Stochastic Komatu–Loewner Evolutions*

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Abstract

Loewner equation is a differential equation for conformal mappings that can be used to describe evolution of a family of simply connected planar domains. It was introduced by C. Loewner in 1923 in his work on the Bieberbach conjecture. Oded Schramm observed and conjectured in 2000 that scaling limit of many two-dimensional lattice models in statistical physics can be described by Loewner evolutions with Brownian motions as the driving function. Many of these conjectures are latter confirmed in a series of joint work by G. Lawler, O. Schramm and W. Werner and by S. Smirnov. On the other hand, Y. Komatu extended Loewner equation to circularly slit annuli in 1950 but in the left derivative sense. The aim of this series of lectures is to survey some recent progress in the study of Komatu-Loewner evolutions and its stochastic counterpart in the canonical slit domains, with emphasis on probabilistic methods.

Contents

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1 Introduction

1.1 Loewner equation and SLE

We begin with a short review on the classical results originating from Loewner [22] of 1923. We use a symbol $\mathbb{H}$ to denote the upper half-plane $\{\, z \in \mathbb{C} \,;\, \Im z > 0 \,\}$ in the complex plane $\mathbb{C}$. Let $\gamma(t)$ be a simple curve such that $\gamma(0) \in \partial\mathbb{H}$ and $\gamma(0,t] \subset \mathbb{H}$. Then the domain $\mathbb{H} \setminus \gamma(0,t]$ is simply connected. By the Riemann mapping theorem, there exists a conformal mapping $g_t \colon \mathbb{H} \setminus \gamma(0,t] \to \mathbb{H}$, which is unique up to some normalization. We adopt the *hydrodynamic normalization at infinity*: $\lim_{z\to\infty}(g_t(z) - z) = 0$. By this normalization, the point at infinity is mapped to itself by g_t, and the other part of "boundary" of $\mathbb{H}\setminus\gamma(0,t]$ is mapped to $\partial\mathbb{H}$. Here, we regard the Jordan arc $\gamma(0,t)$ to be split into its "left" and "right" sides as in Figure 1. In this figure, g_t is extended to a homeomorphism between the red "boundary curves" of $\mathbb{H}\setminus\gamma(0,t]$ and of $\mathbb{H}$. (See e.g. Chapter 14, Section 3 of [13] for the boundary correspondence induced from conformal mappings.) In particular, the limit $\xi(t) := g_t(\gamma(t)) = \lim_{z\to\gamma(t)} g_t(z)$, the image of the tip of the curve γ, always exists. Viewed as a function of t, it will be called the *driving function* below.

In the previous paragraph, $\lim_{z\to\xi} \Im g_t(z) = 0$ if $\xi \in \partial\mathbb{H} \setminus \{\gamma(0)\}$, where $\Im g_t(z)$ is the imaginary part of $g_t(z)$. Hence g_t is extended to an analytic function on $\mathbb{C}\setminus(\gamma[0,t]\cup\{\, z \,;\, \overline{z} \in \gamma[0,t] \,\})$ by the Schwarz reflection principle. On account of the hydrodynamic normalization at ∞, the Laurent expansion

$$g_t(z) = z + \frac{c(t)}{z} + o\left(\frac{1}{z}\right) \quad \text{near } \infty$$

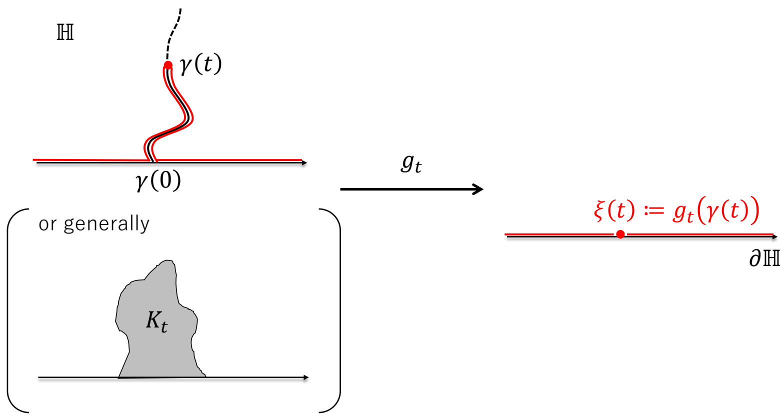

Figure 1: A version of the Riemann mapping theorem.

holds. The coefficient $c(t)$ of the z^{-1} term is positive and measures the "size" of the trace $\gamma(0,t]$. In fact, most of the above-mentioned statements on g_t and $c(t)$ hold if we replace the simple curve γ by a general family of increasing *compact* $\mathbb{H}$*-hulls* K_t. Here, a bounded set $K_t \subset \mathbb{H}$ is called a compact $\mathbb{H}$-hull if $\mathbb{H} \setminus K_t$ is a simply connected domain. See Figure 1. The associated coefficient $c(t)$ is called the *half-plane capacity* of K_t.

For a simple curve γ, the half-plane capacity $c(t)$ is known to be a strictly increasing continuous function of t. If γ is reparametrized so that $c(t) = 2t$, then $g_t(z)$ satisfies the *(chordal) Loewner equation*

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z. \tag{1.1}$$

Here, again $\xi(t) = g_t(\gamma(t))$. Note that, for each fixed initial point $z \in \mathbb{H}$, this is an ordinary differential equation with "unknown variable" $g_t(z)$. Conversely, given an arbitrary continuous function $\xi(t)$ taking values in $\partial\mathbb{H} = \mathbb{R}$, we can solve the initial value problem (1.1) for any $z \in \mathbb{H}$ up to the maximal time $t_z \in (0, \infty]$. Since the right hand side of (1.1) is Lipschitz in $g_t(z)$, such a solution is unique. Let $D_t = \{\, z \in \mathbb{H} \; ; \; t_z > t \,\}$. Then D_t is a simply connected domain, $g_t \colon D_t \to \mathbb{H}, z \mapsto g_t(z)$ is a conformal mapping, and $K_t = \overline{\mathbb{H} \setminus D_t}$ is a compact $\mathbb{H}$-hull that is increasing in t. As a conclusion, we can say that the time-evolution of an increasing family of two-dimensional compact sets K_t is fully described by the one-dimensional real-valued function $\xi(t)$ via (1.1), which will be of great use in the analysis of K_t.

O. Schramm [27] in 2000 investigated the case $\xi(t) = \sqrt{\kappa} B_t \overset{d}{=} B_{\kappa t}$, $\kappa > 0$

in (1.1). Here, B_t is the one-dimensional standard Brownian motion. The resulting maps g_t (and hulls K_t) are called the *stochastic Loewner evolution* or *Schramm–Loewner evolution* with parameter κ. We abbreviate it to SLE_κ. SLE_κ on any simply connected domain D is further defined by pulling the above hulls K_t back to D by an appropriate conformal mapping. SLE_κ so defined is a powerful tool to study two-dimensional critical systems in statistical physics. Indeed, by Schramm [27], Lawler, Schramm and Werner [21], Smirnov [28] and many other authors, it has been proved to be the scaling limit of various lattice models in two dimension. Here is part of the list of the corresponding lattice models:

SLE_2	loop-erased random walk [27, 21]
SLE_8	uniform spanning tree [27, 21]
SLE_6	critical percolation exploration process [28]
SLE_3	critical Ising model [6]
$\mathrm{SLE}_{8/3}$	self-avoiding random walk (conjecture)

1.2 Extension to multiply connected domains

We consider the case in which the domain is multiply connected. In Figure 2, a simple curve γ now lies in an upper half-plane D with two holes removed. Although the domain $D \setminus \gamma(0,t]$ is triply connected, not simply connected, a generalization of Riemann's mapping theorem is still available. There exists a unique conformal mapping $g_t \colon D \setminus \gamma(0,t] \to D_t$ with the hydrodynamic normalization at ∞, and D_t is an upper half-plane with two horizontal slits removed. We call such an upper half-plane with finitely many slits removed a *standard slit domain*. Note that, although standard slit domains are adequate for our theory, there are other choices of "canonical domains" such as circularly slit disks and circularly slit annuli (Figure 3).

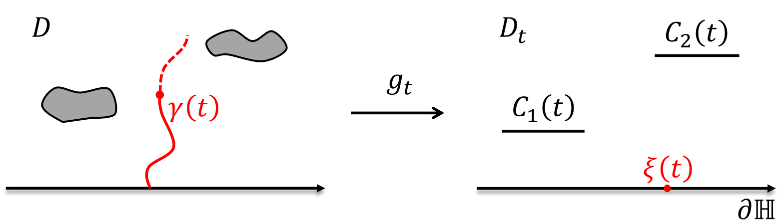

Figure 2: A simple curve γ in a triply connected domain.

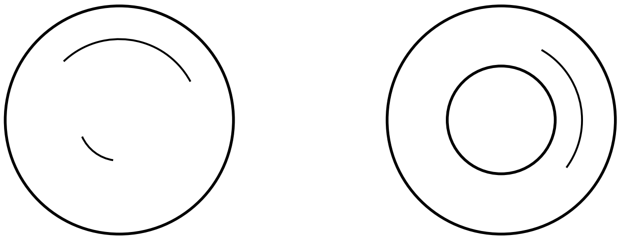

Figure 3: A circularly slit disk and circularly slit annulus.

Different from the simply-connected case, canonical domains are not necessarily conformally equivalent to each other even if they are of the same "type". For example, every doubly connected domain is conformally equivalent to an annulus, and two annuli are conformally equivalent if and only if the ratios between their outer and inner radii are equal. For this reason, in the multiply-connected case above, the characterization of the target domain D_t enters into the picture. As $\gamma(0,t]$ grows in D, the standard slit domain D_t, which is a representative of the conformal equivalence class of $D \setminus \gamma(0,t]$, evolves.

In 1950, Y. Komatu [17] studied the Loewner equation on multiply connected domains. He derived the Loewner-type equation for $g_t(z)$ on circular slit annuli. On the basis of Komatu's idea, Bauer and Friedrich [3] studied the chordal case in which D is a standard slit domain as well as D_t and derived the *(chordal) Komatu–Loewner equation*

$$\partial_t g_t(z) = -2\pi \Psi_{D_t}(g_t(z), \xi(t)). \tag{1.2}$$

They defined the vector field $\Psi_{D_t}(z,\xi)$ on the right-hand side of (1.2) purely in terms of complex analysis. On the other hand, G. Lawler [19] gave a probabilistic description of Ψ_{D_t}. He introduced the *excursion reflected Brownian motion (ERBM)* and identified $\Psi_{D_t}(z,\xi)$ with the *complex Poisson kernel* of ERBM. In these lectures, we shall use the *Brownian motion with darning (BMD)* instead of ERBM. In fact, if the underlying domain is doubly connected, then BMD coincides with ERBM in law (see the paragraph just after Theorem 2.7).

Let us give a close look at (1.2). Its right-hand side depends not only on the driving function $\xi(t)$ but also the target domain D_t via the kernel Ψ_{D_t}. It turns out that given the driving function and an initial standard slit domain D_0, D_t together with $g_t(z)$ is a part of the solution to (1.2). Since D_t is a standard slit domain, it is completely determined by specifying the slits $C_j(t)$, $j = 1, \ldots, N$ ($N = 2$ in Figure 2), and $C_j(t)$'s are determined by the coordinates of their endpoints. Thus, our problem is to find the differential equation for the endpoints of $C_j(t)$. In addition, we have another problem on the t-differentiablity of $g_t(z)$. In the above mentioned works, the equation (1.2) is established only in the sense of left t-derivate of $g_t(z)$. Equations in this sense do not have unique solutions. In order to uniquely characterize the solution, we need to establish (1.2) in a genuine derivative sense.

These lectures are based on the following three recent papers: Chen, Fukushima and Rohde [11], Chen and Fukushima [10], and Chen, Fukushima and Suzuki [12]. We shall see how probabilistic methods, such as BMD, martingale theory and so on, help us to settle the problems in the previous paragraph. Moreover, as an analogue to SLE_κ, we shall define the *stochastic Komatu–Loewner evolution* (SKLE for brevity) and study its property. Finally, note that we do not discuss the convergence of discrete random models to SKLEs. On multiply connected domains, there are only a few results on loop-erased random walks. Other discrete models are yet to be investigated.

2 Conformal mapping and Brownian motion with darning

2.1 Conformal mappings

Let $D \subset \mathbb{C}$ be an open set. A function $f\colon D \to \mathbb{C}$ is *analytic* if the limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists for every $z \in D$. Here are famous properties of analytic functions:

Fact. (1) An analytic function is C^∞.

(2) Let u and v be the real and imaginary parts, respectively, of an analytic function f, that is, $f(z) = u(z) + iv(z)$. By the definition of the derivative f', we have

$$\begin{aligned} f'(z) &= u_x + iv_x = \frac{1}{i}u_y + i \cdot \frac{1}{i}v_y \\ &= v_y - iu_y. \end{aligned}$$

Hence the *Cauchy–Riemann equation* holds:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \text{in } D,$$

which implies $\Delta u = \Delta v = 0$ in D. In other words, u and v are harmonic in D.

Question. What kind of harmonic functions are the real (or imaginary) part of analytic functions in a domain D?

Answer. Those functions which enjoy the *zero period condition* in D.

