Codimension-one foliations with a transversely contracting flow

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Abstract: We show that if a $C^2$ codimension-one foliation on three-dimensional manifold admits a transversely contracting flow, then it must be the unstable foliation of an Anosov flow.

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1 Introduction

Let $M$ be a three-dimensional closed manifold and $\mathcal{F}$ be a codimension-one foliation on $M$. We call a flow $\Phi = \{\Phi^t\}_{t \in \mathbb{R}}$ without stationary points $\mathcal{F}$-transversely contracting if it preserves each leaf of $\mathcal{F}$ and satisfies

$$\lim_{t \to +\infty} \left\| N\Phi^t_{\mathcal{F}}(v) \right\|_{\mathcal{F}} = 0$$

for any $v \in TM/T\mathcal{F}$, where $\left\| \cdot \right\|_{\mathcal{F}}$ is a norm on the normal bundle $TM/T\mathcal{F}$ of the foliation $\mathcal{F}$ and $N\Phi^t_{\mathcal{F}} = \{N\Phi^t_{\mathcal{F}}\}$ is the flow on $TM/T\mathcal{F}$ induced from $\Phi$. Of course, the unstable foliation of an Anosov flow is a foliation with a transversely contracting flow.

Transversely contracting flows appear in the theory of linear deformation of foliations. We say a family $\{\alpha_t\}_{t \in [-1,1]}$ of 1-forms on $M$ is a linear deformation of a foliation $\mathcal{F}$ into contact structures if $\text{Ker} \alpha_0 = T\mathcal{F}$ and $(d/dt)(\alpha_t \wedge d\alpha_t) > 0$. Mitsumatsu [5] observed that if a foliation $\mathcal{F}$ admits such a deformation $\{\alpha_t = \alpha(t, \cdot)\}_{t \in [-1,1]}$ and the intersection of the kernels of $\alpha_0$ and $d\alpha_t/dt|_{t=0}$ defines an $\mathcal{F}$-transversely contracting flow, then $d\alpha$ is a convex symplectic structure on $M \times I$. In [6], he asked whether such a foliation must be the unstable foliation of an Anosov flow (see also Subsection 3.2 of [7]).

In this paper, we show the following result, which gives an affirmative answer to his question in the case of $C^2$ foliations.

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Theorem 1.1. If a $C^2$ foliation $\mathcal{F}$ on a three-dimensional closed manifold admits an $\mathcal{F}$-transversely contracting flow, then it must be the unstable foliation of an Anosov flow.

We say an Anosov flow is algebraic if it is given by the natural action of a one-parameter subgroup of a Lie group $G$ on $G/\Gamma$, where $\Gamma$ is a lattice of $G$. It is known that any three-dimensional algebraic Anosov flow is smoothly conjugate to either the geodesic flow on a closed surface of constant negative curvature or the suspension flow of a hyperbolic toral automorphism, up to finite covering. The former corresponds the case $G = \text{SL}(2, \mathbb{R})$ and the latter corresponds the case that $G$ is a semi-direct product $\mathbb{R} \ltimes \mathbb{R}^2$ associated with an action $t \cdot (x, y) = (e^t x, e^{-t} y)$. By results of Barbot [2, Théorème 5.1] and Ghys [3, Théorème 4.1, 4.7], if the unstable foliation of an Anosov flow is of class $C^\infty$, then it is diffeomorphic to the unstable foliation of an algebraic Anosov flow. In particular, if the above foliation $\mathcal{F}$ is of class $C^\infty$, then it is diffeomorphic to the unstable foliation of an algebraic Anosov flow.

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2 Preliminaries

In this section, we review some basic results on three-dimensional flows with invariant splittings.

We fix a three-dimensional closed manifold $M$ and a $C^r$ flow $\Phi = \{\Phi^t\}_{t \in \mathbb{R}}$ on $M$ with $r \geq 1$. Suppose $\Phi$ has no stationary point. Let $TM$ denote the tangent bundle of $M$ and $D\Phi = \{D\Phi^t\}_{t \in \mathbb{R}}$ the flow on $TM$ defined by the differential of $\Phi$. Let $T\Phi$ be the one-dimensional subbundle of $TM$ tangent to the flow $\Phi$. We fix a norm $\|\cdot\|$ on $TM$.

For a subset $S$ of $M$, we write $\overline{S}$ for the closure of $S$. For $z \in M$, let $O(z)$ denote the orbit $\{\Phi^t(z) \mid t \in \mathbb{R}\}$ and $\omega(z)$ the $\omega$-limit set $\bigcap_{T > 0} \{\Phi^t(z) \mid t > T\}$.

2.1 Dominated splittings

Fix a compact subset $M_0$ of $M$ satisfying $\Phi^t(M_0) \subset M_0$ for any $t \geq 0$.

Lemma 2.1. If a positive-valued continuous function $\alpha$ on $M_0 \times \{t \geq 0\}$ satisfies

$$\alpha(z, s + t) \leq \alpha(\Phi^t(z), t) \cdot \alpha(z, s)$$

for any $z \in M_0$ and $s, t \geq 0$, and $\inf_{t \geq 0} \alpha(z, t) < 1$ for any $z \in M_0$, then there exist $C > 0$ and $\lambda \in (0, 1)$ such that $\alpha(z, t) \leq C \lambda^t$ for any $z \in M$ and $t \geq 0$. 

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Proof. By the compactness of $M_0$, there exist a constant $\lambda_0 \in (0,1)$, a finite open covering $\{U_i\}_{i=1}^{s_*}$ of $M_0$, and a sequence $\{T_i > 0\}_{i=1}^{s_*}$ such that $\alpha(z, T_i) < \lambda_0$ for any $i = 1, \ldots, s_*$ and $z \in U_i$. Put $T_* = \max\{T_i \mid i = 1, \ldots, s_*\}$ and $A_* = \sup\{\alpha(z, t) \mid t \in [0, T_*], z \in M_0\}$. For any $z \in M_0$ and $t > 0$, there exist sequences $\{t_m \geq 0\}_{m=0}^{m_*}$ and $\{i_m\}_{m=0}^{m_*}$ such that $t_0 = 0$, $t_{m_*} \leq t \leq t_{m_*+1}$, $\Phi^{t_m}(z) \in U_{i_m}$ and $t_{m+1} = t_m + T_m$ for any $m$. Since $t \leq m_* T_*$ and $0 \leq t - t_{m_*} \leq T_*$, we have

$$\alpha(z, t) \leq A_* \lambda_0^{m_*-1} \leq A_* \lambda_0^{(t/T_*)-1}. \quad \square$$

