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NOTE ON THE COHOMOLOGY OF FINITE CYCLIC COVERINGS

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Abstract. We introduce the height of a normal cyclic p-fold covering and show a cohomological relation between the base and the total spaces of the covering in terms of the height.

1. Statement of results

The purpose of this note is to show a cohomological property of a normal cyclic p-fold covering with respect to a certain cup-length type invariant of the covering. Let p be a prime and let $E \to B$ be a normal cyclic p-fold covering where B is path connected. Suppose $p = 2$. In [Ko], Kozlov defined the *height* of the covering $\mathrm{h}(E)$ as the maximum n such that $w_1(E)^n \neq 0$, where $w_1(E)$ is the first Stiefel-Whitney class of the covering. By a chain level consideration, he proved

$$H^{\mathrm{h}(E)}(E; \mathbb{Z}/2) \neq 0.$$

This also follows immediately from the Gysin sequence of the double covering $E \to B$. We would like to generalize this result to any prime p. Let p be an arbitrary prime. Let C_p be a cyclic group of order p and let $\rho : B \to BC_p$ be the classifying map of the covering $E \to B$. The *height* of the covering can be generalized as

$$\mathrm{h}(E) = \max\{n \mid \rho^* : H^n(BC_p; \mathbb{Z}/p) \to H^n(B; \mathbb{Z}/p) \text{ is non-trivial}\}.$$

We remark here that the height of a normal cyclic p-fold covering is closely related with the ideal-valued cohomological index theory of Fadell and Husseini [FH1] and hence the Borsuk-Ulam theorem. We will interpret the height in terms of the category weight introduced by Fadell and Husseini [FH2] and studied further by Rudyak [Ru] and Strom [S]. The most difficult point in generalizing the result of Kozlov is the non-existence of the Gysin sequence for the covering $E \to B$ when p is odd. However, we define the corresponding spectral sequence by which we prove:

Theorem 1.1. *Let $E \to B$ be a normal cyclic p-fold covering, where B is path-connected. Then*

$$H^{\mathrm{h}(E)}(E; \mathbb{Z}/p) \neq 0.$$

As an immediate corollary, we have:

Corollary 1.2. *Let $E \to B$ be a normal cyclic p-fold covering, where B is path-connected. If $\mathrm{h}(E) \geq n$ and $H^n(E; \mathbb{Z}/p) = 0$, it holds that $\mathrm{h}(E) \geq n + 1$.*

In section 2, we construct a spectral sequence for a normal cyclic p-fold covering which calculate the mod p cohomology of the total space from the base space whose differential is shown to be given as a certain higher Massey product of Kraines [Kr]. Using this spectral sequence, we prove

Theorem 1.1. In section 3, we interpret the height of a normal cyclic p-fold covering in terms of the category weight introduced by Fadell and Husseini [FH2] and elaborated by [Ru] and [S].

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2. Proof of Theorem 1.1

Throughout this section, let p be an odd prime and the coefficient of cohomology is $\mathbb{Z}/p$.

2.1. **Spectral sequence.** Let $E \to B$ be a normal p-fold covering where B is path-connected. In this subsection, we introduce a spectral sequence which calculates the mod p cohomology of E from B. Analogous spectral sequences were considered in [F] and [Re]. We first set notation. Let $\rho : B \to BC_p$ be the classifying map of the covering $E \to B$. Recall that the mod p cohomology of BC_p is given as

$$H^*(BC_p) = \Lambda(u) \otimes \mathbb{Z}/p[v], \quad \beta u = v, \quad |u| = 1,$$

where β is the Bockstein operation. We denote the cohomology classes $\rho^*(u)$ and $\rho^*(v)$ of B by $\bar{u}$ and $\bar{v}$, respectively. Let $R[C_p]$ denote the group ring of C_p over a ring R. Note that the singular chain complex $S_*(E)$ is a free $\mathbb{Z}[C_p]$-module. We regard $\mathbb{Z}/p[C_p]$ and $\mathbb{Z}/p$ as $\mathbb{Z}[C_p]$-modules by the modulo p reduction and the trivial C_p-action, respectively. Then there are natural isomorphisms

$$H^*(\mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \mathbb{Z}/p[C_p])) \cong H^*(E) \quad \text{and} \quad H^*(\mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \mathbb{Z}/p)) \cong H^*(B). \tag{2.1}$$