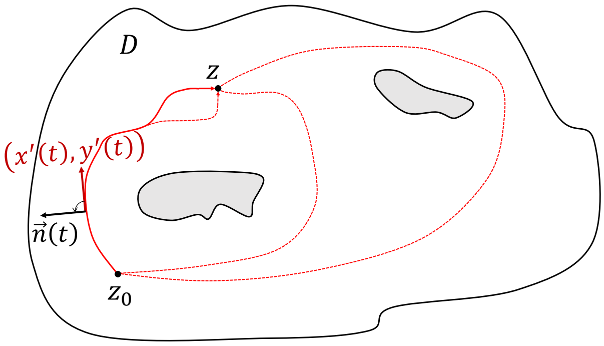

Figure 4: A curve γ, tangent (x', y') and normal $\vec{n}$.

Let us recall how this condition is deduced. Fix a base point $z_0 \in D$. For any $z \in D$, we take a smooth curve γ connecting z_0 to z with $\gamma(t) = (x(t), y(t))$, $0 \le t \le T$. Here, we identify a complex number $x + iy \in \mathbb{C}$ with the two-dimensional vector $(x, y) \in \mathbb{R}^2$. For an analytic function $f = u + iv$ in D, we have

$$\begin{aligned}
u(z) - u(z_0) &= \int_0^T \frac{d}{dt} u(x(t), y(t))\, dt = \int_0^T (u_x x'(t) + u_y y'(t))\, dt \\
&= \int_0^T (u_x, u_y) \cdot (x'(t), y'(t))\, dt = \int_0^T (v_y, -v_x) \cdot (x'(t), y'(t))\, dt \\
&= -\int_0^T (v_x, v_y) \cdot (y'(t), -x'(t))\, dt \\
&= -\int_0^T \nabla v \cdot \overbrace{\left(\frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} - \frac{x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \right)}^{=\vec{n}(t)} d\sigma \\
&= -\int_\gamma \frac{\partial v}{\partial n}\, d\sigma. \qquad (2.1)
\end{aligned}$$

Here, $d\sigma = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$ is the arc length measure, and $\vec{n}(t)$ is the unit normal vector of γ. Hence, if γ is a closed smooth curve in D, then

$$\int_\gamma \frac{\partial v}{\partial n}\, d\sigma = 0.$$

This integral is called the period of v (around the hole surrounded by γ). Thus, v has zero period.

Conversely, if v is a harmonic function in D satisfying the zero period condition, we can use (2.1) to define a function $u(z)$. By letting γ approach to z horizontally and vertically as in Figure 4, we obtain $u_x = v_y$ and $u_y = -v_x$. Hence $f = u + iv$ is analytic and unique up to an additive real constant.

If D is simply connected, then the zero period condition always holds by the Green–Gauss formula

$$\int_D \nabla v \cdot \nabla \varphi + \int_D \Delta v \cdot \varphi = \int_{\partial D} \frac{\partial v}{\partial n} \varphi \, d\sigma.$$

Question. Find an infinitesimal generator $\mathcal{L}$ or a diffusion X_t in D so that its harmonic functions on D are harmonic functions in the classcal sense and have the zero period condition.

Answer. *Brownian motion with darning* (abbreviated as BMD).

Reflected Brownian motion (RBM for short) has the same property owing to the Neumann boundary condition, but we rule it out here. Indeed, the Neumann condition is too strong to capture the property of analytic functions in D. Moreover, we should note the following observation as well: Let g_t be the conformal mapping onto a standard slit domain in Figure 2. Then, the imaginary part $v = \Im g_t$ takes a constant value on each K_i. We shall see below that such a boundary value property is reflected in (the domain of) $\mathcal{L}$ or X_t.

2.2 Brownian motion with darning

What is *Brownian motion with darning* (补丁布朗运动)? “Darning” (补丁) means “mending holes”. In what follows, we “darn” the holes of a space and define a “Brownian motion” on this space.

Let E be a domain in $\mathbb{R}^d$. Until the end of the next subsection, we work with a general dimension d. Let $K_1, K_2, \dots, K_N$ be N disjoint non-polar compact subsets of E. In two-dimension, the condition for a connected compact set to be non-polar is that it contains more than one point. We set $D = E \setminus \bigcup_{j=1}^N K_j$. Note that D can be disconnected. We identify each K_i with a single point a_i^* and consider the new state space $D^* = D \cup \{a_1^*, \dots, a_N^*\}$ equipped with the quotient topology[1]. Let m be the Lebesgue measure

[1]In other words, a fundamental neighborhoods system of each point a_i^* is given by $\{(U \setminus K_i) \cup \{a_i^*\} \,;\, U \text{ is an open subset of } E \text{ containing } K_i\}$. In particular, D^* is a locally compact, Hausdorff, and second countable space and thus is metrizable. (However,

in D, and we define a measure m^* on D^* by $m^*(A) = m(A \cap D)$, i.e., $m^*(\{a_1^*, \dots, a_N^*\}) = 0$.

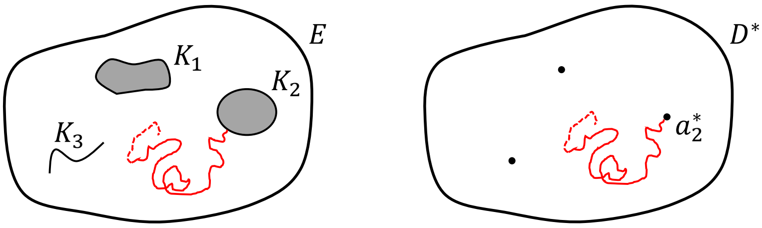

Figure 5: The quotient space D^* of E and the BMD on it.

Definition 2.1. A Brownian motion with darning (BMD) X_t^* is an m^*-symmetric diffusion on D^* such that

1) Its part process in D is the killed Brownian motion in D;

2) It admits no killings on $\{a_1^*, \dots, a_N^*\}$.

Remark 2.2. 1) "m^*-symmetric" means the transition semigroup of X_t^* is symmetric in $L^2(D^*, m^*)$.

2) "No killings on $\{a_1^*, \dots, a_N^*\}$" means $\mathbb{P}\left(X_{\zeta-}^* = a_i^*,\ \zeta < \infty\right) = 0$ for all i. Here, ζ is the lifetime of X_t^*.

3) Since $K^* = \{a_1^*, \dots, a_N^*\}$ satisfies $m^*(K^*) = 0$, X_t^* spends zero time on K^*.

We list some examples:

Example 2.3. Let $E = \mathbb{R}^2$, $N = 1$ and K_1 be a compact set with connected boundary. Then $D^* = (\mathbb{R}^2 \setminus K_1) \cup \{a_1^*\}$ is still homeomorphic to the plane.

Example 2.4. Let $E = \mathbb{R}^2$, $N = 1$ and K_1 be an annulus. Then $D^* = (\mathbb{R}^2 \setminus K_1) \cup \{a_1^*\}$ is homeomorphic to the plane with a sphere sitting on top of it. See Figure 6.

Example 2.5. Let $E = \mathbb{R}$, $N = 1$ and $K_1 = [0, 1/3] \cup [2/3, 1]$. Then $D^* = (\mathbb{R} \setminus K_1) \cup \{a_1^*\}$ is homeomorphic to a knotted curve. See Figure 7.

The following two theorems tell us that BMD is an answer to the question at the end of Section 2.1:

(D^*, m^*) does not have some of the nice geometric properties which often appear in the study of metric measure spaces. For example, it does not enjoy the volume doubling property with respect to the intrinsic metric.)

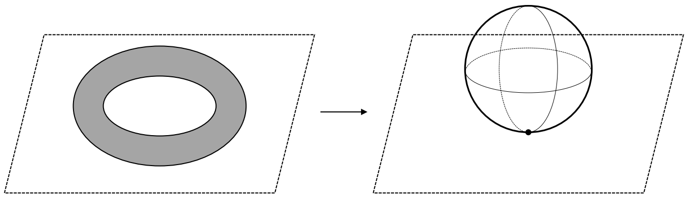

Figure 6: A sphere touches the plane at the origin.

Figure 7: A knotted curve.

Theorem 2.6 (Chen and Fukushima [8, Theorem 7.7.3])**.** *BMD exists and is unique in law.*

Theorem 2.7. *A continuous ($d = 2$) or quasi-continuous ($d \geq 3$) function on D^* is X^*-harmonic if and only if it is a harmonic function in the classical sense in D and has zero period condition.*[2]

Concerning the uniqueness in Theorem 2.6, we note that, if $E = \mathbb{C}$ and $K = \overline{\mathbb{D}}$, then BMD coincides with *excursion reflected Brownian motion* (ERBM for short) introduced by Lawler [19]. Since its generator as a strong Feller process is specified by Drenning [14], we can prove that BMD and ERBM has the same law. This case corresponds to the *one-point extension* of Brownian motion. See [8, Chapter 7] and [9].

To prove Theorem 2.6, we shall take a Dirichlet form approach. Imagine that there is an electronic network with some potential. Then collapsing K_i into a_i^* corresponds to shorting the network. In the Dirichlet form approach, this intuition is realized as follows: We start at standard Brownian motion on $\mathbb{R}^d$. As is well known, its generator is $(1/2)\Delta$, and its Dirichlet form $(\mathbf{D}, \mathcal{F}^{\mathbb{R}^d})$ is given by

$$\mathbf{D}(u,v) \left(= \left(-\frac{1}{2}\Delta u, v \right)_{L^2(\mathbb{R}^d)} \right) = \frac{1}{2} \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, dx,$$
$$\mathcal{F}^{\mathbb{R}^d} = W^{1,2}(\mathbb{R}^d) = \{\, f \in L^2 \,;\, \nabla f \in L^2 \,\}.$$

The same Dirichlet integral corresponds to absorbing Brownian motion X^E

[2]We revisit this theorem in Theorem 2.9.

on E, but its domain of definition is replaced by

$$\mathcal{F}^E \left(= \{ f \in W^{1,2}(\mathbb{R}^d) \; ; \; f = 0 \text{ on } E^c \} \right) = \overline{C_c^\infty(E)}^{\mathbf{D}_1}.$$

Here, $\mathbf{D}_1 = \mathbf{D} + (\cdot,\cdot)_{L^2}$. Now, the bilinear form $(\mathcal{E}, \mathcal{F}^*)$ defined in the following way represents the "short circuit":

$$\mathcal{F}^* = \{ f \in \mathcal{F}^E \; ; \; f = \text{const. on } K_i, \; 1 \le i \le N \} \subset L^2(D^*, m^*)$$

and

$$\begin{aligned} \mathcal{E}(f,f) = \mathbf{D}(f,f) &= \frac{1}{2}\int_D |\nabla f|^2 \, dx + \underbrace{\frac{1}{2}\int_{\bigcup_i K_i} |\nabla f|^2 \, dx}_{=0} \\ &= \frac{1}{2}\int_D |\nabla f|^2 \, dx. \end{aligned}$$

The bilinear form $(\mathcal{E}, \mathcal{F}^*)$ is actually a regular (and strongly local) Dirichlet form on $L^2(D^*, m^*)$, and so there is a unique diffusion X_t^* associated with it. This process is exactly BMD. For the law uniqueness, see e.g. [11, Theorem 2.2].

$\mathcal{F}^*$ is also written as follows:

$$\begin{aligned} \mathcal{F}^* &= \{ u \in W_0^{1,2}(E) \; ; \; u = \text{const. on } K_i \text{ q.e.} \} \\ &= W_0^{1,2}(D) + \text{span}\{u_1, \ldots, u_N\}, \end{aligned} \tag{2.2}$$

where $u_i(x) = \mathbb{E}_x \left[e^{-\sigma_K} \; ; \; X_{\sigma_K}^E \in K_i \right]$ and $K = \bigcup_i K_i$. (Throughout these lectures, the symbol σ_B stands for the first hitting time of a stochastic process, specified by the context, to a set B.) Each u_i is a continuous function on D with boundary values

$$u_i(x) = \begin{cases} 1 & \text{on } K_i \\ 0 & \text{on } K_j \text{ with } j \neq i \\ 0 & \text{on } \partial E \end{cases}$$

and satisfies $(1-\Delta)u_i = 0$. In addition, if we set $\varphi_i(x) = \mathbb{P}_x\left(X_{\sigma_K}^E \in K_i\right)$, then $u_i - \varphi_i = 0$ on $K \cup \partial E$. We have used u_i rather than φ_i in (2.2) because u_i belongs to $L^2(D^*, m^*)$ while φ_i does not.

Let $(A^*, D(A^*))$ be the L^2-generator of X^* (or $(\mathcal{E}, \mathcal{F}^*)$). The condition for u to be an element of $D(A^*)$ is that there exists $f \in L^2(D^*, m^*)$ such that

$\mathcal{E}(u, v) = -(f, v)_{L^2}$ for all $v \in \mathcal{F}^*$. (In this case $A^* u = f$.) Hence we have

$$
\begin{aligned}
& u \in D(A^*) \\
\iff & \frac{1}{2} \int_D \nabla u \cdot \nabla v \, dx = -\int_D f v \, dx, \quad \forall v \in C_c^\infty(D) \cup \{u_1, \dots, u_N\} \\
\iff & \begin{cases} \dfrac{1}{2} \Delta u = f \quad \text{in } D \text{ as a distribution} \\ \dfrac{1}{2} \displaystyle\int_D \nabla u \cdot \nabla u_i \, dx = -\int_D f u_i \, dx, \quad i = 1, \dots, N. \end{cases}
\end{aligned}
\tag{2.3}
$$

Now, we define the *flux* at a_i^* by

$$
\mathcal{N}(u)(a_i^*) \mathrel{\widehat{=}} \int_D (\nabla u \cdot \nabla u_i + u_i \Delta u) \, dx.
$$

Then the second identity in (2.3) is equivalent to

$$
\mathcal{N}(u)(a_i^*) = 0.
$$

We note that, when ∂K_i is smooth,

$$
\mathcal{N}(u)(a_i^*) = \int_{\partial K_i} \frac{\partial u}{\partial n} \, d\sigma
$$

by the Green–Gauss formula. (Compare this condition with RBM. The Neumann boundary condition is given by $\partial u / \partial n = 0$ on the boundary.)