Let $TM|_{M_0}$ denote the restriction of $TM$ on $M_0$. We say a subbundle $E$ of $TM|_{M_0}$ is $\Phi$-invariant if $D\Phi(E(z)) = E(\Phi(z))$ for any $z \in M_0$ and $t \geq 0$. For a $\Phi$-invariant subbundle $E$ of $TM|_{M_0}$, the flow $D\Phi$ induces a semi-flow $N\Phi E$ on $(TM|_{M_0})/E$ and the norm $\|\|\|$ induces a norm $\|\|_{\Phi}$ on $(TM|_{M_0})/E$. Notice that $\Phi$ is a $\Phi$-invariant subbundle of $TM$. We simply write $N\Phi$ for the flow $N\Phi|_{TM}$ and $\|\|_\Phi$ for the norm $\|\|_{\Phi}$. Define two functions $\mu_E$ and $\mu_{\Phi}^2$ on $M_0 \times \{t \geq 0\}$ by

$$\mu_E(z, t) = \sup\{\|N\Phi^t(v)\|_\Phi \mid v \in E/T\Phi(z), \|v\|_\Phi \leq 1\},$$

$$\mu_{\Phi}^2(z, t) = \sup\{\|N\Phi^t(v)\|_{\Phi}^2 \mid v \in (TM|_{M_0})/E(z), \|v\|_E \leq 1\}.$$  

Remark that $\mu_E$ and $\mu_{\Phi}^2$ satisfy the inequality (1).

A decomposition $TM|_{M_0} = E^- + E^+$ of $TM|_{M_0}$ is called a (non-trivial) dominated splitting for $\Phi$ if $(E^-, E^+)$ is a pair of $\Phi$-invariant two-dimensional subbundles with $E^- \cap E^+ = T\Phi|_{M_0}$ and there exist two constants $C > 0$ and $\lambda \in (0,1)$ such that

$$\mu_{E^-}(z, t) \cdot \mu_{E^+}(z, t)^{-1} < C\lambda^t$$

for any $z \in M_0$ and any $t > 0$. Remark that the definition does not depend on the choice of the norm on $TM$.

Lemma 2.2. A dominated splitting $TM|_{M_0} = E^- + E^+$ satisfies the followings:

1. $E^-$ is uniquely determined and is continuous.

2. If $\inf_{t \geq 0} \mu_{E^-}(z, t) < 1$ for any $z \in M$, then $E^-$ is a $C^1$ subbundle of $TM|_{M_0}$.

Proof. The proof is the same as the case of a hyperbolic splitting. See e.g. [4]. For the second assertion, we remark that $E^-$ is codimension one and $N\Phi^t$ is a uniformly exponential contraction on $E^-/T\Phi$ by Lemma 2.1. \square

Proposition 2.3. The followings are equivalent for a continuous $\Phi$-invariant two-dimensional subbundle $E$ of $TM|_{M_0}$:

1. There exists a two-dimensional subbundle $E^*$ of $TM|_{M_0}$ such that $TM|_{M_0} = E^* + E$ is a dominated splitting.
2. $\inf_{t>0}(\mu_E^+(z,t) \cdot \mu_E(z,t)^{-1}) < 1$ for any $z \in M_0$.

Proof. It is trivial that the former implies the latter. Suppose the latter holds. The proof is by a standard argument using the contracting mapping principle. By Lemma 2.1, there exist $C > 0$ and $\lambda \in (0,1)$ such that $\mu_E^+(z,t) \cdot \mu_E(z,t)^{-1} < C \lambda^t$ for any $z \in M_0$ and $t \geq 0$. Put $E_+ = E/T\Phi$. Let $E_-$ be the orthogonal complement of $E_+$ in $(TM/T\Phi)|_{M_0}$ and fix an orthonormal framing $(v_-, v_+)$ associated to the splitting $(TM/T\Phi)|_{M_0}$. For each $z \in M_0$, let $\left( \begin{array}{cc} a(z,t) & 0 \\ b(z,t) & c(z,t) \end{array} \right)$ be the matrix representation of $N\Phi^t$ with respect to the framing $(v_-, v_+)$. Let $\Gamma(M_0)$ be the Banach space of bounded functions on $M_0$ with a norm $\| \alpha \| = \sup \{|\alpha(z)| | z \in M_0\}$. We define a semi-flow $\Gamma_\Phi = \{\Gamma^t_\Phi\}$ on $\Gamma(M_0)$ by

$$\Gamma^t_\Phi(\alpha)(z) = c(z,t)^{-1}\{a(\Phi^t(\alpha))a(z,t) - b(z,t)\}.$$ 

Since $|a(z,t)| = \mu_E^+(z,t)$ and $|c(z,t)| = \mu_E(z,t)$, we obtain that $||\Gamma^t_\Phi(\alpha) - \Gamma^s_\Phi(\alpha')||_F \leq C\lambda^{t-s}\|\alpha - \alpha'\|_F$ for any $\alpha, \alpha' \in \Gamma(M_0)$ and $t \geq s$. By the contracting mapping principle, there exists a unique fixed point $\alpha_\Phi$ of $\Gamma_\Phi$. Let $E^s$ be the two-dimensional subbundle of $TM$ such that $T\Phi \subset E^s$ and $E^s/T\Phi(z)$ is generated by $v_-(z) + \alpha_\Phi(z)v_+(z)$ for any $z \in M_0$. It is easy to check that $TM|_{M_0} = E^s + E$ is a dominated splitting. \hfill $\Box$

We say a compact subset $\Lambda$ of $M$ is $\Phi$-invariant if $\Phi^t(\Lambda) = \Lambda$ for any $t \in \mathbb{R}$. By Lemma 2.2, if $\Lambda$ admits a dominated splitting, then it is unique and continuous.

A $\Phi$-invariant torus $T_0$ is called normally attracting if it admits a dominated splitting $TM|_{T_0} = E^- + TT_0$ such that $\lim_{t \to -\infty} \mu_{E^-}(z) = 0$ for any $z \in T_0$. A normally repelling torus is a normally attracting torus for $\Phi^{-1}$, where $\Phi^{-1}$ is the time-reverse of $\Phi$. We call a $\Phi$-invariant torus $T$ irrational if the restriction of $\Phi$ on $T$ is topologically conjugate to an irrational linear flow.

Lemma 2.4. Any $\Phi$-invariant compact set with a dominated splitting contains at most finitely many irrational tori and they are normally attracting or repelling.

Proof. It is a consequence of Proposition 3.9 of [1]. \hfill $\Box$

Remark that the uniqueness of a dominated splitting implies that any normally attracting irrational torus in $\Lambda$ is tangent to $E^+$. Later, we use the following structure theorem of invariant sets with a dominated splitting due to Arroyo and Rodriguez Hertz.

Proposition 2.5 (Theorem 3.8 of [1]). Suppose that $\Phi$ is of class $C^2$ and $\Lambda$ is a $\Phi$-invariant compact set with a dominated splitting. If all periodic orbits in $\Lambda$ are of saddle-type and $\Lambda$ contains no irrational tori, then $\Lambda$ is a hyperbolic invariant set of saddle-type.
2.2 Projectively Anosov flows

We say $\Phi$ is a projectively Anosov flow (or simply a $\mathbb{P}A$ flow) if it admits a dominated splitting $TM = E^s + E^u$ on the whole manifold $M$. We call the splitting a $\mathbb{P}A$ splitting. A flow is called non-degenerate if all periodic orbits are hyperbolic.

