Kyoto University

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by

Shin-ichi Ohta and Miklós Pálfia

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DISCRETE-TIME GRADIENT FLOWS AND LAW OF LARGE NUMBERS IN ALEXANDROV SPACES

SHIN-ICHI OHTA AND MIKLÓS PÁLFIA

ABSTRACT. We develop the theory of discrete-time gradient flows for convex functions on Alexandrov spaces with arbitrary upper or lower curvature bounds. We employ different resolvent maps in the upper and lower curvature bound cases to construct such a flow, and show its convergence to a minimizer of the potential function. We also prove a stochastic version, a generalized law of large numbers for convex function valued random variables, this version seems new even in the Euclidean setting. These results generalize those in nonpositively curved spaces (partly for squared distance functions) due to Bačák, Sturm and others, and the lower curvature bound case seems entirely new.

1. INTRODUCTION

In this paper, we consider discrete-time gradient flows for convex functions on *Alexandrov spaces* (X, d) with arbitrary upper or lower curvature bounds. An Alexandrov space is a metric space whose sectional curvature is bounded above or below by some constant in the sense of triangle comparison theorem (see Section 2). The *discrete-time gradient flow* is introduced with an appropriate notion of *resolvent operator* $J_\lambda^f : X \to X$ defined for a fixed geodesically convex function f and a positive number $\lambda > 0$. The operator J_λ^f provides a gradient descent step towards the set of minimizers of f. Under upper and lower curvature bounds, we define our J_λ^f differently. In the case of upper curvature bounds, we employ the standard *Moreau–Yosida*

resolvent:

$$(1.1) \qquad J_\lambda^f(x) := \arg\min_{y \in G} \left\{ f(y) + \frac{1}{2\lambda} d(x,y)^2 \right\}$$

for a closed geodesically convex set G containing a nonempty sublevel set of f. In the case of lower curvature bounds, we define

$$(1.2) \qquad J_\lambda^f(x) := g\text{-}\exp(\lambda \nabla(-f)(x)),$$

where g-exp is the gradient exponential map and $\nabla(-f)(x)$ denotes the gradient vector of $-f$ (see [33, 34] and Sections 3, 4 for these notions). Before discussing the reason why we use these different resolvents, we present our results in this paper.

With these mappings at hand, we define the sequence

$$x_{k+1} := J_{\lambda_k}^f(x_k)$$

for $k \geq 0$ with an arbitrary starting point x_0 and for an a priori given positive sequence $\lambda_k > 0$. We prove the convergence of x_k to a minimizer of f under various, plausible conditions on f and the sequence λ_k. In particular, we generalize the classical results in [12] to arbitrary Alexandrov spaces. Furthermore, our results generalize the ones recently given in [5] for *NPC spaces* (Alexandrov spaces with upper curvature bound by 0) to arbitrary Alexandrov spaces. In the upper curvature bound case, we allow X to be infinite dimensional, while in the lower curvature bound case we formulate our results for finite dimensions for technical reasons, although our techniques would work in infinite dimensions equally well. The most general known results in the literature, according to our knowledge, consider NPC spaces and Riemannian manifolds with nonpositive sectional curvature, see for example [7, 15, 23] just to mention a few among the numerous results.

We also consider the case of $f(x) = \sum_{i=1}^n f_i(x)$, where f_i are also gedoesically convex functions. Then, under the assumption of $\sum_{k=0}^\infty \lambda_k^2 < +\infty$ and the Lipschitz continuity of f_i, we prove that the sequence generated by

$$(1.3) \qquad x_{k+1} := J_{\lambda_k}^{f_1} \circ \cdots \circ J_{\lambda_k}^{f_n}(x_k)$$

converges to a minimizer of f in any Alexandrov space. On the one hand, this result generalizes the ones given for Euclidean spaces in [9, 8, 28, 29] and for NPC spaces in [6]. On the other hand, this is also a generalization of the "no dice" approximation

result given in NPC spaces for the *Karcher mean*, which is the minimizer of $f(x) = \sum_{i=1}^{n} w_i d(x, a_i)^2$ with fixed points $a_i \in X$, in [24, 16]. The Karcher mean [18], or more generally the *p-mean* obtained as the unique minimizer of $f(x) = \sum_{i=1}^{n} w_i d(x, a_i)^p$ for $p \in [1, +\infty)$, is of great interest, see for example [3, 4, 10, 11, 22, 19, 20]. Our general approximation results applied for p-means carry over to positively curved Riemannian setting, for example, compact Lie groups with bi-invariant Riemannian metrics [3, 4, 27, 19, 20].

We also prove a stochastic version of the convergence of the discrete-time flow given in (1.3). In this setting, we assume that

$$f(x) := \int_{F_K(G)} h(x) d\mu(h),$$

where μ is a probability measure supported over the vector lattice of lower semi-continuous, K-convex functions $F_K(G)$ over G, with $K > 0$. Then we prove a *law of large numbers* result for the stochastic sequence

$$x_{k+1} := J_{\lambda_k}^{f_k}(x_k),$$

where f_k is a sequence of independent, identically distributed $F_K(G)$-valued random variables with distribution μ. That is to say, we prove that $x_k \to \mathbb{E}\mu$ almost surely, where $\mathbb{E}\mu$ is the (unique) minimizer of $f(x) = \int_{F_K(G)} h(x) d\mu(h)$. This result provides the Euclidean case of law of large numbers as a special case, and generalizes a result of Sturm [38, 39], given again for the Karcher mean on NPC spaces, to arbitrary Alexandrov spaces and arbitrary convex functions. Sturm [36, 37, 38, 40] used his result in his stochastic approach to the theory of harmonic maps between metric spaces. Also the result of Sturm became useful for the Karcher mean in the case of the NPC space of positive definite matrices [22, 24, 16].

The definitions and properties of J_λ^f distinguish two different kinds of approaches in the lower and upper curvature bound cases. In [2] among many others, for setting up the minimizing movements, the original resolvent (1.1) given in [12] is being used that we also adopt in the upper curvature bound case. Besides technical reasons, the usefulness of (1.1) in discrete-time gradient flows is due to the fact that Alexandrov spaces with upper curvature bounds are simply connected and have unique minimal geodesics in balls with designated radii. We cannot expect these properties in the

lower curvature bound case, the injectivity radius can be 0 even locally. Then it is difficult to control the behavior of discrete-time flows and there are no investigation in this direction as far as the authors know, while continuous-time gradient flows are intensively studied in [33, 34, 25, 31].

To overcome this difficulty, we introduce the other (but natural) construction (1.2) relying on gradient vectors directly. This makes an interesting contrast with (1.1):

$$(1.4) \qquad\qquad \log_{x_{k+1}} x_k = \lambda \nabla f(x_{k+1}),$$

$$(1.5) \qquad\qquad \log_{x_k} x_{k+1} = \lambda \nabla(-f)(x_k),$$

in the upper and lower curvature bounds, respectively, where $\log_x y$ is the direction from x to y. In other words, these two flows provided in the opposite curvature bounds are in reverse relation. In the upper curvature bound case, we take the backward flow for the convex function f, while we take the forward flow for the concave function $-f$ in the lower curvature bound case. In Euclidean spaces, both methods work equivalently well [9, 8, 28, 29]. In general, it seems that the curvature bound determines whether a proximal step (1.1) or a gradient step (1.2) is more suitable from the analytic point of view of discrete flows. For instance, the convexity of squared distance functions, which is the very definition of upper curvature bounds, can give a contraction estimate of discrete-time gradient flows together with (1.4) (estimate $d(x_{k+1}, y_{k+1})$ from above by using $d(x_k, y_k)$ and the convexity of f along a geodesic between x_{k+1} and y_{k+1}). Similarly, the concavity of squared distance functions is useful only with (1.5) (via the convexity of f along a geodesic between x_k and y_k).