2.3 Harmonic functions of X^*

Let $O \subset D^*$ be an open set. We say that a function u is X^*-*harmonic* in O if, for any relatively compact connected open subset O_1 of O,

$$
\mathbb{E}_x \left[|u(X^*_{\tau_{O_1}})| \right] < \infty^3 \quad \text{and} \quad u(x) = \mathbb{E}_x \left[u(X^*_{\tau_{O_1}}) \right]
$$

hold for all $x \in O_1$. (The equivalent condition is that $u \in \mathcal{F}^*_{\mathrm{loc}}(O)$ and that $\mathcal{E}(u, \varphi) = 0$ for all $\varphi \in \mathcal{F}^*_O$.)

Fact. *Suppose that u is X^*-harmonic in $O \subset D^*$ and $a_j^* \in O$. Then*

$$
\lim_{x \in O \cap D, x \to z} u(x) = u(a_j^*)
$$

holds for q.e. $z \in K_j \cap \partial(O \cap D)$.

[3] Here, the symbol τ_B stands for the first exit time from a set B.

"Proof".

$$u(x) = \underbrace{\big(u(x) - u(a_j^*)\varphi_j(x)\big)}_{\text{“}=0 \text{ on } K_j\text{”}} + u(a_j^*)\varphi_j(x). \qquad \square$$

Lemma 2.8. *Suppose that v is X^*-harmonic in $O_1 \subset D^*$. Then for any relatively compact connected open subset O_2 of O_1, there exists $f \in L^\infty(O_1)$ with* $\operatorname{supp}[f] \cap O_2 = \emptyset$ *such that $v = G^*_{O_1} f$ holds in O_2.*

This lemma is a concrete form of the classical fact that every harmonic function in an open set is represented as the potential of some function whose support is outside this open set.

Proof of Lemma 2.8. Let $\varphi \in C_c^\infty(O_1)$ be such that $\varphi = 1$ on O_2. Since $\varphi v = v$ on O_2, we have

$$f = -\frac{1}{2}\Delta(\varphi v) \in L^\infty, \quad \operatorname{supp}[f] \subset O_1 \setminus O_2.$$

Here, without loss of generality, we may and do assume that $(O_1 \setminus O_2) \cap \{a_1^*, \ldots, a_N^*\} = \emptyset$. $G^*_{O_1} f$ belongs to $\mathcal{F}^*$ and is X^*-harmonic in O_2, and $w = \varphi v - G^*_{O_1} f$ is harmonic in $O_1 \setminus O_2$. Hence w is X^*-harmonic in O_1. Now, as $w = 0$ on ∂O_1, we have $w = 0$ in O_1 by the maximal principle. In particular, $v = \varphi v = G^*_{O_1} f$ in O_2. $\square$

We now look at Theorem 2.7 again.

Theorem 2.9. *An $\mathcal{E}$-q.c. function v is X^*-harmonic in an open set $O \subset D^*$ if and only if v is (classically) harmonic in $O \cap D$ and has zero period property at any $a_i^* \in O$.*

If all the points of $K = \bigcup_i K_i$ are regular to itself, i.e., $K \subset K^r$, then "$\mathcal{E}$-q.c." can be replaced by "continuous". In two-dimension, this is always true (see Theorem 7.2 in Chapter 2 of [24]).

Proof of Theorem 2.9. Without loss of generality, we may and do suppose that O contains exactly one a_1^*. For convenience, let us assume $K_1 \subset K_1^r$. Under this assumption, $u_1(x) = \mathbb{E}_x\left[e^{-\sigma_K}\,;\, X^E_{\sigma_K} \in K_1\right]$ is continuous at a_1^* and thus on O.

"Only if" part. We set $\eta_\varepsilon = \partial\{u_1 > 1 - \varepsilon\}$. Note that $\{u_1 > 1-\varepsilon\} \searrow K_1$ as $\varepsilon \searrow 0$.

We fix an $\varepsilon_0 > 0$ small enough. By Lemma 2.8, we can find a function f with $\operatorname{supp}[f] \cap \{u_1 > 1 - \varepsilon_0\} = \emptyset$ such that $v = G^*_O f$ holds. We can also find a decreasing sequence $(\varepsilon_n)_{n=1,2,\ldots}$ with $\varepsilon_0 > \varepsilon_n \searrow 0$ such that every η_{ε_n} is a smooth closed curve by Sard's theorem. Thus, the period of v at a_1^* is given by

$$\begin{aligned}
\lim_{n\to\infty}\int_{\eta_{\varepsilon_n}}\frac{\partial v}{\partial n}\,d\sigma &= \lim_{n\to\infty}\frac{1}{1-\varepsilon_n}\int_{\eta_{\varepsilon_n}}\frac{\partial v}{\partial n}u_1\,d\sigma \\
&= \lim_{n\to\infty}\frac{1}{1-\varepsilon_n}\int_{O\setminus\mathrm{int}(\eta_{\varepsilon_n})}\left(\nabla G_O^* f\cdot\nabla u_1+\Delta G_O^* f\cdot u_1\right)\,dx \\
&= \int_{O\setminus K_1}\left(\nabla G_O^* f\cdot\nabla u_1+\Delta G_O^* f\cdot u_1\right)\,dx \\
&= \mathcal{N}(G_O^* f)(a_1^*)=0.
\end{aligned}$$

Here, we have used the Green–Gauss formula.

"If" part. Since this can be proved by a similar reasoning, we omit the proof here. See [7] for the detail. □

2.4 Probabilistic representation of conformal mappings

We return to the case $d=2$ and identify $\mathbb{R}^2$ with $\mathbb{C}$. Recall from Section 2.1 that, if v has zero period around each K_j, then a *harmonic conjugate* u can be defined by

$$u(z)-u(z_0)=\int_\gamma\frac{\partial v}{\partial n}\,d\sigma,\quad z\in D.$$

Here, $z_0\in D$ is fixed, and γ is a curve connecting z_0 to z in D. $f(z)=u(z)+iv(z)$ is an analytic function in D. Conversely, if $f=u+iv$ is analytic, then v enjoys the zero period property. Let us apply this relationship to the following situation: Let F be a compact $\mathbb{H}$-hull (i.e., $\mathbb{H}\setminus F$ is simply connected) in a triply connected domain $D=\mathbb{H}\setminus(K_1\cup K_2)$ (see Figure 8). There exists a (unique) conformal mapping f from D onto a standard slit domain (with the hydrodynamic normalization at ∞). Then $v^*(z)=\Im f(z)$ is harmonic in D and has zero period around K_1 and K_2. Moreover, $v^*\equiv$ const. on each ∂K_i (by the boundary correspondence). Thus, v^* can be regarded as a continuous function on $D^*\setminus F$ and is X^*-harmonic for the BMD X^* on D^*. This observation suggests that v^* should have some probabilistic representation in terms of BMD.

When the domain has no holes, the above-mentioned idea is known to be viable. Let X be the BM in $\mathbb{H}$. The hitting probability to $\Gamma_r=\{\,z=x+ir\,;\,x\in\mathbb{R}\,\}$ with $r>0$ is obtained from the gambler's ruin estimate

$$h_r(z)=\mathbb{P}_z\left(\sigma_{\Gamma_r}<\infty\right)=\frac{y}{r}.$$

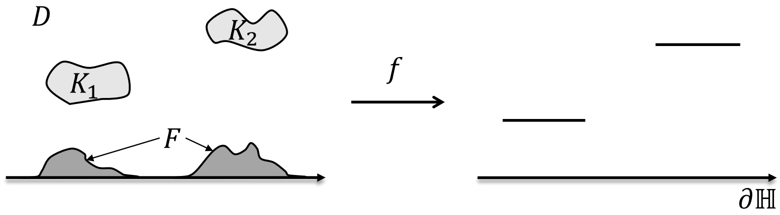

Figure 8: A conformal mapping f that "flatten" a compact $\mathbb{H}$-hull F.

We put $v_r(z) = \mathbb{P}_z(\sigma_{\Gamma_r} < \sigma_F)$ as well. See Figure 9. To obtain a harmonic function with boundary value 0 on F, we amplify v_r by multiplying r, as suggested by the expression of h_r, and take the limit:

Claim. $\lim_{r\to\infty} r v_r(z) = v(z)$ *exists and is harmonic in* $\mathbb{H}\setminus F$.

Indeed, since the strong Markov property implies

$$r h_r(z) = r v_r(z) + \mathbb{E}_z\left[r h_r(X_{\sigma_F})\,;\,\sigma_F < \sigma_{\Gamma_r}\right],$$

v is expressed as

$$v(z) = \lim_{r\to\infty} r v_r(z) = y - \mathbb{E}_z\left[\Im X_{\sigma_F}\,;\,\sigma_F < \infty\right]. \tag{2.4}$$

Now, let u be a harmonic conjugate of v in the simply connected domain $\mathbb{H}\setminus F$. $f(z) = u(z) + iv(z)$ is analytic in $\mathbb{H}\setminus F$. Thus, if f is one-to-one, then $f\colon \mathbb{H}\setminus F \to \mathbb{H}$ is conformal.[4] Indeed, since $v(z) \sim y$ near ∞, it is one-to-one near ∞. Then so is it on $\mathbb{H}\setminus F$ by the degree theorem, which is later described.

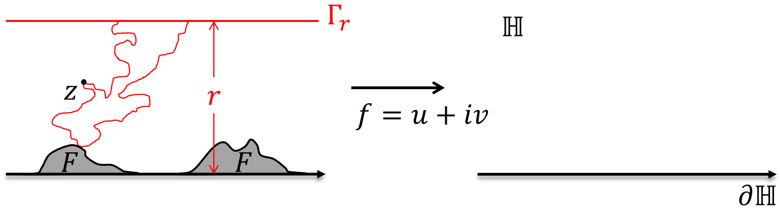

Figure 9: The set Γ_r and Brownian motion X.

The same procedure goes well even in the case of multiple connectivity. We denote by $\mathbb{P}^*$ the law of BMD in $D = \mathbb{H}\setminus\bigcup_{j=1}^N K_j$ or, more precisely, in $D^* = D \cup \{a_1^*, \dots, a_N^*\}$.

[4]In this case, we can easily show the surjectivity of f, noting that $\partial f(\mathbb{H}\setminus F)$ must be obtained by $\partial\mathbb{H}\cup\{\infty\}$ on account of (2.4).

Theorem 2.10. *Define $v^*(z) = \lim_{r\to\infty} r\mathbb{P}^*_z(\sigma_r < \sigma_F)$.*

1) *v^* is well-defined on $D^* \setminus F$ and Z^*-harmonic.*[5] *It follows that*

$$v^*(z) = v(z) + \sum_{j=1}^{N} \mathbb{P}^{\mathbb{H}}_z\left(\sigma_K < \sigma_F,\ Z^{\mathbb{H}}_{\sigma_K} \in K_j\right) v^*(a_j^*), \quad z \in D \setminus F.$$

Here, $v(z) = \Im z - \mathbb{E}^{\mathbb{H}}_z[\Im Z^{\mathbb{H}}_{\sigma_{F\cup K}}\ ;\ \sigma_{F\cup K} < \infty]$.

2) *v^* has a unique harmonic conjugate u^* such that $f(z) = u^*(z) + iv^*(z)$ is analytic in D and enjoys*

$$f(z) = z + \frac{a}{z} + o\left(\frac{1}{z}\right) \quad \textit{near } \infty$$

for some $a \geq 0$.

We sketch out the proof of Theorem 2.10. Put

$$\begin{aligned} v_r^*(z) &\mathrel{\widehat{=}} \mathbb{P}^*_z(\sigma_{\Gamma_r} < \sigma_F) \\ &= \mathbb{P}^{\mathbb{H}}_z(\sigma_{\Gamma_r} < \sigma_{F\cup K}) + \sum_{j=1}^{N} \mathbb{P}^*_z(\sigma_{K^*} < \sigma_{\Gamma_r \cup F},\ Z^*_{\sigma_{K^*}} = a_j^*) v_r^*(a_j^*). \end{aligned} \tag{2.5}$$

By the same proof as that of the previous claim, $r\mathbb{P}^{\mathbb{H}}_z(\sigma_{\Gamma_r} < \sigma_{F\cup K})$ converges to $v(z)$ in Theorem 2.10 as $r \to \infty$. We can also show the convergence of $rv_r^*(a_j^*)$ as follows: Let η_i be a smooth simple closed curve surrounding K_i and ν_i be the harmonic measure on η_i of Z^* starting from a_i^* (see Figure 10). Taking the integral in (2.5) on η_i with respect to ν_i, we have

$$\begin{aligned} v_r^*(a_i^*) = &\int_{\eta_i} \mathbb{P}^{\mathbb{H}}_z(\sigma_{\Gamma_r} < \sigma_{F\cup K})\, \nu_i(dz) \\ &+ \sum_{j=1}^{N} \int_{\eta_i} \mathbb{P}^*_z(\sigma_{K^*} < \sigma_{\Gamma_r \cup F},\ Z^*_{\sigma_{K^*}} = a_j^*)\, \nu_i(dz) \cdot v_r^*(a_j^*). \end{aligned}$$

Using this system of linear equations of $v_r^*(a_i^*)$, $i = 1, \ldots, N$, we can prove that $rv_r^*(a_i^*)$ converges as $r \to \infty$. See [11, Lemma 10.2] for the detail.