Let $\Omega(\Phi)$ denote the non-wandering set of $\Phi$. The following is an immediate consequence of Lemma 2.4 and Proposition 2.5.

**Proposition 2.6.** Suppose that $\Phi$ is a $C^2$ non-degenerate $\mathbb{P}A$ flow. Then, there exists a decomposition $\Omega(\Phi) = \Omega_0 \cup \Omega_1 \cup \Omega_2$ of $\Omega(\Phi)$ into $\Phi$-invariant compact sets such that

1. $\Omega_1$ is a hyperbolic set of saddle-type,
2. $\Omega_0$ is the union of finitely many attracting periodic orbits and finitely many $\Phi$-invariant normally attracting irrational tori, and
3. $\Omega_2$ is the union of finitely many repelling periodic orbits and finitely many $\Phi$-invariant normally repelling irrational tori.

We define the stable set $W^{ss}(z)$ and the weak stable set $W^s(z)$ of $z \in M$ by

$$W^{ss}(z) = \left\{ z' \in M \mid \lim_{t \to \infty} d(\Phi^t(z), \Phi^t(z')) = 0 \right\},$$

$$W^s(z) = \bigcup_{t \in \mathbb{R}} W^{ss}(\Phi^t(z)),$$

where $d(\cdot, \cdot)$ is the distance induced from a norm $\| \cdot \|$ on $TM$. We also define the unstable set $W^{uu}(z)$ and the weak unstable set $W^u(z)$ of $z \in M$ by $W^{uu}(z) = W^{ss}(z; \Phi^{-1})$ and $W^u(z) = W^s(z; \Phi^{-1})$. By the stable manifold theorem, for a point $z$ in a hyperbolic set of saddle type, $W^s(z)$ is diffeomorphic to an open annulus if it contains a periodic orbit, and is diffeomorphic to the plane otherwise.

For a $\Phi$-invariant compact subset $\Lambda$ of $M$, we also define the stable set $W^s(\Lambda)$ and the unstable set $W^u(\Lambda)$ by

$$W^s(\Lambda) = \left\{ z' \in M \mid \lim_{t \to \infty} \inf_{z \in \Lambda} d(\Phi^t(z'), z) = 0 \right\},$$

and $W^u(\Lambda) = W^s(\Lambda; \Phi^{-1})$. It is known that $W^s(\Lambda) = \bigcup_{z \in \Lambda} W^s(z)$ and $W^u(\Lambda) = \bigcup_{z \in \Lambda} W^u(z)$ if $\Lambda$ is a hyperbolic set. In particular, we have

$$M = \Omega_2 \cup W^s(\Omega_0) \cup \left( \bigcup_{z \in \Omega_1} W^s(z) \right) = \Omega_0 \cup W^u(\Omega_2) \cup \left( \bigcup_{z \in \Omega_1} W^u(z) \right). \quad (2)$$

For a foliation $\mathcal{F}$ on $M$, let $\mathcal{F}(z)$ denote the leaf of $\mathcal{F}$ through $z \in M$. 

Lemma 2.7. Suppose that $\Phi$ is a $C^2$ non-degenerate $\mathbb{P}A$ flow with a $\mathbb{P}A$ splitting $TM = E^s + E^u$. Then, $E^s$ defines a $C^1$ foliation $\mathcal{F}^s$ on $M \setminus \Omega_2$ and $W^s(z)$ is a connected component of $\mathcal{F}^s(z) \setminus \Omega_2$ for any $z \in \Omega_1$.

Similarly, $E^u$ defines a $C^1$ foliation $\mathcal{F}^u$ on $M \setminus \Omega_0$ and $W^u(z)$ is a connected component of $\mathcal{F}^u(z) \setminus \Omega_0$ for any $z \in \Omega_1$.

Proof. Take an open neighborhood $U$ of $\Omega_2$ so that $\Phi^{-t}(U) \subset U$ for any $t > 0$ and $\bigcap_{t>0} \Phi^{-t}(U) = \Omega_2$. Since $\omega(z, \Phi) \subset \Omega_0 \cup \Omega_1$, we have $\lim_{t \to -\infty} \mu_{E^s}(z, t) = 0$ for any $z \in M \setminus U$. By Lemma 2.2(2) we obtain that $E^s$ is of class $C^1$ on $M \setminus U$. The invariance of $E^s$ implies that $E^s$ is of class $C^1$ on $M \setminus \Omega_2$.

To show the latter assertion, we claim that $W^s(z)$ is an open subset of $\mathcal{F}^s(z) \setminus \Omega_2$ for any $z \in \Omega_1$. Fix $z \in \Omega_1$. By the local stable manifold theorem, there exists a two-dimensional injectively immersed submanifold $V$ such that $\Phi^t(V) \subset V$ for any $t > 0$, $\bigcap_{t>0} \Phi^{-t}(V) = \mathcal{O}(z)$, $\bigcup_{t>0} \Phi^{-t}(V) = W^s(z)$, and $V$ is uniformly transverse to $E^s$. It is easy to verify that the domination property of the splitting $TM = E^s + E^u$ implies that $V$ must be tangent to $E^s$. Therefore, $W^s(z)$ also is tangent to $E^s$. Since $W^s(z)$ and $\mathcal{F}^s(z)$ are two-dimensional, it implies the claim.

Since $W^s(\Omega_0)$ is an open subset of $M$ and either $W^s(z) = W^s(z')$ or $W^s(z) \cap W^s(z') = \emptyset$ for any $z, z' \in M$, the claim and the equation (2) imply that $W^s(z)$ is a connected component of $\mathcal{F}^s(z) \setminus \Omega_2$ for any $z \in \Omega_1$.

We can obtain the assertion for $E^u$ by replacing $\Phi$ with $\Phi^{-1}$. □

3 Flows with invariant foliations

Let $M$ be a three-dimensional closed manifold and $\mathcal{F}$ be a codimension-one foliation on $M$. Let $T\mathcal{F}$ denote the tangent bundle of the foliation $\mathcal{F}$. For $r \geq 1$, let $\mathcal{X}^r(M)$ be the space of $C^r$-flows on $M$ with the $C^r$-topology and $\mathcal{X}^r(\mathcal{F})$ the subspace of $\mathcal{X}^r(M)$ consisting of $C^r$-flows that preserve each leaf of $\mathcal{F}$. Remark that $\mathcal{X}^r(\mathcal{F})$ is a locally path-connected space.

