The ubiquitous hyperfinite II_1 factor , lecture 1

Kyoto U. & RIMS, April 2019

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Generalities on von Neumann algebras

A **von Neumann (vN) algebra** is a *-algebra of operators acting on a Hilbert space, $M \subset \mathcal{B}(\mathcal{H})$, that contains $1 = id_{\mathcal{H}}$ and satisfies any of the following equivalent conditions:

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• A vN algebra M is closed to polar decomposition and Borel functional calculus. Also, if $\{x_i\}_i \subset (M_+)_1$ is an increasing net, then $\sup_i x_i \in M$, and if $\{p_j\}_j \subset M$ are mutually orthogonal projections, then $\sum_j p_j \in M$.

Examples

- $\mathcal{B}(\mathcal{H})$ itself is a vN algebra.

- Let (X, μ) be a standard Borel probability measure space (pmp). Then the function algebra $L^\infty X = L^\infty(X, \mu)$ with its essential sup-norm $\| \ \|_\infty$, can be represented as a *-algebra of operators on the Hilbert space $L^2X = L^2(X, \mu)$, as follows: for each $x \in L^\infty X$, let $\lambda(x) \in \mathcal{B}(L^2X)$ denote the operator of (left) multiplication by x on L^2X, i.e., $\lambda(x)(\xi) = x\xi$, $\forall \xi \in L^2X$. Then $x \mapsto \lambda(x)$ is clearly a *-algebra morphism with $\|\lambda(x)\|_{\mathcal{B}(L^2X)} = \|x\|_\infty$, $\forall x$. Its image $A \subset \mathcal{B}(L^2X)$ satisfies $A' = A$, in other words A is a maximal abelian *-subalgebra (MASA) in $\mathcal{B}(L^2X)$.

Indeed, if $T \in A'$ then let $\xi = T(1) \in L^2X$. Denote by $\lambda(\xi) : L^2X \to L^1X$ the operator of (left) multiplication by ξ, which by Cauchy-Schwartz is bounded by $\|\xi\|_2$. But $T : L^2X \to L^2X \subset L^1X$ is also bounded as an operator into L^1X, and $\lambda(\xi), T$ coincide on the $\| \ \|_2$-dense subspace $L^\infty X \subset L^2X$ (*Exercise!*) Thus, $\lambda(\xi) = T$ on all L^2, forcing $\xi \in L^\infty X$ (*Exercise!*).

This shows that A is a vN algebra (by vN's bicommutant thm).

A key example: the hyperfinite II_1 factor

A vN algebra M is called a **factor** if its center, $\mathcal{Z}(M) := M' \cap M$, is trivial, $\mathcal{Z}(M) = \mathbb{C}1$.

• Let R_0 be the algebraic infinite tensor product $\mathbb{M}_2(\mathbb{C})^{\otimes\infty}$, viewed as inductive limit of the increasing sequence of algebras $\mathbb{M}_{2^n}(\mathbb{C}) = \mathbb{M}_2(\mathbb{C})^{\otimes n}$, via the embeddings $x \mapsto x \otimes 1_{\mathbb{M}_2}$. Endow R_0 with the norm $\|x\| = \|x\|_{\mathbb{M}_{2^n}}$, if $x \in \mathbb{M}_{2^n} \subset R_0$, which is clearly a well defined operator norm, i.e., satisfies $\|x^*x\| = \|x\|_2$. One also endows R_0 with the functional $\tau(x) = Tr(x)/2^n$, for $x \in \mathbb{M}_{2^n}$, which is well defined, positive ($\tau(x^*x) \geq 0, \forall x$) and satisfies $\tau(xy) = \tau(yx), \forall x, y \in R_0$, $\tau(1) = 1$, i.e., it is a **trace state**. Define the Hilbert space $L^2(R_0)$ as the completion of R_0 with respect to the Hilbert-norm $\|y\|_2 = \tau(y^*y)^{1/2}$, $y \in R_0$, and denote $\hat{R}_0$ the copy of R_0 as a subspace of $L^2(R_0)$.