Next, we show that the analytic function $f(z)$ in Theorem 2.10 is one-to-one and onto.

[5] Only here and in Section 6.1, we change the notation according to [11, 12]. $Z^{\mathbb{H}} = ((\mathbb{P}^{\mathbb{H}}_z)_{z\in\mathbb{H}}, (Z^{\mathbb{H}}_t)_{t\geq 0})$ is the absorbing BM in $\mathbb{H}$, and $Z^* = ((\mathbb{P}^*_z)_{z\in D^*}, (Z^*_t)_{t\geq 0})$ is the BMD in D^*.

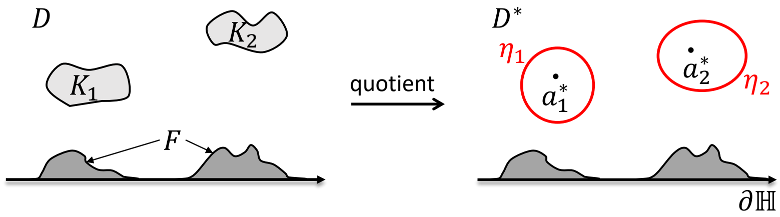

Figure 10: A simple closed curve η_i around a_i^*.

Definition 2.11. Let X, Y be topological spaces. A continuous mapping $f\colon X \to Y$ is said to be *proper* if the pre-images of compact sets are compact.

Lemma 2.12. *Let D_1, D_2 be connected open subsets of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and f be an analytic function in D_1. If f is a proper map from D_1 to D_2, then there is a finite number d such that each $w \in D_2$ has precisely d pre-images in D_1, counting multiplicities.*

Theorem 2.13. *Let D_1, D_2 be connected open subsets of $\overline{\mathbb{C}}$ and f be an analytic function in D_1. Assume that D_2^c has empty interior, that $f(\partial D_1) = \partial D_2$ and that there is a point $w_0 \in D_2$ such that $f^{-1}(w_0) = \{z_0\}$, counting multiplicities. Then f is a conformal map from D_1 onto D_2. Here,*

$$f(\partial D_1) = \bigcap_{K \Subset D_1} \overline{f(D_1 \setminus K)}$$

(the set of the limit points of $f(z)$ as $z \to \partial D_1$).

Roughly speaking, this theorem is proved as follows: Using the assumption that D_2^c has empty interior, we can show that $f^{-1}(D_2)$ is a connected open subset of D_1 and that $f\colon f^{-1}(D_2) \to D_2$ is proper. Then it is one-to-one and onto by Lemma 2.12. Finally, it follows that $f^{-1}(D_2) = D_1$. Here, the last claim is not true if the degree is more than one. For example, consider $f(z) = z^2$ with $D_1 = D_2 = \mathbb{C} \setminus [0, 1]$. In this case, $f^{-1}(D_2) = \mathbb{C} \setminus [-1, 1]$.

Theorem 2.14. *The analytic function f in Theorem 2.10 is a conformal mapping from $D \setminus F$ onto a standard slit domain.*

Theorem 2.14 itself is quite classical, but the contribution of our argument lies in the fact that it gives a useful probabilistic representation of conformal mappings.

Remark 2.15. Such a conformal mapping as in Theorem 2.14 is unique, which can be seen as follows: Let $\varphi\colon \mathbb{H} \setminus F \to \mathbb{H}$ be a unique[6] conformal

[6] The unique existence of φ is commonly known. See [18, Proposition 3.36] for instance.

mapping with the hydrodynamic normalization at ∞. Then $\varphi(D) = \mathbb{H} \setminus \bigcup_i \varphi(K_i)$. We can show that a conformal mapping f_0 from $\mathbb{H} \setminus \bigcup_i \varphi(K_i)$ onto a standard slit domain with the hydrodynamic normalization at ∞ exists uniquely by combining [29, Theorem IX.23] with the Schwarz reflection principle. Since $f \circ \varphi^{-1}$ enjoys exactly the same property as f_0, we have $f_0 = f \circ \varphi^{-1}$, i.e., $f = f_0 \circ \varphi$. See Figure 11. This reasoning, in particular, reveals that [23, Remark 2.4] is not very essential to the uniqueness argument.

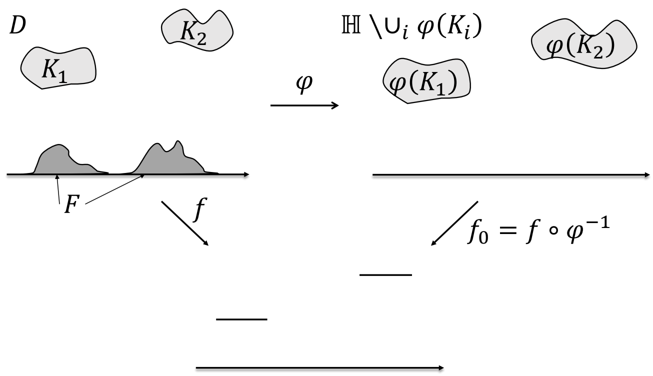

Figure 11: Conformal mappings f, φ and $f_0 = f \circ \varphi^{-1}$.

2.5 Green function and Poisson kernel of BMD

Let $D = E \setminus \bigcup_{j=1}^{N}$, $D^* = D \cup \bigcup_{j=1}^{N} \{a_j^*\}$ and X^* be the BMD in D^*. Here, $E \subsetneq \mathbb{C}$ is a sufficiently nice domain (e.g. $E = \mathbb{H}$).

The Green operator G^* of BMD satisfies, for all $f \in C_c(D)$,

$$\begin{aligned} G^* f(z) &= \mathbb{E}_z \int_0^\infty f(X_t^*)\, dt = G_D f(z) + \mathbb{E}_z[G^* f(X^*_{\sigma_{K^*}}) \,;\, \sigma_{K^*} < \infty] \\ &= G_D f(z) + \sum_{j=1}^{N} G^* f(a_j^*) \underbrace{\mathbb{P}_z(X^*_{\sigma_{K^*}} = a_j^*,\ \sigma_{K^*} < \infty)}_{\varphi_j(z)}. \end{aligned} \tag{2.6}$$

Here, $\varphi_j(z)$ is defined as the harmonic measure of the BM X^E in E on K_j: $\varphi_j(z) = \mathbb{P}_z(X^E_{\sigma_K} \in K_j,\ \sigma_K < \infty)$, $z \in D$. Let a_{ij} be the period of φ_j around K_i and $A = (a_{ij})_{ij}$. Integrate along a smooth simple closed curve η_i around K_i, we have

$$\int_{\eta_i} \frac{\partial G_D f}{\partial n}\, d\sigma + \sum_{j=1}^{N} G^* f(a_j^*) a_{ij} = 0.$$

Here, n stands for the unit normal on η_i pointing toward K_i. Hence

$$\begin{pmatrix} G^*f(a_1^*) \\ \vdots \\ G^*f(a_N^*) \end{pmatrix} = -A^{-1} \begin{pmatrix} \int_{\eta_1} \frac{\partial G_D f}{\partial n}\, d\sigma \\ \vdots \\ \int_{\eta_N} \frac{\partial G_D f}{\partial n}\, d\sigma \end{pmatrix}$$
$$= -A^{-1} \begin{pmatrix} \int_{\partial K_1} \frac{\partial G_D f}{\partial n} \varphi_1\, d\sigma \\ \vdots \\ \int_{\eta_N} \frac{\partial G_D f}{\partial n} \varphi_N\, d\sigma \end{pmatrix} = 2A^{-1} \begin{pmatrix} \langle \varphi_1, f \rangle_{L^2(D)} \\ \vdots \\ \langle \varphi_N, f \rangle_{L^2(D)} \end{pmatrix} \tag{2.7}$$

From (2.6) and (2.7), we obtain the *Green function* of BMD

$$G^*(z,w) = G_D(z,w) + 2\Phi(z)A^{-1}\Phi(w)^{\mathrm{tr}}$$

with $\Phi(z) = (\varphi_1(z), \dots, \varphi_N(z))$. In other words, $G^*(z,w)$ is a kernel which satisfies

$$G^*f(z) = \int_D G^*(z,w) f(w)\, m(dw)$$

for any $f \in C_c(D)$.

Fact. *The period of two functions $y \mapsto G_D(x,y)$ and $y \mapsto G^*(a_i^*, y)$ around K_j are $-2\varphi_j(x)$ and $2\delta_{ij}$, respectively.*

From now on, we assume $E = \mathbb{H}$. We define the *Poisson kernel* $K^*(z,w)$ of BMD by

$$K^*(z,\xi) = -\frac{1}{2}\frac{\partial}{\partial n} G^*(z,\xi) \quad z \in D,\ \xi \in \partial\mathbb{H}.$$

Here , n stands for the unit normal on $\partial\mathbb{H}$ pointing downward.

Lemma 2.16. 1) *For any compact interval $I \subset \partial\mathbb{H}$, $K^*(z,\xi)$ is jointly continuous on $(\overline{\mathbb{H}} \setminus I) \times I$, and $\lim_{z\to\infty} K^*(z,\xi) = 0$ uniformly in $\xi \in I$.*

2) *For $g \in C_b(\partial\mathbb{H})$,*

$$\int_{\partial\mathbb{H}} K^*(z,\xi) g(\xi)\, d\xi = \mathbb{E}_z[g(X^*_{\zeta-})].$$

Proof. (1) See [11, Lemma 5.2].
(2) Let $h(z) = \mathbb{E}_z[g(X^*_{\zeta-})]$. h is harmonic with respect to X^*, and $h = g$ on $\partial\mathbb{H}$. For any $f \in C_c(D)$, we have

$$0 = \mathcal{E}(h, G^*f) = \int_D \nabla h \cdot \nabla G^* f\, dx$$
$$= -\int_D h \Delta G^* f\, dx - \int_{\partial D} h \frac{\partial G^* f}{\partial n}\, d\sigma \tag{2.8}$$

Here, $\Delta G^* f = -2f$ by [11, Lemma 3.3]. In addition, $\partial D = \partial \mathbb{H} \cup \bigcup_i \partial K_i$. Since h is a constant on each ∂K_i, we have

$$\int_{\partial K_i} h \frac{\partial G^* f}{\partial n} \, dx = h(a_i^*) \mathcal{N}(G^* f)(a_i^*) = 0.$$

Thus, it follows from (2.8) that

$$0 = 2 \int_D hf \, dx - \int_{\partial \mathbb{H}} h \frac{\partial G^* f}{\partial n} \, dx.$$

Noting that $h = g$ on $\partial \mathbb{H}$, we have

$$\begin{aligned}\int_D hf \, dx &= \frac{1}{2} \int_{\partial \mathbb{H}} g(x) \frac{\partial G^* f}{\partial n} \, dx = \frac{1}{2} \int_{\partial \mathbb{H}} g(x) \frac{\partial}{\partial n} \int G^*(x, \xi) f(\xi) \, d\xi \, dx \\ &= \int_{\partial \mathbb{H}} g(x) \left(\int K^*(\xi, x) f(\xi) \, d\xi \right) dx.\end{aligned}$$

Let $f_n \in C_c(D)$ be a sequence such that $f_n \, dx \to \delta_z(dx)$. Then

$$h(z) = \int_{\partial \mathbb{H}} g(x) K^*(z, x) \, dx.$$

$\square$

We now define the *complex Poisson kernel* Ψ of BMD. Fix $\xi \in \partial \mathbb{H}$. The function $z \mapsto K^*(z, \xi)$ is X^*-harmonic. Hence, there exists a unique harmonic conjugate $u(z, \xi)$ such that

- $\Psi(z, \xi) := u(z, \xi) + i K^*(z, \xi)$ is analytic;
- $\lim_{z \to \infty} \Psi(z, \xi) = 0$.

Here, we recall the conformal mapping f constructed in Theorems 2.10 and 2.14. Since $\Im f$ is X^*-harmonic as mentioned at the beginning of Section 2.4, we can express $\Im f$ by its boundary value and the Poisson kernel K^* as in Lemma 2.16. Then we can also express the original f by Ψ. This observation is crucial in the next section.