Recall that a flow $\Phi \in \mathcal{X}^r(\mathcal{F})$ is $\mathcal{F}$-transversely contracting if $\lim_{t \to -\infty} \mu^\perp_{T\mathcal{F}}(z, t) = 0$ for any $z \in M$. Let $\mathcal{X}_{tc}^r(\mathcal{F})$ be the subset of $\mathcal{X}^r(\mathcal{F})$ consisting of $\mathcal{F}$-transversely contracting flows and $\mathbb{P}A_{tc}^r(\mathcal{F})$ the subset of $\mathcal{X}_{tc}^r(\mathcal{F})$ consisting of $\mathcal{F}$-transversely contracting $\mathbb{P}A$ flows. Remark that $\mathcal{X}_{tc}^r(\mathcal{F})$ and $\mathbb{P}A_{tc}^r(\mathcal{F})$ are open subsets of $\mathcal{X}^r(\mathcal{F})$ and any flow in $\mathbb{P}A_{tc}^r(\mathcal{F})$ admits a $\mathbb{P}A$-splitting $TM = E^s + E^u$ with $E^u = T\mathcal{F}$.

For a subset $\mathcal{S}$ of $\mathcal{X}^r(\mathcal{F})$, we say that two flows $\Phi$ and $\Phi'$ in $\mathcal{S}$ are $\mathcal{S}$-homotopic if they can be connected by a continuous path in $\mathcal{S}$. By the same argument as the proof of the Kupka-Smale theorem, we can show that non-degenerate flows are generic in $\mathcal{X}^r(\mathcal{F})$. Since $\mathcal{X}_{tc}^r(\mathcal{F})$ (resp. $\mathbb{P}A_{tc}^r(\mathcal{F})$) is an open subset of a locally path-connected space $\mathcal{X}^r(\mathcal{F})$, any $\mathcal{X}_{tc}^r(\mathcal{F})$ (resp. $\mathbb{P}A_{tc}^r(\mathcal{F})$)-homotopy class contains a non-degenerate flow.
3.1 Deformation to a $\mathbb{PA}$ flow

Let $M$ be a three-dimensional closed manifold. In this subsection, we show that any transversely contracting flow can be deformed into a $\mathbb{PA}$ flow. More precisely, we prove the following proposition.

Proposition 3.1. Suppose that $\mathcal{F}$ is a $C^r$ foliation $M$ with $r \geq 2$. Then, any $\mathcal{X}_c(\mathcal{F})$-homotopy class contains a $\mathbb{PA}$ flow.

For flows $\Phi_1$, $\Phi_2$ on a manifold $M$ and a subset $U$ of $M$, we write $\Phi_1|_U = \Phi_2|_U$ if $\Phi_1^t(z) = \Phi_2^t(z)$ for any $t \geq 0$ and $z \in \bigcap_{t \in [0,1]} \Phi_1^{-t}(U)$.

Fix a flow $\Phi$ in $\mathcal{X}_c(\mathcal{F})$. To simplify notations, put $\mu = \mu_{TF}$ and $\mu^\perp = \mu_{TF}$.

Lemma 3.2. For any attracting periodic orbit $\mathcal{O}(z_\ast)$ of $\Phi$ and any neighborhood $U$ of $\mathcal{O}(z_\ast)$, there exists a flow $\Phi_1$ which is $\mathcal{X}_c(\mathcal{F})$-homotopic to $\Phi$ and such that $\Phi_1|_{M \setminus U} = \Phi|_{M \setminus U}$, $W^s(z_\ast; \Phi)$ is $\Phi_1$-invariant, and

$$\lim_{t \to \infty} \mu^\perp(z, t; \Phi_1) \cdot \mu(z, t; \Phi_1)^{-1} = 0$$

for any $z \in W^s(z_\ast; \Phi)$.

Proof. Take a $C^r$ embedding $\psi : [-1,1]^2 \to M$ so that $\psi(0,0) = z_\ast$, $\text{Im} \psi$ is transverse to $\Phi$, and $\psi([-1,1] \times y) \subset \mathcal{F}(\psi(0,y))$ for any $y \in [-1,1]$. There exists $\delta \in (0,1)$ such that a function $\tau(x,y) = \inf \{ t > 0 \mid \Phi^t(\psi(x,y)) \in \text{Im} \psi \}$ is well-defined and of class $C^r$ on $[-\delta, \delta]^2$. We can take $C^r$ functions $f$ on $[-\delta, \delta]^2$ and $g$ on $[-\delta, \delta]$ such that $\Phi^\tau(x,y)(\psi(x,y)) = \psi(f(x,y), g(y))$ for any $(x,y) \in [-\delta, \delta]^2$. Put $U' = \{ \Phi^\tau(\psi(x,y)) \mid (x,y) \in [-\delta, \delta]^2 \}$, $t_\ast = \tau(0,0)$. Remark that $U'$ is a neighborhood of $\mathcal{O}(z_\ast)$, $t_\ast$ is the period of $z_\ast$, $f(0,0) = g(0) = 0$, $|\partial f/\partial x(0,0)| = \mu(z_\ast, t_\ast) < 1$, and $|\partial g/\partial y(0)| = \mu^\perp(z_\ast, t_\ast) < 1$. By replacing $\delta > 0$ with a smaller one, we may assume that $U' \subset U \cap W^s(z_\ast)$ and there exists $\lambda \in (0,1)$ such that $|\partial f/\partial x(x,y)| < \lambda$ and $|\partial g/\partial y(y)| < \lambda$ for any $(x,y) \in [-\delta, \delta]^2$.

Take a function $f_1$ on $[-\delta, \delta]^2$ so that $f_1 = f$ on $[-\delta, \delta]^2 \setminus [-\delta/2, \delta/2]^2$, $|\partial f_1/\partial x(0,0)| > |\partial f/\partial x(0,0)|$, and $|\partial f_1/\partial x(x,y)| < \lambda$ for any $(x,y) \in [-\delta, \delta]^2$. Put $F_\alpha(x,y) = ((1 - \alpha)f(x,y) + \alpha f_1(x,y), g(y))$. Then, we have $F_\alpha^n(x,y) < \lambda^n \lambda^n$ for any $n \geq 0$ and $(x,y) \in [-\delta, \delta]^2$. In particular, $\lim_{n \to \infty} F_\alpha^n(x,y) = (0,0)$. We can take a $\mathcal{X}(\mathcal{F})$-homotopy $\{ \Phi_\alpha \}_{\alpha \in [0,1]}$ such that $\Phi_0 = \Phi$, $\Phi_0|_{M \setminus U} = \Phi|_{M \setminus U}$ and $\Phi_\alpha(x,y)(\psi(x,y)) = \psi \circ F_\alpha(x,y)$ for any $\alpha \in [0,1]$. Then, $z_\ast$ is an attracting periodic orbit of $\Phi_\alpha$ with period $t_\ast$, such that $W^s(z_\ast; \Phi_\alpha) = W^s(z_\ast; \Phi)$ and $\mu^\perp(z_\ast, t_\ast; \Phi_\alpha) = |\partial g/\partial y(0)| < 1$ for any $\alpha \in [0,1]$. Since $\Phi_0|_{M \setminus U} = \Phi|_{M \setminus U}$ and $\Phi$ is $\mathcal{F}$-transversely contracting, it implies that $\lim_{\alpha \to 0} \mu^\perp(z, t; \Phi_\alpha) = 0$ for any $z \in M$ and $\alpha \in [0,1]$. Therefore, $\{ \Phi_\alpha \}_{\alpha \in [0,1]}$ is a $\mathcal{X}_c(\mathcal{F})$-homotopy. We also see

$$\mu(z_\ast, t_\ast; \Phi_1) = |\partial f_1/\partial x(0,0)| > |\partial g/\partial y(0)| = \mu^\perp(z_\ast, t_\ast; \Phi_1).$$

It implies the equation (3). 

\[ \square \]
For a flow $\Psi$ on $M$, let $\text{Per}_0(\Psi)$ denote the union of all attracting periodic orbits of $\Psi$.