2. Alexandrov spaces

We refer to [13] for the basics of metric geometry and Alexandrov spaces. Let (X, d) be a metric space. A continuous curve $\gamma : [0, 1] \to X$ is called a *minimal geodesic* if it satisfies $d(\gamma(s), \gamma(t)) = |s - t| d(\gamma(0), \gamma(1))$ for all $s, t \in [0, 1]$. We say that (X, d) is *geodesic* if any two points $x, y \in X$ admit a minimal geodesic between them. Though minimal geodesics are not necessarily unique, we abuse the notation $x \#_t y$, $t \in [0, 1]$, for denoting a minimal geodesic from x to y. A subset $G \subset X$ is said

to be *geodesically convex* if, for any $x, y \in G$, all minimal geodesics $x\#_t y$ between them are contained in G.

For $\kappa \in \mathbb{R}$, we denote by $\mathbb{M}^2(\kappa)$ a complete, simply connected, 2-dimensional Riemannian manifold of constant sectional curvature κ. For three points $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$, we can take corresponding points $\tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{M}^2(\kappa)$ uniquely up to rigid motions such that

$$d_{\mathbb{M}^2(\kappa)}(\tilde{x}, \tilde{y}) = d(x, y), \quad d_{\mathbb{M}^2(\kappa)}(\tilde{y}, \tilde{z}) = d(y, z), \quad d_{\mathbb{M}^2(\kappa)}(\tilde{z}, \tilde{x}) = d(z, x).$$

We call $\triangle \tilde{x}\tilde{y}\tilde{z}$ a *comparison triangle* of $\triangle xyz$ in $\mathbb{M}^2(\kappa)$.

Definition 2.1. [Alexandrov spaces] A geodesic metric space (X, d) is called an *Alexandrov space of curvature bounded above by κ* if, for any $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$, we have

$$(2.1) \qquad d(y\#_t z, x) \leq d_{\mathbb{M}^2(\kappa)}(\tilde{y}\#_t \tilde{z}, \tilde{x})$$

for any minimal geodesic $y\#_t z$ joining y and z.

Similarly, (X, d) is called an *Alexandrov space of curvature bounded below by κ* if we have

$$(2.2) \qquad d(y\#_t z, x) \geq d_{\mathbb{M}^2(\kappa)}(\tilde{y}\#_t \tilde{z}, \tilde{x})$$

for any minimal geodesic $y\#_t z$.

For instance, if $\kappa = 0$, then (2.1) is calculated as

$$d(x, y\#_t z)^2 \leq (1 - t)d(x, y)^2 + td(x, z)^2 - t(1 - t)d(y, z)^2,$$

and (2.2) is

$$d(x, y\#_t z)^2 \geq (1 - t)d(x, y)^2 + td(x, z)^2 - t(1 - t)d(y, z)^2.$$

By the parallelogram identity, Hilbert spaces have curvature bounded both above and below by 0. Here are some further examples.

Example 2.2. (1) A complete, simply connected Riemannian manifold with the Riemannian distance is an Alexandrov space with curvature bounded above by κ if and only if its sectional curvature is not greater than κ. Typical examples of

nonpositively curved spaces admitting singularities include trees, Euclidean buildings and gluing of nonpositively curved spaces. See [13, §9.1] for further examples.

(2) A complete Riemannian manifold is an Alexandrov space of curvature bounded below by κ if and only if its sectional curvature is not less than κ. Typical examples of nonnegatively curved spaces admitting singularities include the boundaries of convex domains in Euclidean spaces, quotients of nonnegatively curved spaces by isometries (e.g., orbifolds), and the L^2-Wasserstein spaces over nonnegatively curved spaces (see [41, 31]). We refer to [13, §10.2] for further examples.

An important feature of Alexandrov spaces is that angles are well defined between two geodesics γ and η emanating from the same point $\gamma(0) = \eta(0) = x$:

$$\angle_x(\gamma, \eta) := \lim_{s,t \to 0+} \angle \widetilde{\gamma(t)} \tilde{x} \widetilde{\eta(s)},$$

where $\widetilde{\gamma(t)} \tilde{x} \widetilde{\eta(s)}$ is a comparison triangle in $\mathbb{M}^2(\kappa)$. For fixed $x \in X$, we define $\Sigma'_x X$ as the set of unit speed minimal geodesics $\gamma : [0, \delta] \to X$, $\delta > 0$, emanating from x. The angle $\angle_x(\gamma, \eta)$ defines a pseudo-distance on the set $\Sigma'_x X$. The completion of $(\Sigma'_x X / \{\angle_x = 0\}, \angle_x)$ with respect to $\angle_x$ is denoted by $(\Sigma_x X, \angle_x)$, and is called the *space of directions* at $x \in X$.

The *tangent cone* $(C_x X, \sigma_x)$ at $x \in X$ is defined as the Euclidean cone over $(\Sigma_x X, \angle_x)$:

$$C_x X := (\Sigma_x X \times [0, \infty)) / \sim,$$

where $(\gamma, 0) \sim (\eta, 0)$ and

$$\sigma_x((\gamma, s), (\eta, t)) := \sqrt{s^2 + t^2 - 2st \cos \angle_x(\gamma, \eta)}$$

for $(\gamma, s), (\eta, t) \in C_x X$. We denote by o_x the origin $(*, 0) \in C_x X$. For any $u = (\gamma, s), v = (\eta, t) \in C_x X$, we can define their inner product as

$$\langle u, v \rangle := st \cos \angle_x(\gamma, \eta).$$

If X is complete, finite Hausdorff dimensional and has curvature bounded above in the sense of Definition 2.1, then $(\Sigma_x X, \angle_x)$ is an Alexandrov space of curvature bounded above by 1 and $(C_x X, \sigma_x)$ is an Alexandrov space of curvature bounded above by 0. In the case of curvature bounded below, we have the same curvature

bounds for $(\Sigma_x X, \angle_x)$ and $(C_x X, \sigma_x)$, but from below. In the infinite dimensional case, however, this is not the case in general.

By the definition of the angle, we readily have the following (see [13, Corollary 4.5.7]).

Theorem 2.3 (First variation formula). *Let $\gamma : [0, \delta] \to X$ be a geodesic in an Alexandrov space (X, d) with curvature bounded above or below by κ, and assume that X is locally compact in the lower curvature bound case. Put $x := \gamma(0)$ and take $y \in X$ with $d(x, y) < \pi/\sqrt{\kappa}$ if $\kappa > 0$. Then the function $d(t) := d(\gamma(t), y)$ satisfies*

$$(2.3) \qquad \lim_{\epsilon \to 0+} \frac{d(\epsilon) - d(0)}{\epsilon} = -\frac{1}{d(x, y)} \max_{\eta} \langle \gamma'(0), \eta'(0) \rangle,$$

where $\eta : [0, 1] \to X$ runs over all minimal geodesics from x to y.

We remark that η is unique under the upper curvature bound. The inequality '$\leq$' holds in (2.3) for any η in a more general situation without any compactness assumption ([13, Proposition 4.5.2]), and such an inequality is enough in our discussion in the lower curvature bound case. Equality in (2.3) is necessary only in the proof of Lemma 4.6(I).