3 Komatu–Loewner differential equations for multiply connected domains

Let D be a standard slit domain and $\gamma \colon [0, t_\gamma] \to \overline{D}$ be a simple curve such that $\gamma(0) \in \partial \mathbb{H}$ and $\gamma(0, t_\gamma] \subset D$. For each $t \in [0, t_\gamma]$, let g_t be a unique conformal map from $D \setminus \gamma(0,t]$ onto a standard slit domain D_t satisfying

$$g_t(z) = z + \frac{a_t}{z} + o\left(\frac{1}{|z|}\right) \quad \text{at } \infty.$$

For $0 \le s < t \le t_\gamma$, we define

$$g_{t,s}(z) := g_s \circ g_t^{-1}(z) = z + \frac{a_s - a_t}{z} + o\left(\frac{1}{|z|}\right) \quad \text{at } \infty.$$

By the boundary correspondence, the driving function

$$\xi(t) = g_t(\gamma(t)) = \lim_{z \to \gamma(t)} g_t(z) \in \partial\mathbb{H}$$

is well-defined for $t \in [0, t_\gamma]$. Moreover, for $0 \le s < t \le t_\gamma$, there are unique points $\beta_0 = \beta_0(t,s)$ and $\beta_1 = \beta_1(t,s)$ such that

$$\beta_0 < \xi(t) < \beta_1, \quad \text{and} \quad g_{t,s}(\beta_0) = g_{t,s}(\beta_1) = \xi(s).$$

Let $\ell_{t,s} = [\beta_0, \beta_1]$. Then $\Im g_{t,s} > 0$ on $\ell_{t,s}^\circ$. See Figure 12 for the connection among the objects so defined.

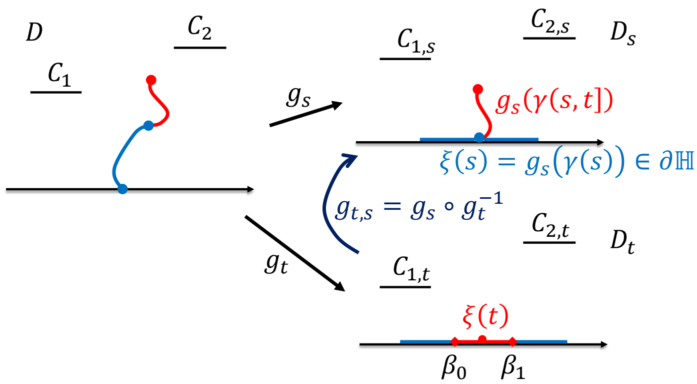

Figure 12: The conformal mappings g_s, g_t and $g_{t,s}$ for $s < t$.

By Schwarz's reflection, we can extend $g_{t,s}$ to an analytic function on $\mathbb{C} \setminus \Gamma_t$. Here, $\Gamma_t = \bigcup_{k=1}^N C_{k,t} \cup \pi\left(\bigcup_{k=1}^N C_{k,t}\right) \cup \ell_{t,s}$, and π is the mirror reflection relative to $\partial\mathbb{H}$. Using Cauchy's integral theorem, we have

$$a_t - a_s = \frac{1}{\pi} \int_{\beta_0}^{\beta_1} \Im g_{t,s}(x + i0+)\, dx \tag{3.1}$$

From this equality, we see that $a_t - a_s > 0$ and $\lim_{s \nearrow t} a_s = a_t$. Note that, in a similar way, we can prove that a_t is right continuous as well.

Let

$$F(z) = g_{t,s}(z) - z = \frac{a_s - a_t}{z} + o\left(\frac{1}{|z|}\right) \quad \text{at } \infty$$

and $f(z) = \Im F(z)$, which is bounded. Then owing to BMD harmonicity,

$$f(z) = \int_{\partial \mathbb{H}} K^*(z, \xi) f(\xi) \, d\xi.$$

Since its harmonic conjugate is unique up to additive real constants,

$$F(z) = \int_{\partial \mathbb{H}} \Psi(z, \xi) f(\xi) \, d\xi + C.$$

In fact, $C = 0$ because $F(z)$ and $\Psi(z, \xi)$ both converge to zero as $z \to \infty$. We now obtain

$$\begin{aligned} g_s \circ g_t^{-1}(z) - z &= \int_{\partial \mathbb{H}} \Psi(z, \xi) \Im g_{t,s}(\xi) \, d\xi, \\ g_s(z) - g_t(z) &= \int_{\partial \mathbb{H}} \Psi(g_t(z), \xi) \Im g_{t,s}(\xi) \, d\xi. \end{aligned}$$

As $s \nearrow t$,

$$\frac{\partial^- g_t(z)}{\partial a_t} = \lim_{s \nearrow t} \frac{g_t(z) - g_s(z)}{a_t - a_s} = -\pi \Psi_{D_t}(g_t(z), \xi(t)). \tag{3.2}$$

Thus, $g_t(z)$ is left differentiable. Here, the subscript D_t of Ψ_{D_t} is put in order to emphasize that it is the complex Poisson kernel for D_t.

To prove the right differentiability, it suffices to show that the right-hand side of (3.2) is continuous in t because, then by [18, Lemma 4.3], the left derivative in (3.2) is indeed a genuine derivative. The continuity is proved through the following steps: First, we show that D_t is "continuous" in t in some sense. For example, since $g_{t,s}(z) \to z$ locally uniformly as $s \nearrow t$ by the above-mentioned reasoning, D_t is "left continuous" in t. Second, we show that Ψ is "Lipschitz continuous" with respect to the variation of standard slit domains. In conclusion, we obtain the continuity of $\Psi_{D_t}(g_t(z), \xi(t))$ in t. As the second step is very lengthy, we omit the detail here. See [11] for the complete proof of

$$\frac{\partial g_t(z)}{\partial a_t} = -\pi \Psi_{D_t}(g_t(z), \xi(t)). \tag{3.3}$$

4 Induced slit motions and Komatu–Loewner evolution

Suppose that $a_t = 2t$ in Section 3. Then (3.3) takes the form

$$\frac{\partial g_t(z)}{\partial t} = -2\pi \Psi_{D_t}(g_t(z), \xi(t)) \tag{4.1}$$

The right-hand side of this equation, the complex Poisson kernel of BMD, depends on D_t. We consider the time-evolution of the slits of this variable domain D_t. The standard slit domain

$$D_t = \mathbb{H} \setminus \bigcup_{j=1}^{N} C_j(t)$$

is determined by the $2N$ endpoints of the slits

$$\{ \, (z_j(t), z_j^r(t)) \; ; \; 1 \le j \le N \, \}$$

(see Figure 13), which can also be regarded as the $3N$-tuple

$$\{ \, (y_j(t), x_j(t), x_j^r(t)) \; ; \; 1 \le j \le N \, \} \in \mathbb{R}^{3N}.$$

We use a symbol $\mathcal{S}$ to denote the set of all the $3N$-tuples

$$\mathbf{s} = (\mathbf{y}, \mathbf{x}, \mathbf{x}^r) \in \mathbb{R}_+^N \times \mathbb{R}^N \times \mathbb{R}^N \subset \mathbb{R}^{3N}$$

that are associated to non-overlapping N-tuples of slits. In other words,

$$\begin{aligned} \mathcal{S} := \{ \, \mathbf{s} = (\mathbf{y}, \mathbf{x}, \mathbf{x}^r) \; ; \; & \mathbf{y}, \mathbf{x}, \mathbf{x}^r \in \mathbb{R}^N, \; \mathbf{y} > 0, \; \mathbf{x} < \mathbf{x}^r, \\ & \text{either } x_j^r < x_k \text{ or } x_k^r < x_j \text{ whenever } y_j = y_k, \; j \neq k \, \}. \end{aligned}$$

Although any permutation of the indexes $\{1, \ldots, N\}$ results in the same N-tuple slits, we regard $\mathcal{S}$ as the space of labelled slits or "labelled" standard slit domains. A distance between such labelled slit domains (or slits themselves) is defined by

$$d(D, \tilde{D}) := \max_{1 \le j \le N} \left(|z_j - \tilde{z}_j| + |z_j^r - \tilde{z}_j^r| \right).$$

As partly mentioned in the final paragraph of Section 3, we can show that

Theorem 4.1 ([11, Theorem 9.1])**.** *$\Psi_{\mathbf{s}}(z, \xi)$ is Lipschitz in $\mathbf{s}$.*

The time-evolution of the slits in (4.1) is given by

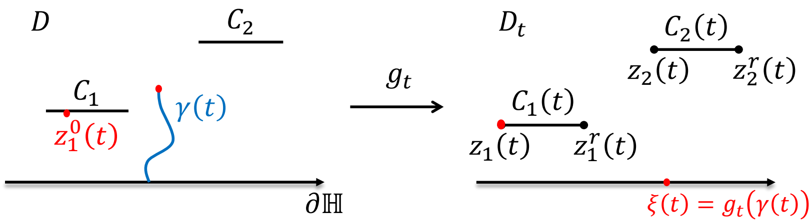

Figure 13: The endpoints $z_j(t)$ and $z_j^r(t)$ of the slit $C_j(t)$ and the pre-image $z_1^0(t) = g_t^{-1}(z_1(t))$.

Theorem 4.2.

$$\begin{aligned} \frac{d}{dt}z_j(t) &= -2\pi\Psi_{\mathbf{s}(t)}(z_j(t), \xi(t)), \\ \frac{d}{dt}z_j^r(t) &= -2\pi\Psi_{\mathbf{s}(t)}(z_j^r(t), \xi(t)). \end{aligned} \tag{4.2}$$

"Proof". Let $z_j^0(t) = g_t^{-1}(z_j(t))$. Differentiating the both side of the identity $z_j(t) = g_t(z_j^0(t))$ yields

$$\begin{aligned} \frac{d}{dt}z_j(t) &= \frac{\partial}{\partial t}g_t(z_j^0(t)) + \underbrace{g_t'(z_j^0(t))}_{=0}\frac{d}{dt}z_j^0(t) \\ &= -2\pi\Psi_{\mathbf{s}(t)}(g_t(z_j^0(t)), \xi(t)) \\ &= -2\pi\Psi_{\mathbf{s}(t)}(z_j(t), \xi(t)). \end{aligned}$$

Here, we have used $g_t'(z_j^0(t)) = 0$, which is because $z_j^0(t)$ is the "double root" of the equation $g_t(z) - z_j(t) = 0$. See [10, Lemma 2.2]. $\square$

We rewrite (4.2) in the real form

$$\begin{aligned} \frac{d}{dt}y_j(t) &= -2\pi\Im\Psi_{\mathbf{s}(t)}(z_j(t), \xi(t)), \\ \frac{d}{dt}x_j(t) &= -2\pi\Re\Psi_{\mathbf{s}(t)}(z_j(t), \xi(t)), \\ \frac{d}{dt}x_j^r(t) &= -2\pi\Re\Psi_{\mathbf{s}(t)}(z_j^r(t), \xi(t)). \end{aligned} \tag{4.3}$$

Since the $\Psi_{\mathbf{s}}(z, \xi)$ is Lipschitz by Theorem 4.1, we can go along a reversed way solving the ODE (4.3) and then (4.1). Before doing this actually, we collect some elementary properties of $\Psi_{\mathbf{s}}(z, \xi)$:

1) $\Psi_{\lambda\mathbf{s}}(\lambda z, \lambda\xi) = \dfrac{1}{\lambda}\Psi_{\mathbf{s}}(z, \xi)$;

2) $\Psi_{\mathbf{s}-\hat{a}}(z-a, \xi-a) = \Psi_{\mathbf{s}}(z, \xi)$, $a \in \mathbb{R}$.

Here,

$$\hat{a} = (\underbrace{a, a, \ldots, a}_{N}, \underbrace{0, 0, \ldots, 0}_{2N}) \in \mathbb{R}^{3N}$$

corresponds to the horizontal translation by a.[7] In particular, we have $\Psi_{\mathbf{s}}(z, \xi) = \Psi_{\mathbf{s}-\hat{\xi}}(z-\xi, 0)$. Taking this identity into account, let

$$\tilde{b}_j(\mathbf{s}) := \begin{cases} \Psi_{\mathbf{s}}(z_j, 0) & 1 \le j \le N \\ \Psi_{\mathbf{s}}(z^r_{j-N}, 0) & N+1 \le j \le 2N. \end{cases}$$

Then $\tilde{b}_j(\mathbf{s})$ is a homogeneous function of $\mathbf{s}$ with degree -1. We further define

$$b_j(\mathbf{s}) := \begin{cases} -2\pi \Im \Psi_{\mathbf{s}}(z_j, 0) & 1 \le j \le N \\ -2\pi \Re \Psi_{\mathbf{s}}(z_{j-N}, 0) & N+1 \le j \le 2N \\ -2\pi \Re \Psi_{\mathbf{s}}(z^r_{j-2N}, 0) & 2N+1 \le j \le 3N. \end{cases}$$

With these symbols, (4.3) is simply written as

$$\frac{d}{dt}\mathbf{s}_j(t) = b_j(\mathbf{s}(t) - \xi(t)). \tag{4.4}$$

Now, given a continuous function $\xi(t) \in \mathbb{R} = \partial\mathbb{H}$, we can solve (4.4) to get $\mathbf{s}_j(t)$ on maximal interval $[0, \zeta)$. These two properties are easily shown:

1) $\mathbf{s}_j(t)$ is continuous in t;

2) $y_j(t)$ is decreasing in t.

Let $r \in [0, \zeta)$ and $z_0 \in D_r = \mathbb{H} \setminus \mathbf{s}(t)$. Here and in what follows, we use the symbol $\mathbf{s}(t)$ to denote not only the coordinate of the endpoints of $C_j(t)$'s but also the set $\bigcup_{j=1}^N C_j(t)$ itself by abuse of notation. We consider the initial value problem

$$\begin{cases} \dfrac{dz(t)}{dt} = -2\pi \Psi_{\mathbf{s}(t)}(z(t), \xi(t)) \\ z(r) = z_0. \end{cases} \tag{4.5}$$

The uniqueness of a local solution to (4.5) holds. We also consider the case $z_0 \in \partial_p \mathbf{s}(r)$. Here, ∂_p stands for the "boundary"[8] with respect to the path distance topology of D_r. Even in this case, we can "solve" (4.5) by taking the Schwarz reflection. If $z_0 \in \partial_p \mathbf{s}(r)$, the solution still stays on $C_j(t)$.