**Lemma 3.3.** $M \setminus W^s(\text{Per}_0(\Phi))$ admits a dominated splitting.

**Proof.** We use some terminology and results in the smooth ergodic theory. Refer to the supplement of [4] for example.

Take $\lambda > 0$ so that
$$\limsup_{t \to +\infty} \frac{1}{t} \log \mu^+(z, t) \leq -2\lambda$$
for any $z \in M$. Let $U_*$ be the set of points $z \in M$ satisfying
$$\liminf_{t \to \infty} \frac{1}{t} \log \mu(z, t) < -\lambda.$$

We will show that $U_* \subset W^s(\text{Per}_0(\Phi))$. By Proposition 2.3, it completes the proof.

Fix $z_* \in U_*$. Let $\{m_t\}_{t \geq 0}$ be a family of Borel probability measures on $M$ satisfying
$$\int_M f \, dm_t = \frac{1}{t} \int_0^t f \circ \Phi^t(z_*) \, dt$$
for any continuous function $f$ on $M$. Choose a sequence $(t_i)_{i \geq 0}$ so that $\lim_{i \to \infty} t_i = \infty$ and $\frac{1}{t_i} \log \mu(z_*, t_i) \leq -\lambda$ for any $i$. Take the weak$^*$-limit $m_*$ of a subsequence of $\{m_{t_i}\}_{i \geq 0}$. Put $f_0(z) = \frac{1}{m} \log \mu(z, t) |_{t=0}$. Then, we have

$$\int_M f_0 \, dm_* \leq \limsup_{i \to \infty} \int_M f_0 \, dm_{t_i} = \limsup_{i \to \infty} \frac{1}{t_i} \log \mu(z_*, t_i) \leq -\lambda.$$ 

By the ergodic decomposition theorem and the Birkhoff ergodic theorem, there exists a $\Phi$-invariant ergodic probability measure $m_e$ satisfying $\text{supp}(m_e) \subset \text{supp}(m_*)$ and

$$\lim_{t \to \infty} \frac{1}{t} \log \mu(z, t) = \int_M f_0 \, dm_e \leq \int_M f_0 \, dm_* \leq -\lambda < 0$$

for $m_*$-almost every $z \in M$, where $\text{supp}(m)$ is the support of a measure $m$. It implies at least one Lyapunov exponent of $m_e$ is negative. If all Lyapunov exponents of $m_e$ are negative, then $\text{supp}(m_e)$ is an attracting periodic orbit by Pesin theory. In this case, we have $z_* \in W^s(\text{Per}_0(\Phi))$ since $\text{supp}(m_e) \cap \omega(z_0, \Phi) \neq \emptyset$.

Assume that one Lyapunov exponent is non-negative. Let $\lambda_- < \lambda_+$ be the pair of Lyapunov exponents and $TM/T\Phi = \mathcal{E}_- \oplus \mathcal{E}_+$ be the Oseledets decomposition associated with $m_e$. Then, we have

$$\lim_{t \to \infty} \frac{1}{t} \log ||N\Phi^t(v)||_\Phi = \lambda_+ \geq 0.$$
for $m_\epsilon$-almost every $z \in M$ and any $v \in (TM/T\Phi)(z) \setminus E_-(z)$. The inequality (4) implies that $E_- = T\mathcal{F}/T\Phi$. Moreover, the Oseledec decomposition theorem also implies
\[
\lim_{t \to +\infty} \frac{1}{t} \log \sin \angle(E_-(\Phi^t(z)), E_+(\Phi^t(z))) = 0
\]
for $m_\epsilon$-almost every $z \in M$, where $\angle(E, E')$ denote the angle of two subspaces $E$ and $E'$ of $TM/T\Phi(z')$ for $z' \in M$.

Let $\pi_{E_-}^\perp$ be the orthogonal projection from $TM/T\Phi$ to the orthogonal complement of $E_-=T\mathcal{F}/T\Phi$. Take a unit vector $v_+ \in E_+(z)$. Since $\mu_{\pi_{E_-}^\perp}^\perp(z, t; \Phi_1) = k_{\pi_{E_-}^\perp}^\perp(v_+) / k_{E_-}^\perp(v_+) k_{\Phi}$, we have
\[
\limsup_{t \to +\infty} \frac{1}{t} \log \sin \angle(E_-(\Phi^t(z)), E_+(\Phi^t(z)))
= \limsup_{t \to +\infty} \frac{1}{t} \log \left( \mu^\perp(z, t) \cdot \|N\Phi_1(v_+)\|_\Phi^{-1} \cdot \|\pi_{E_-}^\perp(v_+)\|_\Phi \right)
\leq -2\lambda - \lambda_+ < 0.
\]
It contradicts the equation (5). 

**Proof of Proposition 3.1.** Let $\Phi$ be a flow in $\mathcal{X}_{\epsilon_0}^r(\mathcal{F})$. By a remark at the beginning of this section, we may assume that $\Phi$ is non-degenerate. By Lemma 3.3, $M \setminus W^s(\text{Per}_0(\Phi))$ admits a dominated splitting. Let $\Omega_0$ be the union of all normally attracting irrational tori. Since $\Phi$ is $\mathcal{F}$-transversely contracting, $\Phi$ has neither repelling periodic orbit nor normally repelling irrational tori. Hence, Proposition 2.5 implies that $\Omega_1 = \Omega(\Phi) \setminus (\text{Per}_0(\Phi) \cup \Omega_0)$ is a hyperbolic invariant set of saddle type. Since $\text{Per}_0(\Phi) \subset \text{Per}_0(\Phi)$ is a subset of $\Omega_1$, it is a hyperbolic invariant set. It implies that this set must be empty. In particular, $\text{Per}_0(\Phi)$ is the union of finitely many orbits.

Fix a neighborhood $U$ of $\text{Per}_0(\Phi)$ so that $U \subset W^s(\text{Per}_0(\Phi))$. By Lemma 3.2, we can take a flow $\Phi_1$ which is $\mathcal{X}_{\epsilon_0}^r(\mathcal{F})$-homotopic to $\Phi$ and satisfies $\Phi_1|_{M \setminus U} = \Phi|_{M \setminus U}$, $W^s(\text{Per}_0(\Phi); \Phi) = \Phi_1$-invariant, and $\lim_{t \to -\infty} \mu^+(z, t; \Phi_1) = 0$ for any $z \in W^s(\text{Per}_0(\Phi); \Phi)$. Then, Proposition 2.3 implies that $\Phi_1$ is a $\mathcal{P}$A flow.