For each $x \in R_0$ define the operator $\lambda(x)$ on $L^2(R_0)$ by $\lambda(x)(\hat{y}) = \hat{xy}$, $\forall y \in R_0$. Note that $R_0 \ni x \mapsto \lambda(x) \in \mathcal{B}(L^2)$ is a *-algebra morphism with $\|\lambda(x)\| = \|x\|$, $\forall x$. Moreover, $\langle \lambda(x)(\hat{1}), \hat{1}\rangle_{L^2} = \tau(x)$.

One similarly defines $\rho(x)$ to be the operator of right multiplication by x on $L^2(R_0)$, for which we have $[\lambda(y), \rho(x)] = 0$, $\forall x, y \in R_0$.

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One can easily see that the vN algebra $R := \overline{\lambda(R_0)}^{so} = \overline{\lambda(R_0)}^{wo}$ is a factor (*Exercise!*). It can alternatively be defined by $R = \rho(R_0)'$ (*Exercise!*). This is the **hyperfinite II_1 factor**.

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Yet another way to define R is as the completion of R_0 in the topology of convergence in the norm $\|x\|_2 = \tau(x^*x)^{1/2}$ of sequences that are bounded in the operator norm (*Exercise!*). Notice that, in both definitions, τ extends to a trace state on R. Note also that if one denotes by $D_0 \subset R_0$ the natural "diagonal subalgebra" (...), then $(D_0, \tau_{|D_0})$ coincides with the algebra of dyadic step functions on $[0,1]$ with the Lebesgue integral. So its closure in R in the above topology, $(D, \tau_{|D})$, is just $(L^\infty([0,1]), \int \, \mathrm{d}\mu)$.

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Note that (R_0, τ) (and thus R) is completely determined by the sequence of partial isometries $v_1 = e^1_{12}$, $v_n = (\Pi_{i=1}^{n-1} e^i_{22}) e^n_{12}$, $n \geq 2$, with $p_n = v_n v_n^*$ satisfying $\tau(p_n) = 2^{-n}$ and $p_n \sim 1 - \sum_{i=1}^n p_i$ (*Exercise!*)

Finite factors: some equivalent characterizations

Theorem A

Let M be a vN factor. The following are equivalent:

1° M is a **finite** vN algebra, i.e., if $p \in \mathcal{P}(M)$ satisfies $p \sim 1 = 1_M$, then $p = 1$ (any isometry in M is necessarily a unitary element).

2° M has a **trace state** τ (i.e., a functional $\tau : M \to \mathbb{C}$ that's positive, $\tau(x^*x) \geq 0$, with $\tau(1) = 1$, and is tracial, $\tau(xy) = \tau(yx), \forall x, y \in M$).

3° M has a trace state τ that's **completely additive**, i.e., $\tau(\Sigma_i p_i) = \Sigma_i \tau(p_i)$, $\forall \{p_i\}_i \subset \mathcal{P}(M)$ mutually orthogonal projections.

4° M has a trace state τ that's **normal**, i.e., $\tau(\sup_i x_i) = \sup_i \tau(x_i)$, $\forall \{x_i\}_i \subset (M_+)_1$ increasing net.

Thus, a vN factor is finite iff it is tracial. Moreover, such a factor has a unique trace state τ, which is automatically normal and faithful, and satisfies $\overline{\mathrm{co}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap \mathbb{C}1 = \{\tau(x)1\}$, $\forall x \in M$.

Some preliminary lemmas

Lemma 1

If a vN factor M has a minimal projections, then $M = \mathcal{B}(\ell^2 I)$, for some I. Moreover, if $M = \mathcal{B}(\ell^2 I)$, then the following are eq.:

1° M has a trace.

2° $|I| < \infty$.

3° M is finite, i.e. $u \in M$, $u^* u = 1 \Rightarrow uu^* = 1$

Proof: Exercise.

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Proof: Exercise.