[7] We sometimes drop the hat symbol and write such an expression as $\mathbf{s} - a (\in \mathcal{S})$ if there is no fear of confusion.

[8] In other words, the "upper" and "lower" sides of $C_j(t)$ are distinguished.

Proposition 4.3 (Proposition 5.4 of [10]). *For any $r \in [0, \zeta)$ and $z_0 \in D_r$, the maximal interval of the existence of the unique solution to* (4.5) *is* $[0, \beta)$ *for some $r < \beta \leq \zeta$. Moreover, if $\beta < \zeta$, then*

$$\lim_{t \nearrow \beta} \Im z(t) = 0 \quad \textit{and} \quad \lim_{t \nearrow \beta} |z(t) - \xi(\beta)| = 0.$$

Why one can extend the solution back to time 0? Suppose that the maximal interval of the solution $\varphi(t)$ is (α, β). $\Im\varphi(t)$ increases as $t \searrow \alpha$. Since $\Psi_{\mathbf{s}}(z, \xi)$ is bounded when z is apart from the boundary $\partial\mathbb{H}$, $\lim_{t \searrow \alpha} \varphi(t) = w$ exists. Moreover, we claim that $w \in D_\alpha = \mathbb{H} \setminus \mathbf{s}(\alpha)$. Indeed, this claim follows from the above-mentioned fact that a solution starting on the slits still stay on the same slit. Hence the solution φ can be extended to $(\alpha - \varepsilon, \alpha)$ as long as $\alpha > 0$. Therefore, $\alpha = 0$.

Theorem 4.4 (Theorem 5.5 of [10]). (i) *For any $z \in D$, there exists a unique solution $g_t(z)$, $t \in [0, t_z)$ of*

$$\frac{\partial}{\partial t} g_t(z) = -2\pi \Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)) \quad \textit{with } g_0(z) = 0,$$

and $g_t(z) \in \mathbb{H} \setminus \mathbf{s}(t) =: D(\mathbf{s}(t))$.

(ii) *Define $F_t := \{ z \in D \,;\, t_z \leq t \}$. Then $D \setminus F_t$ is open, g_t is a conformal mapping from $D \setminus F_t$ onto $D(\mathbf{s}(t))$, and F_t is an $\mathbb{H}$-hull (i.e., $\mathbb{H} \setminus F_t$ is simply connected).*

Proof. (i) follows immediately from the argument above.
(ii) As $\Psi_{\mathbf{s}(t)}(z, \xi)$ is analytic in z and jointly continuous in (t, z, ξ), $g_t(z)$ is continuous in (t, z) and analytic in z by the general theory of ODE. We put $D \setminus F_t = g_t^{-1}(D(\mathbf{s}(t)))$. By the above proposition, $g_t \colon D \setminus F_t \to D(\mathbf{s}(t))$ is one-to-one and onto. Indeed, the "one-to-one" property follows from the uniqueness of solutions; If there are two points z_1 and z_2 such that $g_t(z_1) = g_t(z_2) = z_0 \in D_t$, then it contradicts the uniqueness. The "onto" property can be shown by applying Proposition 4.3 to the backward version of (4.5). □

Theorem 4.5 (Theorem 5.8 of [10]). (i) *The conformal map $g_t(z)$ satisfies the hydrodynamic normalization at ∞:*

$$g_t(z) = z + \frac{a_t}{z} + o\left(\frac{1}{|z|}\right),$$

and $a_t = 2t$.

(ii) *$\{F_t\}_{t \in [0, \zeta)}$ is strictly increasing, and*

$$\bigcap_{\delta > 0} \overline{g_t(F_{t+\delta} \setminus F_t)} = \{\xi(t)\}.$$

5 Stochastic Komatu–Loewner differential equation

In this section, we take the driving function $\xi(t)$ in Section 3 to be a stochastic process. At this time, $\xi(t)$ can be intertwined with $\mathbf{s}(t)$.

5.1 SDE for randomized driving functions and BMD domain constant

What are the natural candidates for random $\xi(t)$?

Motivation. Let $\mathbb{P}_{D,z}$ be the probability measure on the space of simple curves γ (or compact $\mathbb{H}$-hulls) induced from SLE_κ on a simply connected domain D starting at z.[9] Then it enjoys

- domain Markov property and
- conformal transport (invariance).

Analogously, in this subsection, we consider randomized curves with these two properties on standard slit domains and study the law of the associated driving functions.

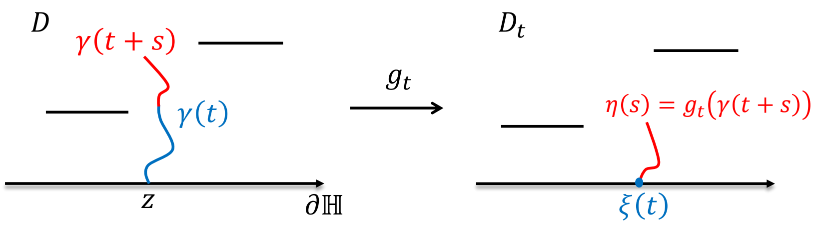

Figure 14: The domain Markov property (combined with the conformal transport) implies that the curve $\eta(s) = g_t(\gamma(t+s))$ has the law $\mathbb{P}_{D_t,\xi(t)}$.

Let D be a standard slit domain. Recall that, for a simple curve $\gamma\colon [0, t_\gamma) \to \overline{D}$ with $\gamma(0) \in \partial\mathbb{H}$ and $\gamma(0, t_\gamma) \subset D$, there exists a unique conformal mapping $g_t\colon D \setminus \gamma(0, t] \to D_t$ with the hydrodynamic normalization $g_t(z) =$

[9] We omit the precise definition of the measurable space on which $\mathbb{P}_{D,z}$ is defined. We have also dropped the one more subscript that should be in the symbol $\mathbb{P}_{D,z}$, which represents the endpoint of the SLE.

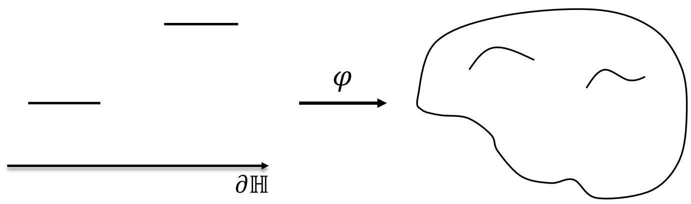

Figure 15: The expected conformal transport is the identity $\mathbb{P}_{\varphi(D),\varphi(z)} = \mathbb{P}_{D,z} \circ \varphi^{-1}$ for a general conformal mapping φ.

$z + a_t/z + o(z^{-1})$ at ∞. We call $a_t(D, \gamma) = \lim_{z\to\infty} z(g_t(z) - z)$ the *half-plane capacity*. Put

$$\Omega(D) = \left\{ \gamma\colon [0, t_\gamma) \to \overline{D}\ ;\ \begin{matrix} \gamma \text{ is simple, } \gamma(0) \in \partial\mathbb{H},\ \gamma(0, t_\gamma) \subset D, \\ 0 < t_\gamma \le \infty,\ a_t(D, \gamma) = 2t \end{matrix} \right\},$$

$$\mathcal{G}_t(D) = \sigma\left(\{\, \gamma(s)\ ;\ 0 \le s \le t \,\}\right),$$

$$\mathcal{G}(D) = \sigma\left(\{\, \gamma(t)\ ;\ t \ge 0 \,\}\right).$$

Let $\mathbb{P}_{D,z}$ be a probability measure on $(\Omega(D), \mathcal{G}(D))$ which enjoys $\mathbb{P}_{D,z}(\gamma(0) = z) = 1$, $z \in \partial\mathbb{H}$, and the following conditions:

(DMP) *Domain Markov property*

For any slit domain $D = \mathbb{H} \setminus \bigcup_{j=1}^N C_j$ and $z \in \partial\mathbb{H}$,

$$\mathbb{P}_{D,z}(\theta_t^{-1} A \mid \mathcal{G}_t(D)) = \mathbb{P}_{g_t(D\setminus\gamma[0,t]), g_t(\gamma(t))}(g_t(A)) \left(= \mathbb{P}_{D_t,\xi(t)}(g_t(A))\right).$$

Here, $A \in \mathcal{G}(D \setminus \gamma(0, t])$, and $\theta_t\colon \gamma \mapsto \gamma(t + \cdot)$ is the shift operator.[10]

(IL) *Invariance under linear (conformal) map*

If $f\colon D \to f(D)$ is a conformal map onto a standard slit domain (such a f is automatically a linear transformation[11]), then

$$\mathbb{P}_{f(D),f(z)} = \mathbb{P}_{D,z} \circ f^{-1}.$$

Put

$$\mathbf{W}_t = (\xi(t), \mathbf{s}(t)) \in \partial\mathbb{H} \times \mathcal{S} \subset \mathbb{R} \times \mathbb{R}^{3N}.$$

Since the pair $(D, z = \gamma(0))$ can be specified by another pair $(\xi = \gamma(0), \mathbf{s})$, which is equal to $\mathbf{W}_0$ by definition, we denote $\mathbb{P}_{D,z}$ by $\mathbb{P}_{(\xi,\mathbf{s})}$ as well.

[10] We have abused the notation a lot. The reader is referred to [10, Section 3] for the precise one. Note that symbols $\dot{\Omega}$ and $\dot{\mathcal{G}}$ are used instead of Ω and $\mathcal{G}$ in that paper.

[11] This formulation avoids the technical issue that may occur if we consider a general conformal mapping φ as in Figure 15. See [10, Remark 3.1].

Theorem 5.1.

$$\mathbb{P}_{(\xi,\mathbf{s})}(\mathbf{W}_0 = (\xi, \mathbf{s})) = 1,$$
$$\mathbb{P}_{(\xi,\mathbf{s})}(\mathbf{W}_{t+s} \in B \mid \mathcal{G}_t(D(\mathbf{s}))) = \mathbb{P}_{\mathbf{W}_t}(\mathbf{W}_s \in B), \quad B \in \mathcal{B}(\partial\mathbb{H} \times \mathcal{S}).$$

Here, $D(\mathbf{s}) = \mathbb{H} \setminus \mathbf{s}$.

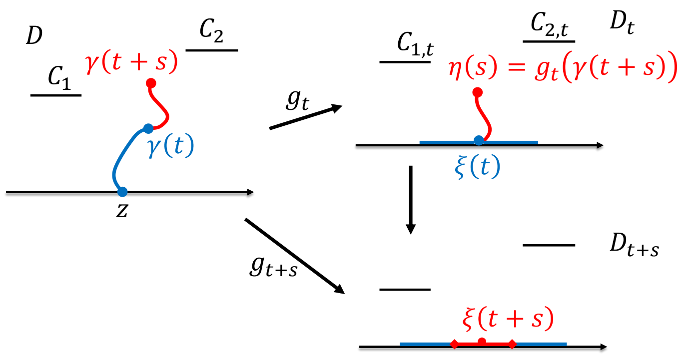

Figure 16: The proof of Theorem 5.1.

Proof. The former identity is trivial. The latter one follows from (DMP). (In Figure 16, the curve $\eta(s) = g_t(\gamma(s))$ has the law $\mathbb{P}_{\mathbf{W}_t}$.) □

By Theorem 5.1, $P_t f(\mathbf{w}) \mathrel{\hat{=}} \mathbb{E}_{\mathbf{w}}[f(\mathbf{W}(t))]$ is a semigroup. Hence, (IL) implies that

$$\{\lambda^{-1}\gamma(\lambda^2 t)\} \text{ under } \mathbb{P}_{\lambda D, \lambda z} \stackrel{d}{=} \{\gamma(t)\} \text{ under } \mathbb{P}_{D,z}. \tag{5.1}$$

Since the half-plane capacity behaves as $a_t(cD, c\gamma) = c^2 a_t(D, \gamma)$, (5.1) yields

$$\{\lambda^{-1}\mathbf{W}_{\lambda^2 t}\} \text{ under } \mathbb{P}_{(\lambda\xi, \lambda\mathbf{s})} \stackrel{d}{=} \{\mathbf{W}_t\} \text{ under } \mathbb{P}_{(\xi,\mathbf{s})}. \tag{5.2}$$

Let $\mathcal{L}$ be the generator of $\{P_t, t \geq 0\}$. Assume the conditions $P_t(C_\infty) \subset C_\infty$ and $C_c^\infty \subset \mathcal{D}(\mathcal{L})$. Then $\mathcal{L}$ is expressed as

$$\mathcal{L} = \frac{1}{2} \sum a_{ij}(\mathbf{w}) \frac{\partial^2}{\partial w_i \partial w_j} + \sum \tilde{b}_j(\mathbf{w}) \frac{\partial}{\partial w_i} + c(\mathbf{w}).$$

We suppose that $c(\mathbf{w}) = 0$, i.e., W_t admits no killing inside $\partial\mathbb{H} \times \mathcal{S}$. The ODE (4.4) implies that

$$a_{ij}(\mathbf{w}) = 0 \quad \text{for } i + j \geq 1.$$

Thus, $\mathbf{W}_t$ enjoys the SDE

$$\begin{aligned} d\xi_t &= \sqrt{a_{00}(\mathbf{W}_t)}\, dB_t + b_0(\mathbf{W}_t)\, dt, \\ d\mathbf{s}_j(t) &= b_j(\mathbf{s} - \xi(t))\, dt, \quad 1 \le j \le 3N. \end{aligned} \tag{5.3}$$

Moreover, by (IL), there are functions a_0 and b_0 on $\mathcal{S}$ such that

$$\sqrt{a_{00}(\mathbf{W}_t)} = a_0(\mathbf{s}(t) - \hat{\xi}(t)), \quad b_0(\mathbf{W}_t) = b_0(\mathbf{s}(t) - \hat{\xi}(t)).$$

Owing to the Brownian scaling (5.2), the function a_0 is homogeneous of degree 0, and b_0 is homogeneous of degree -1.