**3.2 Deformation to an Anosov flow**

The aim of this subsection is to show the following proposition, which completes the proof of Theorem 1.1 with Proposition 3.1.

**Proposition 3.4.** Suppose that $\mathcal{F}$ is a $C^r$ foliation on $M$ with $r \geq 2$. Then, any $\mathcal{P}\mathcal{A}_c^r(\mathcal{F})$-homotopy class contains an Anosov flow.

To simplify the proof, we assume that $\mathcal{F}$ is orientable and transversely orientable. For the other cases, the proof can be done with a small modification. Fix a flow $\Phi$ in $\mathcal{P}\mathcal{A}_c^r(\mathcal{F})$ with $r \geq 2$. By a remark at the beginning of this section, we may assume that $\Phi$ is non-degenerate. Let $\Lambda_\epsilon(\mathcal{F})$ be the union of closed
holonomy of $W$ in trivial holonomy. Fix a simple closed curve $\gamma$ on $\psi$ we can take an embedding $\gamma$, since $W$ is connected, there exists a torus $T_\gamma$ such that $\gamma$ is a locally maximal hyperbolic invariant set and $E^e$ and $E^s$ are mutually transverse, the relation $\leq$ is a partial order. Let $S_-$ be the subset of $S$ consisting of $A_i \in S$ with $W^u(A_i) \cap W^s(A_\gamma(F)) \neq \emptyset$. Notice that if $A_i \in S_-$ and $A_j \leq A_i$, then $A_j \in S_-$. Since $\Phi(\Phi) \subset A_\gamma(F)$, the equation (6) implies that $M = \bigcup_{z \in \Omega_h} W^u(z) \cup A_\gamma(F)$. Since $W^u(A_\gamma(F))$ is an open and proper subset of $M$, the set $S_-$ is non-empty.

Take a minimal element $A_\gamma$ of $S_-$. Then, we have $W^u(A_\gamma) \subset A_- \cup W^s(A_\gamma(F))$. By Proposition 1 of [9], there exists a periodic point $z_h \in A_-$ and a connected component $L$ of $W^u(z_h) \setminus O(z_h)$ such that $L \subset W^s(A_\gamma(F))$. Since $L$ is connected, there exists a torus $T_* \subset A_\gamma(F)$ and a connected component $U$ of $W^s(T_*) \setminus T_*$ such that $L \subset U$. By the normal hyperbolicity of $T_*$, we can take an embedding $\psi_* : \mathbb{T}^2 \times [0, 1] \to W^s(T_*)$ so that $\psi_*(\mathbb{T}^2 \times 0) = T_*$, $\psi_*(\mathbb{T}^2 \times [0, 1]) \subset U$, and $\psi_*(\mathbb{T}^2 \times 1)$ is transverse to $\Phi$.

Let $F_*$ be the restriction of $F$ on $\psi_*(\mathbb{T}^2 \times 1)$. By the classification of $C^2$ Reebless foliation on $\mathbb{T}^2 \times [0, 1]$ due to Moussu and Roussarie [8], $F_*$ must have trivial holonomy. Fix a simple closed curve $\gamma$ in $L$ which is homotopic to $O(z_h)$ in $W^u(z_h)$. Since $F(z_h) = W^u(z_h)$ by Lemma 2.7 and $F'(\gamma) \subset \psi_*(\mathbb{T}^2 \times (0, 1))$ for any sufficiently large $t > 0$, $\psi_*(\mathbb{T}^2 \times 1) \cap L$ is a closed leaf $\gamma'$ of $F_*$ which is homotopic to $O(z_h)$ in $F(z_h)$. Since $z_h$ is a hyperbolic periodic point, the linear holonomy of $F_*$ along $\gamma'$ is non-trivial. It contradicts the result of Moussu and Roussarie. □

Second, we see that each attracting periodic orbit is contained in an invariant closed annulus.

**Lemma 3.6.** For any $z_h \in Per_0(\Phi) \setminus A_\gamma(F)$, there exists an embedded closed annulus $A \subset F(z_h)$ such that boundary components of $A$ are saddle-type periodic orbits in $\Omega_h$ and the interior of $A$ is a subset of $W^s(z_h)$.
Proof. Take an embedded closed annulus $A_0 \subset W^s(z_*) \cap \mathcal{F}(z_*)$ so that $\mathcal{O}(z_*) \subset \text{Int } A_0$ and the boundary components $\gamma_{\pm}$ are transverse to $\Phi$. By Lemma 2.7, $W^u(z)$ is a connected component of $\mathcal{F}(z) \cap \text{Per}_0(\Phi)$ for any $z \in \Omega_b$. The equation (6) implies that $\gamma_{\pm} \subset W^u(z_{\pm})$ for some $z_{\pm} \in \Omega_b$. By the Poincaré-Bendixson theorem, $W^u(z_{\pm})$ is not diffeomorphic to the plane. Hence, it is an open annulus and there exists a periodic point $z_{\pm} \in \Omega_b$ with $W^u(z_{\pm}) = W^u(z_{\pm}')$. Now, it is easy to construct a closed annulus $A_{\pm}$ so that $\partial A_{\pm} = \{\mathcal{O}(z_*), \mathcal{O}(z_{\pm}')\}$ and Int $A_{\pm} \subset W^u(z_{\pm}) \cap W^s(z_*)$. Since $\mathcal{F}(z_*)$ is not a torus, a subset $A = A_+ \cup A_-$ of $\mathcal{F}(z_*)$ is an annulus with Int $A \subset W^s(z_*)$.

The main step of the proof is the following elimination lemma of periodic points.

**Lemma 3.7.** For any $z_* \in \text{Per}_0(\Phi) \setminus \Lambda_*(\mathcal{F})$, there exists a non-degenerate flow $\Phi_*$ in $\mathcal{F}$ which is $\mathcal{F}$-homotopic to $\Phi$ and satisfies $\text{Per}_0(\Phi_*) = \text{Per}_0(\Phi) \setminus \mathcal{O}(z_*)$.