Lemma 2

If M is finite then:
(a) $p, q \in \mathcal{P}(M)$, $p \sim q \Rightarrow 1 - p \sim 1 - q$.
(b) pMp is finite $\forall p \in \mathcal{P}(M)$, i.e., $q \in \mathcal{P}(M)$, $q \leq p$, $q \sim p$, then $q = p$.

Proof: Use the comparison theorem (Exercise).

Lemma 3

If M vN factor with no atoms and $p \in \mathcal{P}(M)$ is so that $\dim(pMp) = \infty$, then $\exists P_0, P_1 \in \mathcal{P}(M)$, $P_0 \sim P_1$, $P_0 + P_1 = p$.

Proof: Consider the family $\mathcal{F} = \{(p_i^0, p_i^1)_i \mid$ with p_i^0, p_j^1 all mutually orthogonal $\leq p$ such that $p_i^0 \sim p_i^1$, $\forall i\}$, with its natural order. Clearly inductively ordered. If $(p_i^0, p_i^1)_{i \in I}$ is a maximal element, then $P_0 = \sum_i p_i^0, P_1 = \sum_i p_i^1$ will do (for if not, then the comparison Thm. gives a contradiction).

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Lemma 4

If M is a factor with no minimal projections, then $\exists \{p_n\}_n \subset \mathcal{P}(M)$ mutually orthogonal such that $p_n \sim 1 - \sum_{i=1}^{n} p_i$, $\forall n$.

Proof: Apply **L3** recursively.

Lemma 5

If M is a finite factor and $\{p_n\}_n \subset \mathcal{P}(M)$ are as in **L4**, then:

(a) If $p \prec p_n$, $\forall n$, then $p = 0$. Equivalently, if $p \neq 0$, then $\exists n$ such that $p_n \prec p$. Moreover, if n is the first integer such that $p_n \prec p$ and $p_n' \leq p$, $p_n' \sim p_n$, then $p - p_n' \prec p_n$.

(b) If $\{q_n\}_n \subset \mathcal{P}(M)$ increasing and $q_n \leq q \in \mathcal{P}(M)$ and $q - q_n \prec p_n$, $\forall n$, then $q_n \nearrow q$ (with so-convergence).

(c) $\sum_n p_n = 1$.

Proof: If $p \simeq p_n' \leq p_n$, $\forall n$, then $P = \sum_n p_n'$ and $P_0 = \sum_k p_{2k+1}'$ satisfy $P_0 < P$ and $P_0 \sim P$, contradicting the finiteness of M. Rest is *Exercise!*

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Lemma 6

Let M be a finite factor without atoms. If $p \in \mathcal{P}(M)$, $\neq 0$, then $\exists$ a unique infinite sequence $1 \leq n_1 < n_2 < \ldots$ such that p decomposes as $p = \sum_{k \geq 1} p'_{n_k}$, for some $\{p'_{n_k}\}_k \subset \mathcal{P}(M)$ with $p'_{n_k} \sim p_{n_k}$, $\forall k$.

Proof: Apply Part (a) of **L5** recursively (*Exercise!*).

If M is a finite factor without atoms, then we let $\dim : \mathcal{P}(M) \to [0,1]$ be defined by $\dim(p) = 0$ if $p = 0$ and $\dim(p) = \sum_{k=1}^{\infty} 2^{-n_k}$, if $p \neq 0$, where $n_1 < n_2 < \dots$, are given by **L4**.

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Lemma 7

$\dim$ satisfies the conditions:

(a) $\dim(p_n) = 2^{-n}$

(b) If $p, q \in \mathcal{P}(M)$ then $p \sim q$ iff $\dim(p) \leq text\dim(q)$

(c) $\dim$ is completely additive: if $q_i \in \mathcal{P}(M)$ are mutually orthogonal, then $\dim(\Sigma_i q_i) = \Sigma_i \dim(q_i)$.

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Proof: *Exercise!*.