We now fix a generator g of C_p and put $\tau = 1 - g \in \mathbb{Z}/p[C_p]$. Observe that $\mathbb{Z}/p[C_p] = \mathbb{Z}/p[\tau]/(\tau^p)$. Consider the filtration

$$0 \subset \tau^{p-1}\mathbb{Z}/p[C_p] \subset \tau^{p-2}\mathbb{Z}/p[C_p] \subset \cdots \subset \tau\mathbb{Z}/p[C_p] \subset \mathbb{Z}/p[C_p].$$

Then there is a spectral sequence (E_r, d_r) associated with the induced filtration of the cochain complex $\mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \mathbb{Z}/p[C_p])$. By (2.1), we have

$$E_1^{s,t} \cong \begin{cases} H^t(B) & 0 \le s \le p-1 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad H^*(E) \tag{2.2}$$

and the degree of the differential d_r is $(-r, 1)$, where the total degree of $E_r^{s,t}$ is t. Let us identify the differential of this spectral sequence. To this end, we calculate the induced coboundary map $\bar{\delta}$ of the associated graded cochain complex

$$\bigoplus_{i=0}^{p-1} \mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \tau^i\mathbb{Z}/p[C_p]/\tau^{i-1}\mathbb{Z}/p[C_p]) \cong \bigoplus_{i=0}^{p-1} \tau^i \mathrm{Hom}_{\mathbb{Z}}(S_*(B), \mathbb{Z}/p).$$

In the special case of the universal bundle $EC_p \to BC_p$, we may put

$$\bar{\delta}(1) = \tau u_1 + \cdots + \tau^{p-1}u_{p-1}, \quad u_i \in \mathrm{Hom}_{\mathbb{Z}}(S_1(B), \mathbb{Z}/p)$$

for $1 \in \mathrm{Hom}_{\mathbb{Z}}(S_0(B), \mathbb{Z}/p)$. Consider the map $E \xrightarrow{\tilde{\rho}\times\pi} EC_p \times B$, where $\tilde{\rho}$ is a lift of ρ and π is the projection. Then we see that

$$\bar{\delta}x = \delta x + \tau\rho^*(u_1)x + \cdots + \tau^{p-1}\rho^*(u_{p-1})x. \tag{2.3}$$

for any $x \in \mathrm{Hom}_{\mathbb{Z}}(S_*(B), \mathbb{Z}/p)$ in general. If $[u_1] = 0$, $1 \in E^{1,0}$ becomes a permanent cycle in the spectral sequence (2.2) for the universal bundle $EC_p \to BC_p$, which contradicts to the contractibility of EC_p. Then by normalizing u if necessary, we may assume

$$[u_1] = u. \tag{2.4}$$

Applying (2.3) in turn to $u_1, \dots, u_{p-1}$, we inductively see from the equality $\bar{\delta}^2 = 0$ that

$$\delta u_i = -\sum_{j<i} u_j u_{i-j} \quad \text{for} \quad i \geq 2. \tag{2.5}$$

Let $\langle x_1, \dots, x_n\rangle_n$ stand for the n-fold Massey product in the sense of Kraines [Kr], where $\langle x_1, x_2\rangle = \pm x_1x_2$. Then by (2.3), (2.4) and (2.5), we obtain that $d_r x$ is represented by an element of $\pm\langle \bar{u}, \dots, \bar{u}, x\rangle_{r+1}$ whose defining system $\{x_{ij}\}_{1\leq i\leq j\leq r+1}$ satisfies $x_{ij} = \rho^*(u_{j-i+1})$ for $j \leq r$, where $x_{i,r+1}$ can be an arbitrary cochain satisfying the condition of defining systems. Hence by [Kr], $\{x_{ij}\}_{1\leq i\leq j\leq r}$ is the pullback of a defining system for

$$\langle u, \dots, u\rangle_k = \begin{cases} \{0\} & k < p \\ \{v\} & k = p. \end{cases} \tag{2.6}$$

Recall the following associativity formula of higher Massey products [May]. Suppose a defining system for $\langle x_1, \dots, x_{n-1}\rangle_{n-1}$ extends to those of $\langle x_{k+1}, \dots, x_n\rangle_{n-k}$. Put $\{x'_{ij}\}_{1\leq i\leq j\leq k+1}$

$$x'_{ij} = \pm x_{ij} \quad \text{for} \quad j \leq k \quad \text{and} \quad x'_{i,k+1} = \sum_{l=k+1}^{n-1} \pm x_{il}x_{ln} \quad \text{for} \quad 2 \leq i \leq k+1. \tag{2.7}$$