**Proof.** By Lemma 3.6, there exists an embedded annulus $A \subset \mathcal{F}(z_*)$ such that $\mathcal{O}(z_*) \subset \text{Int } A \subset W^s(z_*)$ and boundary components of $A$ are periodic orbits in $\Omega_b$. Since $\Phi$ admits neither repelling periodic orbits nor normally repelling irrational invariant tori, Lemma 2.2 implies that $E^s$ is a $C^1$ subbundle of $TM$. Hence, we can take a $C^1$ embedding $\psi : [-2,2]^2 \to M$ so that $\text{Im } \psi \cap \text{Per}_0(\Phi) = \{\psi(0,0)\} = \{z_*\}$, $\psi(0 \times [-1,1]) = A \cap \text{Im } \psi$, $\text{Im } \psi$ is transverse to $\Phi$, $\text{D} \psi(e_x(w)) \in E^s(\psi(w))$ and $\text{D} \psi(e_y(w)) \in E^u(\psi(w))$ for any $w \in [-2,2]^2$, where $(e_x, e_y)$ is the natural orthonormal framing of $T\mathbb{R}^2$. Fix $\delta \in (0,1)$ so that a function $\tau(w) = \inf \{t > 0 \mid \Phi^t \circ \psi(w) \in \text{Im } \psi\}$ is well-defined and of class $C^1$ on $[-\delta, \delta] \times [-1-\delta, 1+\delta]$. Put $I = [-\delta, \delta]$, $J = [-1-\delta, 1+\delta]$, and define a map $F : I \times J \to [-2,2]^2$ by $\psi \circ F(w) = \Phi^w \circ \psi(w)$. Then, there exist functions $f$ on $I$ and $g$ on $J$ such that $F(x,y) = (f(x), g(y))$ for $(x,y) \in I \times J$. Remark that $f(0) = 0, g(y) = y_*$ for $y_* = 0, \pm 1$, $f'(0) < g'(0) < 1$, and $g'(\pm 1) > 1$. By replacing $\delta$ with a smaller one, we may assume that there exists $\lambda > 1$ such that $f'(x) \leq \lambda^{-1}$ for any $x \in I$ and $g'(y) \geq \lambda$ for any $y \in J \setminus [-1,1]$.

Put

$$V_a = \{\Phi^t \circ \psi(w) \mid w \in I^2, t \in [0, \tau(w)]\}$$

$$V(n) = \{\Phi^t \circ \psi(w) \mid w \in I \times g^{-n}(J), t \in [0, \tau(w)]\}$$

for $n \geq 0$. Remark that $V_a \subset W^s(z_*) \cap \left(\bigcap_{n \geq 0} V(n)\right)$ for any $n \geq 0$. We also put $\Lambda_* = \text{Per}_0(\Phi) \cup \Lambda_*(\mathcal{F}) \setminus \mathcal{O}(z_*; \Phi)$. Since $\Lambda_*(\mathcal{F})$ consists of normally attracting tori, we have $\Lambda_*(\mathcal{F}) \cap \mathcal{O}(z_*) = \emptyset$. It implies $\Lambda_*(\mathcal{F}) \cap A = \emptyset$. Recall that $\text{Per}_0(\Phi)$ consists of finitely many orbits and $\text{Per}_0(\Phi) \cap A = \mathcal{O}(z_*)$. By replacing $\delta$ with a smaller one again, we may assume that $\Lambda_* \subset V(0) = \emptyset$.

Take a neighborhood $V_* \subset \mathcal{F}$ of $\Lambda_*$ so that $V_* \subset W^u(\Lambda_*) \cap \text{Per}_0(\Phi)$ and $\Phi^t(V_{\gamma}) \subset V_*$ for any $t \geq 0$. Remark that $\bigcap_{n > 0} \Phi^n(\Lambda_* = \Lambda_* \cap V_* \cup V_*$ for any $t \geq 0$. For any $z \in \mathcal{F}(V_* \cup V_a)$, we have

$$\lim_{t \to +\infty} \mu(z, -t) = 0, \quad \lim_{t \to +\infty} \mu_1(z, -t)^{-1} = 0$$

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since \( \omega(z, \Phi^{-1}) \subset \Omega_b \). By Lemma 2.1 for \( \Phi^{-1} \), there exists \( C_* > 1 \) and \( \lambda_* \in (0, 1) \) such that

\[
\mu(z, t)^{-1} = \mu(\Phi^t(z), -t) \leq C_* \lambda_*^t, \quad (7)
\]

\[
\mu^+(z, t) = \mu^+(\Phi^t(z), -t)^{-1} \leq C_* \lambda_*^t \quad (8)
\]

for any \( t \geq 0 \) and \( z \in M \setminus \Phi^{-I}(V_* \cup V_a) \).

Put \( \lambda_0 = \inf \{ g'(y) \mid y \in J \} \) and \( K = \| D\psi \| \cdot \| D\psi^{-1} \| \). Fix \( n_0 \geq 1 \) so that \( \lambda_0 > 8C_* K \). We take a continuous family \( \{ f_t \}_{t \in [0, 1]} \) of \( C^1 \) function on \( J \) with following conditions:

- \( f_0 = g, \ (f_t)' > 1 \).
- \( f_{\alpha}(J) = g_{\alpha}(J) = \alpha \) and \( (g_{\alpha})' \geq \lambda_0 / 2 \) for any \( \alpha \in [0, 1] \),
- \( (g_{\alpha})'(y) > 1 \) for any \( \alpha \in [0, 1] \) and \( y \in J \setminus \bigcup_{n \geq 0} g_{\alpha}^{-n}(J) \), and
- \( (g_{\alpha})'(y) \geq f'(0) \) for any \( \alpha \in [0, 1] \) and any fixed point \( y_* \) of \( g_{\alpha} \).

![Figure 1: The family \( \{ g_{\alpha} \}_{\alpha \in [0, 1]} \) and the function \( h \)](image)

We also take a smooth even function \( h \) on \( \mathbb{R} \) so that \( h|_{[0, 1]} = 0, h|_{f(J)} = 1 \), and \( h'|_{[0, 1]} \leq 0 \). See Figure 1. Define a map \( F_\alpha \) by \( F_\alpha(x, y) = (f(x), (1 - h(x))g(y) + h(x)g_{\alpha}(y)) \). The family \( \{ F_\alpha \}_{\alpha \in [0, 1]} \) induces a homotopy \( \{ F_\alpha \}_{\alpha \in [0, 1]} \) of flows such that \( F_{\alpha} \mid_{\Lambda^1 V(n_0)} = \Phi \mid_{\Lambda^1 V(n_0)} \) and \( F_{\alpha} \mid_{\Lambda^1 V(n_0)} \psi(w) = \psi \circ F_{\alpha}(w) \) for any \( w \in I \times J \). See Figure 2.

We reduce the proof to the following claim for \( \{ F_{\alpha} \} \), which we show later.

**Claim 3.8.** \( \text{Per}_0(\Phi_1) = \text{Per}_0(\Phi) \setminus \mathcal{O}(z_*) \), \( \{ F_{\alpha} \}_{\alpha \in [0, 1]} \) is a \( \mathcal{P} \Lambda^1_\text{c}(\mathcal{F}) \)-homotopy, and \( M \setminus W^s(\Lambda_*; \Phi_1) \) is a hyperbolic invariant set of saddle-type for \( \Phi_1 \).