3. Convex functions on Alexandrov spaces

Let (X, d) be an Alexandrov space with curvature bounded above or below by $\kappa \in \mathbb{R}$. We say that a function $f : X \to (-\infty, \infty]$ is K-*(geodesically) convex* for $K \in \mathbb{R}$ if

$$(3.1) \qquad f(x \#_t y) \leq (1 - t)f(x) + tf(y) - \frac{K}{2}t(1 - t)d(x, y)^2$$

holds for any $x, y \in X$, $t \in [0, 1]$ and any minimal geodesic $x \#_t y$. The 0-convexity will be simply called the convexity.

Definition 3.1. [Absolute gradients] Let $f : X \to (-\infty, \infty]$ be lower semi-continuous and K-convex. Then the (descending) *absolute gradient* of f at $x \in X$ with $f(x) \neq \infty$ is defined by

$$|\nabla_- f|(x) := \max \left\{ 0, \limsup_{y \to x} \frac{f(x) - f(y)}{d(x, y)} \right\}.$$

Note that $|\nabla_- f|(x) \in [0, \infty]$ and also $|\nabla_- f|(x) = 0$ if $f(x) = \inf_{y \in X} f(y)$.

Definition 3.2. [Directional derivatives] For $f : X \to (-\infty, \infty]$, the *directional derivative* of f at x with $f(x) \neq \infty$ in the direction $v \in C_x X$ is defined as

$$D_x f(v) := \liminf_{(\gamma, s) \to v} \left\{ \lim_{t \to 0+} \frac{f(\gamma(st)) - f(x)}{t} \right\},$$

where $(\gamma, s) \in \Sigma'_x X \times [0, \infty) \subset C_x X$.

The above limit along γ indeed exists for lower semi-continuous, K-convex functions. Note that $D_x f(v) \geq -|\nabla_- f|(x) \cdot |v|$ clearly holds. Typical examples of K-convex (or K-concave) functions are squared distance functions. We set $d_a(x) := d(a, x)$ for $a, x \in X$, and denote closed metric balls by

$$\bar{B}_a(r) := \{x \in X : d(a, x) \leq r\}, \quad a \in X, \ r > 0.$$

Proposition 3.3 (Proposition 3.1 in [30]). *Let (X, d) be an Alexandrov space with curvature bounded above by $\kappa > 0$. Then, for any $a \in X$, the function d_a^2 is K-convex on the (geodesically convex) metric ball $\bar{B}_a(r)$ with $2r = (\pi/2 - \epsilon)/\sqrt{\kappa}$ and $K = (\pi - 2\epsilon) \tan \epsilon$ for arbitrary $\epsilon \in (0, \pi/2)$.*

Proposition 3.4 (Lemma 3.3 in [31]). *Let (X, d) be an Alexandrov space with curvature bounded below by $\kappa < 0$. Then, for any $a \in X$, the function $-d_a^2$ is K-convex on the metric ball $\bar{B}_a(r)$ with $K = -2(1 - \kappa(2r)^2)$ for all $r > 0$.*

Slightly more generally, we can take a geodesically convex set $G \subset X$ with $\operatorname{diam} G \leq 2r$ in Propositions 3.3, 3.4.

In the lower curvature bound case, by comparing the convexity of f and the concavity of the squared distance function, one can find the useful notion of gradient vectors as follows (see [33, 34, 25, 31] for details). Let (X, d) be an Alexandrov space of curvature bounded below and $f : X \to (-\infty, \infty]$ be a lower semi-continuous, K-convex function. Then, at every $x \in X$ with $0 < |\nabla_- f|(x) < \infty$, we can find the unique direction $\gamma \in \Sigma_x X$ such that $D_x f(\gamma) = -|\nabla_- f|(x)$ and

$$(3.2) \qquad\qquad D_x f(\eta) \geq -|\nabla_- f|(x) \langle \gamma, \eta \rangle$$

for all $\eta \in \Sigma_x X$. Thus $\nabla(-f)(x) := (\gamma, |\nabla_- f|(x)) \in C_x X$ can be regarded as the *gradient vector* of $-f$ at x. Set also $\nabla(-f)(x) := o_x$ if $|\nabla_- f|(x) = 0$. The gradient vector will be used to define an appropriate resolvent map for f.

4. Resolvent maps

In this section, we introduce our key tool, the *resolvent map* J_λ^f, to construct discrete-time gradient flows for convex functions. We will adopt different definitions of J_λ^f in the upper and lower curvature bound cases. Throughout the section, let $f : X \to (-\infty, \infty]$ be a convex, lower semi-continuous function not identically $+\infty$, and fix the size $\lambda > 0$ of the discrete-time step and a closed, geodesically convex set $G \subset X$ containing a nonempty sublevel set of f.

First, let (X, d) be a complete Alexandrov space with curvature bounded above by κ. If $\kappa > 0$, then we assume $\operatorname{diam} G < \pi/(2\sqrt{\kappa})$. In this case, we employ the standard resolvent map used in, e.g., [17, 26, 2].

Definition 4.1. [Resolvent map, upper curvature bound case] For each $x \in X$, we define

$$(4.1) \qquad J_\lambda^f(x) := \arg\min_{y \in G} \left\{ f(y) + \frac{1}{2\lambda} d(x, y)^2 \right\}.$$

Lemma 4.2. *Let (X, d), G and f be as above. Then there exists a unique point $y \in G$ attaining the minimum (4.1).*

Proof. By Proposition 3.3 and $\operatorname{diam} G < \pi/(2\sqrt{\kappa})$, the function $y \mapsto f(y) + \frac{1}{2\lambda} d(x, y)^2$ is K-convex on G for some $K > 0$. The rest of the argument can be obtained by a straightforward optimization argument, see for example [39, Proposition 1.7]. $\qquad \square$

For any minimal geodesic $\gamma : [0, \delta] \to G$ with $\gamma(0) = J_\lambda^f(x)$, we deduce from the first variation formula (Theorem 2.3) that

$$(4.2) \qquad D_{J_\lambda^f(x)} f(\dot\gamma(0)) - \frac{1}{\lambda} \langle \dot\eta(0), \dot\gamma(0) \rangle \geq 0,$$

where $\eta : [0, 1] \to X$ is any minimal geodesic from $J_\lambda^f(x)$ to x.

Next we consider the lower curvature bound case. In this setting, the definition provided by (4.1) for $J_\lambda^f(x)$ is not convenient, because the squared distance function is no longer convex, but is concave instead. This concavity leads to, however, the advantage of well defined gradient vectors of $-f$. Then we shall define the resolvent map by using an "exponential map" from $C_x X$ to X. Although we can not simply use geodesics since there may be no geodesic with a given initial direction, the gradient curves of the convex function $-d_x^2$ will do the job.

Let (X, d) be a complete, finite dimensional Alexandrov space of curvature bounded below by κ with $\partial X = \emptyset$. Note that, even for $\kappa < 0$, the function $-d_x^2$ is K-convex on balls $\bar{B}_x(r)$ for some $K = K(\kappa, r) < 0$ by Proposition 3.4. Hence we can construct the *gradient flow* $\Phi : [0, \infty) \times X \to X$ of $-d_x^2$, i.e., each curve $\xi(t) = \Phi(t, y)$ satisfies $\dot{\xi}(t) = \nabla(d_x^2)(\xi(t))$ at almost all $t > 0$. The convexity of $-d_x^2$ ensures the uniqueness and contraction of Φ, see [33, 34]. The *gradient exponential map* $g\text{-}\exp_x : C_x X \to X$ is obtained by a re-parametrized scaling of Φ: Define $g\text{-}\exp_x$ as the limit of the map

$$\Phi(s, \cdot) \circ \mathrm{id}_X : (X, e^s d) \to (X, d)$$

as $s \to \infty$, where $e^s d$ is the scaled distance and $\mathrm{id}_X : (X, e^s d) \to (X, d)$ is the identity map. The gradient exponential map enjoys many nice properties, for instance, the curve $\xi(t) = g\text{-}\exp_x(tv)$ satisfies

$$(4.3) \qquad \dot{\xi}(0) = v, \quad \dot{\xi}(t) = \frac{d(x, \xi(t))}{t} \nabla d_x(\xi(t)), \quad d(x, \xi(t)) \leq t|v|.$$

Moreover, the following useful comparison estimate holds.