Lemma 8 (Radon-Nykodim trick)

Let $\varphi, \psi : \mathcal{P}(M) \to [0,1]$ be completely additive functions, $\varphi \neq 0$, and $\varepsilon > 0$. There exists $p \in \mathcal{P}(M)$ with $\dim(p) = 2^{-n}$ for some $n \geq 1$, and $\theta \geq 0$, such that $\theta\varphi(q) \leq \psi(q) \leq (1+\varepsilon)\theta\varphi(q)$, $\forall q \in \mathcal{P}(pMp)$.

Proof: Denote $\mathcal{F} = \{p \mid \exists n \text{ with } p \sim p_n\}$. Note first we may assume φ faithful: take a maximal family of mutually orthogonal non-zero projections $\{e_i\}_i$ with $\varphi(e_i) = 0$, $\forall i$, then let $f = 1 - \sum_i e_i \neq 0$ (because $\varphi(1) \neq 0$); it follows that φ is faithful on fMf, and by replacing with some $f_0 \leq f$ in $\mathcal{F}$, we may also assume $f \in \mathcal{F}$. Thus, proving the lemma for M is equivalent to proving it for fMf, which amounts to assuming φ faithful.

If $\psi = 0$, then take $\theta = 0$. If $\psi \neq 0$, then by replacing φ by $\varphi(1)^{-1}\varphi$ and ψ by $\psi(1)^{-1}\psi$, we may assume $\varphi(1) = \psi(1) = 1$. Let us show this implies:

(1) $\exists g \in \mathcal{F}$, s.t. $\forall g_0 \in \mathcal{F}$, $g_0 \leq g$, we have $\varphi(g_0) \leq \psi(g_0)$. For if not then

(2) $\forall g \in \mathcal{F}$, $\exists g_0 \in \mathcal{F}$, $g_0 \leq g$ s.t. $\varphi(g_0) > \psi(g_0)$.

Take a maximal family of mut. orth. projections $\{g_i\}_i \subset \mathcal{F}$, with $\varphi(g_i) > \psi(g_i)$, $\forall i$. If $1 - \sum_i g_i \neq 0$, then take $g \in \mathcal{F}$, $g \leq 1 - \sum_i g_i$ (cf. **L5**) and apply (2) to get $g_0 \leq g$, $g_0 \in \mathcal{F}$ with $\varphi(g_0) > \psi(g_0)$, contradicting the maximality. Thus,

$$1 = \varphi(\sum_i g_i) = \sum_i \varphi(g_i) > \sum_i \psi(g_i) = \psi(\sum_i g_i) = \psi(1) = 1,$$

a contradiction. Thus, (1) holds true.

Define $\theta = \sup\{\theta' \mid \theta'\varphi(g_0) \leq \psi(g_0), \forall g_0 \leq g, g_0 \in \mathcal{F}\}$.

Clearly $1 \leq \theta < \infty$ and $\theta\varphi(g_0) \leq \psi(g_0), \forall g_0 \leq g, g_0 \in \mathcal{F}$. Moreover, by def. of θ, there exists $g_0 \in \mathcal{F}$, $g_0 \leq g$, s.t., $\theta\varphi(g_0) > (1+\varepsilon)^{-1}\psi(g_0)$. We now repeat the argument for ψ and $\theta(1+\varepsilon)\varphi$ on $g_0 M g_0$, to prove that

(3) $\exists g' \in \mathcal{F}$, $g' \leq g_0$, such that for all $g_0' \in \mathcal{F}$, $g_0' \leq g_0$, we have $\psi(g_0') \leq \theta(1+\varepsilon)\varphi(g_0')$.

Indeed, for if not, then

(4) $\forall g' \in \mathcal{F}$, $g' \leq g_0$, $\exists g_0' \leq g'$ in $\mathcal{F}$ s.t. $\psi(g_0') > \theta(1+\varepsilon)\varphi(g_0')$.