Suppose the claim holds. Since \( \Phi_1 \) is of class \( C^r \) on \( M \setminus V(n_0) \) and all periodic orbits in \( M \setminus V(n_0) \) are hyperbolic, we can perturb \( \Phi_1 \) into a \( C^r \) non-degenerate flow \( \Phi_* \) which is \( \mathcal{P} \Lambda^1_\text{c}(\mathcal{F}) \)-homotopic to \( \Phi_1 \) (and hence, to \( \Phi \)) and such that \( \Phi_* \mid_{M \setminus V(n_0)} = \Phi_1 \mid_{M \setminus V(n_0)} \).

Notice that if a flow \( \Psi \) on \( M \) satisfies \( \Psi \mid_{M \setminus V(n_0)} = \Phi \mid_{M \setminus V(n_0)} \), then \( \bigcup_{t \geq 0} \Psi^t(V_*) = \Lambda_* \) and \( \bigcup_{t \geq 0} \Psi^{-t}(V_*) = W^s(\Lambda_*; \Psi) \). By the stability of isolated hyperbolic invariant sets, the claim implies that \( M \setminus W^s(\Lambda_*; \Phi_1) \) is a hyperbolic invariant
Figure 2: The family \( \{F_\alpha\}_{\alpha \in [0,1]} \)

set of saddle-type for \( \Phi_* \) if \( \Phi_* \) is sufficiently close to \( \Phi_1 \). In particular, we have \( \text{Per}_0(\Phi_*) = \text{Per}_0(\Phi_1) = \text{Per}_0(\Phi) \setminus \mathcal{O}(z_\ast) \). By approximating a homotopy connecting \( \Phi \) and \( \Phi_* \) by the one in \( \mathcal{P}_n^\alpha(F) \), we complete the proof of the lemma.

**Proof of Claim 3.8.** It is sufficient to show inequalities

\[
\lim_{t \to \infty} \mu^+(z, t; \Phi_\alpha) \cdot \mu(z, t; \Phi_\alpha)^{-1} = 0, \quad \lim_{t \to \infty} \mu^-(z, t; \Phi_\alpha) = 0 \quad (9)
\]

for any \( z \in M \) and \( \alpha \in [0,1] \), and

\[
\lim_{t \to \infty} \mu(z, t; \Phi_1)^{-1} = 0 \quad (10)
\]

for any \( z \in M \setminus W^s(\Lambda_\alpha; \Phi_1) \).

First, we suppose that there exists \( T_1 > 1 \) such that \( \Phi_\alpha^t(z) \notin V(n_0) \) for any \( t \geq T_1 \). In this case, we can show the inequalities (9) for \( \Phi_\alpha^t(z) \) since \( \Phi \) is a flow in \( \mathcal{P}_n^\alpha(F) \) and \( \Phi_\alpha|_{M \setminus V(n_0)} = \Phi|_{M \setminus V(n_0)} \) for any \( \alpha \in [0,1] \). It implies the same inequalities hold for \( z \). Similarly, the inequality (10) holds if \( z \notin W^s(\Lambda_\alpha; \Phi_1) \).

Second, we suppose that there exists \( T_2 > 0 \) such that \( \Phi_\alpha^t(z) \in V(n_0) \) for any \( t \geq T_2 \). Then, there exist \( t > 0 \) and \( w = (x, y) \in I \times g^{-n_0}(J) \) such that \( \Phi_\alpha^t(z) = \psi(w) \) and \( F^\alpha_n(w) \in I \times g^{-n_0}(J) \) for any \( n \geq 0 \). It implies that \( \lim_{n \to -\infty} F^\alpha_n(x, y) = (0, y_\ast) \) for some fixed point \( y_\ast \) of \( g_\alpha \). By the construction of \( g_\alpha \), the inequalities (9) and (10) hold for \( z \).

At last, we suppose that the set \( \{ t \geq 0 \mid \Phi_\alpha^t(z) \in V(n_0) \} \) consists of infinitely many connected components \( \{[t_i, t_i']\}_{i=0}^\infty \) for \( z \in M \). Since \( \Phi_\alpha^t(V_\ast) \subset V_\ast \) for any \( t \geq 0 \) and \( V_\ast \cap V(0) = \emptyset \), we have \( \Phi_\alpha(z) \notin V_\ast \) for any \( t \geq 0 \). We order \( \{[t_i, t_i']\}_{i=0}^\infty \) so that \( t_{i+1} > t_i \) for any \( i \). For each \( i \geq 1 \), there exist \( (x_i, y_i) \in (I \setminus f(I)) \times g^{-n_0}(J) \), \( n_i \geq n_0 \), and \( t_i \in (t_i', t_{i+1}) \) such that \( \Phi_\alpha^t(z) = \psi(x_i, y_i) \), \( \Phi_\alpha^t(z) = \psi \circ F^\alpha_n(x_i, y_i) \), and \( F^\alpha_n(x_i, y_i) \in I \times (J \setminus g^{-1}(J)) \). Notice that \( \Phi^t(z) \notin V(n_0) \) for any \( t \in (t_i', t_{i+1}) \). Since \( n_i \geq n_0 \), we have

\[
\mu^-(\Phi_\alpha^t(z), t_i' - t_i; \Phi_\alpha) \leq K \cdot (f_n)^t \leq K \cdot (s_\alpha)^{-1}.
\]
and
\[ \mu(\Phi^t_\alpha(z), t_i - t_i; \Phi_\alpha) \geq K^{-1}(\lambda_2/2)\lambda^{n_i-1} \geq 4C_\ast. \]

The latter inequality follows from
\[ ||DF_\alpha(e_y(x_i, y_i))|| = (1 - h(x))g'(y_i) + h(x)(g_\alpha)'(y_i) \geq \lambda_2/2 \]
and
\[ ||DF_\alpha(e_y(F^n_\alpha(x_i, y_i)))|| = \begin{cases} (g_\alpha)'(y_i) \geq 1 & (1 \leq n \leq n_i - n_0), \\ (g_\alpha)'(y_i) \geq \lambda & (n_i - n_0 + 1 \leq n \leq n_i). \end{cases} \]

Since \( \Phi_t^i(z) \notin V(n_0) \cup V_\ast \) for any \( t \in (t_i, t_{i+1}) \), the inequalities (7) and (8) imply
\[ \mu(\Phi^t_\alpha(z), t_{i+1} - t_i; \Phi_\alpha) \geq 4, \quad \mu^\perp(\Phi^t_\alpha(z), t_{i+1} - t_i; \Phi_\alpha) \leq 1/8. \] (11)

for any \( i \geq 1 \). Therefore, the inequalities (9) and (10) hold for \( z \).

Now, we prove Proposition 3.4. Any \( \mathbb{P}A_\alpha^r(F) \)-homotopy class contains a non-degenerate flow. By Proposition 2.6, any non-degenerate flow in \( \mathbb{P}A_\alpha^r(F) \) admits only finitely many attracting periodic orbits. Hence, Lemma 3.7 implies that any \( \mathbb{P}A_\alpha^r(F) \)-homotopy class contains a non-degenerate flow \( \Phi \) such that all attracting periodic orbits are contained in \( \Lambda_\ast(F) \). By Lemma 3.5, \( \Phi \) is an Anosov flow.

References