Then $\{x'_{ij}\}_{1\leq i\leq j\leq k+1}$ is a defining system for $\langle x_1, \dots, x_k, \langle x_{k+1}, \dots, x_n\rangle_{n-k}\rangle_{k+1}$ and the resulting element x satisfies

$$x = \pm y x_n$$

for some $y \in \langle x_1, \dots, x_{n-1}\rangle_{n-1}$. Consider the defining system of $\langle \bar{u}, \dots, \bar{u}\rangle_{r+r'}$ given by $\rho^*(u_i)$ for $r + r' \leq p$. By the above observation on $d_{r'}x$, we can extend this defining system to that for $\langle \bar{u}, \dots, \bar{u}, x\rangle_{r'+1}$ as (2.7) so that the resulting element x' represents $d_{r'}x$. Moreover, by (2.6) and the above associativity formula, we have

$$d_r x' = \begin{cases} 0 & r + r' < p \\ \pm\bar{v}x & r + r' = p. \end{cases} \tag{2.8}$$

2.2. **Proof of Theorem 1.1.** We prove the result by calculating the spectral sequence (2.2). We first consider the case $\mathrm{h}(E) = 2m+1$. We can easily see that in the spectral sequence for the universal bundle $EC_p \to BC_p$, it holds that $d_r^{p-1,2m+1}uv^m = 0$ and av^{m+1} according as $r < p-1$ and $r = p-1$, where $a \in (\mathbb{Z}/p)^\times$. Then it follows from naturality of the spectral sequence that

$$d_r^{p-1,2m+1}\bar{u}\bar{v}^m = \rho^*(d_r^{p-1,2m+1}uv^m) = \begin{cases} 0 & r < p-1 \\ \rho^*(av^{m+1}) = 0 & r = p-1, \end{cases}$$

implying that $H^{2m+1}(E) \neq 0$.

We next consider the case $\mathrm{h}(E) = 2m$. Let r be the maximum integer such that $\bar{v}^m \in E_1^{s,2m}$ survives at the E_r-term for all $0 \le s \le p-1$. Suppose that $d_r^{s,2m}\bar{v}^m \neq 0$ for some s. Then we have

$$d_r^{r,2m}\bar{v}^m \neq 0. \tag{2.9}$$

If $\bar{v}^m \in E_1^{r-1,2m}$ survives at the $E_{r'}$-term for $r \le r'$ and satisfies $d_{r'}^{r+r'-1,2m-1}x = \bar{v}^m$ for some x, we have

$$d_r^{r,2m}\bar{v}^m \in \pm\langle \bar{u}, \ldots, \bar{u}, \bar{v}^m\rangle_{r+1}, \quad \bar{v}^m \in \pm\langle \bar{u}, \ldots, \bar{u}, x\rangle_{r'+1} \quad \text{and} \quad r + r' \le p,$$

where defining systems for both higher Massey products are described above. Then it follows from (2.8) that

$$d_r^{r,2m}\bar{v}^m = \begin{cases} 0 & r + r' < p \\ \pm\bar{v}x & r + r' = p \end{cases} \tag{2.10}$$

in the E_r-term. The upper case contradicts to (2.9). Let us consider the lower case. If $r' = 1$, $\bar{u}x = \bar{v}$ and then $\beta(\bar{u}x) = 0$. If $r' \ge 2$, $\bar{u}x = 0$ and so $\beta(\bar{u}x) = 0$. Then in both cases, we have $\bar{v}x = \bar{u}(\beta x)$, and so $\bar{v}x$ turns out to be trivial in the E_r-term, which contradicts to (2.9). Therefore we obtain that $\bar{v}^m \in E_1^{r-1,2m}$ is a permanent cycle, implying that $H^{2m}(E) \neq 0$. Suppose next that $d_r^{s,2m-1}x = \bar{v}^m$ for some s. Then for any $r + r' \le p$, we can choose a representative of $d_{r'}^{r+1,2m}\bar{v}^m$ as above, and hence by an argument similar to the above case, we see that $\bar{v}^m \in E_1^{r+1,2m}$ is a permanent cycle, implying that $H^{2m}(E) \neq 0$. Therefore the proof of Theorem 1.1 is completed.