Lemma 4.3 (Lemma 3.1.2 in [34]). *Put $\xi(t) = g\text{-}\exp_x(tv)$ with $v \in C_x X$. Then, for any $(-K)$-convex function $h : X \to \mathbb{R}$ with $K \geq 0$ and all $t > 0$, we have*

$$h(\xi(t)) \geq h(x) + tD_x h(v) - \frac{K}{2}(t|v|)^2.$$

Remark 4.4. The gradient flow Φ of $-d_x^2$ can be constructed also in proper, infinite dimensional Alexandrov spaces (see [33, Appendix] and [25, 31]). However, the proof of Lemma 4.3 above in [34] essentially requires both $\dim X < \infty$ and $\partial X = \emptyset$. One may consult the argument in [33, Appendix] proving an estimate comparable to Lemma 4.3 (called the *monotonicity* there) along gradient curves parametrized in a different way. However, these curves may be defined only on small intervals.

We are ready to define the resolvent map under lower curvature bound. We abuse the same notation $J_\lambda^f(x)$ as the upper curvature bound case.

Definition 4.5. [Resolvent map, lower curvature bound case] For $x \in X$ with $|\nabla_- f|(x) < \infty$, we define

$$(4.4) \qquad\qquad J_\lambda^f(x) := g\text{-}\exp(\lambda \nabla(-f)(x)),$$

where $\nabla(-f) \in C_x X$ is the gradient vector of $-f$ at x given in Section 3.

The following estimates will play crucial roles in the next section.

Lemma 4.6. *Let (X, d) be a complete Alexandrov space either with curvature bounded above or below by κ.*

(I) *If (X, d) has curvature bounded above by $\kappa > 0$, then also assume $\operatorname{diam} G < \pi/(2\sqrt{\kappa})$. Then we have*

$$(4.5) \qquad d(y, J_\lambda^f(x))^2 \leq d(y, x)^2 - 2\lambda[f(J_\lambda^f(x)) - f(y)]$$

for all $x, y \in G$.

(II) *In the lower curvature bound case, we assume that X is finite dimensional, $\partial X = \emptyset$, and that $\operatorname{diam} G < \infty$ if $\kappa < 0$. Then we have*

$$(4.6) \qquad d(y, J_\lambda^f(x))^2 \leq d(y, x)^2 - 2\lambda[f(x) - f(y)] + \frac{K}{2}(\lambda|\nabla_- f|(x))^2$$

for all $x, y \in G$ satisfying $J_\lambda^f(x) \in G$, where $K = K(\kappa, \operatorname{diam} G) \geq 0$.

Proof. (I) By assumption, the squared distance function is convex (Proposition 3.3). Hence, by Theorem 2.3,

$$d(y, x)^2 \geq d(y, J_\lambda^f(x))^2 - 2\langle \dot\gamma(0), \dot\eta(0) \rangle$$

for $\gamma(t) = J_\lambda^f(x)\#_t y$ and $\eta(t) = J_\lambda^f(x)\#_t x$ (they are unique since $\operatorname{diam} G < \pi/(2\sqrt{\kappa})$). Combine this with (4.2) to get

$$(4.7) \qquad d(y, J_\lambda^f(x))^2 \leq d(y, x)^2 + 2\lambda D_{J_\lambda^f(x)} f(\dot\gamma(0)).$$

Now the convexity of f along γ yields that $D_{J_\lambda^f(x)} f(\dot\gamma(0)) \leq f(y) - f(J_\lambda^f(x))$, so we get from (4.7) that

$$d(y, J_\lambda^f(x))^2 \leq d(y, x)^2 - 2\lambda[f(J_\lambda^f(x)) - f(y)].$$

(II) Put $\xi(\lambda) := g\text{-exp}(\lambda\nabla(-f)(x))$. By Proposition 3.4, the function $-d_y^2$ is $(-K)$-convex on G for some $K = K(\kappa, \operatorname{diam} G) \geq 0$. Thus Lemma 4.3 shows that

$$d(y, \xi(\lambda))^2 \leq d(y, x)^2 + \lambda D_x(d_y^2)(\nabla(-f)(x)) + \frac{K}{2}(\lambda|\nabla_- f|(x))^2.$$

Fixing arbitrary minimal geodesic $\gamma(t) = x\#_t y$, we deduce from the first variation formula (Theorem 2.3) and (3.2) that

$$D_x(d_y^2)(\nabla_- f(x)) \leq -2\langle \dot\gamma(0), \nabla_- f(x) \rangle \leq 2D_x f(\dot\gamma(0)).$$

Finally the convexity of f shows that $f(y) \geq f(x) + D_x f(\dot{\gamma}(0))$. Therefore we obtain

$$d(y, J_\lambda^f(x))^2 \leq d(y, x)^2 - 2\lambda[f(x) - f(y)] + \frac{K}{2}(\lambda|\nabla_- f|(x))^2.$$

$\square$

5. Proximal and sub-gradient methods

The resolvent map J_λ^f can be used to consider proximal point algorithms or, in other words, discrete-time gradient flows for general convex functions in the upper curvature bound case. We start with a basic result that generalizes the one in [5] given in NPC spaces. The algorithm has been used at many places, one of the first occasions was in [12]. The situation is the same as Lemma 4.6(I).

Theorem 5.1. *Let (X, d) be a complete Alexandrov space with curvature bounded above by κ. Let $f : X \to (-\infty, \infty]$ be a convex, lower semi-continuous function and $G \subset X$ be a closed, geodesically convex set containing a sublevel set of f such that $\operatorname{diam} G < \pi/(2\sqrt{\kappa})$ if $\kappa > 0$. Take a positive sequence $\{\lambda_k\}_{k \geq 1}$ with $\sum_{k=1}^\infty \lambda_k = +\infty$. Fix an arbitrary starting point $x_0 \in G$ and put*

$$x_k := J_{\lambda_k}^f(x_{k-1}), \qquad k \geq 1.$$

Then we have $\lim_{k \to \infty} f(x_k) = \inf_{y \in G} f(y)$.

Proof. By the definition (4.1) of J_λ^f, the sequence $f(x_k)$ is monotone non-increasing. Indeed, we have

$$f(J_{\lambda_{k+1}}^f(x_k)) + \frac{1}{2\lambda_{k+1}} d(x_k, J_{\lambda_{k+1}}^f(x_k))^2 \leq f(x_k).$$

Furthermore, by (4.5) in Lemma 4.6, we have for any $y \in G$

$$d(y, x_{k+1})^2 \leq d(y, x_k)^2 - 2\lambda_{k+1}[f(x_{k+1}) - f(y)].$$

This combined with the monotonicity of $f(x_k)$ yields

$$2[f(x_k) - f(y)] \sum_{i=1}^k \lambda_i \leq 2 \sum_{i=1}^k \lambda_i[f(x_i) - f(y)] \leq d(y, x_0)^2 - d(y, x_k)^2,$$

which gives

$$f(x_k) - f(y) \leq \frac{d(y, x_0)^2}{2 \sum_{i=1}^k \lambda_i}.$$

By the choice of λ_k, this implies that $\lim_{k\to\infty} f(x_k) \leq f(y)$ for any $y \in G$. Therefore we obtain $\lim_{k\to\infty} f(x_k) = \inf_{y\in G} f(y)$. □

Remark 5.2. Weak convergence in Alexandrov spaces with upper curvature bounds has been introduced in [14]. The same results for weak convergence as in NPC spaces hold if we restrict the analysis to closed metric balls of diameter at most $\pi/(2\sqrt{\kappa})$. Hence actually one can prove weak convergence to a minimizer (if it exists) in Theorem 5.1 in the same way as in [5] for NPC spaces. Furthermore, if f is K-convex with $K > 0$, then we have (strong) convergence to the unique minimizer y.