But then we take a maximal family of mutually orthogonal $g_i' \leq g_0$ in $\mathcal{F}$, s.t. $\psi(g_i') \geq \theta(1+\varepsilon)\varphi(g_i')$, and using **L5** and (4) above we get $\sum_i g_i' = g_0$. This implies that $\psi(g_0) \geq \theta(1+\varepsilon)\varphi(g_0) > \psi(g_0)$, a contradiction. Thus, (3) above holds true for some $g' \leq g_0$ in $\mathcal{F}$. Taking $p = g'$, we get that any $q \in \mathcal{F}$ under p satisfies both $\theta\varphi(q) \leq \psi(q)$ and $\psi(q) \leq \theta(1+\varepsilon)\varphi(q)$. By complete additivity of φ, ψ and **L6**, we are done.

We now apply **L8** to $\psi = \dim$ and φ a vector state on $M \subset \mathcal{B}(\mathcal{H})$, to get:

Lemma 9

$\forall \varepsilon > 0$, $\exists p \in \mathcal{P}(M)$ with $\dim(p) = 2^{-n}$ for some $n \geq 1$, and a vector (thus normal) state φ_0 on pMp such that, $\forall q \in \mathcal{P}(pMp)$, we have $(1+\varepsilon)^{-1}\varphi_0(q) \leq \dim(q) \leq (1+\varepsilon)\varphi_0(q)$.

Proof: trivial by **L8**

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Proof: trivial by **L8**

Lemma 10

With p, φ_0 as in **L9**, let $v_1 = p, v_2, ..., v_{2^n} \in M$ such that $v_i v_i^* = p$, $\sum_i v_i^* v_i = 1$. Let $\varphi(x) := \sum_{i=1}^{2^n} \varphi_0(v_i x v_i^*)$, $x \in M$. Then φ is a normal state on M satisfying $\varphi(x^*x) \leq (1+\varepsilon)\varphi(xx^*)$, $\forall x \in M$.

Proof: Note first that $\varphi_0(x^*x) \leq (1+\varepsilon)\varphi_0(xx^*)$, $\forall x \in pMp$ (Hint: do it first for x partial isometry, then for x with x^*x having finite spectrum). To deduce the inequality for φ itself, note that $\sum_j v_j^* v_j = 1$ implies that for any $x \in M$ we have

$$\varphi(x^*x) = \sum_i \varphi_0(v_i x^* (\sum_j v_j^* v_j) x v_i^*) = \sum_{i,j} \varphi_0((v_i x^* v_j^*)(v_j x v_i))$$

$$\leq (1+\varepsilon)\sum_{i,j}\varphi_0((v_j x v_i)(v_i x^* v_j^*)) = ... = (1+\varepsilon)\varphi(xx^*).$$

$$\leq (1+\varepsilon)\sum_{i,j}\varphi_0((v_j x v_i)(v_i x^* v_j^*)) = \ldots = (1+\varepsilon)\varphi(xx^*).$$

Lemma 11

If φ is a state on M that satisfies $\varphi(x^*x) \leq (1+\varepsilon)\varphi(xx^*)$, $\forall x \in M$, then $(1+\varepsilon)^{-1}\varphi(p) \leq \dim(p) \leq (1+\varepsilon)\varphi(p)$, $\forall p \in \mathcal{P}(M)$.

Proof: By complete additivity, it is sufficient to prove it for $p \in \mathcal{F}$, for which we have for $v_1, \ldots, v_{2^n}$ as in **L10** $\varphi(p) = \varphi(v_j^* v_j) \leq (1+\varepsilon)\varphi(v_j v_j^*)$, $\forall j$, so that

$$2^n\varphi(p) \leq (1+\varepsilon)\sum_j \varphi(v_j v_j^*) = (1+\varepsilon)2^n \dim(p)$$

and similarly $2^n \dim(p) = 1 \leq (1+\varepsilon)2^n\varphi(p)$.

Proof of Thm A

Define $\tau : M \to \mathbb{C}$ as follows. First, if $x \in (M_+)_1$ then we let $\tau(x) = \tau(\Sigma_n 2^{-n} e_n) = \Sigma_n 2^{-n} \mathrm{dim}(e_n)$, where $x = \Sigma_n 2^{-n} e_n$ is the (unique) dyadic decomposition of $0 \leq x \leq 1$. Extend τ to M_+ by homothety, then further extend to M_h by $\tau(x) = \tau(x_+) - \tau(x_-)$, where for $x = x^* \in M_h$, $x = x_+ - x_-$ is the dec. of x into its positive and negative parts. Finally, extend τ to all M by $\tau(x) = \tau(\mathrm{Re} x) + i\tau(\mathrm{Im} x)$.