In comparison with SLE_κ, it is natural to choose the constant $\sqrt{\kappa}$ as the diffusion coefficient $a_0 = \sqrt{a_{00}}$ in (5.3). What are natural candidates for the drift coefficient b_0? One of them is given as follows: Recall that $\Psi_D(z, \xi)$ is the complex Poisson kernel for BMD in $D = \mathbb{H} \setminus \bigcup_{j=1}^N C_j$. We treat the case $D = \mathbb{H}$ as well by abuse of notation:

$$\Psi_{\mathbb{H}}(z, \xi) = -\frac{1}{\pi} \frac{1}{z - \xi}.$$

Actually, its imaginary part coincides with the usual Poisson kernel for (the absorbing BM in) $\mathbb{H}$:

$$\Im \Psi_{\mathbb{H}}(z, \xi) = \frac{1}{\pi} \frac{y}{(x - \xi)^2 + y^2} = -\frac{1}{\pi} \Im \frac{1}{z - \xi}.$$

For the difference

$$\Psi_D(z, \xi) - \Psi_{\mathbb{H}}(z, \xi) = \Psi_D(z, \xi) + \frac{1}{\pi} \frac{1}{z - \xi},$$

the limit

$$b_{\mathrm{BMD}}(D, \xi) \mathrel{\widehat{=}} 2\pi \lim_{z \to \xi} \left(\Psi_D(z, \xi) + \frac{1}{\pi} \frac{1}{z - \xi} \right)$$

exists.

Fact. $b_{\mathrm{BMD}}(\mathbf{s}) \stackrel{\text{def.}}{=} b_{\mathrm{BMD}}(D(\mathbf{s}), 0)$, *where* $D(\mathbf{s}) = \mathbb{H} \setminus \mathbf{s}$, *is a homogeneous function of degree* -1 *and is Lipschitz continuous in* $\mathcal{S}$.

We call $b_{\mathrm{BMD}}(\mathbf{s})$ the *BMD domain constant*. As seen from its definition, it indicates the discrepancy between D and $\mathbb{H}$. We shall see the reason why the BMD domain constant is a natural candidate for the drift coefficient b_0 in (5.3) in the next subsection through the study of the locality property of SKLE.

5.2 Locality property of SKLE

On the basis of Sections 4 and 5.1, we now introduce SKLE. Let $\alpha(\mathbf{s})$ and $b(\mathbf{s})$ be homogeneous functions[12] of $\mathbf{s} \in \mathcal{S}$ of degree 0 and -1, respectively. In other words, there are functions $\alpha'(\theta)$ and $b'(\theta)$ of $\theta \in \mathbb{S}^{3N-1}$ such that $\alpha(\mathbf{s}) = \alpha'(\mathbf{s}/|\mathbf{s}|)$ and $b(\mathbf{s}) = |\mathbf{s}|^{-1}b(\mathbf{s}/|\mathbf{s}|)$ hold. We determine the driving function $\xi(t)$ and the slits $\mathbf{s}(t)$ by the SDE[13]

$$\begin{cases} d\xi_t = \alpha(\mathbf{s}(t) - \xi(t))\, dB_t + b(\mathbf{s}(t) - \xi(t))\, dt \\ d\mathbf{s}_j(t) = b_j(\mathbf{s} - \xi(t))\, dt, \quad 1 \leq j \leq 3N \end{cases} \tag{5.4}$$

with initial value $(\xi(0), \mathbf{s}(0)) = (\xi, \mathbf{s}) \in \partial\mathbb{H} \times \mathcal{S}$. For the solution $(\xi(t), \mathbf{s}(t))$ to (5.4), we solve

$$\frac{\partial g_t(z)}{\partial t} = -2\pi\Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)), \quad g_0(z) = z,$$

to obtain the $\mathbb{H}$-hull $F_t = \{\, z \in D(\mathbf{s})\ ;\ t_z \leq t \,\}$ with $g_t \colon D(\mathbf{s}) \setminus F_t \to D(\mathbf{s}(t))$ conformal. We call $\{F_t\}$ the *stochastic Komatu–Loewner evolution* driven by (5.4) with coefficients α and b, which is designated by $\mathrm{SKLE}_{\xi,\mathbf{s},\alpha,b}$. Here, note that it depends on the initial data $(\xi, \mathbf{s})$ as well. However, we drop the subscripts ξ and $\mathbf{s}$ when they do not matter.

For an $\mathbb{H}$-hull A in a standard slit domain D, we denote by ϕ_A a unique conformal mapping from $D \setminus A$ onto another standard slit domain with the hydrodynamic normalization at ∞.

Definition 5.2. $\mathrm{SKLE}_{\alpha,b}$ is said to have the *locality property*, if for $\mathrm{SKLE}_{\xi,\mathbf{s},\alpha,b}$ $\{F_t\}$ with initial $(\xi, \mathbf{s}) \in \partial\mathbb{H} \times \mathcal{S}$ and for any $\mathbb{H}$-hull $A \subset D(\mathbf{s})$,

$$\{\, \phi_A(F_t)\ ;\ t < \sigma_A \,\} \overset{d}{=} \mathrm{SKLE}_{\phi_A(\xi), \phi_A(\mathbf{s}), \alpha, b}$$

up to a time-change.

For each t, we put $h_t = \phi_{g_t(A)}$ and $U(t) = h_t(\xi(t))$. We can examine the locality property of $\mathrm{SKLE}_{\alpha,b}$ if the SDE of $U(t)$ can be written down explicitly. Therefore, our goal in this subsection is to study $U(t)$. The key steps are as follows:

(1) Derive the Komatu–Loewner equation for $\tilde{F}_t = \phi_A(F_t)$ in terms of $U(t)$ (in the right derivative sense):

$$\frac{\partial^+ \tilde{g}_t}{\partial \tilde{a}_t} = -\pi\Psi_{\tilde{D}_t}(\tilde{g}_t(z), U(t)). \tag{5.5}$$

[12] We further assume the local Lipschitz continuity of α and b in $\mathbf{s}$. See Eq. (4.1) and Lemma 4.1 of [10].

[13] We change the symbols for the coefficients in (5.3) for notational simplicity.

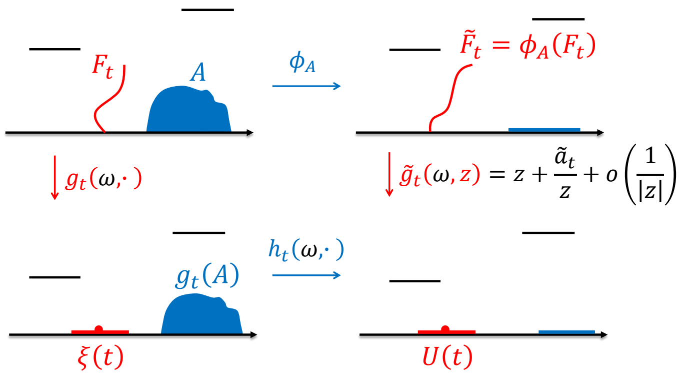

Figure 17: Only is ϕ_A deterministic, and the other g_t, $\tilde{g}_t$ and $h_t = \phi_{g_t(A)}$ are random mappings having the randomness ω.

(2) Show that $h_t(z)$, $h_t'(z)$, h_t'' are jointly continuous in (t, z).[14]

(3) Show that $(t, z) \mapsto \tilde{g}_t(z)$ is jointly continuous. Then $\tilde{D}_t$ is continuous, and $\tilde{a}_t$ is continuous.

Through these steps, we have

$$\frac{\partial \tilde{g}_t}{\partial \tilde{a}_t} = -\pi \Psi_{\tilde{D}_t}(\tilde{g}_t(z), U(t)), \tag{5.6}$$

and the generalized Itô's formula [26, Exercise IV.3.12] applies to $U(t)$:

$$\begin{aligned} dU(t) &= d_t\left(h_t(\xi(t))\right) \\ &= h_t'(\xi(t))((b + b_{\mathrm{BMD}})(\mathbf{s}(t) - \xi(t))\,dt + \frac{1}{2}h_t''(\xi(t))(\alpha(\mathbf{s}(t) - \xi(t))^2 - 6)\,dt \\ &\quad - |h_t'(\xi(t))|^2 b_{\mathrm{BMD}}(h_t(\mathbf{s}(t)) - U(t))\,dt + h_t'(\xi(t))\alpha(\mathbf{s}(t) - \xi(t))\,dB_t \end{aligned} \tag{5.7}$$

In fact, we can prove $\frac{d\tilde{a}_t}{dt} = 2|h_t'(\xi(t))|^2$. (If ϕ_A and hence h_t are linear, then this equality is obvious.) Thus, reparametrizing $\{\tilde{F}_t\}$ so that its half-plane capacity is equal to $2t$ at time t and writing the time-changed version of (5.7), we are led to the following:

Theorem 5.3. *Suppose α is a positive constant.* $\mathrm{SKLE}_{\alpha,-b_{\mathrm{BMD}}}$ *has locality iff $\alpha = \sqrt{6}$.*

Note that $\kappa = \alpha^2$ in comparison with SLE_κ. This theorem is the SKLE version of the celebrated locality property of SLE_6.

[14]Steps (2) and (3) are crucial not just to convert the right derivative in (5.5) into the genuine one but also to apply the generalized Itô's formula. See [12, Remark 2.9].

6 SKLE and SLE

6.1 SKLE v.s. SLE

In this subsection, we study the property of $\mathrm{SKLE}_{\alpha,b}$ $\{F_t\}$ v.s. SLE_κ, where $\kappa = \alpha^2$. To this end, we derive the Loewner, not "Komatu–Loewner", equation for $\{F_t\}$ below. Let $\Phi = \mathrm{id}$, $\Phi_t = g_t^0 \circ \Phi \circ g_t^{-1}$ and $U(t) = \Phi_t(\xi(t))$. g_t^0 enjoys the hydrodynamic normalization

$$g_t^0(z) = z + \frac{a(t)}{z} + o\left(\frac{1}{|z|}\right) \quad \text{at } \infty.$$

Let

$$a(t) = \lim_{z\to\infty} z(g_t^0(z) - z) \left(= \mathrm{Cap}^{\mathbb{H}}(F_t)\right).$$

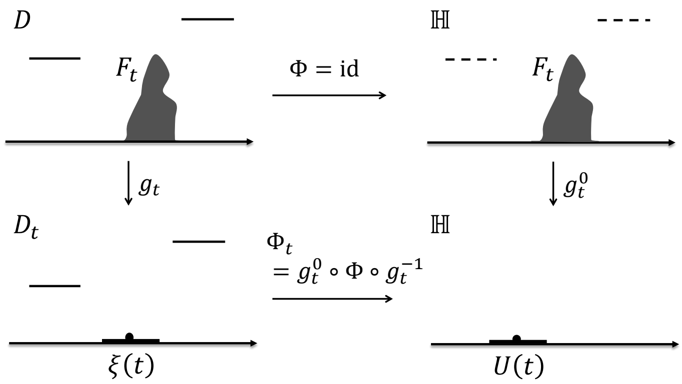

Figure 18: We think of F_t to lie in $\mathbb{H}$ by acting $\Phi = \mathrm{id}|_D : D \hookrightarrow \mathbb{H}$.

Lemma 6.1.

$$\frac{d^+ a(t)}{dt} = 2\Phi_t'(\xi(t))^2$$

Theorem 6.2 ([12, Proposition 2.4])**.** *$g_t^0(z)$ is jointly continuous in (t, z).*

Assume that Theorem 6.2 holds true. If we further show that

Lemma 6.3. *$\Phi_t(z)$, $\Phi_t'(z)$, Φ_t'' are jointly continuous in (t, z),*

then we obtain

Theorem 6.4 ([12, Theorem 2.6]).

$$\frac{dg_t^0(z)}{dt} = \frac{2\Phi_t'(\xi(t))^2}{g_t^0(z) - U(t)},$$

or by time-change

$$\frac{dg_t^0(z)}{da(t)} = \frac{1}{g_t^0(z) - U(t)}.$$

By Itô's formula, $U(t)$ enjoys

$$\begin{aligned} dU(t) &= d(\Phi_t(\xi(t)) \\ &= \Phi_t'(\xi(t))\alpha(\mathbf{s}(t) - \xi(t))\, dB_t + \Phi_t'(\xi(t))(b + b_{\mathrm{BMD}})(\mathbf{s}(t) - \xi(t))\, dt \\ &\quad + \Phi_t''(-3 + \frac{1}{2}\alpha(\mathbf{s}(t) - \xi(t))^2)\, dt. \end{aligned}$$

In particular, for $\mathrm{SKLE}_{\sqrt{6},-b_{\mathrm{BMD}}}$ (i.e., when $\alpha = \sqrt{6}$ and $b = -b_{\mathrm{BMD}}$) we have

$$dU(t) = \Phi_t'(\xi(t))\sqrt{6}\, dB_t.$$

In this way, we get the following two theorems:

Theorem 6.5. $\mathrm{SKLE}_{\sqrt{6},-b_{\mathrm{BMD}}}$ *after reparametrization has the same distribution as* SLE_6 *up to a random time.*

Theorem 6.6. *For* $\mathrm{SKLE}_{\sqrt{\kappa},b}$ *after reparametrization, its Girsanov transform has the same distribution as* SLE_κ *on* $[0, \sigma_n)$*, where* σ_n *is an increasing sequence of random times with* $\sigma_n \nearrow \zeta$*. Here,* ζ *is the lifetime of the process* $(\xi(t), \mathbf{s}(t))$*.*

By Theorem 6.6, $\mathrm{SKLE}_{\sqrt{\kappa},b}$ inherits some geometric properties from SLE_κ. For example, the following is shown in Rohde and Schramm [25, Lemma 7.3]: Let $\kappa \in (4, 8)$ (in fact, $\kappa = 8$ as well) and $t > 0$. Then with probability one, there exists $\varepsilon > 0$ such that $F_t \supset \{\, z \in \mathbb{H}\, ;\, |z| < \varepsilon \,\}$.

