom}(\Gamma_{n,k}, D)$ such that $D_0^{(1)}P^b = I$ for any $P \in U$ and $i = 1, \ldots, n$.

Hence, $D_0^{(1)}P^b_j = I$ for any $j = 1, \ldots, n$ if $P$ is sufficiently $C^3_{\text{loc}}$-close to $P_B$, where $I$ is the identity map on $\mathbb{R}^n$. The following is essentially proved in Lemma 2.3 of [1].

Lemma 2.11. Let $P_*$ be a local action in $\text{Hom}(\Gamma_{n,k}, D)$ such that $D_0^{(1)}P_*^a = k^{-1}I$ and $D_0^{(1)}P_*^b = I$ for any $j = 1, \ldots, n$. Suppose that there exists $\delta > 0$ such that

$$\max_{j=1, \ldots, n} ||A \circ D_0^{(2)}P_*^b_j - 2D_0^{(2)}P_*^b_j \circ (A, I)||^{(2)} \geq \delta ||A||^{(1)},$$

for any linear map $A : \mathbb{R}^n \to \mathbb{R}^n$. Then, $P^a = k^{-1}I$ for any $P$ which is sufficiently $C^2_{\text{loc}}$-close to $P_B$.

Proof. Let $U$ be a $C^2_{\text{loc}}$-open neighborhood of $P_*$ consisting of $P \in U$ such that

$$3||D_0^{(2)}P^b_j - D_0^{(2)}P_*^b_j||^{(2)} + ||D_0^{(1)}P^a - k^{-1}I|| \cdot ||D_0^{(2)}P^b_j||^{(2)} < \delta/2$$
for any \( j = 1, \ldots, n \). Fix \( P \in \mathcal{U} \) and put
\[
A = D_0^{(1)} P^a - k^{-1} I, \\
B_j = D_0^{(2)} P^b_j - D_0^{(2)} P^b_j, \\
C_j = A \circ D_0^{(2)} P^b_j - 2D_0^{(2)} P^b_j \circ (A, I).
\]

We will show that \( A = 0 \). The identity \( P^a \circ P^b_j = P^b_j \circ P^a \) implies that
\[
(k^{-1} I + A) \circ (D_0^{(2)} P^b_j + B_j) = k \cdot (D_0^{(2)} P^b_j + B_j) \circ (k^{-1} I + A, k^{-1} I + A).
\]
Thus, we have that
\[
\|C_j\|^{(2)} = \|A \circ B_j - 2B_j \circ (A, I) - (D_0^{(2)} P^b_j + B_j) \circ (A, A)\|^{(2)} \\
\leq \|A\|^{(1)} \cdot \left( 3\|B_j\|^{(2)} + \|A\|^{(1)} \cdot \|D_0^{(2)} P^b_j\|^{(2)} \right) \\
\leq (\delta/2)\|A\|^{(1)}
\]
for any \( j = 1, \ldots, n \). By assumption, \( \max_{j=1, \ldots, n} \|C_j\|^{(2)} \geq \delta \|A\|^{(1)} \). Hence, we obtain that \( A = 0 \). \( \square \)

We apply the lemma for \( P_B \).

**Lemma 2.12.** The local action \( P_B \) satisfy the assumption of Lemma 2.11. In particular, \( P^a = k^{-1} I \) for any \( P \in \text{Hom}(\Gamma_{n,k}, D) \) which is sufficiently \( C^5_{\text{loc}} \)-close to \( P_B \).

**Proof.** Take a square matrix \( A = (a_{ij}) \) of size \( n \) and put
\[
C_j = A \circ D_0^{(2)} P^b_j - 2D_0^{(2)} P^b_j \circ (A, I) = -2 \left\{ A \circ Q_{v_j} - 2Q_{v_j}(A, I) \right\}.
\]
Then,
\[
C_j(e_i, e_i) = -2\{ A \circ Q_{v_j}(e_i, e_i) - 2Q_{v_j}(Ae_i, e_i) \} \\
= -2\{ A(-v_j e_i) - 2(-v_j a_i e_i) \} \\
= -2v_j \left\{ a_{ii} e_i - \sum_{k \neq i} a_{ki} e_k \right\}.
\]
This implies that \( \|C_j(e_i, e_i)\| = 2|v_j| \cdot \|(a_{ii})_{k=1}^n\| \), and hence,
\[
\max_{j=1, \ldots, n} \|C_j\|^{(2)} \geq 2 \max_{j=1, \ldots, n} |v_j| \cdot \|(a_{ii})_{k=1}^n\|.
\]
Since \( (v_1, \ldots, v_n) \) is a basis of \( \mathbb{R}^n \), there exists \( \delta > 0 \) such that \( \max_{j=1, \ldots, n} |v_j| \geq \delta \) for any \( i = 1, \ldots, n \). We also have \( \|A\|^{(1)} \leq n \max_{i=1, \ldots, n} \|(a_{ki})_{k=1}^n\| \). This implies that \( \max_{j=1, \ldots, n} \|C_j\|^{(2)} \geq (2\delta/n)\|A\|^{(1)} \). \( \square \)
Recall that \( S^{r,n} \) is the vector space of symmetric \( r \)-multilinear maps from \((\mathbb{R}^n)^r \) to \( \mathbb{R}^n \). Elements of \( S^{1,n} \) are just linear endomorphisms of \( \mathbb{R}^n \). For \( Q, Q' \in S^{2,n} \), we define \([Q, Q'] \in S^{1,n}\) by