In the rest of this section, we set up a discrete-time gradient flow converging to a minimizer of a convex function that is the sum of finitely many convex functions. We adjust the setting of Lemma 4.6 to admit such sum of functions.

Definition 5.3. [Proximal Point Algorithm] Let (X, d) be a complete Alexandrov space either with curvature bounded above or below by κ, and $G \subset X$ be a closed, geodesically convex set satisfying the following:

(I) In the upper curvature bound case, $\operatorname{diam} G < \pi/(2\sqrt{\kappa})$ if $\kappa > 0$;

(II) In the lower curvature bound case, $\dim X < \infty$, $\partial X = \emptyset$, and $\operatorname{diam} G < \infty$ if $\kappa < 0$.

Let $f_i : G \to (-\infty, \infty]$ be a convex, lower semi-continuous function for $i = 1, \ldots, n$. Set $f(x) := \sum_{i=1}^{n} f_i(x)$ and suppose that it is not identically $+\infty$. Take a positive sequence $\lambda_k > 0$ such that $\sum_{k=0}^{\infty} \lambda_k = +\infty$ and also $\sum_{k=0}^{\infty} \lambda_k^2 < +\infty$. Given $x_0 \in G$ and for each $k \geq 0$ and $1 \leq i \leq n$, we set

$$x_{kn+i} := J_{\lambda_k}^{f_i}(x_{kn+i-1}),$$

where the resolvent map is defined by (4.1) or (4.4), assuming that $x_m \in G$ for all $m \geq 0$ in the lower curvature bound case.

Before turning to our result on the convergence of the sequences generated in Definition 5.3, we state an elementary lemma from [9, Lemma 3.4] for later use.

Lemma 5.4. *Let $a_k, b_k, c_k \geq 0$ be sequences such that $a_{k+1} \leq a_k - b_k + c_k$ for any $k \geq 1$, and assume $\sum_{k=1}^{\infty} c_k < +\infty$. Then the sequence a_k converges and also $\sum_{k=1}^{\infty} b_k < +\infty$.*

Theorem 5.5. *Let (X, d), $G \subset X$, $f = \sum_{i=1}^{n} f_i$ and $\{\lambda_k\}_{k\geq 0}$ be as in Definition 5.3. Assume further that X is locally compact, f_i is L-Lipschitz for some $L \geq 1$ and all i, and that $\inf_G f$ is attained at some point. Then x_m converges to some minimizer of f in G as $m \to \infty$.*

Proof. Fix a minimizer $y \in G$ of f.

Upper curvature bound case (I): By (4.5) in Lemma 4.6, we have

$$d(y, x_{kn+i})^2 \leq d(y, x_{kn+i-1})^2 - 2\lambda_k[f_i(x_{kn+i}) - f_i(y)].$$

Summing the above for $1 \leq i \leq n$ implies

$$d(y, x_{kn+n})^2 \leq d(y, x_{kn})^2 - 2\lambda_k \sum_{i=1}^{n}[f_i(x_{kn+i}) - f_i(y)],$$

which is equivalent to

$$(5.1) \quad d(y, x_{kn+n})^2 \leq d(y, x_{kn})^2 - 2\lambda_k \sum_{i=1}^{n}[f_i(x_{kn}) - f_i(y)] + 2\lambda_k \sum_{i=1}^{n}[f_i(x_{kn}) - f_i(x_{kn+i})].$$

The next step is to estimate $\sum_{i=1}^{n}[f_i(x_{kn}) - f_i(x_{kn+i})]$ from above. By (4.1), for any $1 \leq j \leq n$, we have

$$f_j(x_{kn+j}) + \frac{1}{2\lambda_k} d(x_{kn+j}, x_{kn+j-1})^2 \leq f_j(x_{kn+j-1}),$$

which yields by using the L-Lipschitz continuity that

$$d(x_{kn+j}, x_{kn+j-1}) \leq 2\lambda_k \frac{f_j(x_{kn+j-1}) - f_j(x_{kn+j})}{d(x_{kn+j}, x_{kn+j-1})} \leq 2\lambda_k L.$$

Since $d(x_{kn}, x_{kn+i}) \leq \sum_{j=1}^{i} d(x_{kn+j-1}, x_{kn+j})$, this gives also that

$$(5.2) \qquad\qquad d(x_{kn}, x_{kn+i}) \leq 2\lambda_k Li.$$

Furthermore, we have

$$\sum_{i=1}^{n}[f_i(x_{kn}) - f_i(x_{kn+i})] = \sum_{i=1}^{n}\sum_{j=1}^{i}[f_i(x_{kn+j-1}) - f_i(x_{kn+j})]$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{i} Ld(x_{kn+j}, x_{kn+j-1})$$

$$\leq \sum_{i=1}^{n} 2\lambda_k L^2 i = \lambda_k L^2 n(n+1).$$

This combined with (5.1) yields

$$(5.3) \qquad d(y, x_{kn+n})^2 \leq d(y, x_{kn})^2 - 2\lambda_k \sum_{i=1}^{n}[f_i(x_{kn}) - f_i(y)] + 2\lambda_k^2 L^2 n(n+1).$$

Since $f(x_{kn}) - f(y) \geq 0$, Lemma 5.4 implies that the sequence $a_k := d(y, x_{kn})^2$ converges and

$$\sum_{k=0}^{\infty} \lambda_k[f(x_{kn}) - f(y)] < +\infty.$$

Hence, by the assumption $\sum_{k=0}^{\infty} \lambda_k = +\infty$, there exists a subsequence $x_{k_l n}$ such that $\lim_{l\to\infty} f(x_{k_l n}) = f(y)$. Since $x_{k_l n}$ is bounded, by local compactness it has a subsequence converging to a point $z \in G$, which by lower semicontinuity of f must be a minimizer of f. Then, by replacing y with z in the above discussion, the sequence $a_k = d(z, x_{kn})^2$ is convergent and has a subsequence converging to 0. Hence the whole sequence a_k converges to 0, i.e., $x_{kn} \to z$ as $k \to \infty$. Moreover, (5.2) gives

$$d(z, x_{kn+i}) \leq d(z, x_{kn}) + d(x_{kn}, x_{kn+i}) \leq d(z, x_{kn}) + 2\lambda_k Li$$

for all $1 \leq i \leq n$. Since we have $\lambda_k \to 0$ by $\sum_{k=0}^{\infty} \lambda_k^2 < +\infty$, we conclude that $x_{kn+i} \to z$ as $k \to \infty$ for all i. Therefore $x_m \to z$ as $m \to \infty$.