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By **L11**, $\forall \varepsilon > 0$, $\exists \varphi$ normal state on M such that $|\tau(p) - \varphi(p)| \leq \varepsilon$, $\forall p \in \mathcal{P}(M)$. By the way τ was defined and the linearity of φ, this implies $|\tau(x) - \varphi(x)| \leq \varepsilon$, $\forall x \in (M_+)_1$, and thus $|\tau(x) - \varphi(x)| \leq 4\varepsilon$, $\forall x \in (M)_1$. This implies $|\tau(x + y) - \tau(x) - \tau(y)| \leq 8\varepsilon$, $\forall x, y \in (M)_1$. Since $\varepsilon > 0$ was arbitrary, this shows that τ is a linear state on M.

By definition of τ, we also have $\tau(uxu^*) = \tau(x)$, $\forall x \in M$, $u \in \mathcal{U}(M)$, so τ is a trace state. From the above argument, it also follows that τ is a norm limit of normal states, which implies τ is normal as well.

Finite vN algebras

Theorem A'

Let M be a vN algebra that's countably decomposable (i.e., any family of mutually orthogonal projections is countable). The following are equivalent:

1° M is a **finite** vN algebra, i.e., if $p \in \mathcal{P}(M)$ satisfies $p \sim 1 = 1_M$, then $p = 1$ (any isometry in M is necessarily a unitary element).

2° M has a faithful normal (equivalently completely additive) trace state τ.

Moreover, if M is finite, then there exists a unique normal faithful **central trace**, i.e., a linear positive map $ctr : M \to \mathcal{Z}(M)$ that satisfies $ctr(1) = 1$, $ctr(z_1 x z_2) = z_1 ctr(x) z_2$, $ctr(xy) = ctr(yx)$, $x, y \in M$, $z_i \in \mathcal{Z}$.

Any trace τ on M is of the form $\tau = \varphi_0 \circ ctr$, for some state φ_0 on $\mathcal{Z}$.

Also, $\overline{\mathrm{co}}\{uxu^* \mid u \in \mathcal{U}(M)\} \cap \mathcal{Z} = \{ctr(x)\}$, $\forall x \in M$.

Proof of $2^\circ \Rightarrow 1^\circ$: If τ is a faithful trace on M and $u^*u = 1$ for some $u \in M$, then $\tau(1 - uu^*) = 1 - \tau(uu^*) = 1 - \tau(u^*u) = 0$, thus $uu^* = 1$.

L^p-spaces from tracial algebras

• A *-operator algebra $M_0 \subset \mathcal{B}(\mathcal{H})$ that's closed in operator norm is called a **C*-algebra**. Can be described abstractly as a Banach algebra M_0 with a *-operation and the norm satisfying the axiom $\|x^*x\| = \|x\|^2$, $\forall x \in M_0$.

• If M_0 is a unital C*-algebra and τ is a faithful trace state on M_0, then for each $p \geq 1$, $\|x\|_p = \tau(|x|^p)^{1/p}$, $x \in M_0$, is a norm on M_0. We denote $L^p M_0$ the completion of $(M_0, \| \ \|_p)$. One has $\|x\|_p \leq \|x\|_q$, $\forall 1 \leq p \leq q \leq \infty$, thus $L^p M_0 \supset L^q M_0$.

Note that $L^2 M_0$ is a Hilbert space with scalar product $\langle x, y \rangle_\tau = \tau(y^* x)$. The map $M_0 \ni x \mapsto \lambda(x) \in \mathcal{B}(L^2)$ defined by $\lambda(x)(\hat{y}) = \hat{xy}$ is a *-algebra isometric representation of M_0 into $\mathcal{B}(L^2)$ with $\tau(x) = \langle \lambda(x)\hat{1}, \hat{1} \rangle_\varphi$. Similarly, $\rho(x)(\hat{y}) = \hat{yx}$ defines an isometric representation of $(M_0)^{op}$ on $L^2 M_0$. One has $[\lambda(x_1), \rho(x_2)] = 0$, $\forall x_i \in M_0$.