We sketch the proof of Theorem 6.2 (the joint continuity of $g_t^0(z)$), which we have skipped above. Firstly, we recall the probabilistic representation

$$\begin{aligned} \Im g_t(z) &= v_t^*(z) \\ &= v_t(z) + \sum_{j=1}^{N} \mathbb{P}_z^{\mathbb{H}} \left(\sigma_K < \sigma_{F_t}\, ;\, Z_{\sigma_K}^{\mathbb{H}} \in C_j \right) v_t^*(c_j^*), \\ v_t(z) &= \Im z - \mathbb{E}_z \left[\Im Z_{\sigma_{K \cup F_t}}^{\mathbb{H}}\, ;\, \sigma_{K \cup F_t} < \infty \right], \end{aligned}$$

from Section 2.4.

Fact.

$$\sup_{0 \le t \le a} v_t^*(c_j^*) < \infty \quad \textit{for each } a > 0.$$

$$v_t^*(c_j^*) > 0 \quad \textit{for every } t > 0.$$

By this fact, there exists a sequence $t_n \nearrow t$ such that $v_{t_n}^*(c_j^*) \to a_j \ge 0$. We claim that $a_j = v_t^*(c_j^*)$. Indeed,

$$\begin{aligned} v_t^*(z) &= \lim_{n\to\infty} v_t^*(z) \\ &= \Im z - \mathbb{E}_z \left[\Im Z^{\mathbb{H}}_{\sigma_{K \cup F_{t-}}} \; ; \sigma_{K \cup F_{t-}} < \infty \right] \\ &\quad + \sum_{j=1}^{N} \mathbb{P}^{\mathbb{H}}_z \left(\sigma_K < \sigma_{F_{t-}}, Z^{\mathbb{H}}_{\sigma_K} \in C_j \right) a_j. \end{aligned} \tag{6.1}$$

Here, we have used the convergence $\sigma_{F_{t_n}} \searrow \sigma_{F_{t-}}$, which follows from $F_{t_n} \nearrow F_{t-}$ and (I.10.4) of Blumenthal–Getoor [4]. Taking $z \to C_j$ in (6.1), actually we get $a_j = v_t^*(c_j^*)$.

Now, we have

$$\begin{aligned} 0 &\ge \mathbb{E}^{\mathbb{H}}_z \left[\Im Z^{\mathbb{H}}_{\sigma_{F_{t-} \cup K}} \; ; \sigma_{F_{t-} \cup K} < \infty \right] - \mathbb{E}^{\mathbb{H}}_z \left[\Im Z^{\mathbb{H}}_{\sigma_{F_t \cup K}} \; ; \sigma_{F_t \cup K} < \infty \right] \\ &= \sum_{j=1}^{N} \underbrace{v_t^*(c_j^*)}_{>0} \underbrace{\left(\mathbb{P}^{\mathbb{H}}_z \left(\sigma_K < \sigma_{F_{t-}}, Z^{\mathbb{H}}_{\sigma_K} \in C_j \right) - \mathbb{P}^{\mathbb{H}}_z \left(\sigma_K < \sigma_{F_t}, Z^{\mathbb{H}}_{\sigma_K} \in C_j \right) \right)}_{=0} \ge 0. \end{aligned}$$

Hence

$$\begin{gathered} \mathbb{P}^{\mathbb{H}}_z(\sigma_K < \sigma_{F_{t-}}) = \mathbb{P}^{\mathbb{H}}_z(\sigma_K < \sigma_{F_t}), \\ \mathbb{E}^{\mathbb{H}}_z \left[\Im Z^{\mathbb{H}}_{\sigma_{F_{t-}}} \; ; \sigma_{F_{t-} \cup K} < \infty \right] = \mathbb{E}^{\mathbb{H}}_z \left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}} \; ; \sigma_{F_t \cup K} < \infty \right]. \end{gathered} \tag{6.2}$$

Using the identity $\Im g_t^0(z) = \Im z - \mathbb{E}^{\mathbb{H}}_z[\Im Z^{\mathbb{H}}_{\sigma_{F_t}} \; ; \sigma_{F_t} < \infty]$, one can show that $t \mapsto \Im g_t^0(z)$ is continuous for every z by decomposing $\mathbb{E}^{\mathbb{H}}_z[\Im Z^{\mathbb{H}}_{\sigma_{F_t}} \; ; \sigma_{F_t} < \infty]$ with an appropriate sequence of stopping times and applying (6.2). (See the last two paragraphs in the proof of [12, Lemma 3.1].) Since $\Im g_t^0(z)$ is harmonic, it is jointly continuous in (t, z). Hence the joint continuity of $g_t^0(z)$ follows from the Cauchy–Riemann relation.

In fact, it is an important problem to replace the right derivative by the genuine derivative. For example, the right derivative version of the Loewner equation in $\mathbb{H}$

$$\begin{cases} \dfrac{\partial^+ g_t(z)}{\partial t} = \dfrac{2}{g_t(z) - \xi(t)} \\ g_0(z) = z \end{cases} \tag{6.3}$$

does NOT uniquely characterize $g_t(z)$.

Why? For $z \in \mathbb{H}$, let t_z be the lifetime of the unique solution $\bar{g}_t(z)$ in the true Loewner equation. Let $\varepsilon < t_z$, and define $g_t(z) = \bar{g}_t(z)$ for $t < \varepsilon$. Take any $z_\varepsilon \in \mathbb{H}$, and define $g_t(z) = \bar{g}_{t-\varepsilon}(z_\varepsilon)$ for $t \in [\varepsilon, \varepsilon + \delta)$. Then $g_t(z)$ also solves (6.3). $\square$

6.2 Komatu–Loewner equation and SKLE in other canonical domains

Up to Section 6.1, we have discussed the Komatu–Loewner equation and SKLE in standard slit domains $\mathbb{H} \setminus \mathbf{s}$. In this subsection, we give a brief overview of these equations in other canonical domains.

(1) Annulus. Let $\mathbb{A}_r = B(0,1) \setminus \overline{B}(0,r)$. Komatu (1943) [16] derived the Komatu–Loewner equation in $\mathbb{A}_r$ in the left derivative sense using Weierstrass functions and Jacobi elliptic functions. Fukushima–Kaneko (2014) [15], the follow up of Chen–Fukushima–Rohde [11], then established the same equation in the true derivative sense in terms of the Villat kernel. Moreover, they show that the Villat kernel is indeed the BMD Schwarz kernel.

Komatu and Fukushima–Kaneko treated the conformal mappings on annuli directly, but we can reduce it to the problem on the infinite strip by Cayley transform

$$\mathbb{D} \to \mathbb{H}, \ z \mapsto i\frac{1+z}{1-z}.$$

In fact, Zhan (2004) [30] considered the annulus SLE by combining such a reduction with the modified Villat kernel, which is the same as the Villat kernel in [15] up to rotation. Also, in a similar way, Lawler (2011) [20] defined SLE in multiply connected domains using Brownian loop measure.

(2) Circularly slit annulus. In a similar way to the annulus case, Komatu (1950) [17], Bauer–Friedrich (2008) [3] and Fukushima–Kaneko (2014) [15] worked with the Komatu–Loewner equation on circularly slit annuli, but all of them only obtain the left derivative.

(3) Circularly slit disk. In this case, Bauer–Friedrich (2006) [2] derived the form of the radial Komatu–Loewner equation and discussed the locality. They conjectured the drift coefficient corresponding to the ansatz having locality for $\kappa = 6$ is given by $-b_{\mathrm{BMD}}$ as in these lectures. Moreover, Böhm–Lauf (2014) [5] established the true Komatu–Loewner equation not only for a single simple curve but also for multiple non-intersecting simple curves. See their paper for the detail.

References

[1] L. V. Ahlfors, *Complex Analysis. An introduction to the theory of analytic functions of one complex variable*, 3rd ed., International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., 1979.

[2] R. O. Bauer and R. M. Friedrich, On radial stochastic Loewner evolution in multiply connected domains, J. Funct. Anal. **237** (2006), 565–588.

[3] R. O. Bauer and R. M. Friedrich, On chordal and bilateral SLE in multiply connected domains, Math. Z. **258** (2008), 241–265.

[4] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory*, Dover, 2007, republication of 1968 edition, Academic Press.

[5] C. Böhm and W. Lauf, A Komatu–Loewner equation for multiple slits, Comput. Methods Funct. Theory **14** (2014), 639–663.

[6] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen and S. Smirnov, Convergence of Ising interfaces to Schramm's SLE curves, C. R. Math. Sci. Paris. **352** (2014), 157–161.

[7] Z.-Q. Chen, *Brownian Motion with Darning*, Lecture notes for the talks given at RIMS, Kyoto University in 2012.

`https://sites.math.washington.edu/~zchen/BMD_lecture.pdf`

[8] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Changes, and Boundary Theory*, Princeton University Press, 2012.

[9] Z.-Q. Chen and M. Fukushima, One-point reflection, Stochastic Process. Appl. **125** (2015), 1368–1393.

[10] Z.-Q. Chen and M. Fukushima, Stochastic Komatu–Loewner evolutions and BMD domain constant, Stochastic Process. Appl. **128** (2018), 545–594.

[11] Z.-Q. Chen, M. Fukushima and S. Rohde, Chordal Komatu–Loewner equation and Brownian motion with darning in multiply connected domains, Trans. Amer. Math. Soc. **368** (2016), 4065–4114.

[12] Z.-Q. Chen, M. Fukushima and H. Suzuki, Stochastic Komatu–Loewner evolutions and SLEs, Stochastic Process. Appl. **127** (2017), 2068–2087.

[13] J. B. Conway, *Functions of One Complex Variable II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995.

[14] S. Drenning, Excursion reflected Brownian motion and Loewner equations in multiply connected domains, Thesis (Ph.D.), The University of Chicago, 2011.

[15] M. Fukushima and H. Kaneko, On Villat's kernels and BMD Schwarz kernels in Komatu–Loewner equations, in: *Stochastic Analysis and Applications 2014*, 327–348, Springer Proc. Math. Stat. **100**, Springer, Cham, 2014.

[16] Y. Komatu, Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten, Proc. Phys. Math. Soc. Japan (3) **25** (1943), 1–42.

[17] Y. Komatu, On conformal slit mapping of multiply-connected domains, Proc. Japan Acad. **26** (1950), 26–31.

[18] G. F. Lawler, *Conformally Invariant Processes in the Plane*, Mathematical Surveys and Monographs **114**, American Mathematical Society, Providence, RI, 2005.

[19] G. F. Lawler, The Laplacian-b random walk and the Schramm–Loewner evolution, Illinois J. Math. **50** (2006), 701–746.

[20] G. F. Lawler, Defining SLE in multiply connected domains with the Brownian loop measure, arXiv:1108.4364v1, 2011.

[21] G. F. Lawler, O. Schramm and W. Werner, Conformal invariance of planar loop-erased random walks and uniform spanning trees, Ann. Probab., **32** (2004), 939–995.

[22] K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I, Mathematiche Annalen, **89** (1923), 103–121.

[23] T. Murayama, Chordal Komatu–Loewner equation for a family of continuously growing hulls, Stochastic Process. Appl. **129** (2019), 2968–2990.

[24] S. C. Port and C. J. Stone, *Brownian Motion and Classical Potential Theory*, Probability and Mathematical Statistics, Academic Press, New-York–London, 1978.

[25] S. Rohde and O. Schramm, Basic properties of SLE, Ann. of Math. **161** (2005), 883–924.

[26] D. Revuz and M. Yor, *Continuous Martingales and Brownian motion*, 3rd ed., Grundlehren der mathematischen Wissenschaften **293**, Springer-Verlag, Berlin, 1999.

[27] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. **118** (2000), 221–288.

[28] S. Smirnov, Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits, C. R. Acad. Sci. Paris, Sér. I Math. **333** (2001), 239–244.

[29] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co., Ltd., Tokyo, 1959.

[30] D. Zhan, Stochastic Loewner evolution in doubly connected domains, Probab. Theory Related Fields **129** (2004), 340–380.