Lower curvature bound case (II): The proof is similar to Case (I). From (4.6) in Lemma 4.6, we get

$$d(y, x_{kn+i})^2 \leq d(y, x_{kn+i-1})^2 - 2\lambda_k[f_i(x_{kn+i-1}) - f_i(y)] + \frac{K}{2}\lambda_k^2 L^2.$$

Summing the above for $1 \leq i \leq n$ yields

$$d(y, x_{kn+n})^2 \leq d(y, x_{kn})^2 - 2\lambda_k \sum_{i=1}^{n}[f_i(x_{kn+i-1}) - f_i(y)] + \frac{K}{2}\lambda_k^2 L^2 n,$$

which is equivalent to

$$d(y, x_{kn+n})^2 \leq d(y, x_{kn})^2 - 2\lambda_k \sum_{i=1}^{n} [f_i(x_{kn}) - f_i(y)]$$

(5.4)

$$+ 2\lambda_k \sum_{i=1}^{n} [f_i(x_{kn}) - f_i(x_{kn+i-1})] + \frac{K}{2}\lambda_k^2 L^2 n.$$

We find by (4.3) and assumption that $d(x_{kn+i-1}, x_{kn+i}) \leq \lambda_k L$, and hence

$$f_i(x_{kn}) - f_i(x_{kn+i-1}) \leq L d(x_{kn}, x_{kn+i-1}) \leq L \sum_{j=1}^{i-1} d(x_{kn+j-1}, x_{kn+j}) \leq \lambda_k L^2 (i-1).$$

Then these bounds combined with (5.4) give

$$d(y, x_{kn+n})^2 \leq d(y, x_{kn})^2 - 2\lambda_k \sum_{i=1}^{n} [f_i(x_{kn}) - f_i(y)] + \lambda_k^2 L^2 n \left(\frac{K}{2} + n - 1 \right).$$

The rest of the argument is identical to Case (I).   □

Remark 5.6. In the lower curvature bound case (II), the assumption that $x_m \in G$ can be met, since $d(y, x_m)$ is bounded as we saw in the proof. Thus, choosing the sequence λ_k appropriately, we can assure that x_m stays inside G.

The above theorem relies on local compactness. In fact, it is known that in the infinite dimensional case we cannot always have convergence under these assumptions [6]. However, if we assume that f is K-convex for positive K, then the assumption of local compactness can be dropped.

Proposition 5.7. *Let* (X, d), $G \subset X$, $f = \sum_{i=1}^{n} f_i$ *be as in Definition* 5.3 *and further assume that* f_i *is* L-*Lipschitz for some* $L \geq 1$ *and all* i, *and that* f *is* K-*convex for some* $K > 0$. *Take* $\lambda_k > 0$ *with* $\lambda_k K < 1$, $\lambda_k \to 0$ *and* $\sum_{k=0}^{\infty} \lambda_k = +\infty$, *and consider a sequence* $\{x_m\}_{m \geq 0}$ *generated by Definition* 5.3. *Then* x_m *converges to the unique minimizer* $y \in G$ *of* f *as* $m \to \infty$.

More concretely, in the upper curvature bound case, $d(x_{kn}, y)^2 \leq a_k$ *holds with* $a_0 := d(x_0, y)^2$ *and* $a_{k+1} := (1 - \lambda_k K)a_k + 2\lambda_k^2 L^2 n(n+1)$ *inductively, that is,*

$$a_{k+1} = \prod_{i=0}^{k} (1 - \lambda_i K) a_0 + 2L^2 n(n+1) \sum_{j=1}^{k} (\lambda_{j-1}^2 \prod_{i=j}^{k} (1 - \lambda_i K) + \lambda_k^2).$$

In the lower curvature bound case, $d(x_{kn}, y)^2 \leq a_k$ similarly holds for $a_0 := d(x_0, y)^2$ and $a_{k+1} := (1 - \lambda_k K)a_k + \lambda_k^2 L^2 n(\frac{K}{2} + n - 1)$ with $K \geq 0$ given in Lemma 4.6(II). Also

$$a_{k+1} = \prod_{i=0}^{k}(1 - \lambda_i K)a_0 + L^2 n \left(\frac{K}{2} + n - 1\right) \sum_{j=1}^{k}(\lambda_{j-1}^2 \prod_{i=j}^{k}(1 - \lambda_i K) + \lambda_k^2)$$

in this case.

Proof. Thanks to the K-convexity with $K > 0$ and the completeness of (X, d), there is a unique minimizer $y \in G$ of f (see, e.g., [39, Proposition 1.7]). For any $x \in G$, by dividing (3.1) with $1 - t$ and letting $t \to 1$, we have

(5.5)
$$\frac{K}{2}d(x, y)^2 \leq f(x) - f(y).$$

Let us consider Case (I), the proof of Case (II) will be similar. By (5.3), we have

$$d(y, x_{kn+n})^2 \leq d(y, x_{kn})^2 - 2\lambda_k[f(x_{kn}) - f(y)] + 2\lambda_k^2 L^2 n(n + 1).$$

Using (5.5), we get

(5.6)
$$d(y, x_{kn+n})^2 \leq (1 - \lambda_k K)d(y, x_{kn})^2 + 2\lambda_k^2 L^2 n(n + 1).$$

Then by induction it is easy to see that $d(x_{kn}, y)^2 \leq a_k$. The explicit formula for a_{k+1} is proved also by induction.

Now we prove $\liminf_{k \to \infty} a_k = 0$ by contradiction. Assume that there are $N \geq 0$ and $c > 0$ such that, for every $k > N$, we have $a_k > c$ and $2L^2 n(n + 1)\lambda_k < cK/2$. Then

$$a_{k+1} = a_k + \lambda_k(2L^2 n(n + 1)\lambda_k - a_k K) \leq a_k - \frac{\lambda_k cK}{2},$$

which is a contradiction, since $\sum_{k=0}^{\infty} \lambda_k = +\infty$. We finally show $\lim_{k \to \infty} a_k = 0$. If $a_k > 2L^2 n(n + 1)\lambda_k/K$, then clearly $a_{k+1} < a_k$. If $a_k \leq 2L^2 n(n + 1)\lambda_k/K$, then

$$a_{k+1} \leq (1 - \lambda_k K)(2L^2 n(n + 1)\lambda_k/K) + 2\lambda_k^2 L^2 n(n + 1) = 2L^2 n(n + 1)\lambda_k/K.$$

Thus we have

$$a_{k+1} \leq \max\{a_k, 2L^2 n(n + 1)\lambda_k/K\},$$

from which we get, for any $l \geq k$,

$$a_{l+1} \leq \max\left\{a_k, (2L^2 n(n + 1)/K) \cdot \max\{\lambda_k, \lambda_{k+1}, \ldots, \lambda_l\}\right\}.$$

Take $\limsup_{l\to\infty}$ and then $\liminf_{k\to\infty}$ of the above to see that $a_k \to 0$. The convergence of the rest of the sequence $d(x_m, y)^2$ to 0 follows from setting up a similar inequality of the form (5.2). $\qquad\square$

For the explicit convergence rate analysis, let us quote a lemma from [28]:

Lemma 5.8. *Let $a_k \geq 0$ be a sequence such that*

$$a_{k+1} \leq \left(1 - \frac{\alpha}{k+1}\right) a_k + \frac{\beta}{(k+1)^2},$$

where $\alpha, \beta > 0$. Then

$$a_k \leq \begin{cases} \frac{1}{(k+2)^\alpha}\left(a_0 + \frac{2^\alpha \beta(2-\alpha)}{1-\alpha}\right) & \text{if } 0 < \alpha < 1; \\ \frac{\beta(1+\log(k+1))}{k+1} & \text{if } \alpha = 1; \\ \frac{1}{(\alpha-1)(k+2)}\left(\beta + \frac{(\alpha-1)a_0 - \beta}{(k+2)^{\alpha-1}}\right) & \text{if } \alpha > 1. \end{cases}$$

From this we obtain that the convergence is sublinear in Proposition 5.7.