More generally, $\|x\| = \sup\{\|xy\|_p \mid \|y\|_p \leq 1\}$. Also, $\|y\|_1 = \sup\{|\tau(xy)| \mid x \in (M)_1\}$. In particular, τ extends to $L^1 M_0$.

Exercise!

Abstract characterizations of finite vN algebras

Theorem B

Let (M, τ) be a unital C*-algebra with a faithful trace state. The following are equivalent:

1° The image of $\lambda : M \to \mathcal{B}(L^2(M, \tau))$ is a vN algebra (i.e., is wo-closed).

2° $\lambda(M) = \rho(M)'$ (equivalently, $\rho(M) = \lambda(M)'$).

3° $(M)_1$ is complete in the norm $\|x\|_{2,\tau}$.

4° As Banach spaces, we have $M = (L^1(M, \tau))^*$, where the duality is given by $(M, L^1 M) \ni (x, Y) \mapsto \tau(xY)$.

Proof: One uses similar arguments as when we represented $L^\infty([0, 1])$ as a vN algebra and as in the construction of R (*Exercise!*).

II_1 factors: definition and basic properties

Definition

An ∞-dim finite factor M (so $M \neq \mathbb{M}_n(\mathbb{C})$, $\forall n$) is called a **II_1 factor**.

• R is a factor, has a trace, and is ∞-dimensional, so it is a II_1 factor.

• The construction of the trace on a non-atomic factor satisfying the finiteness axiom in Thm A is based on splitting recursively 1 dyadically into equivalent projections, with the underlying partial isometries generating the hyperfinite II_1 factor R. Thus, R embeds into any II_1 factor.

• If $A \subset M$ is a maximal abelian *-subalgebra (MASA) in a II_1 factor M, then A is diffuse (i.e., it has no atoms).

• The (unique) trace τ on a II_1 factor M is a dimension function on $\mathcal{P}(M)$, i.e., $\tau(p) = \tau(q)$ iff $p \sim q$, with $\tau(\mathcal{P}(M)) = [0, 1]$ (*continuous dimension*).

• If $B \subset M$ is vN alg, the orth. projection $e_B : L^2M \to \overline{\hat{B}}^{\| \ \|_2} = L^2B$ is positive on $\hat{M} = M$, so it takes M onto B, implementing a cond. expect. $E_B : M \to B$ that satisfies $\tau \circ E_B = \tau$. It is unique with this property.

Amplifications of II_1 factors

• If $n \geq 2$ then $\mathbb{M}_n(M) = \mathbb{M}_n(\mathbb{C}) \otimes M$ is a II_1 factor with trace state $\tau((x_{ij})_{i,j}) = \sum_i \tau(x_{ii})/n$, $\forall (x_{ij})_{i,j} \in \mathbb{M}_n(M)$.

• If $0 \neq p \in \mathcal{P}(M)$, then pMp is a II_1 factor with trace state $\tau(p)^{-1}\tau$, whose isomorphism class only depends on $\tau(p)$.

• Given any $t > 0$, let $n \geq t$ and $p \in \mathcal{P}(\mathbb{M}_n(M))$ be so that $\tau(p) = t/n$. We denote the isomorphism class of $p\mathbb{M}_n(M)p$ by M^t and call it the **amplification of M by t** (*Exercise*: show that this doesn't depend on the choice of n and p.)

• We have $(M^s)^t = M^{st}$, $\forall s, t > 0$ (*Exercise*). One denotes $\mathcal{F}(M) = \{t > 0 \mid M^t \simeq M\}$. Clearly a multiplicative subgroup of $\mathbb{R}_+$, called the **fundamental group of M**. It is an isom. invariant of M.