3. Height and category weight

In this section, we interpret the height of a normal cyclic p-fold covering in terms of the category weight introduced by Fadell and Husseini [FH2] and studied further by Rudyak [Ru] and Strom [S]. As a consequence, the relation between the height of a normal cyclic p-fold covering and the Lusternik-Schnirelmann (L-S, for short) category of the classifying map of the covering becomes clear. Recall that the L-S category of a space X, denoted by $\mathrm{cat}(X)$, is the minimum n such that there is a cover of X by $(n+1)$-open sets each of which is contractible in X. In [BG], the L-S category of a space was generalized to a map: The L-S category of a map $f : X \to Y$, denoted by $\mathrm{cat}(f)$, is the minimum integer n such that there exists an open cover $X = U_0 \cup \cdots \cup U_n$ where the restriction of f to U_i is null-homotopic for all i. Observe that

$$\mathrm{cat}(f) \le \mathrm{cat}(1_X) = \mathrm{cat}(X).$$

It is useful to evaluate $\mathrm{cat}(f)$ by the so-called Ganea spaces. Let $G_n(Y)$ be the n^{th} Ganea space of Y and let $\pi_n : G_n(Y) \to Y$ be the projection. See [CLOT] for definition. We know that $\mathrm{cat}(f) \leq n$ if and only if there is a map $g : X \to G_n(Y)$ satisfying $\pi_n \circ g \simeq f$. The homotopy invariant version of the category weight of a space X due to Rudyak [Ru] and Strom [S] is a lower bound for the L-S category of X which refines the cup-length. As in [IK], cohomologically, the idea of the homotopy invariant version of the category weight due to Rudyak and Strom is summarized as

$$\mathrm{wgt}(X;R) = \max\{n \,|\, \pi_n^* : \overline{H}^*(X;R) \to \overline{H}^*(G_n(X);R) \text{ is injective}\},$$

where R is a ring and $\overline{H}^*$ denotes the reduced cohomology. By definition, $\mathrm{wgt}(X;R)$ is bounded above by $\mathrm{cat}(X)$. Given a map $f : X \to Y$, we can easily generalize the above definition for a space to a map as

$$\begin{aligned}\mathrm{wgt}(f;R) = \max\{n \,|\, &\text{there exists } y \in \overline{H}^*(Y;R) \text{ satisfying } f^*(y) \neq 0,\\ &\text{and } \pi_n^*(z) \neq 0 \text{ whenever } f^*(z) \neq 0 \text{ for } z \in \overline{H}^*(Y;R)\}.\end{aligned}$$

Notice that $\mathrm{wgt}(1_X;R) = \mathrm{wgt}(X;R)$ analogously to the L-S category. Obviously, we have

$$\mathrm{cat}(f) \geq \mathrm{wgt}(f;R).$$

Let us consider the relation between the height of a normal cyclic covering and the category weight. Suppose a space Y is path-connected. In general, since the homotopy fiber of the projection $\pi_n : G_n(Y) \to Y$ has the homotopy type of the join of $(n+1)$-copies of ΩY which is n-connected, the induced map $\pi_n^* : H^k(Y;R) \to H^k(G_n(Y);R)$ is an isomorphism for $k < n$ and is injective for $k = n$. See [CLOT]. We specialize to the case $Y = BC_p$. Recall that $G_n(BC_p)$ has the homotopy type of the quotient of the join of the $(n+1)$-copies of C_p by the diagonal free C_p-action, implying that $G_n(BC_p)$ has the homotopy type of an n-dimensional CW-complex. Then the induced map $\pi_n^* : H^k(BC_p;R) \to H^k(G_n(BC_p);R)$ is the zero map for $k > n$. Summarizing, the induced map $\pi_n^* : H^k(BC_p;\mathbb{Z}/p) \to H^k(G_n(BC_p);\mathbb{Z}/p)$ is injective for $k \leq n$ and is the zero map for $k > n$, and hence for a map $f : X \to BC_p$, we have

$$\mathrm{wgt}(f;\mathbb{Z}/p) = \min\{n \,|\, f^* : H^n(BC_p;\mathbb{Z}/p) \to H^n(X;\mathbb{Z}/p) \text{ is non-trivial}\}.$$

Therefore we obtain:

Proposition 3.1. *Let $E \to B$ be a normal cyclic p-fold covering with the classifying map $\rho : B \to BC_p$, where B is path-connected. Then*

$$\mathrm{h}(E) = \mathrm{wgt}(\rho;\mathbb{Z}/p) \leq \mathrm{cat}(\rho) \leq \mathrm{cat}(B).$$

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