6. Law of large numbers and Jensen's inequality

In this section, we give a stochastic discrete-time gradient flow for arbitrary convex (infinite) combinations of convex functions. We will restrict ourselves to K-convex functions with $K > 0$, however, our proofs can be adapted to the case $K = 0$ in the same manner as we have seen in Theorem 5.5, which is a generalized form of Proposition 5.7 in this sense.

Let $G \subset X$ be a closed, geodesically convex set. For $K > 0$, consider the set of all lower semi-continuous, K-convex functions $f : G \to (-\infty, \infty]$ not identically $+\infty$, denoted by $F_K(G)$. We equip $F_K(G)$ with the norm $\|f\| := \sup_{x\in G} |f(x)|$. Let $\mathfrak{P}(F_K(G))$ denote the set of all real-valued probability measures (with separable support) in $F_K(G)$. Notice that the function $g(x) = \int_{F_K(G)} f(x) d\mu(f)$ is also in $F_K(G)$ for any $\mu \in \mathfrak{P}(F_K(G))$. In particular, the set $F_K(G)$ is a convex cone.

Definition 6.1. [Variance] We define the *variance* of $\mu \in \mathfrak{P}(F_K(G))$ by

$$\text{var}(\mu) := \inf_{x\in G} \int_{F_K(G)} f(x) d\mu(f).$$

Fixed $\mu \in F_K(G)$ can be viewed as the distribution of an $F_K(G)$-valued random variable. In this sense, integration with respect to μ can be viewed as taking expectations:

$$\mathbb{E}\varphi := \int_{F_K(G)} \varphi(f)d\mu(f),$$

where $\varphi : F_K(G) \to \mathbb{R}$ is assumed to be measurable and integrable.

Definition 6.2. [Expectation] Let $\mu \in \mathfrak{P}(F_K(G))$. We define the *expectation* of μ as

$$\mathbb{E}\mu := \arg\min_{x \in G} \int_{F_K(G)} f(x)d\mu(f),$$

which is indeed uniquely determined by the K-convexity of $g(x) = \int_{F_K(G)} f(x)d\mu(f)$.

Note that $g(\mathbb{E}\mu) = \text{var}(\mu)$. Using our new notation, we have a generalization of the *variance inequality* in [39, Proposition 4.4] (see also [40, 32] for the *reverse* variance inequality for squared distance functions under lower curvature bounds). Let L_x denote the evaluation operator at $x \in G$ defined as $L_x f := f(x)$. Clearly L_x is a linear functional on the cone $F_K(G)$.

Proposition 6.3 (Variance inequality). *Let $\mu \in \mathfrak{P}(F_K(G))$. Then, for all $x \in G$, we have*

$$(6.1) \qquad d(x, \mathbb{E}\mu)^2 \leq \frac{2}{K}\mathbb{E}\left(L_x - L_{\mathbb{E}\mu}\right) = \frac{2}{K}\int_{F_K(G)}[f(x) - f(\mathbb{E}\mu)]d\mu(f).$$

Proof. Put $g(x) = \int_{F_K(G)} f(x)d\mu(f)$ and note that $g(\mathbb{E}\mu) = \inf_G g$. Then the claim follows from (5.5). $\qquad\square$

Before proceeding to the law of large numbers result, we need the following lemma.

Lemma 6.4. *Let (X, d) be a complete Alexandrov space with curvature bounded above by κ, and assume $\text{diam } G < \pi/(2\sqrt{\kappa})$ if $\kappa > 0$. Then $f \mapsto J_\lambda^f(x)$ for fixed $x \in G$, as a map $F_K(G) \to X$, is Hölder continuous.*

Proof. Let $f, g \in F_K(G)$ and suppose $\|f - g\| < \epsilon$. Set $\bar{f}(y) := f(y) + \frac{1}{2\lambda}d(y, x)^2$ and $\bar{g}(y) := g(y) + \frac{1}{2\lambda}d(y, x)^2$ for fixed $x \in G$, and notice that $\bar{f}, \bar{g} \in F_K(G)$ and $\|\bar{f} - \bar{g}\| < \epsilon$. Assume without loss of generality that $\bar{f}(J_\lambda^f(x)) \geq \bar{g}(J_\lambda^g(x))$. Then the K-convexity of $\bar{f}$ yields (recall (5.5))

$$f(J_\lambda^g(x)) + \frac{1}{2\lambda}d(J_\lambda^g(x), x)^2 \geq \bar{f}(J_\lambda^f(x)) + \frac{K}{2}d(J_\lambda^g(x), J_\lambda^f(x))^2,$$

which is equivalent to

$$f(J_\lambda^g(x)) - g(J_\lambda^g(x)) \geq \bar{f}(J_\lambda^f(x)) - \bar{g}(J_\lambda^g(x)) + \frac{K}{2}d(J_\lambda^g(x), J_\lambda^f(x))^2.$$

This shows that

$$d(J_\lambda^g(x), J_\lambda^f(x))^2 \leq \frac{2}{K}\epsilon.$$

$\square$

Remark 6.5. Lemma 6.4 ensures us that, in the case of upper curvature bound, the map $f \mapsto d(y, J_\lambda^f(x))^2$ is measurable and integrable for any $x, y \in G$, i.e.,

$$(6.2) \qquad \int_{F_K(G)} d(y, J_\lambda^f(x))^2 d\mu(f)$$

exists. In the case of lower curvature bound, the measurability of $f \mapsto d(y, J_\lambda^f(x))^2$ is nontrivial and verified only in special cases. If μ is finitely supported, then integrability is clear. Also if X is a Euclidean space and μ is supported over differentiable functions, then the measurability follows from the continuity of the gradient vectors of convex functions, see Theorem 25.7 in [35].

In the following, we prove a stochastic variant of Proposition 5.7, which extends the law of large numbers proved in [39, Theorem 4.7] to the case of Alexandrov spaces with arbitrary upper or lower curvature bounds, and arbitrary functions on $F_K(G)$.

Theorem 6.6 (Law of large numbers). *Let (X, d) and $G \subset X$ be as in Definition 5.3. Fix $\mu \in \mathfrak{P}(F_K(G))$ supported on L-Lipschitz functions and let $\{f_k\}_{k \geq 0}$ denote a sequence of independent, identically distributed random variables taking values in $F_K(G)$ with distribution μ. Take a positive sequence $\{\lambda_k\}_{k \geq 0}$ with $\lambda_k K < 1$, $\lambda_k \to 0$ and $\sum_{k=0}^{\infty} \lambda_k = +\infty$. Define the sequence $S_k \in G$ recursively as*

$$S_{k+1} := J_{\lambda_k}^{f_k}(S_k), \quad k \geq 0,$$

with an arbitrary starting point $S_0 \in G$, assuming that $S_k \in G$ for all $k \geq 0$ and the integral in (6.2) exists in the lower curvature bound case. Then $S_k \to \mathbb{E}\mu$ almost surely.