\[
[Q, Q']((\xi_0, \xi_1, \xi_2)) = \sum_{k=0}^{2} Q(\xi_k, Q'(\xi_{k+1}, \xi_{k+2})) - Q'(\xi_k, Q(\xi_{k+1}, \xi_{k+2})),
\]

where we set \( \xi_3 = \xi_0 \) and \( \xi_4 = \xi_1 \). We also define linear maps \( L_B^0 : S^{1,n} \rightarrow (S^{2,n})^n \) and \( L_B^1 : (S^{2,n})^n \rightarrow (S^{3,n})^n(n-1)/2 \) by

\[
L_B^0(A', B') = (A' \circ Q_{v_i} - Q_{v_i} \circ (A', I) - Q_{v_i} \circ (I, A')) + Q_B e_i)_{i=1}^n,
\]

\[
L_B^1(q_1, \ldots, q_n) = ([q_i, Q_{v_i}] - [q_j, Q_{v_i}])_{1 \leq i \leq j \leq n}.
\]

It is not hard to check that \( \text{Im} L_B^0 \subseteq \text{Ker} L_B^1 \). We can obtain the next proposition by the exactly same argument as in p.1841–1844 of [1].

**Proposition 2.13.** If \( \text{Ker} L_B^1 = \text{Im} L_B^0 \), then Theorem 2.9 holds.

We will prove the proposition in the next subsection.

### 2.3 Proof of Proposition 2.13

Recall that \( I = (e_1, \ldots, e_n) \) is the standard basis of \( \mathbb{R}^n \). As shown in Lemma 2.11 of [1], it is enough to prove Proposition 2.13 for the case \( B = I \). Set \( L^0 = L_I^0 \) and \( L^1 = L_I^1 \).

For \( v, w \in \mathbb{R}^n \), let \( \langle v, w \rangle \) be the standard inner product of \( v \) and \( w \), i.e.,

\[
\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i.
\]

Let \( W \) be the subspace of \((S^{2,n})^n\) consisting of elements \( (q_i)_{i=1}^n \) such that \( q_i(e_i, e_i) = 0 \) and \( q_i(e_j, e_j), e_j = 0 \) for any \( i, j = 1, \ldots, n \).

The following formula on \( Q_i \) is useful for computation:

\[
Q_v(e_i, e_j) = Q_{e_i}(v, e_i) = \begin{cases} \langle v, e_i \rangle e_i & (i = j) \\ 0 & (i \neq j) \end{cases}
\]

for \( v \in \mathbb{R}^n \) and \( i, j = 1, \ldots, n \). In particular, \( Q_{e_i}(e_j, e_k) = e_i \) if \( i = j = k \) and \( Q_{e_i}(e_j, e_k) = 0 \) otherwise.

**Lemma 2.14.** \( W + \text{Im} L^0 = (S^{2,n})^n \).

**Proof.** Take \( (q_i)_{i=1}^n \in (S^{2,n})^n \). We put \( a_{ij} = -\langle q_i(e_i, e_j), e_i \rangle, a_{ii} = \langle q_i(e_i, e_i), e_i \rangle \),

and \( b_{ij} = 0, b_{ji} = \langle q_i(e_j, e_j), e_j \rangle \) for mutually distinct \( i, j = 1, \ldots, n \). Let \( A \) and \( B \) be square matrices of size \( n \) whose \((i, j)\)-entries are \( a_{ij} \) and \( b_{ij} \), respectively.

Then, \( L^0(A, B) = \langle q_i^{A,B} \rangle_{i=1}^n \) satisfies that

\[
q_i^{A,B}(e_i, e_i) = A \cdot Q_{e_i}(e_i, e_i) - 2Q_{e_i}(Ae_i, e_i) + Q_{Be_i}(e_i, e_i)
\]

\[
= Ae_i - 2a_{ii} e_i + b_{ii} e_i
\]

\[
= q_i(e_i, e_i),
\]

\[
q_i^{A,B}(e_j, e_j) = A \cdot Q_{e_i}(e_j, e_j) - 2Q_{e_i}(Ae_j, e_j) + Q_{Be_i}(e_j, e_j)
\]

\[
= b_{jj} e_j
\]

\[
= \langle q_i(e_j, e_k), e_j \rangle e_j.
\]
Hence, \( q_i - q_i^{A,B} \) is an element of \( W \). \( \square \)

