Identification from the SFCs of regularly Ogawa integrable random functions

KiyoiKi Hoshino (Osaka prefecture university)*

1. Introduction

It has been discussed the question whether and how a random function is identified from its stochastic Fourier coefficients (SFCs in abbr.). In particular, there are previous works in [3], [1] and [4] in the case that the SFC is given by the Ogawa integral. In [3] and [1], reconstructions from the SFCs of random functions of bounded variation (noncausal finite variation processes) are given by using the law of iterated logarithm for Brownian motion, and in [4], those of certain complex-valued random functions are given by using the cross variation. In this talk, using the cross variation, we present reconstructions from the SFCs of random functions which are Ogawa integrable with respect to regular CONS.

2. Setting

Let \((B_t)_{t \in [0,1]}\) be a Brownian motion on a filtrated probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, P)\) and \((e_n)_{n \in \mathbb{N}}\) a CONS of \(L^2([0,1]; \mathbb{C})\) such that real and imaginary parts of each \(e_n\) are of bounded variation. We say a CONS \((\varphi_m)_{m \in \mathbb{N}}\) of \(L^2([0,1]; \mathbb{R})\) is regular if

\[
\sup_{M \in \mathbb{N}} \left| \sum_{m=1}^{M} \varphi_m \int_{0}^{1} \varphi_m(s) ds \right|_{L^2[0,1]} < \infty.
\]

We denote by \(\int_{0}^{1} d_s B\) the regular Ogawa integral, i.e., Ogawa integral with respect to regular CONS of \(L^2([0,1]; \mathbb{R})\). By \(L^r_1\) we denote the Sobolev space of square integrable \(i\)-parameter Wiener functionals with differentiability index \(r\) with respect to the \(H\)-derivative \(D\). The symbol \(\int_{0}^{1} \delta B\) and \(\int_{0}^{1} d_0 B\) denote the Skorokhod integral \(D^*\) and the Itô integral, respectively (see also Definitions 2.1, 2.3 and 2.4 of [2]). We denote by \(\langle X, Y \rangle_t\) the cross variation (if it exists) at \(t\) of \(X, Y : [0,1] \rightarrow L^0(\Omega)\) and by \([X]_t\) the quadratic variation \(\langle X, X \rangle_t\) (if it exists) at \(t\) of \(X : [0,1] \rightarrow L^0(\Omega)\).

We define the following for \(a \in L^0([0,1] \times \Omega), b \in L^0(\Omega; L^2[0,1])\).

**Definition 1 (SFC-O of stochastic differential)** Suppose \(e_n a\) is regularly Ogawa integrable for every \(n \in \mathbb{N}\). We define the SFC of Ogawa type (SFC-O) \((e_n, d_s Y)\) of the stochastic differential \(d_s Y_t = a(t) d_s B_t + b(t) dt, t \in [0,1]\) with respect to \((e_n)_{n \in \mathbb{N}}\) by \((e_n, d_s Y) := \int_{0}^{1} \hat{e}_n(t) d_s Y_t = \int_{0}^{1} \hat{e}_n(t) a(t) d_s B_t + \int_{0}^{1} \hat{e}_n(t) b(t) dt\). In particular, in the case of \(b = 0\), \((e_n, d_s Y) = (e_n, a d_s B)\) is also called the SFC-O of \(a(t)\).

3. Reconstructions of random functions from SFC-Os

First, as subspaces of \(L^0([0,1] \times \Omega)\), define

\[
\mathcal{A} = \{ A \in L^0([0,1] \times \Omega) \mid \text{Re } A, \text{Im } A \text{ are of bounded variation a.s.} \}.
\]

*e-mail: su301032@edu.osakafu-u.ac.jp*

We note that \( a \in \mathcal{L} \) is a regularly Ogawa integrable random function.

**Proposition 1** Suppose \( a \in \mathcal{L} \), then i.i.p. \( \int_0^1 v_n a \, d\varphi B = 0 \) for every point sequence \((v_n)_{n \in \mathbb{N}}\) of \( BV[0,1] \) which converges to 0 in \( L^2[0,1] \). In particular,

\[
\mathcal{P}((e_n, d_s Y))_{n \in \mathbb{N}}(t) := \text{i.i.p.} \sum_{n=1}^{\infty} \int_0^t e_n(s) \, ds \, (e_n, d_s Y) = Y_t, \quad \forall t \in [0,1],
\]

where \( d_s Y_t = a(t) \, d_s B_t + b(t) \, dt, \; t \in [0,1] \).

Next, as the family of regularly Ogawa integrable random functions identified from the SFC-Os by \( \mathcal{P} \) and the cross variation, set

\[
\mathcal{L}_{PC}^{c,*} = \left\{ a \in L^0([0,1] \times \Omega) \mid \mathcal{P}((e_n, a \, d_s B))_{n \in \mathbb{N}} = \int_0^1 a \, d_s B, \right. \\
\forall s, t \in [0,1] \left[ \int_0^t a \, d_s B \right]_t = \int_0^t |a(u)|^2 \, du, \; \left( \int_0^1 a \, d_s B, \; B_{\cdot s} \right)_t = \int_0^t a(u) \, du \bigg\}.
\]

**Proposition 2** \( \mathcal{A}, \mathcal{M}, \mathcal{W} \subset \mathcal{L}_{PC}^{c,*} \).

**Theorem 1** \( \mathcal{L}_{PC}^{c,*} \) becomes a vector space.

**Corollary 1** Assume \( a(t) \) satisfies Re \( a \), Im \( a \) \( \in \mathcal{L}_{PC}^{c,*} \). Then, letting \( d_s Y_t = a(t) \, d_s B_t + b(t) \, dt, \; t \in [0,1] \), the following hold:

1. \( a(t) \) is identified with \((B_t)_{t \in [0,1]}\) from \((e_n, d_s Y)_{n \in \mathbb{N}}\) by \( a(t) = \frac{d}{dt} \langle \mathcal{P}((e_n, d_s Y)_{n \in \mathbb{N}}), B_t \rangle_t \).
2. \( |\text{Re } a|, \; |\text{Im } a|, \; \text{Re } a \text{ Im } a, \; (\text{sgn } a) a^t \) are identified without \((B_t)_{t \in [0,1]}\) from \((e_n, d_s Y)_{n \in \mathbb{N}}\) by \( Y_t = \mathcal{P}((e_n, d_s Y)_{n \in \mathbb{N}}) \) and the property \( \int_0^t f(s) g(s) \, ds = \langle \int_0^t f \, d_s B, \int_0^t g \, d_s B \rangle_t \) for any \( f, g \in \mathcal{L}_{PC}^{c,*} \).

**Remark 1** These random functions can be identified, even if \((e_n, d_s Y)_{n \in \mathbb{N}}\) lacks its finite elements \((e_n, d_s Y)\).

**Remark 2** The drift term \( b \) is identified from the SFCs as \( a(t) \) is identified.

**Remark 3** If \( a \in \mathcal{L} \), then \( a(t) \) satisfies the assumption of Corollary 1.

**References**


\[\text{sgn } z \text{ equals } 1 \text{ if } 0 \leq \arg z < \pi \text{ and } -1 \text{ if not, } \text{arg } 0 := 0 \text{ for } z \in \mathbb{C}.\]