Proof. We prove only the upper curvature bound case, the lower curvature bound case is similar. By (4.5) in Lemma 4.6, we have

$$d(y, J_{\lambda_k}^{f_k}(x))^2 \le d(y,x)^2 - 2\lambda_k[f_k(J_{\lambda_k}^{f_k}(x)) - f_k(y)]$$

for all $x, y \in G$. Therefore we have

(6.3) $\quad d(\mathbb{E}\mu, S_{k+1})^2 \le d(\mathbb{E}\mu, S_k)^2 - 2\lambda_k[f_k(S_k) - f_k(\mathbb{E}\mu)] + 2\lambda_k[f_k(S_k) - f_k(S_{k+1})].$

By (4.1), we have

$$f_k(S_{k+1}) + \frac{1}{2\lambda_k}d(S_{k+1}, S_k)^2 \le f_k(S_k),$$

which yields by using the L-Lipschitz continuity that

$$d(S_{k+1}, S_k) \le 2\lambda_k \frac{f_k(S_k) - f_k(S_{k+1})}{d(S_{k+1}, S_k)} \le 2\lambda_k L.$$

Thus we obtain

$$f_k(S_k) - f_k(S_{k+1}) \le Ld(S_{k+1}, S_k) \le 2\lambda_k L^2.$$

This combined with (6.3) yields

$$d(\mathbb{E}\mu, S_{k+1})^2 \le d(\mathbb{E}\mu, S_k)^2 - 2\lambda_k[f_k(S_k) - f_k(\mathbb{E}\mu)] + 4\lambda_k^2 L^2.$$

Taking expectations in f_k conditioned on $\mathcal{F}_{k-1} := \{f_1, \dots, f_{k-1}\}$ and using the variance inequality (6.1), we get

$$\mathbb{E}\left(d(\mathbb{E}\mu, S_{k+1})^2 | \mathcal{F}_{k-1}\right) \le d(\mathbb{E}\mu, S_k)^2 - 2\lambda_k \mathbb{E}[f_k(S_k) - f_k(\mathbb{E}\mu)] + 4\lambda_k^2 L^2$$
$$\le d(\mathbb{E}\mu, S_k)^2 - \lambda_k K d(\mathbb{E}\mu, S_k)^2 + 4\lambda_k^2 L^2,$$

and hence

$$\mathbb{E}\left(d(\mathbb{E}\mu, S_{k+1})^2 | \mathcal{F}_{k-1}\right) \le (1 - \lambda_k K)d(\mathbb{E}\mu, S_k)^2 + 4\lambda_k^2 L^2.$$

Taking expectations again yields

$$\mathbb{E}d(\mathbb{E}\mu, S_{k+1})^2 \le (1 - \lambda_k K)\mathbb{E}d(\mathbb{E}\mu, S_k)^2 + 4\lambda_k^2 L^2.$$

From here proving the convergence $\mathbb{E}d(\mathbb{E}\mu, S_{k+1})^2 \to 0$ can be done in the same way as in the proof of Proposition 5.7 after (5.6). To get a convergence rate estimate one can refer to Lemma 5.8. $\quad\square$

Remark 6.7. Suppose that (X, d) has curvature bounded above by $\kappa > 0$. Fix arbitrary $o \in X$ and let $G := \bar{B}_o(r)$ with $2r = (\pi/2 - \epsilon)/\sqrt{\kappa}$ for $\epsilon \in (0, \pi/2)$. Then, by Proposition 3.3, the function $f_a(x) := d(a, x)^2$ with $a \in G$ is K-convex and Lipschitz continuous on G with $K = (\pi - 2\epsilon) \tan \epsilon > 0$. Take $\mu \in \mathfrak{P}(F_K(G))$ such that $\operatorname{supp} \mu \subset \{f_a : a \in G\}$. Then Theorem 6.6 generalizes Sturm's law of large numbers in [38, 39]. In particular, if $\lambda_k := \frac{1}{2k}$, then we have

$$
\begin{aligned}
S_{k+1} = J_{\lambda_k}^{f_k}(S_k) &= \operatorname*{arg\,min}_{z \in G} \left\{ d(a_k, z)^2 + \frac{d(z, S_k)^2}{2\lambda_k} \right\} \\
&= \operatorname*{arg\,min}_{z \in G} \left\{ \frac{2\lambda_k}{1 + 2\lambda_k} d(a_k, z)^2 + \frac{1}{1 + 2\lambda_k} d(z, S_k)^2 \right\} \\
&= S_k \#_{\frac{1}{k+1}} a_k,
\end{aligned}
$$

where a_k is a G-valued random variable with distribution provided by the pushforward measure of μ under the bijective map $f_a \mapsto a$. In this case, one can reproduce the same sublinear order of convergence $O(1/k)$ as in [39]. More generally, one can consider $f_a(x) := d(x, a)^p$ for any $p \in [2, \infty)$, still $f_a(x)$ being K-convex and Lipschitz continuous on the same G, hence Theorem 6.6 can be applied.

It seems reasonable to expect that, in the upper curvature bound case in Remark 6.7, one can take any $2r < \pi/\sqrt{\kappa}$ even though the functions f_a are then not convex on whole G. This is motivated by the results in [1] on the existence and uniqueness of the center of mass in Riemannian manifolds.

Remark 6.8. Theorem 6.6 generalizes the law of large numbers from Euclidean spaces to Alexandrov spaces, moreover, to the case of measures supported over the vector lattice of K-convex Lipschitz functions. We recover the original law of large numbers in Hilbert spaces by choosing X to be a Hilbert space in Remark 6.7. Also the setting in Remark 6.7 is of interest if we choose X to be a sphere, or any compact Lie group with a bi-invariant Riemannian metric.

Using our law of large numbers, we have an alternative proof for Jensen's inequality of Kuwae [21], along the line of the second proof of [39, Theorem 6.2].

Proposition 6.9 (Jensen's inequality)**.** *Let X be a complete Alexandrov space with curvature bound above by $\kappa > 0$, and $G \subset X$ be a closed, geodesically convex set*

with $\operatorname{diam} G < \pi/(2\sqrt{\kappa})$. *Take a probability measure* μ *on* G *and a convex, lower semi-continuous function* $f : G \to \mathbb{R}$. *Then we have*

$$f(\mathbb{E}\mu) \leq \mathbb{E}f,$$

where $\mathbb{E}\mu := \arg\min_{y \in G} \int_G d(a, y)^2 d\mu(a)$ *and* $\mathbb{E}f := \int_G f(a) d\mu(a)$.

Proof. Choose a sequence of independent, identically distributed random variables Y_k with values in G, and with distribution μ. Let $S_k \in G$ be defined as in Remark 6.7, i.e., $S_1 := Y_1$ and $S_{k+1} := S_k \#_{\frac{1}{k+1}} Y_{k+1}$. Similarly, let $Z_k \in \mathbb{R}$ be defined as $Z_1 := f(Y_1)$ and $Z_{k+1} := Z_k \#_{\frac{1}{k+1}} f(Y_{k+1})$. We can explicitly write as

$$Z_{k+1} = Z_k \#_{\frac{1}{k+1}} f(Y_{k+1}) = \frac{k}{k+1} Z_k + \frac{1}{k+1} f(Y_{k+1}).$$

By Theorem 6.6, we have $S_k \to \mathbb{E}\mu$ and $Z_k \to \mathbb{E}(f_*\mu) = \mathbb{E}f$, where $f_*\mu$ denoted the push-forward of μ. We proceed by induction showing

(6.4) $$f(S_k) \leq Z_k.$$

For $k = 1$, this clearly holds. For general $k \geq 1$, we have by induction

$$f(S_{k+1}) = f(S_k \#_{\frac{1}{k+1}} Y_{k+1})$$

$$\leq \frac{k}{k+1} f(S_k) + \frac{1}{k+1} f(Y_{k+1})$$

$$\leq \frac{k}{k+1} Z_k + \frac{1}{k+1} f(Y_{k+1}) = Z_{k+1}$$

showing (6.4). Hence, by the lower semi-continuity of f, we obtain

$$f(\mathbb{E}\mu) \leq \liminf_{k \to \infty} f(S_k) \leq \liminf_{k \to \infty} Z_k = \mathbb{E}f$$

and complete the proof. $\qquad\square$

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DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: sohta@math.kyoto-u.ac.jp

E-mail address: palfia.miklos@aut.bme.hu