**Lemma 2.15.** \( \text{Ker} L^1 \cap W = \{0\} \).

**Proof.** Take \((q_i)_{i=1}^n \in \text{Ker} L^1 \cap W \). Since \((q_i)_{i=1}^n \in W \), we have \( q_i(e_i, e_i) = 0 \) and \( (q_i(e_j, e_j), e_j) = 0 \) for any \( i, j = 1, \ldots, n \). If \( i \neq j \),

\[
[q_i, Q_{e_j}](e_j, e_j, e_j) = 3 \{ q_i(e_j, e_j) - Q_{e_j}(e_j, q_i(e_j, e_j)) \} = 3 \{ q_i(e_j, e_j) - (q_i(e_j, e_j), e_j)e_j \},
\]

\[
= 3q_i(e_j, e_j),
\]

\[
[q_j, Q_{e_i}](e_i, e_j, e_j) = 3 \{ q_j(e_j, e_j) - Q_{e_i}(e_i, q_j(e_j, e_j)) \} = 0.
\]

Since \([q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0\), we obtain that \( q_i(e_j, e_j) = 0 \).

If \( i \neq j \),

\[
[q_i, Q_{e_j}](e_i, e_j, e_j) = q_i(e_i, e_j) - 2Q_{e_j}(e_j, q_i(e_i, e_j)) = q_i(e_i, e_j) - 2(q_i(e_i, e_j), e_j)e_j
\]

\[
[q_j, Q_{e_i}](e_i, e_j, e_j) = -Q_{e_i}(e_i, q_j(e_j, e_j)) = -Q_{e_i}(e_i, 0) = 0.
\]

Since \([q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0\), we obtain that \( q_i(e_i, e_j) = 0 \).

For mutually distinct \( i, j, k = 1, \ldots, n \),

\[
[q_i, Q_{e_j}](e_j, e_k, e_k) = q_i(e_k, e_j) - 2Q_{e_j}(e_j, q_i(e_k, e_k)) = q_i(e_k, e_k) - 2(q_i(e_j, e_k), e_j)e_j
\]

\[
[q_j, Q_{e_i}](e_j, e_k, e_k) = 0.
\]

Since \([q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0\), we obtain that \( q_i(e_j, e_j) = 0 \). Now, we have \( q_i(e_j, e_k) = 0 \) for any \((q_i)_{i=1}^n \in \text{Ker} L^1 \cap W \) and any \( i, j, k = 1, \ldots, n \). \( \square \)

Now, we prove Proposition 2.13. Since \( \text{Im} L^0 \) is a subspace of \( \text{Ker} L^1 \), we have \( (S^{2,n})^n = W \oplus \text{Im} L^0 \) by the above lemmas. By \( \text{Im} L^0 \subset \text{Ker} L^1 \) and \( \text{Ker} L^1 \cap W = \{0\} \) again, we obtain that \( \text{Ker} L^1 = \text{Im} L^0 \).

## 3 Proof of Proposition 1.4

It is easy to see that any linear isomorphism \( g \in G \) of \( \mathbb{R}^n \) can be extended uniquely to a diffeomorphism \( h_g \) of \( T^n = (\mathbb{R} \cup \{\infty\})^n \) and the diffeomorphism \( h_g \) is a conjugacy between \( \rho_B^B \) and \( \rho_B^B \).

Suppose that \( \rho_B \) and \( \rho_B^B \) are smoothly conjugate by a diffeomorphism \( h \). We will show that \( h = h_g \) for some \( g \in G \). The conjugacy \( h \) preserves the unique repelling fixed point \((0, \ldots, 0)\) of \( \rho_B^B \) and \( \rho_B^B \), and their unstable manifold \( \mathbb{R}^n \subset T^n = (\mathbb{R} \cup \{\infty\})^n \). The restriction \( h_\mathbb{R} \) of \( h \) to \( \mathbb{R}^n \) commutes with the linear
map $x \mapsto kx$. As in the proof of Lemma 2.8, the map $h_R$ is linear. Take $(a_{ij})_{i,j=1}^n$ such that $h_R(e_j) = \sum_{i=1}^n a_{ij}e_i$.

We set

$$V_j = \{(x_1, \ldots, x_n) \in \mathbb{T}^n \mid x_j = \infty, x_i \neq \infty \text{ if } i \neq j\}$$

for $i = 1, \ldots, n$. Each $V_i$ is a submanifold of $V_j$ which is diffeomorphic to $\mathbb{R}^{n-1}$. Since $h$ is continuous, we have

$$h(V_j) \subset \bigcap_{a_{ij} \neq 0} V_i.$$ 

Since $h$ is a diffeomorphism of $\mathbb{T}^n$, there exists a unique $\sigma(j) \in \{1, \ldots, n\}$ such that $a_{ij} \neq 0$ for each $j = 1, \ldots, n$. Since the linear transformation $h|_R$ is invertible, $\sigma$ is a permutation of $\{1, \ldots, n\}$. Therefore, $h_R$ is an element of $G$.

References

